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THE EIGENVALUE PROBLEM FOR SUBLINEAR OPERATORS
WITH UNIFORMLY SYMMETRIZABLE DERIVATIVES*

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MASTER

1. Introduction.

In previous papers, the author has derived some of the spectral properties of the nonlinear eigenvalue problem in several concrete cases. The first two of these [refs. 11, 12] dealt with the following problem defined for $0 \leq s \leq 1$:

$$(1) \quad \left(p(s) u_s \right)_s + \lambda^{-1} f(s, u(s)) = 0,$$

$$u(0) - ap(0)u_s(0) = 0, \quad u(1) + bp(1)u_s(1) = 0,$$

where $a, b > 0$, $p(s) > 0$ is differentiable, and $f(s, u)$ is continuous in the strip $0 \leq s \leq 1$, $-\infty < u < +\infty$. Moreover $f(s, u)$ was assumed to satisfy conditions, designated H-1, which made the problem "sublinear" in the first paper, and "superlinear" in the second. Additional conditions, designated H-2 and H-3 defined the assumptions as to asymptotic behavior. It was mentioned that problem (1) is equivalent to a Hammerstein equation:

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$$(2) \quad \lambda u(s) = \int_0^1 G(s,t)f(t,u(t))dt,$$

(with care taken that the operator is defined in the given space), where $G(s,t)$ is the Green's function for the linear differential operator involved in (1).

The next paper dealt only with the sublinear case, [ref. 13], and was an attempt to develop the similar spectral properties expected to exist for Hammerstein equations

$$(3) \quad \lambda x(s) = \int_0^1 K(s,t)f(t,x(t))dt$$

with kernels $K(s,t)$ more general than that appearing in (2). Namely, $K(s,t)$ was assumed to be an "oscillation kernel" [ref. 3, pp. 236-254], a class of kernels which includes Green's functions for Sturm Liouville operators of positive definite type.

The assumption in (3) that $K(s,t)$ be an oscillation kernel is drastic. Though the class of oscillation kernels presumably contains kernels which are not Green's functions, Gantmacher and Krein [ref. 3, pp. 312-325] are not able to give examples. The third paper of the author does give an independent approach to equation (3), though the results are recognized not to be significantly more general than those for (2), and certainly not as complete.

In the present paper the author generalizes the results obtained for (3) in two ways: (a) the theory is presented in a Hilbert Space

setting with an abstract definition of what is meant by a symmetrizable odd nonlinear operator defined in the space with appropriately generalized sublinearity and asymptotic linearity assumptions; (b) it is shown how branches of solutions can be developed without the assumption of an "oscillation operator," which would be the abstract notion corresponding to an integral operator in $L_2(0,1)$ with oscillation kernel. Thus the present paper could be applied to the Hammerstein operator which would result from the following boundary value problem:

$$\text{D.E.} \quad \nabla \cdot p(s,t) \nabla u + \lambda^{-1} f(s,t,u(s,t)) = 0$$

$$\text{B.C.} \quad \frac{du}{dn} + pu = 0 \quad \text{on a closed continuous boundary.}$$

This is the generalization of problem (1) to partial differential equations.

The author wishes to thank Prof. J. Berkowitz of New York University and Prof. I. Kolodner of the University of New Mexico for suggesting the value of an abstract formulation of the nonlinear eigenvalue problem. He also wishes again to acknowledge Prof. Kolodner's work on the dynamics of a rotating heavy string as a source of inspiration in connection with these problems [ref. 7].

In addition to Kolodner's work on the heavy string, H. B. Keller, J. B. Keller and E. L. Reiss [ref. 6] have a nonlinear eigenvalue problem to deal with in their work on buckled plates. M. A. Krasnoselskii in a survey paper [ref. 8] makes many allusions to instances of nonlinear eigenvalue problems in applied work.

2. Spaces and Operators.

Let H be a real separable Hilbert space, and let $T(x)$ be an operator (generally nonlinear) defined for every element $x \in H$ and mapping H into itself. We consider $T(x)$ to be odd: $T(-x) = -T(x), x \in H$. In particular $T(\theta) = \theta$, where $\theta \in H$ is the null element. Moreover we assume $T(x)$ to be bounded, continuous, and compact [ref. 8, p. 350].

$T(x)$ is assumed to be three times Fréchet differentiable [ref. 9, p. 188], which implies for each $x, h \in H$ that we have the Taylor expansion

$$(4) \quad \begin{aligned} T(x+h) - T(x) &= dT(x;h) + \frac{1}{2!} d^2T(x;h,h) \\ &\quad + \frac{1}{3!} d^3T(x;h,h,h) + R_3(x,h) \end{aligned}$$

where $R_3(x,h) = o(\|h\|^3)$. Here $dT(x;h)$, $d^2T(x;h,h)$ and $d^3T(x;h,h,h)$ are the first, second and third differentials. For given $x \in H$ there is assumed to be a sphere $\|h\| \leq \rho$ for which $R_3(\theta,h)$ is differentiable with respect to h .

We write $dT(x;h) = T'(x)h$, where $T'(x)$ is the first derivative of $T(x)$. It is known [ref. 8, p. 360] that the first derivative of a compact continuous operator $T(x)$ is, for given $x \in H$, a compact linear transformation of H into itself.

Let $[H_1 \rightarrow H_2]$ denote the class of all bounded linear operators mapping a Banach space H_1 into a Banach space H_2 . Then $T'(x)$ maps H into $[H \rightarrow H]$. By using the weak form of the differential [ref. 9, p. 184], it can be seen that this mapping is even: $T'(-x) = T'(x)$.

We assume that $T'(x)$, for any given $x \in H$, is fully symmetrizable uniformly [ref. 15, p. 371]. In fact we shall use the stronger assumption that there exists a positive definite bounded linear transformation P such that for given $x \in H$, $PT'(x)$ is a symmetric bounded linear transformation. In the present paper, $PT'(x)$ will be assumed positive definite.

The second differential $d^2T(x; h_1, h_2)$ is a bilinear function of the variables $h_1, h_2 \in H$ for given $x \in H$. By successive differentiation we have: $d^2T(x; h_1, h_2) = dT'(x; h_1)h_2 = T''(x)h_1h_2$, where the multiplication indicated is linear operator multiplication. Here, $dT'(x; h)$ belongs to $[H \rightarrow H]$ for given $x, h \in H$, while for given $x \in H$, $T''(x)$ belongs to $[H \rightarrow (H \rightarrow H)]$, and is called the second derivative.

By using the weak form of the differential of $T'(x)$, which exists whenever the Fréchet differential exists [ref. 9, p. 185], we can ascertain that $T''(x)$, as a mapping of H into $[H \rightarrow (H \rightarrow H)]$, is odd: $T''(-x) = -T''(x)$. Thus in particular $d^2T(\theta; h_1, h_2) = T''(\theta)h_1h_2 = \theta$.

The nonlinear operator $T(x)$ is said to be sublinear provided $PT''(x)x = PdT'(x; x)$ is a negative definite linear transformation of H into itself, for given $x \in H$, $\|x\| \neq 0$.

The third differential $d^3T(x; h_1, h_2, h_3)$ is trilinear in the variables $h_1, h_2, h_3 \in H$. It also can be computed by successive differentiation: $d^3T(x; h_1, h_2, h_3) = d^2T'(x; h_1, h_2)h_3 = dT''(x; h_1)h_2h_3 = T'''(x)h_1h_2h_3$ where the multiplication is again linear operator multiplication. Here,

$d^2T'(x;h_1,h_2)$ belongs to $[H \rightarrow H]$ for given $x, h_1, h_2 \in H$; $dT''(x;h_1)$ is a member of the class $[H \rightarrow (H \rightarrow H)]$ for given $x, h_1 \in H$; $T'''(x)$, the third derivative, belongs to $[H \rightarrow (H \rightarrow \{H \rightarrow H\})]$ for given $x \in H$, [ref. 9, p. 189].

Let $\alpha > 0$ be any number. For given $x \in H$ we have

$$dT'(\alpha x; x) = T''(\alpha x)x = dT''(\theta; \alpha x)x + R(\theta, \alpha x)x$$

where $R(\theta, \alpha x) = o(\alpha \|x\|)$. Since $(PdT'(\alpha x; x)h, h) < 0, \|h\| \neq 0$, by the sublinearity property, we see by taking α small that $(PdT''(\theta; x)h, h) \leq 0$. We assume now that this form is actually negative definite:

$$(PdT''(\theta; x)h, h) = (Pd^2T'(\theta; x, x)h, h) < 0, \quad \|x\| \neq 0, \quad \|h\| \neq 0.$$

It is necessary for $dT(x; h)$ to have some continuity properties. Namely, we assume $dT(x; h)$ to be continuous with respect to x in the norm $\|\cdot\|$ of H , uniformly over the unit sphere $\|h\| = 1$. In other words $T'(x)$ is continuous in the topology of $[H \rightarrow H]$, i.e. the usual norm $\|T'(x)\| = \sup_{\|h\|=1} \|T'(x)h\|$. We suppose that $\|T'(x)\| \leq M_1$, a constant.

Further we assume there exists a bounded linear operator B such that PB is symmetric and positive definite, and such that

$$\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \|T'(x) - B\| = 0.$$

Using the fact that $T(x) = \int_0^1 T'(sx)dsx$ (from the integral form of the remainder in Taylor's series [ref. 9, p. 187, eq. (9)]) we have

$$\|T(x)-Bx\| = \left\| \int_0^1 T'(sx)dsx-Bx \right\| \leq \int_0^1 \|T'(sx)-B\| ds \|x\|, \text{ whence}$$

$$\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \frac{\|T(x)-Bx\|}{\rho} = 0. \text{ This is the usual notion of asymptotic}$$

linearity. It is known that such an operator B is compact [ref. 8, p. 361].

We summarize our assumptions in statements intended to be analogous to those of the author's previous papers:

H-1: $T(x)$ is odd, $T(\theta) = \theta$; derivatives $T'(x)$, $T''(x)$ and $T'''(x)$ exist with $PT'(x)$ positive definite, $PT''(x)x$ negative definite, $PT'''(\theta)xx$ negative definite, all for given $x \in H$. $T'(x)$ is continuous in terms of the bounded linear operator topology. Not independently, we have $\|T'(x)\| \leq M_1$.

H-3: There exists a bounded linear operator B such that PB is positive definite, and

$$\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \|T'(x)-B\| = 0.$$

If we were to write a statement H-2 corresponding to statement H-2 in earlier work, we should merely set B equal to the null operator in H-3. In this paper we are unable to treat this case successfully. Indeed, in the sequel it will be necessary to put temporary restrictions on B.

The type of symmetrizability introduced above for the derivatives $T'(x)$ is the usual left symmetrizability. Hammerstein operators (see eq. (3)), however, have right-symmetrizable derivatives [ref. 11, 12, 13].

In previous work the author has assumed $f(t, x)$ to be monotone increasing in x . One can construct an equivalent eigenvalue problem involving the operator

$$G(y) = f(s, \int_0^1 K(s, t)y(t)dt),$$

with left-symmetrizable derivative by putting $y(s) = f(s, x(s))$. Indeed, making this substitution in eq. (3), we get

$$y(s) = f(s, \lambda^{-1} \int_0^1 K(s, t)y(t)dt)$$

with the parameter λ now appearing nonlinearly. For given fixed $\lambda > 0$, the problem

$$\beta y(s) = f(s, \lambda^{-1} \int_0^1 K(s, t)y(t)dt)$$

is a left-symmetrizable problem as contained herein. By adjusting λ , with $\beta = 1$, the branches of solutions with λ as parameter can be studied. In this sense, the present paper embraces sublinear operators of Hammerstein type.

3. Eigenvalue Problems, and the Extension of Solutions.

We are interested in solutions of the problem

$$(5) \quad T(x) = \lambda x$$

with a sublinear operator $T(x)$ as described in section 2, and with a real parameter λ . This is an eigenvalue problem in the sense that we seek those values of λ such that eq. (5) has a nontrivial solution $x \neq \theta$. Such a solution is an eigenelement corresponding to λ . Often it is convenient to speak of the solution pair (x, λ) .

The linear problem

$$(6) \quad T'(x)h = \mu h$$

defined for any $x \in H$, has a sequence $\{h_i^{(x)}\}$ of eigenelements $h_i^{(x)} \in H$. The corresponding sequence of eigenvalues, $\{\mu_i^{(x)}\}$, tending to the origin of the spectral λ -axis (we assume nondegeneracy) is real, positive, and consists of eigenvalues of index one only, [ref. 15, p. 383, Th. 3].

Problem (5) has the trivial solution for all real values of λ , since $T(\theta) = \theta$. There exist ranges of λ where this is the only solution.

Theorem 1: For $\lambda \geq \mu^{(\theta)}$ and for $\lambda < 0$, problem (5) has only the trivial solution $x = \theta$. Here $\mu_0^{(\theta)}$ is the highest eigenvalue of $T'(\theta)$.

Proof: $T'(x)$ is symmetric in the scalar product $(P \cdot, \cdot)$. Let

$\|\cdot\|_P = \sqrt{(P \cdot, \cdot)}$ and $\|\cdot\|_P = \sup_{\|h\|_P=1} \|\cdot\|_P$. Consider $\lambda \geq \mu_0^{(\theta)}$, and suppose

there exists $x \neq \theta$ satisfying eq. (5). Now $T(x) = \int_0^1 T'(sx)dsx$,

[ref. 9, p. 187, eq. (9)]. Therefore, from (5) we have

$$(7) \quad \|x\|_P \leq \lambda^{-1} \int_0^1 \|T'(sx)\|_P ds \cdot \|x\|_P.$$

Since

$$T'(sx)h - T'(\theta)h = \int_0^1 dT'(tsx; sx)dth,$$

[ref. 9, p. 187, eq. (9)], where $PdT'(tsx; sx)$ is negative definite by $H-1$, we have

$$(PT'(sx)h, h) - (PT'(\theta)h, h) = \int_0^1 (PdT'(tsx; sx)h, h)dt < 0,$$

$$\|h\|_P \neq 0.$$

Therefore

$$\mu_o^{(sx)} = \|T'(sx)\|_P = \max_{h \in H} \frac{(PT'(sx)h, h)}{(Ph, h)} < \max_{h \in H} \frac{(PT'(\theta)h, h)}{(Ph, h)} = \mu_o^{(\theta)},$$

[ref. 15, p. 380, Th. 1], since the maximum is assumed. Hence from eq. (7), $\|x\|_P < \lambda^{-1} \mu_o^{(\theta)} \|x\|_P$, which is a contradiction if $\lambda \geq \mu_o^{(\theta)}$.

Hence $x = \theta$ for $\lambda \geq \mu_o^{(\theta)}$.

To prove the theorem when $\lambda < 0$, suppose again there exists $x \neq \theta$ satisfying eq. (5). Then

$$\begin{aligned}\lambda(Px, x) &= (PT(x), x) = \left(\int_0^1 PT'(sx) ds x, x \right) \\ &= \int_0^1 (PT'(sx)x, x) ds > 0,\end{aligned}$$

since $PT'(sx)$ is positive definite, $0 \leq s \leq 1$. If $\lambda < 0$ this is a contradiction. This completes the proof.

The most interest is in the important spectral range $0 \leq \lambda \leq \mu_0^{(\theta)}$. Here problem (5) is not generally expected to have a unique solution. In the following, we suppose there exists a pair (x^*, λ^*) which satisfies equation (5) and inquire into the possibility of extending the solution to a neighborhood of this pair. The following theorem is an immediate consequence of the generalized implicit function theorem, [ref. 9, p. 194]:

Theorem 2: Let the pair (x^*, λ^*) solve equation (5) in such a way that $[\lambda^* I - T'(x^*)]^{-1}$ exists. Then there exist numbers $d > 0$, $\delta > 0$ and a continuous Hilbert Space valued function $x(\lambda)$ defined on $[\lambda^* - \delta, \lambda^* + \delta]$ such that $x(\lambda^*) = x^*$. $x(\lambda)$ is the solution of eq. 5 for $\lambda^* - \delta \leq \lambda \leq \lambda^* + \delta$, unique in the sphere $\|x - x^*\| \leq d$. Furthermore $[\lambda I - T'(x(\lambda))]^{-1}$ exists on $[\lambda^* - \delta, \lambda^* + \delta]$.

A chain of such extensions is now possible, to the right or left, each extension being centered at an endpoint of the previous extension. There results a continuous sheet of solutions [ref. 4, p. 151, sec. 18] for equation (5) which can have as a finite boundary point, (x_b, λ_b) ,

only a pair such that $[\lambda_b I - T'(x_b)]^{-1}$ does not exist.

Let us now consider a pair (x^*, λ^*) satisfying eq. (5), and such that $[\lambda^* I - T'(x^*)]^{-1}$ does not exist. Thus λ^* is an eigenvalue of the linear operator $T'(x^*)$. Much seems to depend on the multiplicity of this eigenvalue. In order to make the present analysis tractable, we assume that λ^* is of multiplicity one. Hence λ^* is one of the positive numbers $\left\{ \mu_i^{(x^*)} \right\}$, a simple eigenvalue, and the corresponding null space N^* of the operator $[\lambda^* I - T'(x^*)]$ is one-dimensional.

By the symmetrizability assumption, λ^* has Riesz index unity, [ref. 14, pp. 217, 180]. Therefore the range R^* of $[\lambda^* I - T'(x^*)]$ is such that $H = N^* \oplus R^*$. Hence N^* and R^* admit projections E^* and $I - E^*$ respectively.

The "extended psuedo-inverse" of $[\lambda^* I - T'(x^*)]$ is a linear bounded transformation $M \in [H \rightarrow H]$ such that $[\lambda^* I - T'(x^*)]M = M[\lambda^* I - T'(x^*)] = I - E^*$. Thus M is an operator which, restricted to the closed subspace R^* , acts as the inverse of $[\lambda^* I - T'(x^*)]$ so restricted. A necessary and sufficient condition for M to exist is the existence of the projections E^* and $I - E^*$, [ref. 10, pp. 72, 84].

The pair (x^*, λ^*) solves eq. (5); let us try to find a nearby solution pair $(x^* + h, \lambda^* + y)$, where (h, y) satisfies the equation $T(x^* + h) = (\lambda^* + y)(x^* + h)$. Expanding the left side in Taylor's series with remainder (see eq. (4)) and performing some manipulation, there results

$$\begin{aligned} (8) \quad [\lambda^* I - T'(x^*)]h &= -yx^* - yh + \frac{1}{2!} d^2 T(x^*; h, h) + \frac{1}{3!} d^3 T(x^*; h, h, h) + R_3(x^*, h) \\ &= F_y(h) \quad . \end{aligned}$$

Using the notation $F_y(h)$ given in eq. (8), let us consider the following problem:

$$(9) \quad h = MF_y(h) + u$$

where $u = \xi h^* \in N^*$ is arbitrary. Here ξ is a scalar and h^* spans N^* . A local solution can be obtained for eq. (9). Indeed, since $dT(x;h) = T'(x)h$ is continuous in x uniformly over the unit sphere $\|h\| = 1$, a result of Bartle [ref. 1, p. 367, Th 1.1b] can be applied to the function

$$\Phi(h,y) = T(x^* + h) - (\lambda^* + y)(x^* + h)$$

to show that the mapping $F_y(h)$ satisfies the condition

$$\|F_y(h_1) - F_y(h_2)\| \leq C(h_1, h_2, y) \|h_1 - h_2\|, \quad ,$$

where $C(h_1, h_2, y) \geq 0$ goes to zero with $|y|$, $\|h_1\|$, $\|h_2\|$. It is then easy to show that there exist constants $\tau > 0$, $\tau_1 > 0$ and $\tau_2 > 0$ such that if $|y| \leq \tau$, $|\xi| \leq \tau_1$, equation (9) has the solution $x = V_y(u)$, unique in the sphere $\|x\| \leq \tau_2$. The method of contraction mappings [ref. 9, p. 27] is used, and the solution is produced as a sequence of iterates $h_0 = \xi h^* = u$, $h_{n+1} = MF_y(h_n) + u$, $n = 0, 1, 2, \dots$, convergent in norm.

In order that the solution $h = V_y(u)$ of eq. (9) be also a solution of eq. (8), it is necessary and sufficient that $F_y[V_y(\xi h^*)] \in R^*$, or what means the same thing

$$(10) \quad E^* F_y [V_y(\xi h^*)] = 0 \quad .$$

A detailed rationale for this is found in the author's previous paper [ref. 13, p. 583]. Eq. (10), often known as the "bifurcation condition", represents just one scalar equation in the scalar unknown ξ for given values of y . Solution of eq. (10) for values $\xi, |\xi| \leq \tau_1$ with given values $y, |y| \leq \tau$, yields pairs (y, ξ) which, when inserted in the solution $h = v_y(\xi h^*)$ of eq. (9), provide solutions of eq. (8) and therefore of eq. (5).

Following Bartle [ref. 1, p. 372] in a manner exactly analogous to the method of a previous paper of the author [ref. 13, p. 583-4], we express eq. (10) more concretely as

$$\begin{aligned}
 (11) \quad E^*F_y[v_y(\xi h^*)] = & -y (Ph^*, x^*) - y\xi[1 + (h^*, Pd^2T(x^*; h^*, Mx^*))] \\
 & + \frac{1}{2!} \xi^2 (h^*, Pd^2T(x^*; h^*, h^*)) + \frac{1}{3!} \xi^3 (h^*, Pd^3T(x^*; h^*, h^*, h^*)) \\
 & + \sum_{i+j=1}^3 \theta_{ij}(\xi, y) = 0 \quad |y| \leq \tau, \quad |\xi| \leq \tau_1,
 \end{aligned}$$

where $\theta_{ij}(\xi, y) = o(|\xi|^i |y|^j)$ is differentiable in ξ .

Next it is seen that eq. (11) is conformable to the class of equations treated by J. Dieudonne [ref. 2] using the Newton Polygon method. The application of the method here is almost the same as in the author's earlier paper, [ref. 13, p. 584]. Equation (11) often has multiple solutions which produce, in turn, multiple solutions of equation (5) since multiple pairs (ξ, y) are then used in eq. (9) to get solutions

of eq. (5). Hence values $\lambda^* = \mu_p^{(x^*)}$, which are simple eigenvalues of the derivative $T'(x^*)$, are often known as branch points or bifurcation points.

For $x^* = 0$ we have the important case of bifurcation at the origin. In eq. (11) we have $(Ph^*, x^*) = 0$, $(h^* Pd^2 T(0; h^*, Mx^*) = 0$ and $(h^*, Pd^2 T(0; h^*, h^*)) = 0$; the latter since $T''(x)$ is an odd operator as was noted in section 2. Also by statement H - 1 we have $(h^*, Pd^3 T(0; h^*, h^*, h^*)) = (h^*, PT'''(0)h^*h^*h^*) < 0$, $\|h^*\| \neq 0$. The simplified bifurcation equation is similar to that in the author's earlier paper [ref. 13, p. 585, eq. (23)], and all subsequent steps in proving the following analogous theorem [ref. 13, p. 587, Th. 3] are omitted:

Theorem 3: The simple eigenvalues in the eigenvalue sequence $\{\mu_p^{(\theta)}\}$ of the operator $T'(\theta)$ (i.e., eigenvalues of multiplicity one), form the sequence of branch or bifurcation points at the origin for problem (5). Let $\mu_p^{(\theta)}$ be such a point. There exist constants $\tau > 0$, $\tau_1 > 0$, and two branches $x_p^\pm(\lambda)$ of nontrivial solutions, continuous in λ in terms of the norm of H , defined on the interval $\mu_p^{(\theta)} - \tau \leq \lambda \leq \mu_p^{(\theta)}$.

We have $\|x^\pm(\lambda)\| \leq \tau_1$ on this interval, and $\lim_{\lambda \rightarrow \mu_p^{(\theta)}} \|x_p^\pm(\lambda)\| = 0$.

Because $T(x)$ is an odd operator, the two branches $x_p^\pm$ of Theorem 3 differ only in sign.

4. Branches of Solutions and their Properties.

By Theorem 3 there exist two branches, $x_p^+(\lambda)$, $x_p^-(\lambda)$, of nontrivial solutions of problem (5) emanating from a branch point $\lambda = \mu_p^{(\theta)}$, which is a simple eigenvalue for the derivative $T'(\theta)$. These branch functions are defined for $\lambda < \mu_p^{(\theta)}$ in the sublinear case, and are continuous in norm and such that $\lim_{\lambda \rightarrow \mu_p^{(\theta)}} x_p^\pm(\lambda) = \theta$. In other words, $x_p^\pm(\lambda)$ represents a pair of Hilbert space valued functions, each of which maps a certain left neighborhood of $\lambda = \mu_p^{(\theta)}$ of the spectral axis into a certain locus in H passing through $x = \theta$ and as yet confined to a neighborhood thereof.

We propose now to study these branches. Since $T(x)$ is an odd operator: $T(-x) = -T(x)$, for given λ , $-x$ is a solution of problem (5) provided x is a solution. Therefore, $x_p^- = -x_p^+$, and for many purposes we can treat $x_p^\pm = x_p$ as one function.

We introduce a linear bounded operator, defined in H for given $x \in H$, which is useful in studying branches:

$$(12) \quad S(x)h = \int_0^1 dT(\bar{s}x; h) d\bar{s} = \int_0^1 T'(\bar{s}x) d\bar{s} h.$$

Lemma 1: For given $x \in H$, $S(x)$ is completely continuous.

Proof: Let there be given a weakly convergent sequence $\{\varphi_i\}$ of elements in H [ref. 14, p. 198]. We have $\|\varphi_i\| \leq M$ for some constant M . For given s , $0 \leq s \leq 1$, $T'(sx)$ is completely continuous; therefore the sequence of images $\{T'(sx)\varphi_i\}$ converges in norm, [ref. 14, p. 206, def. 3]. Also, $\|T'(sx)\varphi_i\| \leq \|T'(sx)\| M$, $i = 0, 1, 2, \dots$, where $\|T'(sx)\|$ is a summable

function on $0 \leq s \leq 1$. By the abstract Lebesgue Bounded Convergence Theorem [ref. 5, p. 83] we see, therefore, that the sequence of images $\{S(x)\varphi_1\} = \left\{\int_0^1 T'(sx) d\varphi_1\right\}$ also converges in norm, and this proves the lemma.

It is clear that $S(x)$ itself is symmetrizable. Indeed, for $f, g \in H$ we have

$$(PS(x)f, g) = \int_0^1 (PT'(\bar{s}x)f, g) d\bar{s} = \int_0^1 (f, PT'(\bar{s}x)g) d\bar{s} = (f, PS(x)g) \quad .$$

Following the usual practice, all eigenvalue sequences are enumerated counting multiplicities. We call $x_p(\lambda)$, emanating from the simple eigenvalue $\mu_p^{(\theta)}$, the p 'th branch.

Theorem 4: The p 'th branch function $x_p(\lambda)$, for given $\lambda < \mu_p^{(\theta)}$, is identical with the p 'th eigenelement of the operator $S(x_p(\lambda))$ to within a multiplicative constant.

Proof: By a known property [ref. 9, p. 187, eq. (9)], we have

$$T(x) = \int_0^1 T'(\bar{s}x) d\bar{s} x = S(x)x \quad .$$

By Lemma 1, $S(x)$ is completely continuous and symmetrizable. Hence $S(x)$ has a sequence $\{\delta_q^{(x)}\}$ of eigenvalues which are positive and are enumerated, counting multiplicity, in decreasing order. Let the corresponding sequence of P -orthogonal elements [ref. 15, pp. 379-385] be $\{W_q\}$.

Consider $S(x_p)$ with x_p a branch element. As $\lambda \rightarrow \mu_p^{(\theta)}$ we have $x_p(\lambda) \rightarrow \theta$ in norm, by Theorem 3. Therefore $S(x_p) = \int_0^1 T'(\bar{s}x_p) d\bar{s} \rightarrow S(\theta) = T'(\theta)$ in the uniform topology as $\lambda \rightarrow \mu_p^{(\theta)}$. Then $PS(x_p) = PS(\theta) + P[S(x_p) - S(\theta)]$ with $\|S(x_p) - S(\theta)\| \rightarrow 0$ as $\lambda \rightarrow \mu_p^{(\theta)}$. By a known result

[ref. 14, pp. 238, 239] we have $\delta_q(\lambda) \rightarrow \mu_q^{(\theta)}$ as $\lambda \rightarrow \mu_p^{(\theta)}$.

We already know a solution pair (W, δ) for the eigenvalue problem $S(x_p)W = \delta W$, namely $\delta = \lambda$, $W = x_p$. This is because as already noted, $S(x_p)x_p = T(x_p) = \lambda x_p$; this is problem (5) for which (x_p, λ) is a solution pair. Hence λ is among the eigenvalues $\{\delta_q(\lambda)\}$ and $x_p(\lambda)/\|x_p(\lambda)\|$ is among the normalized eigenelements $\{W_q(\lambda)\}$ of the operator $S(x_p(\lambda))$.

Since $\mu_p^{(\theta)}$ is a simple eigenvalue of $T'(\theta)$, as $\lambda \rightarrow \mu_p^{(\theta)}$ only one of the eigenvalues $\{\delta_q(\lambda)\}$ tends to $\mu_p^{(\theta)}$, namely $\delta_p(\lambda)$. Since $\lambda \rightarrow \mu_p^{(\theta)}$ and λ is an eigenvalue, this one must be λ itself; hence $\lambda = \delta_p(\lambda)$.

Thus the eigenelements $x_p(\lambda)$ and $W_p(\lambda)$ are the same up to a multiplicative constant, and the theorem is proved.

Now let (x_p^*, λ^*) be a solution pair of eq. (5) on the p 'th branch. By Theorem 3 there exist such pairs, and by Theorem 4 above, $x_p^*(\lambda^*)$ is identical with $W_p(\lambda^*)$ to within a multiplicative constant.

If one plans to use Theorem 2 to extend $x_p(\lambda)$ as a sheet of solutions, it must be shown that $[\lambda^*I - T'(x_p^*)]^{-1}$ exists and is bounded for a pair (x_p^*, λ^*) on the branch. There are essentially two ways in which $[\lambda^*I - T'(x_p^*)]^{-1}$ might not exist: if $\lambda^* = \mu_q$, $q \geq p$, or if $\lambda^* = \mu_q$, $q < p$. Here μ_q is one of the eigenvalues of the problem

$$(13) \quad T'(x_p^*)h = \mu h, \quad ,$$

enumerated in decreasing order with multiplicities counted. (Eq. (13) is eq. (6) with $x = x_p^*$.) We are in a position now to exclude fully only the first of the above contingencies.

Theorem 5: Let (x_p^*, λ^*) belong to the p 'th branch. Then $\mu_p < \lambda^*$, where μ_p is the p 'th eigenvalue of eq. (13), counting multiplicities.

Proof: By the Courant mini-max principle as applied to symmetrizable operators [ref. 15, p. 384, Th. 4],

$$\mu_p = \min_{\{V_0, \dots, V_{p-1}\}} \max_{h \perp V_0, V_1, \dots, V_{p-1}} \frac{(PT'(x_p^*)_{h,h})}{(Ph,h)},$$

where the minimum is taken over sets of p arbitrary elements of H . Hence

$$\mu_p \leq \max_{h \perp W_0, \dots, W_{p-1}} \frac{(PT'(x_p^*)_{h,h})}{(Ph,h)},$$

where $\{W_0, \dots, W_{p-1}\}$ are the first p eigenelements, counting multiplicities, of the operator $S(x_p^*)$.

For the operator $T'(sx_p^*)$, $0 \leq s \leq 1$, we have [ref. 9, p. 187, eq. (9)]

$$T'(x_p^*)_{h,h} - T'(sx_p^*)_{h,h} = \int_0^1 dT'(\{s + t(1-s)\}x_p^*; (1-s)x_p^*)_{h,h} dt.$$

By statement H - 1, for any $h \in H$, $0 \leq s \leq 1$, $0 \leq t \leq 1$, $\|h\| \neq 0$,

$$\begin{aligned} & (PdT'(\{s + t(1-s)\}x_p^*; (1-s)x_p^*)_{h,h}) \\ &= \frac{1-s}{s+t(1-s)} (PdT'(\{s + t(1-s)\}x_p^*; \{s + t(1-s)\}x_p^*)_{h,h}) \end{aligned}$$

is negative. Therefore, $(PT'(x_p^*)_{h,h}) < (PT'(sx_p^*)_{h,h})$ for any $h \in H$ and

$0 \leq s \leq 1$, and

$$(PT'(x_p^*)_{h,h}) < (P \int_0^1 T'(sx_p^*)_{h,h} ds) = (PS(x_p^*)_{h,h}).$$

Therefore [ref. 15, p. 383, Th.3]

$$\mu_p \leq \max_{h \perp W_0, \dots, W_{p-1}} \frac{(PT'(x_p^*)h, h)}{(Ph, h)} < \max_{h \perp W_0, \dots, W_{p-1}} \frac{(PS(x_p^*)h, h)}{(Ph, h)} = \delta_p$$

since the maximum is assumed. Here δ_p is the p 'th eigenvalue of $S(x_p^*)$ counting multiplicities. The corresponding eigenelement of $S(x_p^*)$, P -orthogonal to every other eigenelement, is $W_p(\lambda^*)$. By Theorem 4, $W_p(\lambda^*)$ is to be identified with $x_p^*(\lambda^*)$; hence $\delta_p = \lambda^*$ since $T(x_p^*) = S(x_p^*) x_p^* = \lambda^* x_p^*$. This completes the proof.

If we could now ascertain that $\lambda^* < \mu_{p-1}$, we could assert that $[\lambda^* I - T'(x_p^*)]^{-1}$ exists. Since $\mu_p^{(\theta)}$ is a simple eigenvalue of $T'(\theta)$, this assertion can be made in a neighborhood of $x = \theta$. Hence given a pair (x_p^*, λ^*) of the p 'th branch, with $\|x_p^*\|$ sufficiently small, there exists by Theorem 2 an interval $[\lambda^* - \delta, \lambda^* + \delta]$ and a continuous function $x_p(\lambda)$ defined thereon such that $x_p(\lambda^*) = x_p^*$. $x_p(\lambda)$ gives a locally unique solution of eq. (5). This leads to the following result which sounds like theorem 4:

Theorem 6: The function $x_p(\lambda)$ for $\lambda \in [\lambda^* - \delta, \lambda^* + \delta]$ is identical with the p 'th eigenelement of the operator $S(x_p(\lambda))$ to within a multiplicative constant.

Proof: As $\lambda \rightarrow \lambda^*$, $\|x_p(\lambda) - x_p^*\| \rightarrow 0$ by continuity. Hence the operator $S(x_p(\lambda)) = \int_0^1 T'(\bar{s}x_p) d\bar{s} \rightarrow S(x_p^*)$ in the uniform topology. By a result employed in proving Theorem 4 [ref. 14, pp. 238, 239] we have $\delta_p(\lambda) \rightarrow \delta_p(\lambda^*)$. Here $\delta_p(\lambda)$ is the p 'th eigenvalue of the operator $S(x_p(\lambda))$ while $\delta_p(\lambda^*)$

is the p 'th eigenvalue of $S(x_p^*)$. Let the corresponding eigenelements be $W_p(\lambda)$ respectively $W_p(\lambda^*)$, P -orthogonal to all other eigenelements of the operators $S(x_p(\lambda))$ respectively $S(x_p(\lambda^*))$. The p 'th eigenelement of $S(x_p(\lambda^*))$ happens to be $x_p^* = x_p(\lambda^*)$ itself (up to a multiplicative constant) corresponding to the eigenvalue $\delta_p(\lambda^*) = \lambda^*$; this is because (x_p^*, λ^*) belongs to the p 'th branch. Also the eigenelement $W_p(\lambda)$ of the operator $S(x_p(\lambda))$, belonging to the eigenvalue $\delta_p(\lambda)$ which converges to $\delta_p(\lambda^*) = \lambda^*$ as $\lambda \rightarrow \lambda^*$, happens to be $x_p(\lambda)$ corresponding to eigenvalue $\delta_p(\lambda) = \lambda$. Therefore $x_p(\lambda) = W_p(\lambda)$ to within a multiplicative constant, and the theorem is proved.

Theorems 5 and 6 have appeared in the context of the assessment of properties of branches of solutions as far as they have been evolved on the basis of Theorem 3. Note, however, that these results are applicable also to branches which have been extended into the large, provided such extension is possible.

5. Extension of Branches of Solutions; a Special Case.

By property H - 3, there exists a bounded linear P-symmetrizable operator B such that $\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \|T'(x) - B\| = 0$, and as a consequence $\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \frac{1}{\rho} \|T(x) - Bx\| = 0$. By a known result [ref. 8, p. 361], B is completely continuous. Therefore [ref. 15, p. 383, Th. 3] there exists a sequence $\{\gamma_n\}$ of positive eigenvalues with $\lim_{n \rightarrow \infty} \gamma_n = 0$ (assuming non-degeneracy).

Lemma 2: $\gamma_n < \mu_n^{(x)}$, $n = 0, 1, \dots$, for given $x \in H$.

Proof: For $0 < s_1 < s_2$, [ref. 9, p. 187, eq. (9)], and $\|h\| \neq 0$

$$\begin{aligned} T'(s_2 x)h - T'(s_1 x)h &= \int_0^1 dT'(s_1 x + t(s_2 - s_1)x; (s_2 - s_1)x) dt h \\ &= \int_0^1 \frac{s_2 - s_1}{s_1 + t(s_2 - s_1)} dT'([s_1 + t(s_2 - s_1)]x; [s_1 + t(s_2 - s_1)]x) dt h \end{aligned}$$

whence, since $(PdT'(y; y)h, h) < 0$ for $y \in H$ by property H - 1, we have $(PT'(s_2 x)h, h) < (PT'(s_1 x)h, h)$. Letting $s_1 = 1$ and $s_2 \rightarrow \infty$ this gives $(PBh, h) < (PT'(x)h, h)$. An application of Zaanen's modified version of Courant's minimax principle as in proving Theorem 5 [ref. 15, p. 384, Th. 4] now yields the result.

For the purposes of this section it is necessary to make the following assumption:

Assumption: The operator B is such that $\mu_p^{(\theta)} \leq \gamma_{p-1} < \mu_{p-1}^{(x)}$.

In this assumption $\mu_p^{(\theta)}$ is the bifurcation point of the p'th branch which we propose to extend from the small into the large. In the exten-

sion of the p 'th branch by methods based on Theorem 2, with this con-

dition we always have $\lambda^* < \mu_p^{(\theta)} \leq \gamma_{p-1} < \mu_{p-1}^{(x_p^*)}$ for any pair (x_p^*, λ^*) .

Theorem 5 then implies that $\mu_p^{(x_p^*)} < \lambda^* < \mu_{p-1}^{(x_p^*)}$, so that λ^* is in the resolvent set, i.e., $[\lambda^* I - T'(x_p^*)]^{-1}$ exists and is bounded.

Note: It may not be necessary to make such an assumption. If $T(x)$ is asymptotically linear in accordance with property H - 3, there may be branches such that $\mu_p^{(\theta)} \leq \gamma_{p-1}$. This may be true for every branch: the sequences $\{\mu_n^{(\theta)}\}$ and $\{\gamma_n\}$ can be interlaced, [ref. 11, p. 131]. Also, in extending the highest branch, $p = 0$, the assumption is obviously unnecessary.

With this assumption, it is possible to use the process mentioned in connection with Theorem 2 to extend the p 'th branch $x_p(\lambda)$ of nontrivial solutions of eq. (5) leftward continuously from its genesis at the branch point $\lambda = \mu_p^{(\theta)}$. Let $(x_p^{(0)}, \lambda^{(0)})$ be an initial pair provided by Theorem 3. There results the sequence of pairs $(x_p^{(1)}, \lambda^{(1)}), \dots, (x_p^{(k)}, \lambda^{(k)}), \dots$, which are left endpoints of overlapping extension intervals. Each interval carries the p 'th branch one step leftward. How far can this process proceed?

Theorem 7: Let $N > 0$ be arbitrary. Under the assumption that $\mu_p^{(\theta)} \leq \gamma_{p-1}$, where γ_{p-1} is the $p - 1$ th eigenvalue of the operator B (see statement H - 3), the stepwise leftward extension of the p 'th branch $x_p(\lambda)$, $\lambda < \mu_p^{(\theta)}$, can be continued until $\|x_p(\lambda)\| > N$.

Proof: The proof is similar to a corresponding result in the author's

recent paper [ref. 13, pp. 593-595]. There are modifications however.

We sketch the proof here.

Suppose the converse. Then $x_p(\lambda)$ is extendable with any number of steps so that $\|x_p(\lambda)\| \leq N$. Let Λ_p be the set of values $\lambda > 0$ for which $x_p(\lambda)$ is uniquely defined. The set Λ_p contains the union of extension intervals: $\bigcup_{k=0}^{\infty} [\lambda^{(k)} - \tau_k, \lambda^{(k)} + \tau_k]$, and by Theorem 3 is closed on the right by the point $\lambda = \mu_p^{(\theta)}$. By Theorems 4 and 6, wherever the stepwise process is used, $x_p = w_p$ to within a multiplicative constant; we consider this property part of the definition of $x_p(\lambda)$. By virtue both of Theorem 2 (in the extension interval) and of Theorem 5, $[\lambda I - T'(x_p)]^{-1}$ exists and is bounded for $\lambda \in \Lambda_p$ (with the exception of $\lambda = \mu_p^{(\theta)}$, the right endpoint).

The set Λ_p is closed relative to $(0, \mu_p^{(\theta)}]$ by virtue of the compactness of $T(x)$, the a priori boundedness of $x_p(\lambda)$, the continuity in norm of $T(x)$, and an argument similar to that in the proof of Theorem 6. It is also open relative to $(0, \mu_p^{(\theta)})$ by Theorem 2 and Theorem 5. Hence $\Lambda_p = (0, \mu_p^{(\theta)})$ and $x_p(\lambda)$ is uniquely extendable leftward to $\lambda = 0$ with $\|x_p(\lambda)\| \leq N$, $0 < \lambda \leq \mu_p^{(\theta)}$.

For any sequence $\lambda_m \rightarrow 0$, a subsequence of $\{\|x_p(\lambda_m)\|\}$ is bounded away from zero; this is considered already extracted [ref. 13, p. 595; ref. 8, p. 393, 394]. Then $\{\|T(x_p(\lambda_m))\|\}$ is bounded away from zero since $T(x) \neq \theta$ for $x \neq \theta$. (If $T(x) = \theta$, $x \neq \theta$, then $0 = (PT(x), x) = (PS(x)x, x) = \int_0^1 (PT'(\bar{s}x)x, x) d\bar{s} > 0$, which is a contradiction.) Then since from problem (5) we have $\|x_p(\lambda_m)\| = \frac{1}{\lambda_m} \|T(x_p(\lambda_m))\| \rightarrow \infty$ as $\lambda_m \rightarrow 0$, the initial assumption that $\|x_p(\lambda)\| \leq N$ is contradicted. The theorem is proved.

Hence there exists some value $\bar{\lambda} \geq 0$ such that $\lim_{\lambda \rightarrow \bar{\lambda}} \|x_p(\lambda)\| = \infty$,
 $\lambda > \bar{\lambda}$.

Theorem 8: Under the assumption that $\mu_o^{(\theta)} \leq \gamma_{p-1}$, where γ_{p-1} is the $p-1$ 'th eigenvalue of the operator B (see statement H - 3), $\lim_{\lambda \rightarrow \lambda_p} \|x_p(\lambda)\| = \infty$.

Proof: Using equation (5) and the operator B , we can arrive at the following inequality:

$$(14) \quad 1 \leq \|(\lambda I - B)^{-1}\| \cdot \frac{1}{\|x_p\|} \|T(x_p) - Bx_p\|, \quad \|x_p\| \neq 0,$$

[ref. 13, p. 595, eq. (35)]. Since $\lim_{\rho \rightarrow \infty} \sup_{\|x\|=\rho} \frac{1}{\rho} \|T(x) - Bx\| = 0$ it is apparent from (14) that the value $\bar{\lambda}$ where we have $\lim_{\lambda \rightarrow \bar{\lambda}} \|x_p(\lambda)\| = \infty$ must

be an eigenvalue of B , i.e., one of the numbers $\{\gamma_n\}$. Since $\lambda < \gamma_{p-1}$ by assumption and $\gamma_p \leq \mu_p < \lambda$ by Theorem 5, we must either have $\bar{\lambda} = \gamma_{p-1}$ or $\bar{\lambda} = \gamma_p$. We cannot have $\bar{\lambda} = \gamma_{p-1}$ however since extension proceeds leftward from $\lambda = \mu_p^{(\theta)} \leq \gamma_{p-1}$. Hence $\bar{\lambda} = \gamma_p$, and the theorem is proved.

6. Extension of Branches of Solutions: the General Case.

In section 5 we had complete success in extending the p 'th branch, emanating from a bifurcation point $\mu_p^{(\theta)}$ which is a simple eigenvalue of $T'(\theta)$, for $\lambda \leq \mu_p^{(\theta)}$. We had to assume however that the $p-1$ 'th eigenvalue γ_{p-1} of the operator B of statement H - 3 was such that $\mu_p^{(\theta)} \leq \gamma_{p-1}$.

Theorem 9: In the general case of sublinear completely continuous uniformly symmetrizable operators satisfying statements H - 1 and H - 3, the extension of a branch can be carried out to a monotone increasing sequence of norm levels, $\{C_m\}$.

Proof: We can attach an asymptotic behavior for $T(x)$ which satisfies the assumption of section 5. Indeed, let B_1 be a linear bounded P -symmetrizable operator on H with positive eigenvalues $\{\gamma_n^{(1)}\}$, $\gamma_n^{(1)} < \mu_n^{(\theta)}$, $n = 0, 1, 2, \dots$, and for which $\gamma_{p-1}^{(1)} = \mu_p^{(\theta)}$ with $\gamma_{p-1}^{(1)}$, $\gamma_p^{(1)}$ simple. (Such an operator is constructed as follows: Let $\{\varphi_n\}$ be a complete sequence orthonormal in the P -norm $\sqrt{(P^*, \cdot)}$. Let $\{\gamma_n^{(1)}\}$ be a sequence of positive numbers satisfying the above requirements. Then let $B_1 \varphi =$

$$\sum_{n=0}^{\infty} \gamma_n^{(1)} (P\varphi, \varphi_n) \varphi_n \text{ for arbitrary } \varphi \in H.)$$

Now let $\Delta > 0$, $C_1 > 0$ be numbers with C_1 adjusted so that $\|x\| \leq C_1 + \Delta$ is the largest sphere for which $\gamma_{p-1}^{(1)} \leq \mu_{p-1}^{(x)} \leq \mu_{p-1}^{(\theta)}$; here $\mu_{p-1}^{(x)}$ is the $p-1$ 'th eigenvalue of $T'(x)$, $\|x\| \leq C_1 + \Delta$. With Δ small enough for this requirement, let it be fixed. (In proof of Lemma 2, put $s_1 = 0$, $s_2 = 1$.)

Does there exist such a sphere? In other words, is θ an interior point of the set $\{x | x \in H, \gamma_{p-1}^{(1)} \leq \mu_{p-1}^{(x)} \leq \mu_{p-1}^{(\theta)}\}$? If this were not true,

there would exist a sequence $\{x_n\}$ of points $x_n \in H$ such that $x_n \rightarrow \theta$ and $\mu_{p-1}^{(x_n)} < \gamma_{p-1}^{(1)}$. By a previously used result however [ref. 14, pp. 238, 239], $\mu_{p-1}^{(x)}$ is continuous in x in norm. Hence as $n \rightarrow \infty$ we have $\mu_{p-1}^{(x_n)} \rightarrow \mu_{p-1}^{(\theta)}$, which is a contradiction.

Let $f(t)$ be a real differentiable function for $t \geq 0$ with the following properties: $f(t) \equiv 0$, $t \leq C_1$; $f(t) \equiv 1$, $t > C_1 + \Delta$; $f'(C_1) = f'(C_1 + \Delta) = 0$, and $f'(t) > 0$, $C_1 < t < C_1 + \Delta$.

Now let us form the bounded linear operator $T_1'(x)$, defined on H for each fixed $x \in H$:

$$T_1'(x) = \begin{cases} T'(x) & \|x\| \leq C_1 \\ [1 - f(\|x\|)]T'(x) + f(\|x\|)B_1 & C_1 \leq \|x\| \leq C_1 + \Delta \\ B_1 & \|x\| \geq C_1 + \Delta \end{cases}.$$

For the first differential of this operator ($\epsilon[H \rightarrow H]$ for given $x, h \in H$) we have

$$dT_1'(x;h) = \begin{cases} dT'(x;h) & \|x\| \leq C_1 \\ [1 - f(\|x\|)]dT'(x;h) - f'(\|x\|)\frac{(h,x)}{\|x\|}[T'(x) - B_1] & C_1 \leq \|x\| \leq C_1 + \Delta \\ \theta h = \theta & \|x\| \geq C_1 + \Delta \end{cases}.$$

Since $T(x)$ is sublinear, $PdT'(x;x)$ is negative definite (see statement H - 1); since $P[T'(x) - B_1]$ is symmetric and $0 < \gamma_n^{(1)} \leq \mu_n^{(x)}$, $n = 0, 1, \dots$, we have $P[T'(x) - B_1]$ positive definite. Hence $PdT_1'(x;x)$ is negative definite for $\|x\| < C_1 + \Delta$, and it is the null operator for $\|x\| \geq C_1 + \Delta$.

Let us now form the nonlinear operator

$$T_1(x) = \int_0^1 T_1'(sx) x ds \quad .$$

Computing the derivative,

$$\begin{aligned} T_1(x+h) - T_1(x) &= \int_0^1 T_1'(sx+sh) (x+h) ds - \int_0^1 T_1'(sx) (x+h) ds \\ &\quad + \int_0^1 T_1'(sx) (x+h) ds - \int_0^1 T_1'(sx) x ds \\ &= \int_0^1 dT_1'(sx; sh) ds x + \int_0^1 T_1'(sx) h ds + o(x, h) \\ &= T_1'(x) h + o(x, h) \quad . \end{aligned}$$

where $T_1'(x)$ is the derivative [ref. 9, p. 183] of $T_1(x)$ as expected.

$T_1(x)$ is sublinear for $\|x\| \leq C_1 + \Delta$.

All considerations of section 5 are now applied to $T_1(x)$. By Theorem 7 there exists a branch $x_p^{(1)}(\lambda)$ of solutions for the equation $\lambda x = T_1(x)$, branching from the trivial solution at $\lambda = \mu_p^{(\theta)}$, and defined for $\gamma_p^{(1)} \leq \lambda \leq \mu_p^{(\theta)}$. By Theorem 8, $\lim_{\lambda \rightarrow \gamma_p^{(1)}} \|x_p^{(1)}(\lambda)\| = \infty$. We have $T_1(x) \equiv T(x)$ for $\|x\| \leq C_1$ so that $x_p^{(1)}(\lambda) \equiv x_p(\lambda)$ for $\|x_p^{(1)}(\lambda)\| \leq C_1$.

By continuity there exists a parameter value $\lambda^{(1)}$ such that $\|x_p^{(1)}(\lambda^{(1)})\| = C_1$. Let B_2 be a linear bounded P-symmetrizable operator

on H with positive eigenvalues $\{\gamma_n^{(2)}\}$, $\gamma_n^{(2)} < \gamma_n^{(1)} < \mu_n^{(\theta)}$, $n = 0, 1, 2, \dots$, and for which $\gamma_{p-1}^{(2)}, \gamma_p^{(2)}$ are simple with $\gamma_{p-1}^{(2)} = \lambda^{(1)}$. (The construction is the same as for B_1 .) With the same number $\Delta > 0$ adjust $C_2 > 0$ such that $\|x\| \leq C_2 + \Delta$ is the largest sphere for which $\gamma_{p-1}^{(2)} \leq \mu_{p-1}^{(x)} \leq \mu_{p-1}^{(\theta)}$. Due to the monotonicity of $\mu_{p-1}^{(sx)}$ as a function of s for any $x \in H$ (see proof of Lemma 2), we have $C_2 > C_1$, and C_2 is uniquely determined.

Now form the bounded linear operator, defined on H for each fixed $x \in H$:

$$T_2'(x) = \begin{cases} T'(x) & \|x\| \leq C_2 \\ [1 - f(\|x\|)]T'(x) + f(\|x\|)B_2 & C_2 \leq \|x\| \leq C_2 + \Delta \\ B_2 & \|x\| \geq C_2 + \Delta \end{cases},$$

and the nonlinear operator

$$T_2(x) = \int_0^1 T_2'(sx) x ds$$

which coincides with $T(x)$ for $\|x\| \leq C_2$. $T_2(x)$ is a sublinear operator for $\|x\| \leq C_2 + \Delta$ and is the null operator for $\|x\| \geq C_2 + \Delta$. By Theorem 7, there exists a branch $x_p^{(2)}(\lambda)$ of solutions of the equation $\lambda x = T_2(x)$, branching from the trivial solution at $\lambda = \mu_p^{(\theta)}$ and defined for $\gamma_p^{(2)} < \lambda < \mu_p^{(\theta)}$. By Theorem 8, $\lim_{\lambda \rightarrow \gamma_p^{(2)}} \|x_p^{(2)}(\lambda)\| = \infty$. We have $x_p^{(2)}(\lambda) \equiv x_p(\lambda)$ for $\|x_p^{(2)}(\lambda)\| \leq C_2$.

By continuity there exists a value $\lambda = \lambda^{(2)}$ such that $\|x_p^{(2)}(\lambda^{(2)})\| = C_2$. Let B_3 be an operator with positive eigenvalues $\{\gamma_n^{(3)}\}$, $\gamma_n^{(3)} <$

$\gamma_n^{(2)} < \gamma_n^{(1)} < \mu_n^{(\theta)}$, $n = 0, 1, 2, \dots$, (the construction is the same as for B_1, B_2), for which $\gamma_{p-1}^{(3)}, \gamma_p^{(3)}$ are simple, and $\gamma_{p-1}^{(3)} = \lambda^{(2)}$. From here we carry on in like manner, repetitively. It is clear how the completed induction would go.

For a given $\Delta > 0$, a given function $f(t)$, and a given sequence $\{\varphi_n\}$ for forming the operators $\{B_n\}$, this process gives rise to numerical sequences: $\{\lambda^{(m)}\}$, $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)} > \dots$, and $\{C_m\}$, $C_1 < C_2 < C_3 < \dots$. This completes the proof of Theorem 9.

The sequence of positive numbers $\{\lambda^{(m)}\}$ obviously converges to a non-negative limit. The same ought not to be said for $\{C_m\}$. Divergence would be desired for $\{C_m\}$, since this would indicate that, in the sub-linear case, extension of branches can be carried to arbitrarily high norm levels. This would completely generalize previous results [rev. 11]. At present a proof of divergence for the monotone sequence $\{C_m\}$ is not at hand.

7. A Basic Problem.

In section 5, the branch $x_p(\lambda)$ was completely developed under the assumption that $\mu_p^{(\theta)} \leq \gamma_{p-1} < \mu_{p-1}^{(x)}$. Section 6 was an attempt to use this special result of section 5 in order to develop the branch $x_p(\lambda)$ in case the assumption should not be valid.

Fundamentally, the difficulty is as follows: In Theorem 5 we were able to show that $\mu_p < \lambda^*$, where μ_p is the p 'th eigenvalue, simple or not, for problem (13), and (x_p^*, λ^*) is a representative pair of the p 'th branch. We were not able to obtain a corresponding measure of control over μ_{p-1} , the $(p-1)$ th eigenvalue of eq. (13). Needed is a theorem stating that $\lambda^* < \mu_{p-1}$; for if $\lambda^* = \mu_{p-1}$ for any pair (x_p^*, λ^*) of the p 'th branch, that pair is a boundary point as far as extension by means of Theorem 2 is concerned. $\lambda^* = \mu_{p-1}$ would be a bifurcation point to which the considerations involving eq. (10) would have to be applied. Particularly if μ_{p-1} were a multiple eigenvalue of eq. (13), but also in the case of simplicity where eq. (10) leads to eq. (11), the welter of possibilities does not lead to an attractive study.

Apparently with the nonlinear Sturm-Liouville problem (1) [ref. 11, 12] the question is resolved in using the initial value problem for the differential equation in (1) as a tool for studying branches. The problem did arise however with eq. (3) [ref. 13].

The author does not know whether or not it is reasonable for the problems $T(x^*) = \lambda^* x^*$ and $T'(x^*)h = \lambda^* h$ to have nontrivial solutions x^*, h for the same scalar λ^* .

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