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OPTIMAL FEEDBACK CONTROL OF THE RADIOACTIVE VOLTAGE STANDARD

TECHNICAL REPORT EE-150

by

David William Braudaway

March 1968

This report consists essentially of a doctoral dissertation submitted by David William Braudaway and supervised by R. R. Mohler.

Prepared for Sandia Corporation
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ABSTRACT

Problems in maintaining a reference voltage using standard cells lead to the need for alternate techniques of producing a highly predictable voltage. Chosen for development is a technique which relates voltage to the activity of a radioactive source. By use of this technique, the value of the voltage is essentially as predictable as is the source activity. Careful choice of the radioactive source permits realization of the desired goal of one-part-in-ten-million predictability.

A two-limb calorimeter is used to realize the desired function. Heat flow in each limb produces a temperature rise across the limb air gaps. In one limb, the temperature rise caused by the radioactive source is sampled by a pair of temperature-sensitive resistors. In the other limb, a second pair of resistors samples the temperature rise caused by the resistive heater. These four temperature-sensing resistors are connected to form a bridge which is balanced when the heat outputs of the sources are matched. While the bridge output is a measure of system response and power balance, the output voltage is produced by a sampling resistor connected in series with the heater. Because the thermal time constants of the system are long, automatic control

of the balanced condition is desired.

Before the control can be developed, however, a mathematical model of the system behavior is necessary. Analysis of the bridge characteristics and the limb thermal behavior leads to linear-approximation models for each of these subsystems. In order to complete the system model, consideration of the characteristics of a galvanometer-amplifier is also necessary. By careful selection of these characteristics, a system is produced in which Brownian noise effects and response effects of the galvanometer are negligible. From this analysis the system is modeled by a linear second-order equation.

System operation is evaluated in terms of a quadratic performance index for each of two criteria. The primary criterion is employed when use of the system output is necessary. Under this criterion the two power levels must be held in balance and the rate of change of heater current must be minimized. Where the system output is not used, the secondary criterion is employed in which weighting of the rate of change of heater current is reduced, producing a faster response. By use of the maximum principle a realizable optimal feedback control is developed for each of the two quadratic criteria. Use of optimization techniques assures that the produced voltage is nearly as predictable as is the radioactive source activity. This condition makes practical use of the device as a standard.

PREFACE

The quest for new and different techniques of establishing and maintaining the electrical values is continual. Among the techniques desired are those for producing highly predictable sources of voltage. Described in this dissertation is a working device developed around one such technique, the research done to define the system and the synthesis of the automatic control which makes it practical. In this system, the reproducibility of the voltage output is derived from the predictability of the decay of a radioactive heat source.

As an introduction, a brief background of the history of electrical standards is presented in Chapter 1. Included are the characteristics of present voltage-standard devices and some of the reasons alternate techniques are desired.

Chapter 2 contains the basic theory of the radioactive voltage standard and the design details of the working device which lead to the description of the system in mathematical terms. Included are the salient characteristics of the radioactive heat source and an introduction to the problems which make automatic control of the system desirable. Related information on

the temperature-controlled-bath environment for the device is contained in Appendix A. In Appendix B, a more complete analysis of the radioactive source is presented, which includes the techniques necessary to predict its power output.

The mathematical models of the calorimeter, the temperature-sensing bridge and the galvanometer which make up the total system are developed in Chapter 3. These models are combined into the system model which is required before control of the output voltage can be synthesized. Appendix C contains the development of the thermal constants for the calorimeter model.

From the system dynamical equation, the control policy is developed in Chapter 4. Two sets of control criteria are considered. The developed control produces a rapid approach during warmup and a minimum output variation during operation. In addition, circuitry necessary to realize the control is described.

Finally, in Chapter 5, a summary of theoretical and practical results is presented. Emphasis is on the recommendation for future calorimetric design.

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CHAPTER 1

INTRODUCTION

A. History of Standards

The history of electrical measurements is closely tied to the history of electrical standards. In the early days, experimenters used a convenient length of wire as a resistance standard and some voltaic cell as a voltage standard (References 1 and 2). As experiments grew in complexity and as the number of experimenters increased, these simple standards were no longer adequate. There was need for stable, interchangeable standards to enhance repeatability of results and to permit easy comparison of work done by different experimenters. A continued search has been carried on for physical phenomena which would exhibit sufficient repeatability and ease of establishment to permit their use as practical standards.

Absolute determinations of the electrical quantities can be made in terms of the fundamental physical quantities of length, mass and time. These complex comparisons are carried out periodically by national laboratories such as the National Bureau of Standards (NBS). Between determinations the electrical values are preserved by using banks of Thomas and NBS type resistors and Weston cells (References 3 and 4). Similar

banks form the reference standards for non-national standards laboratories. As NBS periodically re-establishes the electrical values, so must the non-national laboratories periodically re-establish their electrical values in terms of the legal or national value. Resistors of the Thomas and NBS types have demonstrated their relative immunity to shock in transit and they serve well as transportable standards. Cells, however, are chemical devices which have shown relatively low tolerance to the thermal and the mechanical shock encountered in transportation.

B. Present Voltage Standards

The Weston cadmium cell was patented by Dr. Edward Weston in 1892, and it was later adopted as the voltage standard (References 4 and 5). This cell has since served as both a working and a reference standard. The cell is constructed as diagrammed in Figure 1.1, and it is available in two forms (depending on the saturation of the electrolyte). Pure mercury is used as the positive electrode, over which is placed a mercurous sulfate depolarizer. The negative electrode is of 10 per cent cadmium-mercury amalgam. The electrolyte is a saturated or unsaturated aqueous solution of cadmium sulfate, which usually has added sulfuric acid. In manufacture, only the highest purity chemicals are used, and they are proportioned to enhance the basic

Figure 1.1 The Weston Cadmium Cell
 (Note: The unsaturated form does not contain the excess of CdSO_4 crystals)

stability of the design.

As the chemicals react in the cell, additional cadmium sulfate is formed. This increases the concentration level of cadmium sulfate in the electrolyte of the unsaturated cells. As the electrolyte concentration increases the cell voltage decreases. Initial voltage is about 1.0193 volts and it decreases 20 to 40 microvolts per year. This form of the cell is made with septa,¹ and it can be conveniently handled without damage. The cell voltage has a temperature coefficient of about 5 microvolts per degree Kelvin which makes the cell suited to general laboratory use without extreme temperature control. In time, the unsaturated cell usually develops a hysteresis characteristic which produces large voltage excursions upon temperature shock. This behavior eventually renders the cell unsatisfactory for continued use. The unsaturated cell generally can be relied on to within 0.01 per cent for periods up to one year from the standardization date.

The electrolyte of the saturated cell cannot change its concentration providing the cell is held at a fixed temperature. As a result, this form of the cell has a very constant voltage when maintained in a controlled environment (about 1.0183 volts at 293° Kelvin). Ther-

¹Septa are porous plugs which are used in each limb of the cell to keep the chemicals in place.

mal behavior of the saturated cell is complicated by the slow response of the electrolyte concentration to temperature change. The temperature coefficient of voltage is about 50 microvolts per degree Kelvin, which makes temperature control necessary. Because of chemical contamination problems, these cells were formerly made entirely without septa and they could be ruined by tipping. High purity, inert materials are now available and some saturated cells are manufactured using these as septa.

The saturated cell has proven very repeatable over relatively long periods of time when it is maintained in controlled environment. The voltage generated is subject to variations which occur naturally in the chemical process. Transportation of saturated cells often produces changes in the voltage of individual cells, but the average is usually reliable if the group contains a sufficient number of cells (six or more). Recent improvements in temperature-controlled baths have made possible transportation of saturated cells without removing the cells from the controlled environment. This technique limits possible temperature shock which is believed to be the most significant upsetting factor. Demands on the bath for environmental control are quite severe and they necessitate special handling.

As the demand for higher accuracy has increased, the saturated cell has been pressed to keep pace.

Carefully maintained, these cells are fulfilling the requirement for repeatability of a few microvolts (a few parts per million) over long periods of time. They have thus far met the challenge of newer sources of stable voltage.

C. Possible Alternate Voltage Standards

Because of the nature of standard cells, it is desired that alternate sources of highly predictable voltage be developed. Such sources would augment the banks of saturated cells now in use. They would be especially useful in determining the good from the bad during periods when numbers of cells are failing. Ideally, such devices might assume the task of transferring the value from NBS to the local standards laboratory without severe transportation problems.

One check on the stability of standard cells has been made at NBS using the gyromagnetic ratio of the proton as the stable factor (Reference 6). This technique relates the frequency of precession of protons in a magnetic field to the strength of the field. The current which produces the magnetic field is also passed through a known resistance across which the voltage is developed. At the sensitivity levels required, the field is readily disturbed by apparatus located nearby. For a stability check, fixed apparatus produces no net error; however, moving magnetic disturbances at con-

siderable distance complicate operation. The requirement of a fixed magnetic environment cannot be met easily and it is beyond the reach of most standards laboratories.

The zener diode supply offers promise as a standard cell substitute. Performance of the zener diode supply is dependent not only on the diode itself, but also on the associated electronic circuitry. Tests on various commercial supplies have indicated that they are subject to significant instability after they are exposed to the shock, temperature and humidity encountered in shipping. These supplies do not possess sufficient stability at the present state of development for replacement of standard cells at all accuracy levels. Further development of the zener diode and its associated electronic circuitry is necessary before general application to the less than 10-microvolt stability range is practical.

Two methods of developing a highly predictable voltage source have been suggested by K. C. Jordan of Mound Laboratory. Both methods use calorimetric techniques, and they produce a voltage whose value decays in direct relationship to the decay of a radioactive heat source (Reference 7). The predictability is derived from the known behavior of the radioactive source. To ensure precise predictability, considerable care in the choice of the heat source is required.

The first or direct method employs a thermopile with one set of junctions heated by a radioactive source; while the opposite set of junctions is maintained at a constant temperature. Conduction and radiation of heat from the source through the system produces the temperature difference necessary to develop the voltage. Because of system nonlinearities the voltage is dependent on variations in temperature (several parts in 10^5 for a 0.01° Kelvin controlled bath). Further, the voltage is subject to minor variations in the thermopile junction set with time. Finally, the magnitude of the output voltage is not readily adjustable to compensate for source decay. For these reasons this technique is unsuited to the highest-precision applications.

In the second or comparison method the temperature rise due to the heat flow from a radioactive source is compared with the temperature rise due to the heat flow from a resistive heater. Heat from both sources flows to a sink through similar insulators. Temperature-sensitive resistors are located at fixed points in the insulators and they are connected to form a comparison bridge. Ideally, when the bridge is balanced the heat flow from the two sources is identical. Because of the balanced configuration, temperature variations in the sink are reduced to a low-order effect. The temperature-sensing resistors can be wound from high-purity wire which has excellent long-term stability. Further, if

wire from the same batch is used for all resistors the drift effects will be similar, and the bridge behavior will be nearly constant with time.

This type of calorimeter has been generally used to determine the activity level and decay rate of radioactive isotopes. The current to the heater is passed through an external series resistor across which the desired voltage is produced. By making this series resistor variable in small increments, the decay of the source strength can be readily compensated. For these reasons, the comparison calorimeter offers promise for use in establishing a highly predictable voltage source. It is this specific approach which has been chosen for development into a working device.

Figure 1.2 presents the basic system concept. Indicated by dashed lines is a further refinement where the bridge output voltage is fed back following amplification and filtering to control the power input to the heater. This is used to balance the system automatically or to control the system.

There are several reasons why control of the system is desired. First, before the voltage produced can be used, the system must be brought to balance. Further, this balance must be maintained throughout the time the voltage is being used. Finally, the response time constants of thermal systems of this type are quite long, and psychological problems complicate manual balancing.

Figure 1.2 The Radioactive Voltage Standard Basic Concept

of systems with such behavior.

For the reasons listed it is desired to operate the calorimeter system under automatic control. Needed are minimal time to balance on startup and minimal disturbance of the voltage output during measurement periods.

Before the control can be synthesized, the dynamical characteristics of the calorimeter and its associated circuitry must be developed. A physical description of the calorimeter is presented in Chapter 2 which is the basis for the thermal and the electrical analyses that follow in Chapter 3.

CHAPTER 2

CONSTRUCTION OF THE COMPARISON CALORIMETER

A. The Basic Concept

The basic concept of the comparison calorimeter is illustrated in Figure 1.2. Flow of heat from the radioactive heat source through an insulator produces a temperature difference across the insulator. Similarly, flow of heat from the resistive heat source produces a temperature difference across a second insulator. These temperatures are sampled at fixed points in the insulators, and their relative equality is indicated by the temperature-balance sensor.

Each source-insulator-sensor subsystem is called a limb. Ideally, when both limbs are identical, a balance is obtained when the heat flow from the radioactive source exactly matches the heat flow from the resistive heater. Variations in the sink temperature which are common to both limbs affect both the same. These effects cancel in the temperature-balance sensor. Because the temperature difference between the two limbs is sampled, temperature variations which are not common to both limbs produce effects which do not cancel.

Shown dashed in Figure 1.2 is the basic feedback interconnection for control of the current to the resistive heat source which is necessary for automatic

control of the balanced condition. The output of the system is the voltage drop across a current-sampling resistor which is connected in series with the resistive heater. The voltage across the sampling resistor decays as the radioactive source decays. If the characteristics of the radioactive source are precisely known, the voltage can be determined from its initial value less the decay during the time since the value was established.

In practice, the two limbs need not be, and they cannot be, identical. They can, however, be made quite similar in their response characteristics and their temperature outputs. Because there are nonlinearities in the limb response characteristics, even small differences between the limbs result in some sensitivity to variations in the sink temperature. This sensitivity is reduced to a lower order as a result of the balanced configuration. Because of this sensitivity, it is necessary to control not only the relative temperature between the two calorimeter limbs, but also the sink temperature.

B. The Sink and Associated Constant-Temperature Bath

The two limbs are cut into a sink made of a solid block of copper which is finned and ducted to permit maximum heat transfer to a temperature-controlled bath. Figure 2.1 is a photograph of the sink which shows the fins. The fins tie the two limbs together thermally

Figure 2.1. Comparison Calorimeter Heat Sink

to minimize any differential temperature. Ducts are cut through the block from front to rear to permit maximum passage of coolant. Thermometer wells are located in the sink near both limbs. A third thermometer well is located in the central channel which is used to return interconnecting leads to the top of the sink.

The sink block is designed to fit in the inner channel of a laminar-flow bath. The bath coolant temperature is controlled to better than 0.001° Kelvin on a short-term (24-hour) basis. The bath and its control are described briefly in Appendix A. The bath is shown in Figure 2.2. Provision is made to operate the bath either with water or with oil as the coolant.

In normal calorimetric practice, water has been used as coolant because of its high thermal-transfer capacity. Water readily absorbs gases and dissolves salts. In time it builds up a high contamination level of corrosive chemicals. Oil has a low thermal-transfer capacity, but the buildup of corrosive contaminants in the oil is very slow. For this reason oil is the preferred coolant. Initial results indicate that because of the finned sink design, adequate cooling is obtained with oil.

C. Limb Construction

The limb is the heart of the calorimetric system. Figure 2.3 shows one limb of the calorimeter in cutaway section which reveals pertinent construction features.

Figure 2.2. The Temperature-Controlled Bath

Figure 2.3 Cutaway Sketch Showing Details of the Calorimeter

Following normal practice, concentric, cylindrical limb components are employed. The radioactive power source or a blank of similar thermal characteristics is centrally located. The blank is used to match the thermal response of the radioactively powered limb and the heated limb. Without the blank the heated limb responds faster because of the lower total heat capacity. Concentrically around the source is a heat-conducting cylinder of aluminum which serves as a support for the source and as the winding form for the resistive heater. Aluminum is used because of its high thermal conductivity and its low heat capacity.

Around the heat-conducting cylinder is a concentric insulating cylinder which serves as the support for the heat-conducting cylinder and as the winding form for the temperature-sensing resistors. Pyrex glass is used for the insulating cylinder because of its easy formability to tight tolerances, its immunity to deterioration under radiation and its low expansion coefficient. Forming is accomplished by first shrinking an oversized blank onto a precision cylindrical form, then grinding the outside surface to the required dimensions.

All cylindrical mating surfaces are machined to a sliding fit to create a reproducible geometry. Both limbs are constructed as nearly identical as possible. Either limb can be used with a heater power source or with a radioactive power source. Figure 2.4 is a

Figure 2.4. Conducting and Insulating Cylinders

photograph of the actual conducting and insulating cylinders which shows the lead dress.

D. Heater Construction

Two heater resistors (with a resistance of about 800 ohms each) are wound in a single, quadrafilar layer on each heat-spreading cylinder. Series or parallel connection can be used to produce a heater with a more desirable resistance. Quadrafilar winding permits use of a single layer and a single lead-wire exit point. Because of the small effective area encompassed by the windings, this technique also minimizes the inductance.

Evanohm¹ resistance wire is used in preference to other high-stability-alloy resistance wires, such as manganin (References 3 and 8). Advantages of Evanohm are its low temperature coefficient, its high resistivity and its extremely low thermoelectric emf versus copper (less than $0.2 \mu\text{V}/^\circ\text{Kelvin}$ in the normal temperature range). The insulation is glass fabric impregnated with a high-temperature enamel. To produce good resistance stability, the finished resistors are aged above 423°Kelvin for 48 hours following normal practice (Reference 3). The actual wire is selected from several similar batches on the basis of temperature stability in the 293.15° to $303.15^\circ\text{Kelvin}$ range. Temperature coefficients for each batch are

¹Registered Trade Mark, Wilbur B. Driver Co.

developed from measurements of sample resistors at three temperatures in the specified range (limited by equipment available). The resistance of the selected wire can be accurately approximated by

$$R = R_{298.15} [1 + 0.0000052(T - 298.15) - 0.00000005(T - 298.15)^2] \quad (2.1)$$

where

T = the temperature in $^{\circ}$ Kelvin,

R = the resistance at any temperature and

$R_{298.15}$ = the resistance at 298.15° Kelvin.

Lead-wires of copper are attached to the heater wires by silver soldering to minimize junction creep. Junction creep, which occurs primarily with soft solder, causes a change of the active length of the resistance wire and its resistance with time. The heaters are wound under minimum tension and are cushioned against the form by the insulation. No attempt is made to match the expansion characteristics of the wire and the form, since this limits the choice of materials with the desired thermal characteristics. The care given the heater windings and their terminations is necessary to produce resistors which exhibit low drift with time. Future designs could consider the possibility of matching the expansion coefficients of the wire and the form.

E. Construction of the Temperature-Sensitive Resistors

Two temperature-sensitive resistors of aged pure nickel wire are wound in a single quadrafilar layer on each insulating cylinder (in a manner similar to the heaters). Factors such as insulation, winding tension, lead-wire connection and aging are identical with those of the heater windings.

To achieve high sensitivity, a high-temperature coefficient material is required. Platinum (0.004 ohm/ohm °Kelvin) and nickel (0.006 ohm/ohm °Kelvin) have the highest temperature coefficients of all pure metals (Reference 9). Platinum is a noble metal which does not easily form chemical compounds, and it is used in most precision resistance thermometers. Nickel is finding increased application in low-cost, high-sensitivity apparatus for the measurement and for the control of temperature. Nickel is used in this application because of its higher temperature sensitivity, although experience has indicated that it may be less stable than platinum.

Analysis of the actual nickel wire shows that its nickel content is in excess of 99.7 per cent. The resistive temperature coefficients of this wire are determined by measurement of a sample resistor at several temperatures between 294.15° and 318.15° Kelvin. The resistance of this wire can be accurately approximated by

$$R = R_{301.15} [1 + 0.005100522(T - 301.15) + 0.000006227978(T - 301.15)^2] \quad (2.2)$$

where

T = the temperature in $^{\circ}$ Kelvin,

R = the resistance at any temperature and

$R_{301.15}$ = the resistance at 301.15° Kelvin.

The four temperature-sensitive resistors are adjusted to near equal value (approximately 200 ohms) for connection in a bridge as bridge arms. Equal bridge arms are desired to produce a null output when both power sources are inactive.

There are several techniques for adjusting resistors. The resistance can be raised either by adding a small series trimming resistor or by abrading some of the wire. The resistance can be lowered slightly either by shunting it with another resistor or by removing a length of the wire. Since there are two resistors on each insulator, abrasion would raise the value of both, and it cannot be employed. Because of the high temperature sensitivity the resistors cannot be satisfactorily measured or adjusted in a normal laboratory environment. For this reason, adjustment by shortening the wire length is impractical. Finally, high-value resistors with sufficient stability are not readily available, and adjustment by shunting is also impractical. The addition of small series trimming resistors is accomplished

using available materials. These resistors are of manganin wire wound on ceramic forms. Lead-wires are attached by silver soldering and the finished resistors are aged at 423° Kelvin for 48 hours. Finally, the trimmers are coated with a heavy layer of ceresin wax² for protection.

F. The Temperature-Balance Sensor

The four temperature-sensitive resistors are connected with their trimming resistors (where required) and with a link resistor to form a bridge as is indicated in Figure 2.5. This bridge is the temperature-balance sensor. The two resistors in each limb are connected in opposite bridge arms to produce maximum bridge output for a given temperature change. Manganin wire is used for the 0.4-ohm link resistor. Figure 2.6 shows the wiring in the base of the calorimeter where the bridge interconnections are made. It shows the link resistor and the two trimming resistors that are required.

One corner³ of the bridge is not predetermined, as is shown in Figure 2.5. Leads are brought out from

² Ceresin wax is a synthetic compound which remains semi-soft in the normal temperature range and which has an extremely low electrical leakage.

³ The term "bridge corner" refers to the circuit nodes at which interconnection of the bridge arms, the power supply and the null detector is accomplished. Because practical conductor materials have some resistance the corners are often difficult to identify precisely.

Figure 2.5 Bridge Interconnection Diagram

Figure 2.6. Calorimeter Base Wiring

both ends of the low-resistance link and these are connected to the ends of a precision slide-wire. By use of this circuit, the true bridge corner can be determined from outside the calorimeter when neither power source is active. As the sliding contact is moved, the corner can be effectively shifted from one side of the link to the opposite side. The bridge will be balanced when the slide-wire is at the true corner. If the bridge arms drift at different rates, the corner will change position on the slide-wire. This shift permits easy evaluation of the bridge stability.

Except for the trimming and the link resistors, the bridge arms are wound from the same batch of nickel wire and they are expected to drift at the same rate. Further, the value of the nickel resistors is large compared to the link and trimming resistors (200 ohms compared to 3 ohms in the worst case). Because of this, drift in the value of the bridge arms is predominantly that of the nickel resistors. The action of the bridge circuit cancels out uniform drift in all arms, and a stable balance point is expected with time. Initial tests confirm this behavior.

G. The Radioactive Source

Ideally, when a balanced condition is obtained in the temperature-balance sensor, the heater power and the radioactive source power are simply related. Under this

condition

$$P_h = K_1 P_s \quad (2.3)$$

where

P_h = the heater power dissipation,

P_s = the power output of the radioactive source and

K_1 = the ratio between the heater power and the radioactive power when the calorimeter is balanced.

This ratio includes the power-to-temperature conversion characteristics of the limbs and the temperature-to-voltage conversion characteristics of the bridge. In this equation, it is assumed that the calorimeter transient effects are negligibly small so that the ratio can be accurately approximated by a constant.

The heater power is a function of the heater resistance and the heater current which is related to the voltage drop across the series sampling resistor. This voltage drop is the output of the system. In mathematical notation,

$$P_h = (I_h)^2 R_h = \left\{ \frac{V_o}{R_s} \right\}^2 R_h \quad (2.4)$$

where

P_h = the heater power (as given above),

I_h = the heater current,

R_h = the heater resistance,

V_o = the system output voltage and

R_s = the resistance of the series sampling resistor.

Substitution of Equation 2.4 into Equation 2.3 yields

$$V_o = R_s \sqrt{\frac{K_1 P_s}{R_h}} \quad (2.5)$$

The active quantity of a single radioisotope decays according to the exponential decay law. Further, the average energy released per decay event is a constant. Because of this behavior, the power output of a single radioisotope source will decay according to the same exponential decay law as the active quantity. For such a source,

$$P_s = P_o e^{-\frac{t}{\gamma_1}} \quad \text{for } t > 0 \quad (2.6)$$

where

P_s = the effective power output of the source,

P_o = the value of P_s at $t=0$,

γ_1 = the source time constant and

t = time.

If the source response time is much longer than that of the calorimeter, the ratio K_1 in the above equations can still be approximated by a constant and

$$V_o = R_s \sqrt{\frac{K_1 P_o}{R_h}} e^{-\frac{t}{2\gamma_1}} \quad (2.7)$$

This equation gives the output voltage under idealized conditions which are only approximated in the operation of the actual system. Either directly or by implication, however, many of the desired radioactive-source characteristics can be deduced from the idealized behavior.

First, compared to the response time of the system, the time constant of the output voltage must be long. Equation 2.7 shows that when the system is maintained in balance the time constant of the voltage is twice that of the radioactive source. This effectively reduces the requirement on the source time constant by a factor of two.

Second, the entire output of the radioactive source should be heat. Practical radioisotopes emit alpha particles or beta particles which can be absorbed by a suitable container. In addition to these, gamma rays are emitted which cannot be as easily absorbed and they represent some lost power. Alpha particles are the easiest to absorb and their kinetic energy can be completely turned to heat.

Many of the more desirable source materials yield an occasional fission. Much of the energy released by fission is turned to heat, but it is an upsetting factor. Fission is dependent on the material configuration, the mass of the material and the confinement. By consideration of these factors, the fission problem can be reduced.

Third, it is desired to keep the radiation level of the source at a minimum so that it can be handled without severe personnel restrictions. Higher-energy gamma rays penetrate the container readily. The gamma rays need to be kept few in number and low in energy. A further problem is that the desirable source materials are biological poisons. This necessitates careful enclosure of the material in order to prevent a health hazard.

Finally, a source is desired which is composed of a single radioisotope and which decays to a stable element. Suitable radioisotopes generally decay in a family of steps in which each step has a different time constant and a different decay mechanism. Because isotopes with the same atomic number are extremely difficult to separate, there are always unwanted impurities. These impurities also have different decay mechanisms and different time constants than the desired material. The precise expression of the power output of a practical source is complicated by these additional radioisotopes. While the complete characterization is difficult, the decay of a practical source often can be adequately approximated by a single exponential term over a limited time period.

Equation 2.7 and the assumptions made in developing it do not give the desired power level of the source.

Logically the source should have a high power level in order to produce a high voltage output from the temperature-balance sensor for a given unbalance. Experience with precision resistors has indicated that a 10° rise above 301.15° Kelvin is permissible where low drift is desired. Tests on the calorimeter show that this condition can be met for both heater and temperature-sensitive resistors with a maximum source power of 3 watts.

The selected source is of plutonium 238 which decays by alpha emission. Accompanying the alpha emission is the emission of low-energy gamma rays. At the surface of the source the gamma radiation is low enough that the source can be safely handled without special precautions. The active material is in the form of microspheres of plutonium dioxide which are less easily assimilated by living tissue. The material is mixed with powdered aluminum oxide and the combination is sealed in the inner of three nested, stainless-steel canisters. Each canister is sealed in turn to prevent any possible release of the active material and the subsequent personnel hazard.

The plutonium 238 is sufficiently active that only about 2.5 grams of it is required. Because of the small mass and because the material is distributed throughout the aluminum-oxide powder, the possibility of neutron capture is minimized. As a result, power from fission

is also minimized. The power level of the source is 1.2 watts which produces a temperature rise of about 3° Kelvin in the limb. This time is well within the 10° maximum temperature change permitted.

Some power is also dissipated in the limbs by the temperature-balance sensor. To keep this power low and to ensure minimum drift in the nickel resistors, the bridge is operated at 0.2 watt maximum (bridge current equals 31.6 milliamperes maximum). Similar effects are produced by this heat source in both limbs. Because of the balanced condition, these effects tend to cancel.

The alpha particles produced as the source decays are the nuclei of helium atoms. The helium formed is in the gaseous state. Internal source pressure, which increases as the source decays and more helium is formed, will eventually necessitate replacement of the source. It is estimated that the source will remain safe to handle for at least ten years from the date of assembly.

Several other radioisotopes of plutonium are present in the source. These isotopes and those of their decay families complicate the prediction of the power level. The time constant of plutonium 238 is about 130 years (the half-life is about 90 years) and it produces a system voltage output with a time constant of 260 years. This time constant is quite long compared to

those of the calorimetric system. A discussion of the source material, the additional isotopes present and the necessary prediction techniques is contained in Appendix B. Shown in Figure 2.7 are the source and the source holder which is required to permit insertion and removal of the source from the calorimeter.

H. Anticipated Operation

Operation of the radioactive voltage source will be by a boot-strap technique. Comparison of the calorimeter output voltage to a known voltage will be done periodically. From these comparisons the effective source strength and the decay time constants can be determined. After the first pair of determinations, a prediction can be made to cover the next period. Data gathered over each succeeding time period will be used to refine the predictability of the output voltage.

There are practical limits to the accuracy with which the initial value of the voltage can be determined and with which the critical resistors can be measured. From the various data obtained, the limits of predictability will be calculated. It is estimated that several years of operation will be required before definitive values can be assigned. Once established, the radioactive voltage source can be used to "remember" an assigned voltage and to predict its change with time.

Figure 2.7. The Source Holder and the Radioactive Source

Before the output voltage can be used, the calorimeter must first be brought to balance. Then the voltage can be compared with some other voltage, a process which usually requires that a second balance is obtained. The thermal time constants of the calorimeter complicate the adjustment of the heater power by manual means. In addition, the 130-year time constant of the source necessitates readjustment of the system every five minutes in order to keep the voltage within one part in ten million. This repeatability is required if the voltage produced is to be comparable with the best available from saturated standard cells. For these reasons it is required that the system be automatically controlled.

In Equation 2.7, K_1 is defined as the constant which relates the two power levels in the system when the temperature-sensitive bridge is balanced. It is the balancing process that is subject to automatic control. In operation, it is neither necessary nor possible to bring the system to the true balance point. Thermal-noise voltage and residual, uncompensated thermoelectric emf's prevent the attainment of a perfect null.

An error of two parts per million in the balance, however, means an error of only one part per million in the output voltage. This improvement in error between power and voltage can be demonstrated as follows. The

voltage can be represented by

$$V = V_n(1 + \epsilon_1) \quad (2.8)$$

where

V = the actual voltage,

V_n = the nominal voltage and

ϵ_1 = the normalized error.

The power is a function of the voltage for a given resistor or

$$P = \frac{V^2}{R} = \frac{V_n^2}{R} (1 + \epsilon_1)^2 = \frac{V_n^2}{R} (1 + 2\epsilon_1 + \epsilon_1^2) \quad (2.9)$$

where

P = the actual power,

R = the resistance, and

the voltage terms are as given above. The nominal power (P_n) is

$$P_n = \frac{V_n^2}{R} \quad (2.10)$$

$$\text{The error of this power} = 2\epsilon_1 + \epsilon_1^2 \quad (2.11)$$

When $\epsilon_1 < 1$, then $\epsilon_1^2 \ll 1$ and ϵ_1^2 can be neglected.

For a two-part-per-million error in the balance,

$\epsilon_1 = 10^{-6}$ and $\epsilon_1^2 = 10^{-12}$ which is negligible compared to ϵ_1 .

When the radioactive source is placed in its limb a delay is initiated which lasts until the temperature

in that limb reaches an acceptable level near its final value. The initial source temperature might be above or below the sink temperature and this energy must be drained off or supplied by the system. The delay corresponds to the requirement in Equation 2.7 that K_1 had reached a point sufficiently close to its final value. During the delay, a balanced state does not mean that the voltage output is necessarily within the desired limit of the predictable value. It is not practical to automatically control the characteristics of this delay.

Under normal operating procedures the radioactive source will not be removed from the calorimeter. The delay is a problem only when removal is necessary; otherwise the control problem is simplified. On start-up or from undesired offset, it is required that an acceptable balance be reached as quickly as practical. A different control criterion is required when the balance is approached close enough that the output voltage can be utilized. Under this condition the heater power or current must not undergo significant shift if the system is within the required limit from balance.

The two operating criteria can be formulated in a performance index which is to be minimized. The optimal control technique can yield the control policy necessary to minimize these requirements if such a control exists. Before the control-theory optimization

techniques can be employed, however, the dynamical equations defining the behavior of the calorimeter must be developed. In Chapter 3 mathematical models are established which explain the observed behavior of the calorimeter.

CHAPTER 3

DEVELOPMENT OF THE MATHEMATICAL MODELS

A. Division of the System into Subsystems

In Chapter 2 the physical construction of the radioactive voltage standard is discussed. Problems experienced with manual control of the system lead to the desire for automatic and, specifically, for optimal control. Synthesis of the control cannot proceed, however, until the dynamical behavior of the system is formulated mathematically. In this chapter, mathematical models are developed which adequately explain the observed behavior of the calorimeter. For convenience, the system is first divided into a set of subsystems, each of which is then considered independently.

Figure 3.1 shows the practical division of the calorimetric system into subsystems. The key to this division lies in the operation of the temperature-balance sensor. Of the four temperature-sensitive resistors which form the heart of the temperature-balance sensor, two are located in each of the limbs. The thermal characteristics of these resistors are physically a part of the thermal characteristics of the limb. These resistors are electrically connected into a bridge. The bridge produces an output voltage which is related

Figure 3.1 Division of the Calorimetric System Into Subsystems

to the temperature in each of the two limbs. Effectively, a temperature-to-voltage conversion is performed which is independent of the thermal characteristics of the resistors. The output impedance of the bridge is considered as a part of the galvanometer subsystem. Thus, the idealized voltage-to-temperature conversion characteristic of the bridge forms the first subsystem.

Each of the limbs forms a subsystem which includes the thermal effects of the heater windings, the radioactive source or a blank (when one is used) and its two temperature-sensitive resistors. The normal input to each of these limbs (or subsystems) is a heat flow from either the radioactive heat source or the power dissipated in the limb's resistive heater. The subsystem output is a temperature which is related to the heat input and to the sink temperature by the dynamical behavior of the limb. Because the two limbs are similar in construction, their characteristics are approximated by a single analysis.

In order to realize the intended function of the radioactive voltage standard, a fourth subsystem is considered. Near balance, the voltage output from the temperature-balance sensor is low. For example, a one-part-in-ten-million shift from balance produces only about 3 nanovolts offset in the bridge output. To achieve the desired control, voltages of this magnitude must be detected. Because of this, a device with a

galvanometric-type input is used. Ideally suited to this application is the light-beam galvanometer-amplifier. In this device, light is reflected from a mirror on the galvanometer moving coil to a photosensitive transducer. By use of the light beam, nearly complete isolation between the galvanometer input and the photo-sensor output is achieved. In addition, a large amplification of the input voltage is produced. The dynamical characteristics of the galvanometer-amplifier in combination with the bridge output impedance contribute to the behavior of the total system. This combination is analyzed as the fourth subsystem.

The actual system response is determined from recorder plots of the bridge output following step power inputs to the limbs. Limited readability on the plots prevents analysis to better than about ± 2 per cent of the amplitude spread. Alternate techniques, such as determination of the system frequency response, are impractical for various reasons. First, accurate equipment for the generation and measurement of voltages in the desired frequency range is not available (about 0.001 Hz to 1 Hz). Second, the large input to output attenuation of the system complicates precise frequency response measurements.

It is desired that the ratio between heater power and the radioactive power is controlled to about two.

parts in ten million for a one-part-in-ten-million control of the output voltage (see Chapter 2, Section H). As the radioactive source decays (about 1.3 parts in 10^8 per minute) the heater power must be adjusted to remain within the desired limit.

The output of the bridge is a measure of the balance between the heater and radioactive power levels, and it can be fed back to control the heater power. In such a control, the dynamical response of the system is effective. At power balance, the bridge output is zero and in normal use offsets from this condition are small. Because these offsets are small, the 2 per cent accuracy is adequate to describe the limb characteristics.

Models of the four subsystems are developed from a combination of the predicted theoretical performance and the actual measured response. As discussed in Section C of this chapter, unknown component effects require the use of both techniques. Theoretical analysis also produces information useful in future development of systems of this type.

The bridge is used to measure the output temperature of the limbs. For this reason and because the bridge output impedance is required in analyzing the dynamic contribution of the galvanometer, the bridge model is developed first.

B. Development of the Bridge Model

A Wheatstone bridge is formed by interconnecting the four temperature-sensitive nickel resistors with appropriate trimming resistors and with a link resistor. Details of the construction of the temperature-sensitive resistors and of the bridge are contained in Chapter 2, Sections E and F.

Figure 3.2a shows the basic circuit. At the top are the low-resistance link and the external slide-wire which is connected in parallel with it. Power to the bridge is supplied through the slide-wire contact. The link resistor and the slide-wire form a three-terminal, delta-connected resistor set. With the delta configuration, the bridge corner cannot be directly visualized. By making a Δ -Y transformation, the delta resistor set can be replaced by an equivalent, wye resistor set. With the wye configuration, the bridge corner is readily apparent, as can be seen on Figure 3.2b.

The link has a low resistance. Because of this the wye resistors in series with the two bridge arms also have a low resistance. Bridge power is supplied through the third wye resistor. Depending on the position of the slide-wire moving contact, the value of this resistor varies from near zero to about half of the slide-wire resistance. Effects of the slide-wire on the bridge voltage and current are eliminated by operating

a) The actual bridge circuit

Δ -Y
EQUIVALENT
CIRCUITS

BRIDGE
POWER
SOURCE

b) The transformed bridge circuit

Figure 3.2 The Bridge

the bridge from a constant-current source. Regulation of this source is required to provide constant sensitivity.

One form of the equation for the no-load bridge output voltage is

$$V=I \left[R_2 \left\{ \frac{R_1+R_4}{R_1+R_2+R_3+R_4} \right\} - R_4 \left\{ \frac{R_2+R_3}{R_1+R_2+R_3+R_4} \right\} \right] \quad (3.1)$$

where

V = the output voltage,

I = the bridge current and

R_1 , R_2 , R_3 and R_4 = the bridge arm resistors which are indicated on Figure 3.2b.

The terms inside the parentheses are the respective ratios of the currents in resistors R_2 and R_4 to the total bridge current. The value of each resistor is the sum of the resistance of the appropriate nickel resistor, the resistance of its trimmer and the wye contribution of the link, if any.

The quadrafilar lay (used in winding the nickel wire) produces pairs of resistors in each limb which have approximately equal wire lengths and resistances. A pair of trimming resistors is required for the nickel resistors in one limb in order to match their values to those in the second limb. Assuming equal nickel resistors in each limb, equal trimming resistors and equal link contributions to R_1 and R_3 ,

$$R_2 = R_1 - R_\ell \quad (3.2)$$

and

$$R_4 = R_3 - R_\ell \quad (3.3)$$

where

R_1 , R_2 , R_3 and R_4 are as above and

R_ℓ = the link-resistance contribution.

Substitution of Equations 3.2 and 3.3 into Equation 3.1 produces the simplified output-voltage relationship

$$V = \frac{I}{2}(R_2 - R_4) \quad (3.4)$$

The conditions specified in developing Equation 3.4 are realized to a reasonable approximation in the construction of the actual calorimeter. The calculated error introduced because the assumed conditions are not exactly met is less than 0.02 per cent, which is negligible.

When the bridge is operated with a constant-current supply, the output voltage is a linear function of resistors R_2 and R_4 , as is indicated by Equation 3.4. Because of the particular bridge circuit used, linear behavior can be realized only with a constant-current supply. The bridge current is controlled to 30 milliamperes which is less than the 31.6 milliamperes maximum specified in Chapter 2, Section G.

Equation 2.2 gives the resistance of the nickel resistors as a function of temperature. Inserting the 301.15° Kelvin value of the nickel resistors in Equation

2.2 and adding the necessary trimming resistance yields

$$R_2 = 2.0 + 204.2[1 + 0.005100522(\Delta T_1) + 0.000006227978(\Delta T_1)^2] \quad (3.5)$$

and

$$R_4 = 206.2[1 + 0.005100522(\Delta T_2) + 0.000006227978(\Delta T_2)^2] \quad (3.6)$$

where

R_2 and R_4 are as given above,

$\Delta T_1 = (T_1 - 301.15)$, the temperature offset from 301.15° Kelvin in one limb and

$\Delta T_2 = (T_2 - 301.15)$, the temperature offset from 301.15° Kelvin in the second limb.

Substituting the Equations 3.5 and 3.6 into Equation 3.4 and inserting the magnitude of the bridge current yields the idealized temperature-to-voltage conversion

$$V = 0.01562(\Delta T_1) + 0.00001908(\Delta T_1)^2 - 0.01578(\Delta T_2) - 0.00001926(\Delta T_2)^2 \quad (3.7)$$

In the operation of the calorimeter, the maximum temperature rise in the limbs with the 1.2-watt source is less than 3° Kelvin. With this temperature rise, the squared term contributes only about 0.4 per cent to the bridge output voltage. For any appreciable temperature difference, the galvanometer will be effectively saturated. In addition, there is a practical bound on the power that can be fed to the heater. Thus, the

output expressed by Equation 3.7 is useful only when the two temperatures differ by much less than 3° Kelvin. Under this condition, effects of the second-order terms can be neglected. Neglecting these terms, the temperature-to-voltage conversion is

$$V = 0.01562(\Delta T_1) - 0.01578(\Delta T_2) \quad (3.8)$$

In the limbs, different portions of the nickel resistors sense slightly different temperatures. The effective temperature sensed is

$$\Delta T = \sum_i K_i \Delta T_i \quad (3.9)$$

where

ΔT_i = the temperature affecting the fraction K_i of the winding ($\sum_i K_i = 1$) and $i = a, b, \dots$, the identifying notation for different fractions of the winding.

Substitution of Equation 3.9 into Equation 3.8 yields

$$V = 0.01562 \sum_i K_i \Delta T_{1i} - 0.01578 \sum_j K_j \Delta T_{2j} \quad (3.10)$$

where i is used as the identifying notation for fractions of the winding in limb 1 and j is used similarly for limb 2. This form of the temperature-to-voltage conversion equation is the most useful in developing the dynamical system response.

All of these equations are for the condition of no

output load, which is the idealized case. In terms of a Thevenin-equivalent circuit, it is the voltage term which has been developed. The effective impedance looking into the bridge from its output affects the dynamical behavior of the galvanometer, which is discussed in Section D of this chapter. For the low-temperature differences experienced in the calorimeter, the bridge resistors remain essentially constant at about 206 ohms. Because all bridge arms have the same resistance, the approximate output impedance is a resistive 206 ohms.

C. Development of the Limb Models

In Chapter 2, Section A, the basic heat-flow concept is presented. Details of the calorimeter (which is designed to approximate this behavior) are given throughout the remainder of that chapter. As presented, the heat-flow concept is idealized; the actual behavior of the calorimeter is more complex. Mathematical models are developed in this section which are sufficient to explain the observed thermal behavior of the limb subsystems. Because the two calorimeter limbs are similar, a single analysis is used to develop the models for both.

The limb behavior is determined by a combination of theoretical computation and analysis of the measured response. For several reasons, this technique gives the most-complete description of these subsystems. First, the thermal effects of the insulation on the Evanohm and

the nickel wires can only be approximated theoretically. Second, sufficiently accurate measurement of some of the dimensions is impractical. Examples of this include the clearance air gap between sliding cylindrical surfaces and the effective air gap where one surface is formed by a winding. Further, the thermal conductivities, the specific heat capacities and the surface emissivities are known only imprecisely for the materials used.

Among the problems encountered in developing the limb model is the necessity of significantly correcting initial estimates of the behavior. It is also necessary to separate the actual calorimeter response from the total measured response which includes characteristics introduced by the test apparatus. In order to simplify the presentation, equipment response characteristics are eliminated and only the actual limb response is presented.

Sketched in Figure 2.3 is one limb of the calorimeter in cutaway section. From this sketch and from the limb construction details in Chapter 2, most of the limb characteristics can be deduced. The aluminum heat-spreading cylinder and the source are isolated axially from the sink by a large air gap. Radially, however, the path to the sink through the concentric insulating media is short. Thus, it is assumed that all of the heat flows through the concentric insulating media.

In Figure 3.3 is a simplified limb cutaway which is sketched radially through the central axis and which shows the two significant heat-flow paths. The thermal properties of the calorimetric components are distributed throughout their volume. A complete analysis employing the distributed parameters is complicated by the non-linear radiation effects. There are, however, large differences in the heat flow between the radial and the axial directions. In addition, the significant heat capacitance is located in regions of high conductivity. Thus, the effects of the distributed parameters can be approximated adequately by lumped equivalent components. The use of lumped parameters simplifies the mathematical description of the limb subsystems. In Figure 3.4 the lumped-parameter block diagram of the two heat-flow paths is shown.

Conductivity in the aluminum heat-spreading cylinder and the radioactive source is high compared to the conductivity of the insulating media which form the heat-flow paths. Thus, the heat-spreading cylinder and the radioactive source are modeled as lumped heat capacities with infinite internal conductivity. The heat-spreading cylinder acts as a uniform temperature source to the rest of the limb.

For heat flow in the radial direction, the glass cylinder presents a large surface area, but the heat flows through only a small thickness of the material.

Figure 3.3 Heat Flow in the Calorimeter Limb

Figure 3.4 Lumped Parameter Heat-flow Diagram

Because of the large area-to-thickness ratio, conductivity through the insulating cylinder in the radial direction is large. In this direction the insulator conductivity is assumed infinite. In the axial direction, however, the area-to-thickness ratio is small and the conductivity is likewise small. Because of this, axial insulator conductivity is neglected.

For both the heater and the temperature-sensing windings, the insulation thickness is small and its conductivity is large, as are the conductivities of the wires themselves. The heat capacity of the windings is small compared to the heat capacity of their associated cylinders. These effects are combined with those of the aluminum cylinder or the appropriate portion of the glass cylinder. The thermal effects of the wire and the insulation alike are assumed to be the same as those of nickel (nickel is the predominant material in the heater and the only material in the temperature-sensing wire).

The source is represented as a lump with heat capacity C_1 on Figure 3.4. Similarly, the aluminum heat-spreading cylinder and the heater are modeled as the single heat capacity C_2 . In addition, the heat capacity of the central portion of the insulating cylinder and its winding is represented by C_3 .

The conductivity of air in the gaps is sufficiently low that it cannot be considered infinite in the radial

direction as is done with the glass. In the axial direction, however, the air gap conductivity is negligible. Because of this, the heat flow is divided into the two paths indicated on Figure 3.3. End effects around these paths are neglected.

In the secondary path, heat flows directly to the insulating cylinder through the ends of the heat-spreading cylinder. The heat capacity of this portion of the insulating cylinder is small compared to that of the heat-spreading cylinder and it is neglected. This heat then flows to the sink across an air gap by radiation and conduction (G_1 and H_1 on Figure 3.4, respectively).

In the main path, heat flows from the heat-spreading cylinder through an air gap by radiation and by conduction to the insulating cylinder. The radiation and conduction in this air gap are represented by blocks G_2 and H_2 respectively on Figure 3.4. This heat then flows across the central portion of the insulating cylinder and through a similar portion of a second air gap by radiation and conduction. This second radiation and conduction are represented by blocks G_3 and H_3 respectively on Figure 3.4.

The temperature of the sink is closely controlled, as is mentioned in Chapter 2, Section B, and as is discussed in Appendix A. For this reason, the sink temperature is assumed constant at 301.15° Kelvin.

Referring to Figure 3.3 and Figure 3.4, the flow of heat from the source to the sink is readily visualized. Heat flows from the source or from the resistive heater in three ways. First, some heat flows into (or from) the source-heat-spreading-cylinder heat capacity proportional to the derivative of its temperature ($C_1 + C_2$ or just C_2 if the source is not present). Second, some heat flows across the air gap between the heat-spreading cylinder and the central portion of the insulating cylinder by radiation (G_2) and by conduction (H_2). This flow is dependent on the temperatures of the two cylinders and it is in the main heat-flow path. Third, in the secondary heat-flow path some heat flows from the ends of the heat-spreading cylinder directly to the sink by conduction (H_1) and by radiation (G_1). The sum of these three heat flows must equal the heat supplied by the source or the heater.

After the heat flows across the first gap in the main path it flows in two additional ways. First, some of it flows to (or from) the heat capacity (C_3) of the insulating cylinder, depending on the rate of change of the cylinder temperature. The remainder flows to the sink across the final air gap in the main path by radiation (G_3) and conduction (H_3). The sum of these two equals the heat-flow input to this section. The heat flow described above is expressed mathematically by

$$\begin{aligned}
Q(t) = & (C_1 + C_2) \frac{dT_a}{dt} + H_2(T_a - T_b) \\
& + G_2 [(T_a)^4 - (T_b)^4] + H_1(T_a - T_o) \\
& + G_1 [(T_a)^4 - (T_o)^4] \quad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
H_2(T_a - T_b) + G_2 [(T_a)^4 - (T_b)^4] = \\
C_3 \frac{dT_b}{dt} + H_3(T_b - T_o) + G_3 [(T_b)^4 - (T_o)^4] \quad (3.12)
\end{aligned}$$

where

$Q(t)$ = the heat flow,

C_1 , C_2 and C_3 = heat capacities of the source, the heat-spreading cylinder (with the heater) and the central glass insulator section (with its windings), respectively,

H_1 , H_2 and H_3 = the conductance terms associated with their respective air gaps,

G_1 , G_2 and G_3 = the radiation factors associated with their respective air gaps and

T_a , T_b and T_o = the temperature of the source, the temperature-sensitive windings and the sink, respectively, in °Kelvin.

The thermal constants are developed in Appendix C.

In terms of the difference temperatures which are used in Equation 3.10, Equations 3.11 and 3.12 can be rewritten

$$\begin{aligned}
Q(t) = & (C_1 + C_2) \frac{d(\Delta T_a)}{dt} + H_2 (\Delta T_a - \Delta T_b) \\
& + G_2 \left[(\Delta T_a + T_o)^4 - (\Delta T_b + T_o)^4 \right] \\
& + H_1 (\Delta T_a) + G_1 \left[(\Delta T_a + T_o)^4 - (T_o)^4 \right] \quad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
H_2 (\Delta T_a - \Delta T_b) + G_2 \left[(\Delta T_a + T_o)^4 - (\Delta T_b + T_o)^4 \right] = \\
C_3 \frac{d(\Delta T_b)}{dt} + H_3 (\Delta T_b) + G_3 \left[(\Delta T_b + T_o)^4 - (T_o)^4 \right] \quad (3.14)
\end{aligned}$$

In these equations, as in Equation 3.10,

$$\Delta T = (T - 301.15^\circ \text{ Kelvin}) \quad (3.15)$$

where

$T_o = 301.15^\circ \text{ Kelvin}$, which is the bath temperature.

The radiation terms can be further simplified. A generalized form of the radiation heat flow is

$$Q_g(t) = G \left[(\Delta T_d + T_o)^4 - (\Delta T_e + T_o)^4 \right] \quad (3.16)$$

where

G = the radiation factor,

ΔT_d = the rise above bath temperature of one surface,

ΔT_e = the rise above bath temperature of the second surface and

$T_o = 301.15^\circ \text{ Kelvin}$, which is the bath temperature.

Normalization of the two temperature terms yields

$$Q_g(t) = G(T_o)^4 \left[(1 + \delta_d)^4 - (1 + \delta_e)^4 \right] \quad (3.17)$$

where all terms are as above except

$$\delta_d = \frac{\Delta T_d}{T_o} \quad \text{and}$$

$$\delta_e = \frac{\Delta T_e}{T_o} .$$

Expansion of the normalized temperature terms by multiplication yields

$$Q_g(t) = 4G(T_o)^4 \left[(\delta_d - \delta_e) + 3/2(\delta_d^2 - \delta_e^2) + (\delta_d^3 - \delta_e^3) + 1/4(\delta_d^4 - \delta_e^4) \right] . \quad (3.18)$$

This same result is obtained by expanding Equation 3.16 in a Taylor's series for a function of two variables. In this expansion, $T_{do} = T_{eo} = T_o$, which leaves the Q or heat-flow term unaffected by the basic expansion temperature.

In the calorimetric system, the maximum ΔT is less than 3° Kelvin, which means that both δ_d and δ_e are less than 1 per cent in magnitude. Because of this, the second-order terms affect the heat flow only about 1.5 per cent and the effects of the higher-order terms are negligible.

For each heat flow by radiation there is a corresponding heat flow by conduction. In this system the

conduction terms are about ten times as large as the radiation terms when air is the medium in the limbs. Thus, the effect of the second-order radiation terms is reduced to only about 0.15 per cent, which is negligible. Under this condition, Equation 3.18 simplifies to

$$Q_g(t) = 4G(T_o)^4(\delta_d - \delta_e) \quad (3.19)$$

or, reinserting the ΔT values,

$$Q_g(t) = 4G(T_o)^3(\Delta T_d - \Delta T_e). \quad (3.20)$$

Because T_o is a constant and only first-order difference-temperature terms are present, these equations are linear. The term $4G(T_o)^3$ behaves in the same manner as a conductance. By inserting appropriate temperatures and radiation constants in Equation 3.20, all of the radiation terms in Equations 3.13 and 3.14 can be linearized. Linearization of Equations 3.13 and 3.14 yields

$$Q(t) = A_1 \frac{d(\Delta T_a)}{dt} + (B_1 + B_2)(\Delta T_a) - B_2(\Delta T_b) \quad (3.21)$$

and

$$B_2(\Delta T_a) = A_2 \frac{d(\Delta T_b)}{dt} + (B_2 + B_3)(\Delta T_b) \quad (3.22)$$

where the combined lumped parameters are

$$A_1 = C_1 + C_2,$$

$$A_2 = C_3,$$

$$B_1 = H_1 + 4G_1(T_O)^3,$$

$$B_2 = H_2 + 4G_2(T_O)^3 \text{ and}$$

$$B_3 = H_3 + 4G_3(T_O)^3.$$

The A terms are heat capacities while the B terms are, effectively, conductances. The values of these terms are computed in Appendix C.

The simplified block diagram is shown in Figure 3.5(a) and the electrical analog of this thermal model is shown in Figure 3.5(b). The current input is the analog of the heat input, while the voltages at the two circuit nodes are the analog of the temperatures. In the electrical circuit, the capacitances are the analog of the heat capacities and the conductances are the analog of the thermal conductance terms. This electrical circuit is helpful in visualizing the behavior of the limb sub-systems.

In the transform domain (with no energy stored in the heat capacities initially) the Laplace transforms of Equations 3.21 and 3.22 are

$$\Delta T_a(s) = \frac{\left(\frac{1}{A_1} \right) \left(s + \frac{B_2 + B_3}{A_2} \right) Q(s)}{s^2 + \left(\frac{B_1 + B_2}{A_1} + \frac{B_2 + B_3}{A_2} \right) s + \frac{B_1 B_2 + B_1 B_3 + B_2 B_3}{A_1 A_2}} \quad (3.23)$$

and

a) Lumped parameter heat-flow diagram

b) Electrical analog of the lumped parameter heat-flow diagram

Figure 3.5 Simplified Limb Representation

$$\Delta T_b(s) = \frac{\left(\frac{B_2}{A_1 A_2} \right) Q(s)}{s^2 + \left(\frac{B_1+B_2}{A_1} + \frac{B_2+B_3}{A_2} \right) s + \frac{B_1 B_2 + B_1 B_3 + B_2 B_3}{A_1 A_2}} \quad (3.24)$$

These two equations can be combined using the values of K_i specified in Equation 3.10,

$$\sum_i K_i \Delta T_i(s) = \frac{\left(\frac{K_a}{A_1} \right) \left(s + \frac{B_2 + K_b B_2 / K_a + B_3}{A_2} \right) Q(s)}{s^2 + \left(\frac{B_1+B_2}{A_1} + \frac{B_2+B_3}{A_2} \right) s + \frac{B_1 B_2 + B_2 B_3 + B_1 B_3}{A_1 A_2}} \quad (3.25)$$

Other K_i in the subsystem are at $\Delta T_i = 0$.

The actual values of the conductances, heat capacities and the winding fractions can now be inserted from Appendix C for each of two cases. In the first case the radioactive source or a blank with similar heat capacity is used. For this case

$$\sum_i K_i \Delta T_i(s) = \frac{Q(s) (1.2704 \times 10^{-4}) (s + 18.210)}{(s + 1.4517) (s + 0.07079)} \quad (3.26)$$

The second case of interest is where the source and source holder are not used. For this case

$$\sum_i K_i \Delta T_i(s) = \frac{Q(s) (2.399 \times 10^{-4}) (s + 18.210)}{(s + 1.7247) (s + 0.11254)} \quad (3.27)$$

In both of these equations time is in minutes and Q is required to be in joules per minute for T_i in $^{\circ}$ Kelvin. The models represented by Equations 3.26 and 3.27 fit the

average observed response of both limbs within the possible error of the measurements, which is about 2 per cent.

Where required, the effective temperature output of the limb can be computed for other heat-source masses. Values of the heat capacity necessary for term A_1 in Equation 3.25 must be computed for each case as is done for those presented.

D. The Galvanometer Model

Because very small voltage shifts in the output of the bridge must be sensed, a device with a galvanometric type input is required (as introduced in Section A of this chapter). The dynamical characteristics of this device are a part of the total system response. These characteristics are analyzed in this section as the fourth subsystem.

Determination of the required sensitivity is accomplished by analysis of the limb response to its input power. The power level of the actual radioactive source is about one watt (Chapter 2, Section G). For such a power, the temperature rise in the limb is about 2° Kelvin and the corresponding bridge output is about 30 millivolts (shown by Equation 3.10). For a one-part-in-ten-million offset in the output voltage, two parts in ten million of the source power (or the bridge output due to the limb temperature rise) must be detected

with certainty (shown in Section H of Chapter 2).

Thus the detector must be able to resolve at least 6 nanovolts. For control to the desired limit, somewhat lower voltage offsets than this must be resolved. A practical sensitivity limit is about one part in ten million of the maximum input or 3 nanovolts for the one-watt source power.

Ideal for this application is the light-beam, galvanometer-amplifier whose output follows its input but with a large amplification. In this device, light is reflected from a mirror on the galvanometer moving-coil to a photosensitive transducer. By means of the light-beam-photo-sensor mechanism, a nearly complete input-output isolation is produced in addition to a high gain.

The angular rotation characteristics of moving-coil galvanometers have been thoroughly investigated (References 10, 11 and 12). Over the range in which the output voltage of the photo-sensor is proportional to the angular coil deflection, this behavior is expressed by

$$\frac{d^2 V_{go}(t)}{dt^2} + K_2 \frac{dV_{go}(t)}{dt} + K_3 V_{go}(t) = K_4 V_{gi}(t) \quad (3.28)$$

where

V_{go} = the device output voltage,

V_{gi} = the input voltage,

K_2 = a constant which is related to the electrical resistance seen by the galvanometer and to

other damping factors,

K_3 = a constant which is related to the counter-torque characteristics of the coil suspension and

K_4 = a constant which is related to the device gain and to the torque on the coil due to current flowing in it.

In the Laplace-transform domain,

$$V_{g\theta}(s) = \frac{K_4 V_{gi}(s)}{s^2 + K_2 s + K_3} \quad (3.29)$$

In order to prevent undue response to some particular noise frequency range, the galvanometer must be operated either in the critically damped or in the overdamped condition. Further, it is desired that the galvanometer affect the overall system characteristics as little as possible. This condition is met when the galvanometer has the fastest response. Operation at critical damping produces the fastest response of any non-underdamped condition. For these reasons the galvanometer is critically damped. This behavior is achieved when Equation 3.29 has a double pole. Thus,

$$V_{g\theta}(s) = \frac{K_4 V_{gi}(s)}{(s+K_5)^2} \quad (3.30)$$

where

$$K_5 = \frac{K_2}{2} = \sqrt{K_3} = \frac{2\pi}{t_o} \quad (t_o \text{ is the undamped or free})$$

galvanometer period).

In the light-beam-photo-sensor-galvanometer system, the useable resolution is limited by residual thermoelectric emf's in the circuit and by thermal agitation noise. When making sub-microvolt measurements, care must be taken to insure that these limiting factors do not produce a meaningless response.

In construction of the bridge, copper lead-wires are used in the critical circuits. Necessary junctions are made either in the sink base or in close thermal contact with their opposite member in order to minimize the temperature difference between thermocouple pairs. These techniques minimize the residual thermoelectric emf's. Tests indicate that the thermoelectric emf's in the bridge circuit are acceptably low.

Thermal agitation noise appears as a voltage output from the galvanometer-amplifier which is difficult to distinguish from wanted signals of similar magnitude (References 11, and 13). Careful selection of the galvanometer characteristics assures that the noise output is sufficiently improbable that it can be neglected. A criterion is developed in Reference 11 which approximates the smallest emf that can be detected with certainty by a single indication. This criterion is

$$V_{gi} \text{ min} \approx 4 \times 10^{-10} \sqrt{\frac{R_c}{t_o}} \quad (3.31)$$

where

R_c = the circuit resistance at critical damping

and

t_0 = the galvanometer free period in seconds.

The development of this equation is based on the probability of an indication being due to thermal noise of about 0.0006. Because the resistors in this circuit are all of solid wire and approach the ideal-resistor behavior closely, no other sources of thermal noise are present.

As developed in Section B of this chapter, the bridge output resistance is about 206 ohms. This is the resistance which is seen by the galvanometer, and it is the external-critical-damping resistance (CDRX) required by the galvanometer for critical damped operation. The galvanometer coil resistance is about 44 ohms, making the circuit resistance about 250 ohms.

For a 250-ohm R_c and a 3-nanovolt sensitivity, the requirement specified in Equation 3.31 is met by a galvanometer with a 4.44-second free period. Thus, the specifications of the desired galvanometer are

$$t_0 = 4.44 \text{ seconds,} \quad (3.32)$$

$$\text{CDRX} = 206 \text{ ohms and} \quad (3.33)$$

$$\text{galvanometer coil resistance} \approx 44 \text{ ohms.} \quad (3.34)$$

These requirements are readily met by available galvanometer designs. Careful construction of the light

path yields essentially linear operation over several orders of magnitude and a voltage gain of above 10^6 . Use of this design criterion insures that thermal noise is not a problem in the system.

For the galvanometer described, the output voltage equation becomes

$$V_{go}(s) = \frac{7194K_y V_{gi}(s)}{(s + 84.82)^2} \quad (3.35)$$

where

K_y = the galvanometer gain ($K_3/7194$), a voltage ratio, and

the time is changed to minutes in order to match the units used in the limb models.

As the input voltage to the galvanometer increases, the output will become nonlinear, and for a small further increase it will saturate. Under this condition no information is available from the galvanometer concerning the state of the output except that the input is above the saturation level. Additional increase in the input voltage can damage the galvanometer, and this must be avoided.

Several techniques can be employed to circumvent the galvanometer saturation problem, most of which require a switching of the galvanometer function before saturation is reached. First, the galvanometer sensitivity can be decreased by use of a suitable Ayrton

shunt which retains the proper CDRX while reducing the effective galvanometer input voltage (Reference 11). Second, reduction of bridge operating current also reduces galvanometer sensitivity. Third, the galvanometer can be replaced with a lower-sensitivity detector-amplifier. These three techniques require switching of the circuit function at suitable preset levels. Alternately, a nonlinear feedback can be employed to limit the galvanometer sensitivity as a function of input level. The necessary feedback or switching path around the galvanometer is shown dashed on Figure 3.1 as an additional control input. All of these techniques prevent saturation and yield information on the balanced condition of the bridge for any possible input.

The change of system characteristics by switching to a lower sensitivity or by using a different detector causes only slight complication of the control problem. Needed in the control synthesis are the characteristics near balance where the heater power is not constrained to operation on a boundary. Thus, only the characteristics of the galvanometer for low, non-saturated inputs are presented.

E. The Complete System Dynamical Equations

The models of the four subsystems are developed in Sections B, C and D of this chapter. In this section

these models are combined to produce the model of the complete system. In the process of combination some further simplification is possible.

The behavior of the radioactively powered limb is given by Equation 3.26 (in the Laplace-transform domain with no initial energy storage). Poles for this system have time constants of 0.6888 minute and 14.126 minutes.

In contrast, the heat input to this limb also yields a decaying exponential time response but with an effective time constant of about 130 years (see Chapter 2, Section G, and Appendix B). When the source is inserted into the limb, the dynamic response is initiated. The response terms related to the limb characteristics decay exponentially towards their final value from which they differ negligibly after some time has passed. Following this initial time period, the limb output behaves in a manner similar to that of the source. Thus, the radioactively powered limb output can be approximated by a constant term multiplied by the source response characteristics. Because the effective source time constant is long, the constant term is approximated by solving for the final value of the limb response to a unity step function source. In mathematical notation,

$$K_i \Delta T_i(s) = 0.02251 Q_s(s) \quad (3.36)$$

where

0.02251 = the final value of Equation 3.26 to
a unity step in $^{\circ}\text{Kelvin minutes joules}^{-1}$
and

$Q_s(s)$ = the Laplace transform of the source
output in joules per minute.

In operation of the calorimeter, the source is not removed from the limb under normal conditions. Thus, Equation 3.36 represents the only useful output of the radioactively powered limb. It is impractical to control the delay initiated on insertion of the source. The complete dynamic response of the heater-powered limb, however, is effective in determining the system output.

The Laplace transform of the system response is obtained by combining Equations 3.36, 3.27, 3.10 and 3.35. The heater-powered limb normally does not have the heat capacity matching blank and it is represented by Equation 3.27. Limb 1 in Equation 3.10 is chosen as the heated limb while limb 2 is radioactively powered for convenience. The combination of these equations yields

$$V_{go}(s) = \left[\frac{7194K_6}{(s + 84.82)^2} \right] \times \left[\frac{(3.747 \times 10^{-6})(s + 18.210) Q_h(s)}{(s + 1.7247)(s + 0.11254)} - (3.552 \times 10^{-4}) Q_s(s) \right] \quad (3.37)$$

where

$V_{go}(s)$ = the transformed galvanometer-amplifier output voltage,

K_6 = the galvanometer voltage gain,

$Q_s(s)$ = the transform of the source power in joules per minute and

$Q_h(s)$ = the transform of the heater power in joules per minute.

In this equation, energy stored initially in the heater-powered limb is ignored.

Analysis of the system response shows that it is dominated by the pole at -0.11254 . The galvanometer poles, which are about 800 times further from the origin than this pole, contribute little to the system response. For example, only 0.02 per cent of the response to a step function and only a few per cent of the response to any normal input are supplied by the galvanometer. Physically, the galvanometer response is sufficiently fast that its output behaves effec-

tively like the system output. For this reason the effects of the galvanometer poles are neglected. In addition, replacement of the galvanometer by any amplifier having faster poles and a different gain merely changes the gain term K_6 . The transform of the simplified equation is

$$V_{go}(s) = \frac{(3.747 \times 10^{-6})(s + 18.210) K_6 Q_h(s)}{(s + 1.7247)(s + 0.11254)} - 3.552 \times 10^{-4} K_6 Q_s(s). \quad (3.38)$$

From this, the corresponding differential equation describing the system response is readily obtained and is

$$\frac{d^2 V_{go}}{dt^2} + 1.8372 \frac{dV_{go}}{dt} + 0.19410 V_{go} = 3.747 \times 10^{-6} K_6 \frac{dQ_h}{dt} + 6.823 \times 10^{-5} (Q_h - 1.0104 Q_s). \quad (3.39)$$

Because of the long time constant of the chosen source compared to the system time constants, derivatives of Q_s , the source power, are negligible. Equation 3.31 describes the behavior of the radioactive voltage source for heater-source inputs under normal operation. This equation is the basis around which the control is developed in Chapter 4.

CHAPTER 4

DEVELOPMENT OF THE CONTROL

A. The Control Problem

The control required for the radioactive voltage standard is unusual in that the system output voltage is developed across a resistor through which the system input current flows (see Figure 2.5). As modeled by Equation 3.39, the galvanometer output provides information on the relative balance between the heater power and the radioactive-source power. Although the galvanometer output is not the desired system output, it is useful in controlling the system behavior. Controls for each of two criteria are required as is introduced in Chapter 2, Section H. A detailed description of these criteria, suitable performance indices and the control problem are presented in this chapter.

After the two power inputs are brought to near balance, the system is operated under the primary control, in which a useful system output is produced. Under this criterion there are two requirements of the control policy. First, it is necessary that the control hold the galvanometer output near zero. This condition insures that the two power inputs remain in balance. Second and equally important, it is necessary that the heater current (or power) be held close to the

value obtained when the two power sources are in balance. This condition guarantees low drift in the system output voltage (which is developed from the heater current).

As even small offsets from the desired terminal states preclude obtaining a useful system output, the primary criterion is on the terminal states where a useful output is produced. Because rapid changes in the system input (heater power) are prevented under this criterion, response of the system to undesired offsets is comparatively slow. For this reason, the primary control is employed only when the system is sufficiently near balance and use of the output is necessary.

Significant offsets in the balanced condition are occasionally experienced (such as during start-up of the system). During periods of such imbalance, the heater power is not directly related to the radioactive-source power. Balance between these two powers is required before a useful system output is produced (see Chapter 2, Section H). Preferred during such periods is a secondary control policy which brings the system to near the power balance rapidly and then holds it in this condition. In order to accomplish this, it is necessary to release the tight control on the rate of change of the heater power. As with the primary criterion, the secondary criterion is on the terminal states of the

system. Because of the more rapid recovery of the power balance following unwanted offsets, the secondary control is employed whenever use of the system output is unnecessary.

A complication occurs because of the boundedness of the heater power. For practical reasons the heater current is limited to the range between zero and I_h max. (see Chapter 2, Section G). This condition suggests the use of a time-optimal or bang-bang control under the secondary criterion where rapid response is required. In some cases, such a control can also be used to minimize a quadratic performance index such as the integral of the galvanometer output (Reference 14).

While time-optimal control insures minimum time to reach the terminal states, it does not guarantee that these states will then be maintained. In this system, it is required that the terminal states be maintained once they are reached. This requirement leads to the possibility of a dual-mode control in which a time-optimal sequence would be followed by a terminal state control. The behavior of the radioactive voltage standard system, however, is known only to about 2 per cent (see Chapter 3, Section A). Because of this, considerable error is likely in the terminal states following a bang-bang switching sequence. For these reasons, use of bang-bang sequences in either time-optimal or multiple-mode control is impractical.

Combination of the two control criteria into a single time-varying performance index is also attractive. The principal advantage of such a combination is that the control is automatically changed from the secondary or fast control to the primary control where a useful output is produced. Such a control is useful when utilization of the output is made at a fixed time following start-up or the occurrence of some unwanted offset. For such a control to be optimal, it is, unfortunately, required that there is a knowledge of future states, including possible offsets (Reference 15). Because knowledge of the future is necessarily limited, this type of control cannot be used.

The dynamic behavior of the radioactive voltage standard is modeled mathematically by Equation 3.39 for any acceptable power input. This equation linearly relates the system power input to the galvanometer output, but the heater current is not linearly related to the heater power. Thus, the system performs a non-linear transformation which complicates realization of most desired control policies. Because the heater current is restricted to one polarity, this transformation has one-to-one correspondence. Exclusive of start-up where the heater power and the radioactive-source power are initially unrelated, anticipated offsets from the desired balanced condition are small. In terms of the system behavior, such offsets require only small

correction of the heater power. Small changes in power can be approximated linearly by small changes in the heater current. Thus, optimal linear feedback type controls are readily realized for small offsets and can be developed for both the primary and the secondary control criteria.

In the mathematical analysis leading to the development of the system dynamical equation (Equation 3.39) many approximations are necessary. Because of these approximations, some imprecision exists in the description of the system's behavior, and any control policy is necessarily sub-optimal. Linear feedback controls approach the optimal performance closely, however, as the galvanometer output approaches zero. As this is a practical control problem, considerable emphasis is placed on the known behavior of the system and also on the realizability of the developed control.

Because of the power-to-current nonlinearity and the power boundedness, any optimal linear feedback control becomes non-optimal for large unbalanced conditions. As indicated above, more applicable feedback policies are not feasible due to dynamical model imprecision and to lack of knowledge of the future. By permitting the gain in the feedback to decrease as a function of the magnitude of the offset, an acceptable control is realized for large offsets. This modified

linear feedback control is optimal for small offsets, and it is acceptable at the extremes of the offset range. For these reasons, the selected control technique is the modified optimal linear feedback for both the primary and the secondary control criteria.

Feedback control which is determined for a performance index over the infinite interval brings the desired states to their terminal value at infinite time. The control requirement in this problem is not that the terminal is reached, but that the states are within a suitable range of the terminal states. Because this condition is reached in finite time, feedback control is satisfactory.

B. The Performance Index

With the primary control, it is desired that a combination of terms be minimized as is introduced in the first section of this chapter. First, it is required that the galvanometer output be brought to a minimum and then held in that condition. Second, in order to hold the heater current constant, it is required that the rate of change of this current likewise be minimized and then held in that condition. These two terms are formulated into an integral performance index. To prevent the possibility of positive and negative portions of response cancelling over time, the square of each term is used. Further, the use of squared

terms weights larger offsets more heavily than smaller offsets. Because a useful output is produced whenever the controlled states are within a range of these terminal states, this weighting is desirable. Mathematically,

$$J = \frac{1}{2} \int_0^{\infty} (x_1^2 + ru^2) dt \quad (4.1)$$

where

J = the cost to be minimized,

x_1 = the galvanometer output (V_{go}) in volts,

u = the control variable which is the rate of change of heater current,

r = a weighting factor with units of ohms² minutes² and

t = time.

This integral is formulated over the infinite interval in order to insure that the terms are maintained near zero when they have reached that condition.

The secondary control is utilized when large offsets are likely to exist in the galvanometer output, such as occur on start-up. With this control it is required that the galvanometer output be brought to near zero rapidly, then kept in that condition. Under normal circumstances, this control is desired until it is necessary to utilize the system output.

Because of the restriction on the rate of change of heater current, the primary control response is slower.

As no useful system output is produced with the secondary criterion, this restriction can be relaxed. By permitting r in Equation 4.1 to become small, the desired condition is achieved. Thus, Equation 4.1 serves as the performance index for both types of operation.

The factor r determines the relative weighting of the galvanometer output term and the control variable term in Equation 4.1. For each criterion, r is determined by equating these two terms for values of x_1 and u which represent equivalent system effects. Mathematically,

$$(\Delta x_1)^2 = r(\Delta u)^2 \quad (4.2)$$

where

Δx_1 = the galvanometer output for the appropriate offset from balance and

Δu = the corresponding heater current rate offset from zero.

Operation under the primary criteria is based on an offset in the system output voltage of one part in ten million. Equivalently, because a maximum of about one minute is required in use of the system output, a maximum rate of change of one part in ten million per minute is required. A two-part-in-ten-million galvanometer output corresponds to a similar power change (see Equation 3.39). This also corresponds to a one-part-in-ten-million change in the system output voltage (see Chapter 2, Section G).

From the average power input (1.150 watts) and the heater resistance (393.3 ohms), the required heater current rate is readily determined. Inserting the values for the primary criterion in Equation 4.2 yields

$$r = 0.8046 K_6^2 \text{ ohms}^2 \text{ minutes}^2. \quad (4.3)$$

System conditions at the point of switching between the galvanometer-amplifier and the auxiliary amplifier are used to establish the secondary criterion. It is desired that switching occur where the heater power is not on a boundary (0 or 3 watts). Further, it is required that switching between the two amplifiers occur when they both are within their linear, acceptable range. Saturation of the galvanometer-amplifier occurs at an input in excess of one microvolt. The secondary amplifier is useable for inputs below one microvolt, which forms the desired switch point. To fulfill these conditions, a value of 3.362 microvolts is chosen to correspond to 33.62 milliamperes. This current is the extra current required to drive the heater from balanced condition to the upper boundary. The 3.362-microvolt value is well above the one-microvolt switch point. For practical switching, the current rate is limited to 33.62 milliamperes per minute. Insertion of these conditions yields the secondary weighting function

$$r = 10^{-8} K_6^2 \text{ ohms}^2 \text{ minutes}^2. \quad (4.4)$$

C. Development of the Optimal Control

In order to realize the intended function of the radioactive voltage standard, optimization of both its design and its performance are desired. The performance index (Equation 4.1) is a measure of the system behavior which is subject to the action of the control. Optimization techniques yield a control policy by which this performance index is minimized and, thus, the system performance is optimized. For the performance index of Equation 4.1, the optimal feedback control is readily developed. Linear systems which have a quadratic performance index have been thoroughly investigated (Reference 15).

Before the control problem is solved, however, it is necessary to linearize the system dynamical equation (Equation 3.39) around the heater current and convert it to a more useable form. The heater power is

$$Q_h = 60 I_h^2 R_h \quad (4.5)$$

where

Q_h = the heater power in joules per minute,

I_h = the heater current in amperes and

R_h = the heater resistance in ohms.

The factor 60 in this equation is necessary to convert watts to joules per minute. The rate of power change is

$$\frac{dQ_h}{dt} = 120 R_h I_h \frac{dI_h}{dt} \quad (4.6)$$

For small offsets I_h behaves approximately as a constant, I_{ho} . Inserting I_{ho} and substituting the controller, u , for the rate of change of heater current yields

$$\frac{dQ_h}{dt} = (120 R_h I_{ho}) u \quad (4.7)$$

The system dynamical equation (Equation 3.39) can now be put into form for the control problem.

This equation is

$$\begin{aligned} \frac{d^2 x_1}{dt^2} + 1.8372 \frac{dx_1}{dt} + 0.19410 x_1 &= 3.747 \times 10^{-6} K_6 \frac{dQ_h}{dt} \\ &+ 6.823 \times 10^{-5} K_6 (Q_h - 1.0104 Q_s) \end{aligned} \quad (4.8)$$

where V_{go} is replaced by x_1 for convenience. To preclude having an equation with integral as well as differential terms when u is inserted, it is necessary to differentiate both sides of Equation 4.8. Differentiating and inserting u yields

$$\begin{aligned} \frac{d^3 x_1}{dt^3} + 1.8372 \frac{d^2 x_1}{dt^2} + 0.19410 \frac{dx_1}{dt} \\ = 9.564 \times 10^{-3} K_6 \frac{du}{dt} + 1.7415 \times 10^{-1} K_6 u - 6.894 \times 10^{-5} K_6 \frac{dQ_s}{dt} \end{aligned} \quad (4.9)$$

where I_{ho} is replaced with the current necessary for a 1.150-watt source (0.05408 amp), and R_h is replaced by the

actual heater resistance which is 393.3 ohms. The value of 1.150 watts for the source is the average over its useful lifetime. For different source strengths the values of the source power and I_{ho} must be changed accordingly.

The value of the derivative of radioactive source power depends on the decay rate of the source. For sources with long half-lives, this rate is approximately

$$\frac{dQ_s}{dt} = \frac{-60P_o}{\tau} \quad (4.10)$$

where

P_o = the source power level in watts and

τ = the effective source time constant in minutes.

When the source decay rate is sufficiently high as to be significant, Q_s can be combined with u by making a linear transformation of variables to u' such that

$$u' = u + 0.023752 \frac{P_o}{\tau} \quad (4.11)$$

For the selected plutonium 238 source, the effective time constant of about 90 years is sufficiently long that the decay effect is only a fraction of the minimum value of u . For this reason, the decay term is neglected. In this approximation, source power drift is treated as a small unwanted offset which is to be controlled out.

Elimination of the source decay term from Equation 4.6 yields

$$\frac{d^3 x_1}{dt^3} + a \frac{d^2 x_1}{dt^2} + b \frac{dx_1}{dt} = c \frac{du}{dt} + du \quad (4.12)$$

where

$$a = 1.8372 \text{ minutes}^{-1},$$

$$b = 0.19410 \text{ minutes}^{-2},$$

$$c = 9.564 \times 10^{-3} K_6 \text{ ohms minutes}^{-1} \text{ and}$$

$$d = 1.7415 \times 10^{-1} K_6 \text{ ohms minutes}^{-2}.$$

These units correspond to x_1 in volts and u in amperes per minute. Equation 4.12 is readily converted to a set of state equations which are

$$\frac{dx_1}{dt} = x_2, \quad (4.13)$$

$$\frac{dx_2}{dt} = x_3 + cu \quad (4.14)$$

and

$$\frac{dx_3}{dt} = -ax_3 - bx_2 + eu \quad (4.15)$$

where $e = d - ac = 1.5658 \times 10^{-1} K_6 \text{ ohms minutes}^{-2}$.

The state variables x_2 and x_3 are defined by these relationships. x_2 is the rate of change of the galvanometer output x_1 , or the velocity of this output. x_3 , however, is not exactly the acceleration of the galvanometer output, although the two are related. This state differs from the acceleration by the additive control term cu .

Equations 4.10 through 4.12 can be combined further by use of vector notation. The vector relationship is

$$\frac{d\mathbf{x}}{dt} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \quad (4.16)$$

where, in this case,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (4.17)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b & -a \end{bmatrix}, \quad (4.18)$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ c \\ e \end{bmatrix} \quad (4.19)$$

and

$$\mathbf{u} = u. \quad (4.20)$$

There is only one control variable, u .

In vector notation, the performance index is

$$J = \frac{1}{2} \int_0^{\infty} \left(\langle \mathbf{x}, \mathbf{Q}\mathbf{x} \rangle + \langle \mathbf{u}, \mathbf{R}\mathbf{u} \rangle \right) dt \quad (4.21)$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (4.22)$$

$$\mathbf{R} = r \quad (4.23)$$

and $\mathbf{x}$ and $\mathbf{u}$ are as given above.

In such systems the feedback control is readily determined providing several conditions are met. For a more

complete description of the requirements for optimal control of such systems, the reader should refer to pages 752 through 791 of Reference 15. The first condition is that the control variable must not be constrained. This condition implies that any required control variable magnitude is available. Because the control variable also appears in the performance index, however, large values are weighted more heavily than small values. Although the heater power is bound, its state is not represented in the problem formulation. In addition, there are no direct constraints on the rate of change of the heater current. Thus, for small offsets which require only small changes in the heater power that do not approach its bound, the control variable is unconstrained.

The second condition is that the matrix Q be positive semidefinite. This condition is necessary to guarantee that the controlled system is stable. Because there is only one non-zero term in Q , and it is a positive constant on the diagonal, this condition is verified.

Third, in order to minimize J (see Equation 4.21), it is required that the R matrix, which in this case is the constant r , be positive definite. Also, the positive definite condition requires the existence of a finite control. Because r is a positive constant related to the effective system gain, this condition is also verified.

For an optimal feedback control, it is also required that the system is linear and that all of its states are both observable and controllable. The galvanometer output (the state x_1) is the only measurable state. In addition to being linear, however, the system as modeled is also time-invariant. By using these properties, the missing states are generated by successive time differentiation of the x_1 state and combination with the control variable. Thus, the system is in effect, observable. In addition, similar reasoning can be used to show that the system is controllable. The ability to recover missing states, inherent in all linear time-invariant systems, is used in the synthesis of the control after the necessary formulation of the feedback is accomplished.

For systems with quadratic performance indices which satisfy the above conditions, it is shown by Athans and Falb (Reference 15) that the optimal feedback control is given by

$$\underline{u} = -R^{-1} B' K \underline{x} \quad (4.24)$$

where K , a positive definite symmetric matrix, is the solution of the matrix Riccati equation

$$\frac{dK}{dt} = -K'A - A'K + KBR^{-1}B'K - Q. \quad (4.25)$$

In these two equations the ' notation is used to represent the transpose of the matrix. The unity negative exponential is used to signify the inverse of the matrix. For systems which are time-invariant and for which time increases without limit, the K matrix is not time-dependent; i. e.,

$$\frac{dK}{dt} = 0. \quad (4.26)$$

In this case, solution is required of a set of nonlinear simultaneous equations in k.

In expanded form, the control equation and the set of simultaneous equations are

$$u = - \left(\frac{1}{r} \right) \left[(ck_{12}+ek_{13})x_1 + (ck_{22}+ek_{23})x_2 + (ck_{23}+ek_{33})x_3 \right], \quad (4.27)$$

$$\left(\frac{1}{r} \right) (ck_{12}+ek_{13})^2 = 1, \quad (4.28)$$

$$\left(\frac{1}{r} \right) (ck_{12}+ek_{13})(ck_{22}+ek_{23}) - k_{11}+bk_{13} = 0, \quad (4.29)$$

$$\left(\frac{1}{r} \right) (ck_{12}+ek_{13})(ck_{23}+ek_{33}) - k_{12}+ak_{13} = 0, \quad (4.30)$$

$$\left(\frac{1}{r} \right) (ck_{22}+ek_{23})^2 - 2k_{12}+2bk_{23} = 0, \quad (4.31)$$

$$\left(\frac{1}{r} \right) (ck_{22}+ek_{23})(ck_{23}+ek_{33}) - k_{13} - k_{22} + ak_{23} + bk_{33} = 0 \quad (4.32)$$

and

$$\left(\frac{1}{r} \right) (ck_{23}+ek_{33})^2 - 2k_{23}+2ak_{33} = 0. \quad (4.33)$$

Solution of these equations is complicated by their nonlinear nature, but it can be accomplished by using iterative approximation techniques. Listed in Table 4.1 are the values obtained by such a technique for both the primary and the secondary criteria.

Table 4.1

THE K MATRIX VALUES

<u>Term Identification</u>	<u>Value for Primary Criterion</u>	<u>Value for Secondary Criterion</u>	<u>Units</u>
r	$0.8046K_6^2$	$10^{-8}K_6^2$	ohms ² minutes ²
k ₁₁	4.820	1.2307×10^{-1}	minutes
k ₁₂	11.167	7.575×10^{-3}	minutes ²
k ₁₃	5.047	1.7599×10^{-4}	minutes ³
k ₂₂	40.03	7.742×10^{-4}	minutes ³
k ₂₃	19.553	3.1288×10^{-5}	minutes ⁴
k ₃₃	9.666	2.7200×10^{-6}	minutes ⁵

As in the solution of most nonlinear equations, more than one set of answers is possible. In this case, however, it is required that the K matrix is positive

definite. This requirement limits the acceptable solution to a single value set. Examination of the K matrices formed from the values in Table 4.1 shows that they are positive definite and that this is the desired solution.

The units listed in Table 4.1 are uncommon even though they occur as a direct consequence of the system and feedback optimization equations. Insertion of the listed values into Equation 4.27 produces the control variable in the correct units of amperes per minute. Under the primary criterion,

$$u = - \frac{1}{K_6} \left(1.1149x_1 + 4.281x_2 + 2.1135x_3 \right). \quad (4.34)$$

Under the secondary criterion,

$$u = - \frac{1}{K_6} \left(1.0000 \times 10^4 x_1 + 1.2304 \times 10^3 x_2 + 7.251 \times 10^1 x_3 \right). \quad (4.35)$$

Because some of the equipment necessary to automatically control the radioactive voltage standard is unavailable, verification of optimality of the feedback control is accomplished by alternate means. A preliminary check on the control optimization is obtained by comparison with the control for more simple systems. Specific systems of interest are developed from Equation 3.38 which is the transform of the system response. First, the effects of the secondary pole and the zero

are ignored, and, second, just the effects of the zero are ignored.

There are several results of the mathematical analysis of the three described control problems. First, examination of the control relationships shows that the feedback multiplier for the x_1 state must be the same for all three cases. Indeed, this is the result obtained. Although the feedback multipliers of the x_2 and x_3 states vary significantly in magnitude for the three cases, effects on the performance index are small. Use of the feedback derived for absence of the secondary pole and zero in the complete system produces an increased performance index of only about 3 per cent for the primary criterion. Similar results for the secondary criterion produce an increase in the performance index of about 10 per cent over the optimal value. Thus, the system could be reasonably approximated by a single pole system. In addition, these results, though approximate, support the assumption that the effects of the relatively remote galvanometer poles could be neglected (see Chapter 3, Section E).

A complete verification of optimality of the feedback control is accomplished using an analog computer simulation. Shown in Figure 4.1 is the functional circuit diagram used to study system response and the performance index. For the primary criterion, the constants represented by small letters are precisely the

Figure 4.1 Functional Simulation of the System Under Feedback Control

same as those of the basic problem (given by Equations 4.3 and 4.12). The constants f , g and h are the respective state multipliers for the feedback given by Equation 4.34 with a K_6 of unity. For operation under the secondary criterion these constants represent the same system constants but with a large weighting required to keep behavior within computer bounds. Under this condition, f , g and h represent the respective state multipliers in Equation 4.35 for a K_6 of unity.

Under the primary criterion, Figure 4.2 gives the optimal system response to a step offset in the galvanometer output. Because the galvanometer output (the state x_1) is the only measurable state, the step input represents a practical offset in the actual system. Figure 4.3 gives the corresponding performance index buildup versus time. Shown is the optimal curve and, in the inset, differences caused for large time by varying the f , g and h constants. For short time periods following an offset, the buildup of the performance index is not the minimum of all possible values. During such periods, the error components (x_1 and u) are, likewise, not near zero. As time increases, however, x_1 and u approach zero and the performance index approaches a constant which is clearly the minimal value. Because the performance index is not evaluated at a specific terminal time, but rather on the infinite interval, this is the anticipated result. As is shown by Figures 4.2 and 4.3, the requirement that

Figure 4.2 Optimal System Response with Primary Criterion

Figure 4.3 The Performance Index with Primary Criterion

the states are held near zero is achieved once such a condition is reached. Thus, Equation 4.34 is the optimal feedback control for the system. Of specific interest is the relative insensitivity of the performance index to variations of h , the multiplier in the feedback for the state x_3 . This insensitivity is a further indication that the system could be reasonably approximated by a single pole model.

Figures 4.4 and 4.5 show the system response and performance index for the secondary criterion in a manner similar to that for the primary criterion. Of specific interest in these figures is the more rapid speed of response to the offset compared to that under the primary criterion. This is the principle requirement of the secondary control. Because the performance index is minimized for the feedback specified by Equation 4.35, this equation gives the optimal control for the secondary criterion.

D. Realization of the Control

The radioactive voltage standard system has only one input, the heater power, which can also be represented in terms of the heater current or voltage. In addition, the system has a single galvanometer-amplifier or auxiliary amplifier output which indicates the states of the system. This output voltage is a convenient signal and it represents the state x_1 directly. Before

Figure 4.4 Optimal System Response with Secondary Criterion

Figure 4.5 The Performance Index with Secondary Criterion

the primary and the secondary feedback circuits can be synthesized, however, similar signals must be developed from x_1 to represent the missing states. When the controls are formed, they must be converted into acceptable signals to feed the heater and the series sampling resistor.

Because the system is linear and time-invariant, signals which represent the missing states are readily developed. These states (x_2 and x_3) are defined in terms of the galvanometer-amplifier output (x_1) and the controller (u) by Equations 4.13 and 4.14. Substituting these relationships into Equations 4.34 and 4.35 produces the control as a function of only the galvanometer-amplifier output. For the primary criterion

$$u = - \left(\frac{1}{K_6} \right) \left(1.1379x_1 + 4.369 \frac{dx_1}{dt} + 2.1571 \frac{d^2x_1}{dt^2} \right). \quad (4.36)$$

In a similar manner, for the secondary criterion

$$u = - \left(\frac{1}{K_6} \right) \left(3.2625 \times 10^4 x_1 + 4.014 \times 10^3 \frac{dx_1}{dt} + 2.365 \times 10^2 \frac{d^2x_1}{dt^2} \right). \quad (4.37)$$

The control variable cannot be fed back directly, however, as the actual system input is a power or current to the heater. By definition (see Section B of this chapter) the control variable is the rate of change of the heater

current. Thus, the heater current can be formed by integration of the control variable.

Mathematically, the heater currents under primary and secondary criteria are, respectively,

$$I_h = - \left(\frac{1}{K_6} \right) \left(1.1379 \int_0^t x_1 d\omega + 4.369x_1 + 2.1571 \frac{dx_1}{dt} \right) + K_7 \quad (4.38)$$

and

$$I_h = - \left(\frac{1}{K_6} \right) \left(3.2625 \times 10^4 \int_0^t x_1 d\omega + 4.014 \times 10^3 x_1 + 2.3656 \times 10^2 \frac{dx_1}{dt} \right) + K'_7 \quad (4.39)$$

where

ω is used as a dummy variable and

K_7 and K'_7 are the constants of integration.

Because of the integral terms and the constants of integration, the actual balancing current is developed as a function of time, and the desired terminal can be closely approached.

The output of either the galvanometer-amplifier or the auxiliary amplifier is a voltage. For this reason and because the system input is described as a current, a voltage-to-current conversion must be made in developing the control. It is convenient to use a voltage-in, voltage-out type of feedback to simplify the circuitry.

The system input voltage is formed in part by the voltage drop across the heater. This voltage is related linearly to the heater current by the heater resistance which is a constant. The remaining portion of the input voltage is dropped across the sampling resistor which develops the system output. To compensate decay in the radioactive-source power level, the value of the sampling resistor is adjusted periodically so that the output voltage is relatively fixed in magnitude. For control purposes, however, the value of this resistor behaves as a constant. Thus, the current input to the system is simply related to the voltage input by the total system input resistance. In terms of a voltage input rather than a current input, Equations 4.38 and 4.39 become, respectively,

$$V_i = - \left(\frac{1}{K_6} \right) \left(468.9 \int_0^t x_1 d\omega + 1800.5x_1 + 888.9 \frac{dx_1}{dt} \right) + 412.1 K_7 \quad (4.40)$$

and

$$V_i = - \left(\frac{1}{K_6} \right) \left(1.3445 \times 10^7 \int_0^t x_1 d\omega + 1.6542 \times 10^6 x_1 + 9.749 \times 10^4 \frac{dx_1}{dt} \right) + 412.1 K'_7 \quad (4.41)$$

where V_i = the system input voltage (or the control output voltage).

In developing these equations, it is assumed that the system output voltage is 1.0180 volts which is near the voltage of saturated standard cells. This choice of value permits direct comparison of the system output with that of the cells. By this technique, comparison accuracy is enhanced. Selection of different values of the system output voltage requires re-determination of the numerical constants of Equations 4.40 and 4.41 in terms of the appropriate total input resistance.

The feedback function specified by Equations 4.40 and 4.41 can be closely approximated by several circuit techniques using readily available equipment. Although circuit details differ, all of these techniques are related because of the functional requirements of the control.

First, and perhaps simplest to realize, is the use of a commercially available continuous-action process-controller. This device accepts a voltage input and produces an output consisting of the required integral, proportional and derivative components of the feedback. The contribution of each component is adjustable over wide limits.

In another approach, the feedback function is generated using computer-type operational amplifiers which are connected to match the required control characteristics. The control circuitry necessary to this

approach can occupy a relatively small volume and it is more suitable to portable use. Shown in Figure 4.6 is a generalized system circuit which meets the basic requirements of the control. Analysis of the feedback functions illustrates some of the problems in achieving the actual control.

The integral component of the control is developed using an operational amplifier with capacitive feedback. Because of the long time constants in the system, the feedback capacitor must be capable of retaining a charge for very long periods of time. This requires that the feedback capacitor have a very small leakage and that the amplifier have a very large input impedance in addition to a high gain. These requirements restrict the choice of the amplifier and its associated components.

Under the secondary control, the multipliers of the proportional and differential components of the feedback are much smaller than the integral component (see Equation 4.41). In this case, these functions can be closely approximated using a passive network of resistance-capacitance type components. Under the primary criterion, however, the proportional and derivative feedback components have multipliers which are larger than that of the integral (see Equation 4.40). In this case, the derivative must be generated by an active circuit, a function which is difficult to achieve

Figure 4.6 A Circuit for Developing the Optimal Feedback Control

practically. As is indicated on the performance index plot, Figure 4.3, the system response under the primary control is least sensitive to the derivative component (the x_3 component) of feedback. For this reason, the performance requirement on this derivative function can be reduced to make its realization practical.

While the generation of the feedback is not without difficulty, the system response is slow and the response of the required amplifiers can be reduced accordingly. This simplifies realization of the control. As is introduced in Section A of this chapter, many approximations are necessary which limit the practical control to a sub-optimal performance. At high offset levels the power input is on the boundary and the actual control has only small effect. Though imperfect, the feedback approaches the optimal closely as the bridge output approaches zero.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

A. The Basic System

Investigation into the possible techniques by which predictability of a voltage can be related to the activity of a radioactive source yields several results. These results and the conclusions derived from them are listed below.

1. Development of the design concept required to realize such a voltage is the first result. The goals of the design are outlined in Chapter 1, and the details of the actual design are presented in Chapter 2. Each step in this design is one of optimization which is necessary because of the high predictability requirement. Even though the demands on the system performance are high, the investigation shows that the desired behavior can be achieved.

2. Formulation of a suitable mathematical model is necessary before an automatic control can be synthesized for the system. This plant identification is the second result of the investigation and it leads to techniques by which the system performance can be further improved. These techniques are outlined in Section B of this chapter.

3. The control which makes practical the use of the

device as a standard is presented in Chapter 4 of this dissertation. Development of this control is the principal result of the investigation into the system behavior. By utilizing optimization techniques, the control assures that the system output voltage is essentially as predictable as is the radioactive source decay.

The control policy developed for use with the radioactive voltage standard can be readily adapted for use with other systems. Any system in which the useful output is obtained from the steady-state input following balance of system conditions can make use of this type of control. One possible application is in the control of the calorimeters which are used in the determination of radioactive-source strengths and half-lives. Steps similar to those outlined in Chapter 4 are necessary in the synthesis of the control.

B. Calorimeter Thermal Analysis

Analysis of the physical behavior of the terms in the simplified calorimeter thermal equations (Equations 3.21 and 3.22) yields two techniques for changing the system response. The effective conductivity terms in these equations are dominated by conduction through the air gaps (see Chapter 3, Section C). This conduction varies as a function of the elements which make up the gas in the gaps and also as a function of the gas pressure. Partial evacuation of the calorimeter or use of a

gas other than air permits variation of these conductivity values. Lowering of the conductivities produces an increase in the temperature output for a given input power level and an increase in the system response time. As the response time is already long, any further increase offsets the advantage of increased sensitivity.

In the practical system, minor differences are likely between the air-gap conductivity values for the two limbs. Because of these differences, small changes in atmospheric pressure can cause relatively large changes in the total heat flow. For this reason, it is recommended that any transportable version of the radioactive voltage standard be operated in an hermetically sealed environment.

Variation of the central heat capacity in the heated limb (A_1 in the simplified thermal equations) is an additional method of changing the response characteristics. In contrast to the results of changing the conductivity terms, variation of this heat capacity does not change the sensitivity of the temperature output to heat.

Lumped heat capacities which are approximated at the heat source and temperature-sensing resistors produce the two major time constants in the limb response (see Chapter 3, Section C). Because of the conductivity through the intervening air gap, the sensing windings are at a lower temperature than the source.

By a redesign of the limb structure a simplified system is produced which can be approximated by a single

lumped heat capacity term and a single conductivity term. In addition, the temperature sensitivity is increased and the time-response model is simplified to contain only one major time constant. This simplification reduces the complexity of the required feedback control to one requiring only proportional and integral functions. The increase in the temperature sensitivity produces an improvement in the realizable system accuracy.

These improvements can be accomplished by making two basic changes in the limb structure. First, it is necessary that there is only one air gap in the limb. This air gap produces the single conductivity term. Second, it is required that the radioactive source or heater and the heat-sensing resistors are connected by a high-conductivity path. This path assures that the sensing windings are at the source temperature and that the system can be approximated by a single lumped heat capacity.

The changes described produce a modified system which has simplified characteristics. Because the control is simpler, the system is more readily adapted for transportable use. For this reason, and because of the higher sensitivity to temperature, the modified structure is recommended for future construction.

APPENDIX A

THE CONTROLLED-TEMPERATURE BATH

In Chapter 2, Section B, the requirement for a constant-temperature environment for the calorimeter sink is discussed. Briefly mentioned are the characteristics of the bath which is used to establish this environment.

The sink block of the calorimeter is designed to fit into the inner channel of the temperature-controlled, laminar-flow bath. Figure A.1 is a top view of the bath which shows the sink position and the coolant-flow pattern. Figure 2.2 shows the complete bath.

The outside shell of the bath is an open-top, rectangular tank which has a fifty-gallon coolant capacity. This shell is insulated from ambient temperature on the sides and the bottom by layers of fiberboard. A metal covering is provided over the insulation to present a good appearance and to simplify cleaning of the apparatus. The depth of the tank is sufficient to contain the vertical height of the sink block.

The sink is located at one end of an inner channel which is formed in a U shape from a single sheet of heavy gage metal. The bottom of the inner channel is

Figure A.1 The Bath Plan

insulated from the shell by a sheet of low-conductivity plastic. The sides of the U form baffles which separate the bath into inner and outer channels.

A circulator is located at the inlet to the inner channel which is at the opposite end from the sink. The circulator is made from a ten-inch, three-blade fan, and it is driven through a set of bevel gears by a low-speed motor. Transfer of heat to the coolant is prevented by locating the motor outside the bath. Flow of the coolant is indicated in Figure A.1. The circulator pumps the coolant through a cylindrical duct which is in the center of a planar baffle. This baffle is perpendicular to the oil flow at the inlet end of the inner channel, and it is used to control the coolant turbulence. The coolant flows through the inner channel to the sink past a pair of probes which sense its temperature.

After the coolant flows through the sink, it flows into the outer channels following a path formed by a pair of curved baffle plates. In the outer channels the coolant is heated and it returns to the circulator following a path established by a second pair of curved baffle plates.

The layout of the bath is such that the sink and the temperature-sensitive probes in the inner channel are isolated by the coolant in the outer channels. These outer channels act as an insulating buffer be-

tween the bath shell and the inner channel. Because it is thermally isolated from normal environment by the outer channels, very close control of the inner-channel temperature is possible.

The rate of circulation can be altered by changing the fan blade pitch. Three blades with different pitches are provided which give approximate pumping rates of 15, 40 and 100 gallons per minute. There is sufficient turbulence in the inner channel to affect a thorough mixing of the coolant, which reduces the possibility of temperature gradients.

Nickel resistors form the sensing elements of the two probes and they are connected in the opposite arms of a bridge circuit. The other two arms of the bridge are made of manganin resistors, one of which is adjustable. The adjustable resistor permits selection of the temperature at which the bridge is balanced. Diagrammed in Figure A.2 are the bridge and the associated bath-control circuitry. The two manganin bridge arms form the set-point unit which can be seen at the top of Figure A.3.

The output from the bridge is amplified through a high-gain amplifier. This amplified voltage feeds the input of a Current-Adjusting Type Controller (C.A.T.). The C.A.T. permits matching of the feedback characteristics to the bath response.

Current from the C.A.T. is fed to a power supply

Figure A.2 The Bath-Temperature Control

Figure A.3. The Top of the Bath

which has been modified to produce a current-amplification function. In response to the C.A.T. output, power from the supply is fed to the bath heaters which are made of groups of wire-wound resistors.

Figure A.4 shows the set of control instruments. The control system from the probes to the C.A.T. was made especially for this application by the Leeds and Northrup Company. Some of their own equipment uses similar temperature-control apparatus.

The bath has been operated with refined mineral oil as the coolant and it has demonstrated temperature stability of about $\pm 0.0003^{\circ}$ Kelvin on a short-term basis. With refined mineral oil the maximum pumping rate is used and the oil is recirculated twice each minute.

The temperature coefficient of manganin is about $0.000012 \text{ ohm/ohm } ^{\circ}\text{Kelvin}$, while that of nickel is about $0.006 \text{ ohm/ohm } ^{\circ}\text{Kelvin}$. The ratio between these is 500/1. Ambient-temperature variations of 0.3° Kelvin produce 0.0006° Kelvin variations in the bath. These also have a ratio of 500/1. Because both ratios are the same, it appears that the bath variation is due to coupled, room-ambient variation. Figure A.5 is a portion of a plot of the control signal which is derived from the bridge unbalance. Sketched on the figure for comparison is a plot of the room-temperature variation.

Figure A.4. The Bath Control Instruments

Figure A.5 Short-Term Bath-Temperature Variation

The set-point unit is located in the normal ambient environment and it serves as the input source for variation in ambient temperature. Effects of such a variation can be reduced by operating the set-point unit in a controlled environment, such as in the bath itself.

The sink block mass is 150 pounds. Because of the high heat capacity of the sink, temperature variations in the bath are significantly reduced. Temperature variations such as are shown in Figure A.5 are not discernible in the sink temperature.

As is discussed in Chapter 2, Section B, refined mineral oil was selected as the coolant because of its low build-up of corrosive contaminants. The oil has a low thermal-transfer capacity. Should a high thermal transfer become necessary, the oil can be replaced by water. The entire bath design is such that either oil or water may be used as the coolant.

The bath and its associated temperature-control circuitry are complex in both function and design. Where high temperature stability is required, this complexity cannot be avoided. The short-term temperature stability achieved by the bath is better by a factor of about 10 than the actual bath temperature can be measured with certainty. Temperature measurements at this level of accuracy require the use of a precision platinum resistance

thermometer and suitable high-precision resistance-measuring equipment.

APPENDIX B

THE RADIOACTIVE HEAT SOURCE

The idealized heat-source characteristics are covered briefly in Chapter 2, Section G. Presented also are the salient characteristics of plutonium 238 which is the radioisotope chosen for the heat source.

Plutonium 238 is man-made from uranium in the neutron flux inside a reactor. In the build-up of the desired material, quantities of other radioisotopes are also produced. Those which have different atomic numbers than plutonium are readily eliminated by chemical separation techniques. The unwanted isotopes of plutonium, however, cannot be separated chemically. Isotope separation techniques (such as the mass-spectrometric method) can be used to enrich the desired material. This enrichment is limited by the high cost and the low yield of the process. Because of these factors, the actual source contains significant quantities of the various plutonium isotopes as are listed in Table B.1 (Reference 16). Listed also in Table B.1 are the half-lives and the specific activities for the various isotopes (Reference 17).

The decay product of each of the source isotopes is also a radioisotope. In time these will decay

Table B.1

RELATIVE CONTRIBUTION OF SOURCE ISOTOPES

<u>Isotope</u>	<u>Relative Amount by Weight (Ref. 16)</u>	<u>Decay Mechanism</u>	<u>Decay Half-Life in Years (Ref. 17)</u>	<u>Decay Time Constant in Years (calculated)</u>	<u>Specific Activity Curies/Gm (Ref. 17)</u>
^{238}Pu	82.8%	α	8.64×10^1	1.25×10^2	1.74×10^1
^{239}Pu	13.0%	α	2.44×10^4	3.52×10^4	6.14×10^{-2}
^{240}Pu	3.0%	α	6.24×10^3	9.01×10^3	2.39×10^{-1}
^{241}Pu	1.0%	β	1.33×10^1	1.92×10^1	1.12×10^2
^{242}Pu	0.1%	α	3.79×10^5	5.47×10^5	3.90×10^{-3}
All others	0.1%				

through a succession of steps to stable elements. Each step has a different energy release per decay event and a different half-life. Table B.2 gives the first decay product, the mechanism of the second decay step and the second-step decay characteristics. Listed at the end of Table B.2 is the third step for plutonium-241 (the second decay step for americium-241). Because of the short half-lives of the isotopes involved, this third step must be considered. During the useful life of the source, contributions from other steps in the decay families are negligible.

Unfortunately, the relative mass of the contributing isotopes, their half-lives and their specific activities are known only approximately (a few tenths of a per cent to a few per cent). Analysis of the contribution of each of the materials, however, leads to the development of an adequate decay model. To achieve the desired result, the source power output must be known to one part in ten million or better throughout some prediction period (relative to the power output at the period inception).

All of the radioisotopes decay according to the exponential decay-law. For the isotopes present in the source initially, the activity is

$$A = \frac{N_{0a}}{\gamma_a} e^{-\frac{t}{T_a}} \quad (B.1)$$

Table B.2.

SECOND-STEP DECAY CHARACTERISTICS

Initial Isotope	First Decay Product	Second Decay Mechanism	Half-Life of the Second Decay in Years (Ref. 17)	Decay Time Constant in Years (calculated)	Specific Activity Curies/Gm (Ref. 17)
$^{238}_{94}\text{Pu}$	$^{234}_{92}\text{U}$	α	2.48×10^5	3.58×10^5	6.17×10^{-3}
$^{239}_{94}\text{Pu}$	$^{235}_{92}\text{U}$	α	7.13×10^8	1.03×10^9	2.14×10^{-6}
$^{240}_{94}\text{Pu}$	$^{236}_{92}\text{U}$	α	2.39×10^7	3.45×10^7	6.34×10^{-5}
$^{241}_{94}\text{Pu}$	$^{241}_{95}\text{Am}$	α	4.58×10^2	6.61×10^2	3.24×10^{-0}
$^{242}_{94}\text{Pu}$	$^{238}_{92}\text{U}$	α	4.51×10^9	6.51×10^9	3.33×10^{-7}
<hr/>					
$^{241}_{94}\text{Pu}$ ↓	$^{237}_{93}\text{Np}$	α	2.14×10^6	3.09×10^6	7.05×10^{-4}

where

N_{oa} = the number of atoms initially,

γ_a = the decay time constant and

t = time.

The ratio N_{oa}/γ_a is the initial activity.

For any second decay step, the activity is

$$A = \frac{N_{ob}}{\gamma_b - \gamma_c} \left(e^{-\frac{t}{\gamma_b}} - e^{-\frac{t}{\gamma_c}} \right) \quad (B.2)$$

where

N_{ob} = the number of atoms of the parent initially,

γ_b = the time constant of the first decay step,

γ_c = the time constant of the second step and

t = time.

$N_{ob}/(\gamma_b - \gamma_c)$ is the equivalent of the initial activity for these isotopes.

The power contributed by each radioisotope is proportional to the product of its activity and the energy release per decay event. Although not identical, the total energy released per decay event by all of the radioisotopes listed in Tables B.1 and B.2 is approximately the same (about 5 mev). In the alpha decay steps, nearly all of the energy is carried by the alpha particles which are captured and their energy is transformed to heat. For these steps, the residual energy is carried by soft gamma rays which are nearly all captured inside the source and which

also yield heat. In the beta decay of plutonium 241, however, only about 0.02 mev is carried by the beta particle and the rest is carried by the associated gamma ray. The beta particles are captured and their energy becomes heat. Because of the high energy of the associated gamma rays, many escape and these contribute significantly to the external radiation of the source.

Expansion of Equations B.1 and B.2 in Taylor series yields

$$A = \frac{N_{oa}}{\gamma_a} \left[1 - \frac{t}{\gamma_a} + \frac{t^2}{2!\gamma_a^2} - \frac{t^3}{3!\gamma_a^3} + \dots \right] \quad (B.3)$$

and

$$A = \frac{N_{ob}}{(\gamma_c - \gamma_b)} \left[t \left(\frac{1}{\gamma_b} - \frac{1}{\gamma_c} \right) - \frac{t^2}{2!} \left(\frac{1}{\gamma_b^2} - \frac{1}{\gamma_c^2} \right) + \frac{t^3}{3!} \left(\frac{1}{\gamma_b^3} - \frac{1}{\gamma_c^3} \right) - \dots \right] \quad (B.4)$$

In terms of a normalized multiplier for plutonium 238, the relative power contributions of each isotope are given in Table B.3. The contributions of the unlisted isotopes can be neglected throughout the expected useful source life of ten years.

Over short periods of time, such as a month, only the first three terms of the series are effective in the power prediction. In this case the source power

Table B.3

RELATIVE POWER CONTRIBUTION BY THE SOURCE ISOTOPES

<u>Effective Contribution (starting from t=0)</u>						
<u>Radioisotope</u>	<u>Constant</u>	<u>1st Order</u>	<u>2nd Order</u>	<u>3rd Order</u>	<u>4th Order</u>	<u>5th Order</u>
${}^{238}\text{Pu}$	1.000	8.0×10^{-3}	3.2×10^{-5}	1.7×10^{-7}	1.7×10^{-10}	0
${}^{239}\text{Pu}$	5.78×10^{-4}	1.6×10^{-8}	0	0	0	0
${}^{240}\text{Pu}$	4.98×10^{-4}	5.5×10^{-8}	0	0	0	0
${}^{241}\text{Pu}$	3.12×10^{-4}	1.6×10^{-5}	4.2×10^{-7}	3.7×10^{-9}	9.5×10^{-11}	0
${}^{242}\text{Pu}$	2.71×10^{-8}	0	0	0	0	0
${}^{234}\text{U}$	0	-2.8×10^{-8}	0	0	0	0
${}^{241}\text{Am}$	0	-1.2×10^{-4}	-3.1×10^{-6}	-5.5×10^{-8}	-7.3×10^{-10}	-9.4×10^{-12}
Approximate Total Effect (t=0)	1.001	7.9×10^{-3}	2.9×10^{-5}	1.2×10^{-7}	-4.7×10^{-10}	-9.4×10^{-12}

can be approximated by

$$P_s = P_o e^{-\frac{t}{\gamma_d}} \quad (B.5)$$

where

P_s = the effective source power,

P_o = the source power at the beginning of the prediction period,

γ_d = the effective time constant and

t = time.

This is the simplest prediction possible.

Over periods of a year the prediction of the source power is more complicated. For this case the first four terms of the series are effective. Evaluation of the magnitude of these terms indicates that the power can be approximated to the desired precision by

$$P_s = P_o \left(e^{-\frac{t}{\gamma_e}} + K_e t \right) \quad (B.6)$$

where

P_o , P_s and t are given above,

γ_e = the decay time constant and

K_e = an additional linear decay constant.

At any time during the useful source life the power output can be developed from a polynomial as is indicated by the data in Table B.3.

The precise constants required in these equations must be developed in the operation of the system. As described in Chapter 2, Section H, operation of the calorimeter is by a boot-strap technique to determine the necessary decay constants. Computer programs are available which simplify the determination of these constants.

APPENDIX C

COMPUTATION OF THE THERMAL CONSTANTS FOR THE CALORIMETER LIMBS

1. General

In Chapter 3, Section C, the mathematical models of the calorimeter limbs that employ lumped thermal parameters are developed. The numerical values of the winding fractions, the heat capacities, the conductances and the linearized radiation constants used in these models are calculated in this appendix.

As is discussed in Chapter 3, Section C, the limb behavior is determined by a combination of theoretical computation and analysis of the system response. The equations used in calculating the lumped thermal constants are available from numerous standard heat-flow texts; e. g., see Reference 18. System dimensions are determined either by physical measurements or by analysis of the observed response. In all cases, the thermal terms produced are within the range anticipated.

2. Calculation of the Winding Fractions

As they appear in Chapter 3, Sections B and C, the winding fractions (K_i) are ratios of an active length of the temperature-sensitive resistors to the total length.

$$K_i = \frac{\ell_i}{\ell_t} \quad (C.1)$$

where

K_i = the winding fraction,

ℓ_i = the corresponding portion of the total winding length and

ℓ_t = the total winding length.

In the limb each portion of the winding is exposed to a specific temperature which is identified in Equation 3.10 by a K_i or K_j with a corresponding ΔT_{1i} or ΔT_{2j} . Because both limbs are similar, the i and the j , the K_i and the K_j are also similar. The ΔT 's are not necessarily similar, however. The winding fractions are given in Table C.1.

Table C.1

WINDING FRACTIONS

Fraction Identification	ℓ_i in Inches*	Winding Fraction Value
K_a	0.510	0.04533
K_b	9.74	0.86578
K_c	1.00	0.08889
Total	11.25	1.00000

* Determined by measurement of the actual calorimeter.

K_a is the fraction of the sensing windings which is at the temperature of the source. K_b is the fraction of

these windings which senses the temperature along the central portion of the insulating cylinder. Finally, K_c is the portion of the sensing windings which is at the sink temperature. As required in Equation 3.9, the sum of the K_i equals 1.0000.

3. Calculation of the Heat Capacities

The lumped heat capacities in the system are computed from either

$$C = \rho v c \quad (C.2)$$

or

$$C = cm \quad (C.3)$$

where

C = the lumped heat capacity,

ρ = the specific density,

v = the volume,

c = the specific heat capacity and

m = the mass represented by volume v .

When the heat capacity is made up of lumped contributions from different materials,

$$C = \sum_i c_i m_i + \sum_j \rho_j c_j v_j \quad (C.4)$$

where i and j identify the contributing materials. Using Equation C.4, the heat capacities of the system are computed. The contributing factors and the heat capacities are listed in Table C.2.

Table C.2-

THE LIMB HEAT CAPACITIES

Heat Capacity Identification	c_a in Calories per Gram ° Kelvin	c_b in Calories per Gram ° Kelvin	m_a in Grams*	m_b in Grams*	v_b in Cubic Centimeters*	ρ_b in Grams per Cubic Centimeter	Heat Capacity in Joules per ° Kelvin
C ₁	0.215 (Ref. 19)	0.118 (Ref. 20)	12.75	339.76	-	-	167.91
C ₂	0.215 (Ref. 19)	0.105 (Ref. 19)	119.1	21.6	-	-	188.94
C ₃	0.215 (Ref. 19)	0.2 (Ref. 21)	22.5	-	67.2	2.3 (Ref. 21)	139.41

*Determined by measurement of the actual calorimeter.

C_1 is the heat capacity of the source. In this term, the heat capacity of the actual source is assumed to behave as the case material, AISI 301 stainless steel. This is summed with the heat capacity of the source holder which is of aluminum. The subscript a in Table C.2 refers to the source holder and subscript b refers to the source.

C_2 is the heat capacity of the heat-spreading cylinder summed with the heat capacity of the heater. The heat-spreading cylinder is of aluminum (subscript a in Table C.2) and the heater is assumed to behave as nickel (subscript b in Table C.2).

C_3 is the heat capacity of the central portion of the insulating cylinder and the associated temperature-sensing windings. The heat capacity of the glass cylinder (subscript b in Table C.2) is calculated from the volume of the glass. This is summed with the heat capacity of the windings, which are assumed to behave as nickel (subscript c in Table C.2). The glass used is code 7740 Pyrex.

For convenience, these heat-capacity terms are converted to the mechanical equivalent of heat by using the constant, 4.1858 joules = 1 calorie (at 288.15° Kelvin), (Reference 22).

4. Calculation of the Conductance Constants

For conduction in uniform, cylindrical, concentric layers with no fringing,

$$H = \frac{2\pi K \ell}{\ln \frac{r_2}{r_1}} \quad (C.5)$$

where

H = the conductance,

K = the conductivity of air (2.6×10^{-4} watts/centimeter
°Kelvin), (Reference 22),

ℓ = the axial length,

r_1 = the inner radius and

r_2 = the outer radius.

The values of conductance are tabulated along with their contributing factors in Table C.3.

Conductance H_1 is for the air in the secondary heat-flow path between the temperature-sensitive windings and the sink.

In the main heat-flow path there are two air gaps and the conductances through these are H_2 and H_3 . H_2 is the conductance through the air between the heater winding and the insulating cylinder. H_3 is the conductance between the main portion of the temperature-sensing winding and the sink.

Because the effective radius to the outside of the heater winding and the temperature-sensing windings

Table C.3

THE CONDUCTIVITY TERMS

<u>Conductivity Identification</u>	<u>l in Inches*</u>	<u>r₁ in Inches**</u>	<u>r₂ in Inches*</u>	<u>Conductivity in Joules per Minute ° Kelvin</u>
H ₁	0.51	0.791	0.846	1.889
H ₂	9.74	0.6758	0.690	116.27
H ₃	9.74	0.791	0.846	36.07

*Determined by measurement of the actual calorimeter.

**Determined by analysis of the calorimeter response.

cannot be directly measured, the r_1 terms are determined by calculation from the measured response. The values listed are between the maximum radius measured and a radius determined by subtracting the wire radius from this value. These two radii form the limits of the expected range.

5. Calculation of the Radiation Constants

For cylindrical, concentric layers with no fringing, the radiation factor is

$$G = \frac{\sigma}{\left(\frac{1}{A_1}\right)\left(\frac{1}{e_1}\right) + \left(\frac{1}{A_2}\right)\left(\frac{1}{e_2} - 1\right)} \quad (C.6)$$

or for the linearized case it is

$$4G(T_0)^3 = \frac{4\sigma(T_0)^3}{\left(\frac{1}{A_1}\right)\left(\frac{1}{e_1}\right) + \left(\frac{1}{A_2}\right)\left(\frac{1}{e_2} - 1\right)} \quad (C.7)$$

where

G = the radiation constant,

σ = the Stefan-Boltzmann constant (5.6697×10^{-5} erg/centimeters² °Kelvin⁴ seconds), (Reference 22),

A_1 = the area of the emitting surface which has surface emissivity e_1 ,

A_2 = the area of the receiving surface which has surface emissivity e_2 and

T_0 = the sink temperature in °Kelvin (301.15).

Values of the calculated radiation constants are given in Table C.4.

The constants G_1 , G_2 and G_3 correspond to radiation across the same air gaps for which conductances H_1 , H_2 and H_3 are calculated in Section 4 of this appendix. Values of the emissivities are estimated from their normal range for the materials used (Reference 18). The A_1 areas are all effective areas on winding surfaces. These areas are calculated using the radii determined in Section 4.

6. Combination of the Thermal Constants

The various thermal constants developed in this appendix are combined to form the simplified thermal constants required in Section C of Chapter 3. These simplified constants are formed as defined for Equations 3.21 and 3.22. Specifically,

$$B_1 = H_1 + 4G_1(T_O)^3 = 2.014, \quad (C.8)$$

$$B_2 = H_2 + 4G_2(T_O)^3 = 124.40 \text{ and} \quad (C.9)$$

$$B_3 = H_3 + 4G_3(T_O)^3 = 38.46. \quad (C.10)$$

All of these three terms have the units of conductance, joules/minute $^{\circ}\text{Kelvin}$. Also,

$$A_2 = C_3 = 139.13 \text{ joules}/^{\circ}\text{Kelvin}. \quad (C.11)$$

Table C.4

THE LINEARIZED RADIATION CONSTANTS

<u>Constant Identification</u>	<u>A₁ in Centimeters^{2**}</u>	<u>A₂ in Centimeters^{2*}</u>	<u>e₁ Estimated</u>	<u>e₂ Estimated</u>	<u>Linearized Radiation Constant in Joules per Minute Kelvin</u>
4G ₁ (T _o) ³	16.35	17.49	0.9	0.2	0.1253
4G ₂ (T _o) ³	266.8	272.4	0.9	0.9	8.128
4G ₃ (T _o) ³	312.3	334.0	0.9	0.2	2.393

*Determined by measurement of the actual calorimeter.

**Calculated from values determined by analysis of the calorimeter response.

Finally, A_1 takes on two values depending on whether the heat capacity of the source (or a blank) is used.

When the source is used,

$$A_1 = C_1 + C_2 = 167.91 + 188.94 = 356.9 \text{ joules/}^\circ\text{Kelvin.} \quad (\text{C.12})$$

When the source is not used,

$$A_1 = C_2 = 188.94 \text{ joules/}^\circ\text{Kelvin.} \quad (\text{C.13})$$

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CURRICULUM VITAE

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