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DEVELOPMENT OF A MODIFIED CHAPMAN-ENSKOG
SOLUTION OF THE LENARD-BALESCU EQUATION
FOR FULLY IONIZED TWO-TEMPERATURE PLASMAS

Prepared by
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June 1969

MASTER

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Sandia Laboratories

under

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ABSTRACT

The Lenard-Balescu equation is utilized to determine the characteristic transfer frequencies associated with a tenuous test particle stream interacting with a two-temperature Maxwellian hydrogen plasma. A set of ordered equations are then obtained with the appropriate ordering parameter

$\epsilon = \left(\frac{M_e}{M_i} \right)^{\frac{1}{2}}$. Utilizing this ordering parameter, a modification of the classical Chapman-Enskog method of solution is systematically developed.

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SUMMARY

The Lenard-Balescu equation, the interaction integral of which includes the influence of long-range collective effects, is utilized to determine the characteristic transfer frequencies associated with a tenuous "test particle" stream interacting with a two-temperature Maxwellian hydrogen plasma. Like-particle energy and momentum transfer frequencies, and unlike-particle energy and momentum transfer frequencies are determined for the range of T_e/T_i from 1 to 100. On comparison with the results of other investigators, the electron-electron transfer frequencies are observed to include long-range collective corrections ranging up to 40%.

These large collective corrections make evident the necessity of employing the Lenard-Balescu kinetic equation to describe the behavior of two-temperature fully ionized plasmas. The "test particle" frequencies are accepted, for ordering purposes, as being good representations of the corresponding frequencies associated with the various Lenard-Balescu interaction integrals. The ordered equation which results makes evident the appropriateness of the ordering parameter $\epsilon = (m_e/m_i)^{1/2}$ for these systems (using this parameter an energy decoupling results which is, of course, consistent with physical reality). Utilizing this ordering parameter, a modification of the "classical" Chapman-Enskog method of solution is systematically developed. This method of solution is sufficiently general that it is applicable to all

stable fully ionized plasma systems of small mass ratio. The formal solutions obtained are found to be surprisingly simple and a brief discussion is given on the determination of physical properties of interest from these solutions.

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CHAPTER I

INTRODUCTION

Fully ionized laboratory plasma systems are typically two temperature since the energy exchange mechanisms are such that one of the constituents is preferentially heated, and by virtue of the large difference in mass between ions and electrons the energy exchange between constituents is, relatively ineffective. The determination of a mathematical description of the behavior of a two-temperature plasma is complicated by the fact that long-range collective interactions are important necessitating the use of the Lenard-Balescu kinetic equation. Complications also arise due to the fact that the "classical" Chapman-Enskog method is adequate only when both like and unlike particle interactions dominate the behavior, which is, of course, not the case for two-temperature systems. Numerous investigators [References 1,2,3,4,5] have attempted to develop methods which could adequately describe the behavior of this important class of systems; however, the mathematical methods utilized are justified only on the basis of intuitive arguments, and the studies carried out are incomplete.

A general discussion of plasma kinetic theory is given in Chapter II which is intended to make evident the advantages and limitations of both the various kinetic equations and the methods of solution which are currently used. The need to use the Lenard-Balescu kinetic equation to describe the evolution of the one-particle distribution functions for two-temperature plasmas is established. The "classical" Chapman-Enskog method

and the Grad method of solution are discussed and determined to be inappropriate for these systems.

A general discussion of the "test particle" method of computing momentum and energy transfer frequencies is given in Chapter III. Using the Lenard-Balescu kinetic equation to describe the evolution of the "test particle" distribution function, the "test particle" method is employed to compute like-particle and unlike-particle energy and momentum transfer frequencies over the range of T_e/T_i from 1 to 100. Each frequency is exhibited to be the sum of two terms. The first term takes account of the collision interactions within a Debye sphere and hence agrees with the results of Montgomery and Tidman [6] who compute some of these times using the Fokker-Planck equation with a Debye cut-off potential.¹ The second term takes account of long-range collective effects as would be anticipated since the Lenard-Balescu equation differs from the Fokker-Planck equation in that long-range collective effects are accounted for through the inclusion of the dielectric function $[\epsilon^+(\vec{\ell}, \omega)]$. The long-range collective term ranges in magnitude up to 40% of the usual collision term.

In Chapter IV the Lenard-Balescu kinetic equation is employed to describe the behavior of two-temperature fully ionized plasmas. The large collective correction to certain transfer frequencies and the discussion of the Lenard-Balescu equation

¹In Thompson [7] it is shown that the Landau and Fokker-Planck equations are mathematically identical.

given in Chapter II dictate that this kinetic equation be used for such systems. The "test particle" frequencies are accepted, for ordering purposes, as being good representations of the corresponding frequencies associated with the various Lenard-Balescu interaction integrals. Using these "test particle" frequencies, an expansion parameter $\epsilon = (m_e/m_i)^{1/2}$ is identified as the appropriate expansion parameter for a two-temperature plasma. The Lenard-Balescu equation is ordered in ϵ and the resulting zeroth and first-order equations solved by a mathematical procedure similar to the "classical" Chapman-Enskog approach. In obtaining the formal solutions for the one-particle distribution functions, terms of $O(\epsilon^2)$ and higher are ignored. The formal solutions obtained exhibit a surprisingly simple form. A brief discussion is given on the determination of physical properties of interest utilizing these solutions.

It will be noted that although the results in Chapter IV are exhibited for a hydrogen plasma with $T_e > T_i$ to facilitate comparison with the results of other investigators, the mathematical method is valid for all plasmas in which the constituent mass ratio is small.

CHAPTER II

PLASMA KINETIC THEORY

Introduction

In the first section of this chapter a brief discussion is given of the various kinetic equations which are utilized in describing fully ionized plasmas with emphasis placed on the advantages and limitations of each equation. The Lenard-Balescu kinetic equation and Landau kinetic equation will be discussed, in more depth than the other kinetic equations, since these two kinetic equations are found to include the information contained in all the other kinetic equations considered.

In the second section of this chapter, the two most widely used methods of solution of kinetic equations, the "classical" Chapman-Enskog method and the Grad method, are discussed.

Kinetic Equations

It is usually assumed that the state of a multispecies plasma can be adequately described by the one-particle distribution functions. Corresponding to each species of the plasma, b , there is a distribution function $f_b(\vec{r}, \vec{c}, t)$ such that the number of particles n_b of type b with position coordinates in the range d^3r about $\vec{r}$ and with velocity coordinates in the range d^3c about $\vec{c}$ is $f_b(\vec{r}, \vec{c}, t) d^3r d^3c$. The integral of $f_b(\vec{r}, \vec{c}, t)$ over all velocities is therefore

$$n_b(\vec{r}, t) = \int f_b(\vec{r}, \vec{c}, t) d^3c$$

If these distribution functions are known, it is then possible to compute macroscopic properties such as heat flux and electrical current. For the nonequilibrium fully ionized hydrogen plasma of interest, the evolution of the system is described by the changes in the one-particle distribution functions f_b . These functions obey a kinetic equation of the form.

$$\frac{\partial f_b}{\partial t} + \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \frac{\vec{F}_b}{m_b} \cdot \frac{\partial f_b}{\partial \vec{c}} = \sum_{a=e,i} J(f_b, f_a) \quad (2-1)$$

$\vec{F}_b$ is the vector force acting on the species of type b. This force is assumed to be much smaller than the forces acting on the particle during encounters with other particles. The subscripts b and a will be used throughout to designate electron or ion quantities. The letter i is used to denote ion quantities and the letter e is used to denote electron quantities. The letters α and β will be used throughout as tensor subscripts. $J(f_b, f_a)$ represents the time rate of change of f_b due to the interaction of particles of type b, with particles of type a.

The kinetic equations most widely used to describe plasma systems are (1) the Boltzmann equation, (2) the Fokker-Planck equation, (3) the Landau equation, and (4) the Lenard-Balescu equation. It has been shown that for fully ionized plasmas, the Fokker-Planck equation is mathematically identical to the Landau equation [7]. It has also been shown that, since the effects of large deflection binary collisions are negligible, the Boltzmann equation and the Fokker-Planck equation are mathematically equivalent [8]. In light of these equivalences, it will suffice to discuss only the Lenard-Balescu and the Landau kinetic equations.

The Lenard-Balescu kinetic equation as derived from the Liouville equation [9,10,11], yields a more complete description of fully ionized plasma systems than the Landau kinetic equation [12]. This is due to the fact that the interaction integral of the Lenard-Balescu kinetic equation has the capability of correctly treating not only the interactions between particles with a separation less than the Debye length but also the long-range collective interactions. In fact, the Landau kinetic equation can be obtained from the Lenard-Balescu kinetic equation by making a simplifying approximation.

The Lenard-Balescu kinetic equation is given by

$$\frac{\partial f_b}{\partial t} + \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \frac{\vec{F}_b}{m_b} \cdot \frac{\partial f_b}{\partial \vec{c}} = \sum_{a=e,i} \frac{8\pi^4 e^4}{m_b} \int_0^{\ell_0} d^3 \ell \int d^3 c' \vec{\ell} \cdot \frac{\partial}{\partial \vec{c}} \left| \frac{v_{\ell}}{\epsilon^+(\vec{\ell}, U)} \right|^2$$

$$\delta(\vec{\ell} \cdot \vec{g}) \left\{ \frac{\vec{\ell}}{m_b} \cdot \frac{\partial}{\partial \vec{c}} - \frac{\vec{\ell}}{m_a} \cdot \frac{\partial}{\partial \vec{c}'} \right\} f_b(\vec{c}) f_a(\vec{c}') \quad (2-2)$$

where

$$\epsilon^+(\vec{\ell}, U) = 1 + i \sum_{a=e,i} \frac{4\pi^2 e^2}{m_a \ell^2} \int d^3 c' \delta_-(\vec{\ell} \cdot \vec{y}) \vec{\ell} \cdot \frac{\partial f_a(\vec{c}')}{\partial \vec{c}'}$$

$$\vec{g} = \vec{c} - \vec{c}'$$

$$\delta_-(y) = \delta(y) - \frac{1}{\pi} \text{Pr}\left(\frac{1}{y}\right)$$

$$U = \vec{\ell} \cdot \vec{c} / \ell$$

$$\ell_0 = kT_e / e^2$$

$$e v_{\ell} = e / 2\pi^2 \ell^2$$

The quantity, $e/2\pi^2 \ell^2$, is the Fourier transform of the, e/r , coulomb potential; $\vec{\ell}$ is the wave vector; δ is the Dirac delta function; and the symbol Pr indicates the principal value when

placed under the integral sign. The upper limit of the integration is ℓ_0 which is the inverse of the distance of closest approach of thermal electrons. The value of the integral depends very weakly on the cut-off value ℓ_0 since wave numbers larger than ℓ_0 correspond to large angle deflections, which as previously indicated, have negligible effect in fully ionized systems. The function $\epsilon^+(\vec{\mathcal{L}}, U)$ is the dielectric function which is a macroscopic property that characterizes the response of the plasma to a small perturbation as exhibited in Appendix A. It is the inclusion of this function which allows one to account for long-range collective effects properly.

A comparison of the Landau interaction integral with the Lenard-Balescu interaction integral reveals that if $\epsilon^+(\vec{\mathcal{L}}, U)=1$, in the latter, the two integrals are identical. Hence, the Landau integral cannot adequately account for long-range interactions. This inadequacy, in fact, results in a divergence of the Landau integral. In order to prevent this divergence, and to crudely account for the long-range collective effects, the wave number integration is terminated at a lower limit corresponding to the Debye wave number, ℓ_D , [6,13]. This procedure is equivalent to assuming that the effective interaction potential is a pure coulomb potential for interparticle spacings less than a Debye length and that the effective interaction potential is zero for interparticle spacings greater than the Debye length. This potential will be referred to as the Debye cut-off coulomb potential in this work. As an alternative to this procedure, the divergence can be prevented by replacing the coulomb potential, e/r , with the Debye screened coulomb potential

$(e/r) \exp(-r/\lambda)$. The Debye length, $\lambda = 1/\ell_D$, is given by

$$\frac{1}{\lambda} = \left(\sum_{b=e,i} \frac{4\pi n_b e^2}{kT_b} \right)^{1/2} \quad (2-3)$$

where k and T_b are the Boltzmann constant and the temperature of the species of type b .

Important physical insight can be obtained by comparing the Landau and Lenard-Balescu interaction integrals. As pointed out in the work of Balescu [13] one can say that formally the Lenard-Balescu interaction integral is equivalent to a Landau interaction integral for a plasma in which the particles interact through the effective potential $\chi(\vec{\mathcal{L}}, U)$ which is given by

$$\chi(\vec{\mathcal{L}}, U) = e \left| \frac{v_{\mathcal{L}}}{\epsilon^+(\vec{\mathcal{L}}, U)} \right| \quad (2-4)$$

Obviously, if $\epsilon^+ = 1$, then χ becomes the Fourier transform of the coulomb potential. If the limit $U \rightarrow 0$ is taken, then

$$|\epsilon^+| = 1 + 1/\lambda^2 \ell^2 \quad (2-5)$$

and χ becomes the Fourier transform of Debye screened coulomb potential.

As shown in Appendix A, the velocity dependent dielectric function is derived by assuming a small perturbation to the plasma. This dielectric function is a property of the plasma medium and takes into account the response of the plasma particles to a small perturbation. Specifically, it can account for the collective response of the plasma to a moving test charge. In view of these remarks, it should be obvious why $\chi(\vec{\mathcal{L}}, U)$ has

been referred to as the effective interaction potential. This effective potential will be referred to as the dynamically screened coulomb potential throughout the remainder of this report.

In certain plasma systems, this dynamically screened coulomb potential can become quite large compared to the Debye cut-off coulomb potential or the Debye screened coulomb potential. This occurs because ϵ^+ can assume small values for a range of values of the velocity parallel to the wave vector (i.e. $U = \vec{k} \cdot \vec{c} / \ell$). One such system in which this enhancement of the effective potential is particularly important is a plasma in which, to first approximation, the ion and electron distributions are Maxwellian with temperatures T_i and T_e and in which the temperature ratio T_e/T_i is greater than one. For these systems, it is then obvious that the Lenard-Balescu interaction integral must be used.

Since the Lenard-Balescu equation is of primary interest, the limitations of this equation will be enumerated. In order that the Lenard-Balescu equation retain its validity, the plasma must be such that the following conditions are satisfied: (1) three-particle correlation effects must be negligible, (2) the adiabatic hypothesis must be satisfied (i.e., the two-particle correlation function must relax in a time short compared to the rate of change of the one-particle distribution function [14]), (3) the characteristic distance associated with the change in the one-particle distribution function must be less than the particle mean free path, (4) the energy gain of a particle due to external forces in one mean free path must be

much less than the energy transferred in interactions with other particles, and (5) the effect of close encounters between particles must be negligible. Fortunately, these conditions are satisfied for a wide class of plasma systems.

Methods of Solving the Kinetic Equations

With regard to methods of solution, there are two basic methods worthy of consideration. These are the Grad method [15, 16] and the Chapman-Enskog method [17,18]. Using the Grad method, it is possible to expand the nonequilibrium one-particle distribution function of each constituent of the plasma in three-dimensional Hermite polynomials about the constituent Maxwellian. Since there is no requirement that the constituents be at the same temperature, the Grad method can be used as the method of solution for the two-temperature plasma. Transport coefficients computed by investigators [19,20] using the Grad method, indicate, however, that it is not possible to reproduce the widely accepted results obtained by Spitzer and Harm [21] who used a numerical technique and the Fokker-Planck equation. It would appear feasible, by using the Grad technique, to treat strongly driven systems in which the time and space variations of the one-particle distribution function and the effects of external fields are the same order as particle interactions in influencing the behavior of the system. To date, this apparently greater generality of the Grad method as compared to the Chapman-Enskog method is of purely academic interest since in strongly driven systems the conditions (2) and (3) above would surely be violated. Further, the Grad method necessitates a truncation of

the representation of the distribution function with no means of evaluating the degree of error introduced by the truncation.

The Chapman-Enskog method, on the other hand, involves an ordering of the terms of the kinetic equation and the generation of a solution consistent with this ordering. Assuming the terms are ordered in accord with physical reality, the solution generated is asymptotically convergent to a solution of the kinetic equation [22,23]. As shown in the work of Devoto [24], the results of Spitzer and Harm [21] can be obtained by employing the Chapman-Enskog method of solution.

A brief discussion of the "classical" Chapman-Enskog method of obtaining the solution of a kinetic equation will be given to exhibit the salient characteristics of the technique. The ordering of the kinetic equation is obtained by introducing a characteristic macroscopic time t_0 and characteristic microscopic times τ_{bb} and τ_{ba} . Using these characteristic times, the kinetic equation for a fully ionized hydrogen plasma takes the form

$$Df_b = \frac{t_0}{\tau_{bb}} J(f_b, f_b) + \frac{t_0}{\tau_{ba}} J(f_b, f_a) \quad (2-6)$$

$$(a, b = e, i; a \neq b)$$

where

$$Df_b \equiv \frac{\partial f_b}{\partial t} + \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \frac{\vec{F}_b}{m_a} \cdot \frac{\partial f_b}{\partial \vec{c}}$$

The time t_0 is the characteristic macroscopic time associated with the time variation of the quantities appearing on the left hand side of the kinetic equation. The time τ_{bb} is the characteristic microscopic time associated with $J(f_b, f_b)$ and the

time τ_{ba} is the characteristic microscopic time associated with $J(f_b, f_a)$. In the "classical" method, it is assumed that

$$t_o/\tau_{bb} \sim t_o/\tau_{ba} \sim \frac{1}{\epsilon}$$

The ordering parameter ϵ is, therefore, the ratio of either of the characteristic microscopic times to the characteristic macroscopic time. With this definition of ϵ , the kinetic equation can be written

$$Df_b = \frac{1}{\epsilon} \sum_{a=e,i} J(f_b, f_a) \quad (2-7)$$

If it is assumed that f_b can be represented by the expansion

$$f_b = \sum_{n=0}^{\infty} \epsilon^n f_b^n \quad (2-8)$$

then it is possible to split the kinetic equation into a series of equations corresponding to the various order of ϵ . The $O(1/\epsilon)$ equation is

$$\sum_{a=e,i} J(f_b^0, f_a^0) = 0 \quad (2-9)$$

and the $O(1)$ equation is

$$Df_b^0 = \sum_{a=e,i} J(f_b^0, f_a^1) + J(f_b^1, f_a^0) \quad (2-10)$$

If f_b^1 is written in the form $f_b^1 = f_b^0 \phi_b$, then the $O(1)$ equation becomes

$$Df_b^0 = \sum_{a=i,e} J(f_b^0, f_a^0 \phi_a) + J(f_b^0 \phi_b, f_a^0) \quad (2-11)$$

These are the equations of the "classical" Chapman-Enskog technique which, if solved, yield f_b^0 and f_b^1 , the zeroth and first approximations to the one-particle distribution function.

The assumption that all ratios t_o/τ_{ab} appearing in a given constituent equation are $O(\frac{1}{\epsilon})$ constrains the zeroth order one-particle distribution functions to be Maxwellian at the same temperature T for every constituent no matter what form of the interaction integral is utilized. Since this thesis is concerned with two-temperature fully ionized plasma systems, it is obvious that the "classical" Chapman-Enskog method of solutions cannot be utilized.

CHAPTER III

CALCULATION OF ENERGY AND MOMENTUM TRANSFER FREQUENCIES

Formulation of the "Test Particle" Method¹

Both the momentum transfer and energy transfer frequencies for both electron and ion "test particles" scattering within a background plasma, in which the electron temperature T_e is greater than or equal to the ion temperature T_i , are calculated and exhibited in this chapter. To be more specific, the background plasma is taken to be a field-free², fully ionized, hydrogen plasma in which the electron and ion distribution functions are assumed to be Maxwellian with temperatures T_e and T_i respectively. This background plasma is assumed to co-exist at time $t=0$ with a monoenergetic stream of "test particles" whose distribution is

$$f_t(\vec{C}) = n_t \delta(\vec{C} - \vec{C}_0)$$

where, the subscript t (denoting "test particle") may correspond to either e or i . In this study C is assumed to be $(2kT_i/m_i)^{1/2}$ and $(2kT_e/m_e)^{1/2}$ for ion and electron "test particles" respectively.

¹This formulation follows closely that outlined in Montgomery and Tidman [6]. The formulation here is different in that T_e and T_i are not required to be equal and in that the Lenard-Balescu interaction integral is utilized instead of the Fokker-Planck interaction integral.

²It is possible to allow an applied electric field if the particle motion is not markedly affected by the field in a time corresponding to the time between interactions.

The "test particle" stream, whose number density is n_t , is assumed to be extremely tenuous (i.e., $n_t \ll n_e, n_i$). In this case, to first order in n_t , the "test particles" interact only with the electrons and ions of the background plasma. Also, the probability of interaction with any particle twice is negligible. These "test particles" in effect see an essentially uniform plasma in which the electrons and ions retain their Maxwellian distributions at temperatures T_e and T_i for all time. The instantaneous time derivatives of the velocity moments of f_t at $t=0$ can thus be obtained from the velocity moments of

$$\frac{\partial f_t}{\partial t} = J_{LB}(f_t, f_e) + J_{LB}(f_t, f_i) + J_{LB}(f_t, f_t) \quad (3-1)$$

If $\delta(\vec{C}-\vec{C})n_t$ is substituted for f_t in the above equation and terms containing n_t^2 are ignored, then the equation may be written

$$\frac{\partial \delta(\vec{C}-\vec{C})}{\partial t} = \sum_{a=e,i} J_{LB}(\delta(\vec{C}-\vec{C}), f_a) \quad (3-2)$$

where the symbol J_{LB} denotes the Lenard-Balescu interaction integral. It should be noted that the Lenard-Balescu interaction integral is being used to represent the particle interactions.

The deflection time or momentum transfer time¹ for a "test

¹Strictly speaking the τ 's introduced here are not times since they can assume negative values. A negative value, of course, corresponds physically to an impedance of the transfer of the quantity. Taking absolute values of these quantities will yield a physically meaningful time.

particle" of velocity $\vec{C}$ is defined as

$$\tau_M = C^2 / (\partial C_L^2 / \partial t)_{t=0} \quad (3-3)$$

where

$$(\partial C_L^2 / \partial t)_{t=0}$$

is the rate at which the square of the velocity of the stream perpendicular to the direction $\vec{C}/C$ is increasing at $t=0$. This time is, roughly speaking the characteristic time in which the "test particle" stream is deflected through an angle of order $\pi/2$. The absolute value of the inverse of this time is referred to as the momentum transfer frequency. To illustrate the notation used herein, the momentum transfer frequency obtained by considering the deflection of an electron test stream due to interactions with only the ions in the background plasma, is referred to as the electron-ion momentum transfer frequency and is denoted by ν_{Mei} ; the corresponding transfer time is denoted by τ_{Mei} . With these definitions there are four momentum transfer frequencies for two constituent systems.

If one defines

$$E \equiv m_t C^2 / 2$$

another transfer time, usually referred to as the energy exchange time, can be written

$$\tau_E \equiv -E / (\partial E / \partial t)_{t=0} \quad (3-4)$$

where

$$(\partial E / \partial t)_{t=0}$$

is the rate of change of energy of the stream of "test particles" of velocity $\vec{C}$ at $t=0$. The absolute value of the inverse of this time is referred to as the energy transfer frequency. The notation used is similar to that used for momentum transfer frequencies (i.e., ν_{Eei} is the electron-ion energy transfer frequency and τ_{Eei} is the corresponding transfer time).

The values of $(\partial C_{\perp}^2 / \partial t)_{t=0}$ and $(\partial E / \partial t)_{t=0}$ are obtained by taking the appropriate velocity moments of Eq. (3-2), and assuming that at $t=0$

$$C_y = C_z = 0; \quad C_x = C_{\parallel} = C$$

Having obtained these moments, it is a simple matter to compute τ_M and τ_E .

Review of Previous Work.

"Test particle" transfer frequencies or transfer times have been calculated by several authors [6,25,26]. By making use of a temperature relaxation model, other authors [27,28,29] have obtained relaxation times or frequencies closely related to some of the "test particle" frequencies calculated herein.

The works referred to above are of three types: (1) "Test particle" calculations which assume that the effective interaction potential is a Debye cut-off coulomb potential. (2) "Test particle" calculations which assume that the effective interaction potential is a dynamically screened coulomb

potential. (3) Temperature relaxation calculations which assume that the effective interaction potential is a dynamically screened coulomb potential; these works exhibit temperature relaxation times.

The work of Montgomery and Tidman [6] and that of Spitzer [25] are typical of the works of type (1) in that the effect of dynamical screening is not included in the calculation of "test particle" transfer times or frequencies. In addition, the temperature ratio T_e/T_i is assumed to be small.

To the author's knowledge, the work of Tidman and Eviatar [26] is the only work of type (2) that has been published. In this work, the effect of dynamical screening is included in the calculations. However, for the two-temperature plasma only the electron-electron momentum transfer time is computed; and unfortunately the "resonance approximation" is used for evaluating an integral required in the computation of this transfer time. (It will be demonstrated in this report that the "resonance approximation" is inexact for the range of temperature ratios of interest.)

The works of type (3), of which the work of Wu, et al. [27] is the most complete, include the effects of dynamical screening in their calculations. The remarks which follow apply specifically to the work of Wu, et al. The results include the electron temperature relaxation time and the ion temperature relaxation time. In each case, the constituent under consideration is assumed to be represented by an anisotropic temperature distribution at time $t=0$. Also included is the temperature

relaxation time for electrons and ions initially at different temperatures T_e and T_i . The integrations involved in the computation of these times are performed numerically, and a wide range of the temperature ratio T_e/T_i is considered. For the range of T_e/T_i between 1 and 100; however, the only values exhibited are for $T_e/T_i=1$ and $T_e/T_i=100$.

In order to facilitate a comparison of the methods and results of the above works with those of this chapter, the following comments pertaining to this chapter are included. The Lenard-Balescu interaction integral is used to represent particle interactions; therefore, the effect of dynamical screening on the transfer frequencies is included. The values of the various "test particle" transfer frequencies are obtained using numerical methods to evaluate integrals that are not amenable to analytic evaluation. The values of the frequencies are exhibited for eleven values of T_e/T_i ranging from 1 to 100. The values of all possible momentum and energy transfer frequencies for a fully ionized hydrogen plasma are calculated. As mentioned above, the evaluation of certain integrals needed for the determination of the transfer frequencies is performed numerically; therefore, the "resonance approximation" for these integrals, which is both inexact and misleading for temperature ratios less than or equal to 100, is avoided.

Development of Equations

In this section the scattering of the "test particles" discussed in the previous section, will be investigated in detail. A direct linearization of the Lenard-Balescu interaction integral in the "test particle" stream distribution function f_t yields an equation for the evolution of the distribution f_t which exhibits the mathematical form of a Fokker-Planck equation. It is evident that the Fokker-Planck coefficients and transfer frequencies obtained from the use of this equation include the effects of dynamic screening.

The linearized Lenard-Balescu equation for the "test particle" problem is, as indicated by Eq. (3-2)

$$\begin{aligned} \frac{\partial \delta(\vec{c}-\vec{c}')}{\partial t} = \frac{8\pi^4 e^4}{m_t} \sum_{a=e,i} \int_0^{l_0} d^3 \ell \int d^3 c' \vec{\ell} \cdot \frac{\partial}{\partial \vec{c}} \frac{v_{\ell}^2}{|\epsilon_0^+(\ell, U)|^2} \\ \delta(\vec{\ell} \cdot \vec{g}) \left\{ \frac{\vec{\ell}}{m_t} \cdot \frac{\partial}{\partial \vec{c}} - \frac{\vec{\ell}}{m_a} \cdot \frac{\partial}{\partial \vec{c}'} \right\} \delta(\vec{c}-\vec{c}') f_a(\vec{c}') \end{aligned} \quad (3-5)$$

where the f_a are the Maxwellian distributions of the background plasma. With the exception of $\epsilon_0^+(\ell, U)$ all other symbols are defined in Eq. (2-2). The symbol $\epsilon_0^+(\ell, U)$ is used to denote the plasma dielectric function as computed for Maxwellian distributions of electrons and ions with temperatures T_e and T_i . The form $\epsilon_0^+(\ell, U)$ for such distributions is exhibited in the work of Wu, et al [30]. This form is

$$\epsilon_0^+(\ell, U) = 1 + \frac{\ell_D^2}{\ell^2} \left\{ A(U) + iB(U) \right\} \quad (3-6)$$

Making the following definitions

$$\eta \equiv T_e/T_i$$

$$\rho \equiv (m_e/nm_i)^{1/2}$$

$$y \equiv U/(2kT_i/m_e)^{1/2}$$

the functions $A(y)$ and $B(y)$ may be written

$$A(y) = 1 + \frac{1}{1+\eta} \left\{ \eta y \operatorname{Re}[Z(y)] + \rho y \operatorname{Re}[Z(\rho y)] \right\}$$

$$B(y) = \frac{\eta y}{1+\eta} \operatorname{Im}[Z(y)] + \frac{\rho y}{1+\eta} \operatorname{Im}[Z(\rho y)]$$

The plasma dispersion function $Z(y)$ is tabulated and discussed by Fried and Conte [31]. $\operatorname{Re}[Z(y)]$ is the real part of $Z(y)$ and $\operatorname{Im}[Z(y)]$ is the imaginary part.

$$Z(y) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{dx e^{-x^2}}{x-y-i\Delta} \quad (3-7)$$

The plasma dielectric constant is derived and discussed in Appendix A.

Introducing

$$F_a(U) = \int d^3c' \delta(U - \vec{l} \cdot \vec{c}'/l) f_a(\vec{c}')$$

$$\delta(\vec{l} \cdot \vec{g}) = \delta(U - \vec{l} \cdot \vec{c}'/l)/l$$

the linearized Lenard-Balescu equation, Eq.(3-5) in standard tensor notation, assumes the following convenient form.

$$\begin{aligned} \frac{\partial \delta(\vec{c} - \vec{c}')}{\partial t} &= \frac{2e^4}{m_t^2} \sum_{a=e,i} \frac{\partial}{\partial c_\alpha} \int \frac{d^3l}{l^5} \frac{l_\alpha l_\beta}{|\epsilon_o^+(l, U)|^2} \frac{\partial}{\partial c_\beta} \delta(\vec{c} - \vec{c}') F_a(U) \\ &- \frac{2e^4}{m_t} \sum_{a=e,i} \frac{1}{m_a} \frac{\partial}{\partial c_\alpha} \int \frac{d^3l}{l^4} \frac{l_\alpha}{|\epsilon_o^+(l, U)|^2} \delta(\vec{c} - \vec{c}') \frac{\partial}{\partial U} F_a(U) \end{aligned} \quad (3-8)$$

It is advantageous to cast Eq. (3-8) in the mathematical form of a Fokker-Planck equation. To expedite this transformation, the following definitions are made

$$T_{\alpha\beta}(\vec{c}) \equiv \sum_{a=e,i} \frac{4e^4}{m_t^2} \int d^3\ell \frac{\ell_\alpha \ell_\beta F_a(U)}{\ell^5 |\epsilon_O^+(\ell, U)|^2} \quad (3-9)$$

$$B_\alpha(\vec{c}) = \sum_{a=e,i} \frac{2e^4}{m_t m_a} \int d^3\ell \frac{\ell_\alpha \partial F_a(U) / \partial U}{\ell^4 |\epsilon_O^+(\ell, U)|^2} \quad (3-10)$$

Employing these definitions Eq. (3-8) becomes

$$\frac{\partial \delta(\vec{c}-\vec{c}')}{\partial t} = \frac{1}{2} \frac{\partial}{\partial c_\alpha} \left\{ T_{\alpha\beta}(\vec{c}) \frac{\partial}{\partial c_\beta} \delta(\vec{c}-\vec{c}') \right\} - \frac{\partial}{\partial c_\alpha} B_\alpha(\vec{c}) \delta(\vec{c}-\vec{c}') \quad (3-11)$$

Noting that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial c_\alpha} \left\{ T_{\alpha\beta}(\vec{c}) \frac{\partial}{\partial c_\beta} \delta(\vec{c}-\vec{c}') \right\} &= \frac{1}{2} \frac{\partial}{\partial c_\alpha} \frac{\partial}{\partial c_\beta} \left\{ T_{\alpha\beta}(\vec{c}) \delta(\vec{c}-\vec{c}') \right\} \\ - \frac{1}{2} \frac{\partial}{\partial c_\alpha} \left\{ \frac{T_{\alpha\beta}(\vec{c})}{\partial c_\beta} \delta(\vec{c}-\vec{c}') \right\} &\quad (3-12) \end{aligned}$$

Eq. (3-11) may be written

$$\begin{aligned} \frac{\partial \delta(\vec{c}-\vec{c}')}{\partial t} &= - \frac{\partial}{\partial c_\alpha} \left\{ \left(B_\alpha(\vec{c}) + \frac{1}{2} \frac{\partial T_{\alpha\beta}(\vec{c})}{\partial c_\beta} \right) \delta(\vec{c}-\vec{c}') \right\} \\ + \frac{1}{2} \frac{\partial^2}{\partial c_\alpha \partial c_\beta} \left\{ T_{\alpha\beta}(\vec{c}) \delta(\vec{c}-\vec{c}') \right\} &\quad (3-13) \end{aligned}$$

Making the identification

$$\mathcal{F}_\alpha(\vec{c}) = B_\alpha(\vec{c}) + \frac{1}{2} \frac{\partial T_{\alpha\beta}(\vec{c})}{\partial c_\beta} \quad (3-14)$$

or equivalently

$$\begin{aligned} \mathcal{F}_\alpha(\vec{c}) = \sum_{a=e,i} \frac{2e^4}{m_t} \left\{ \frac{1}{m_a} \int \frac{d^3 \ell}{\ell^4} \frac{\ell_\alpha}{|\epsilon_0^+(\ell, U)|^2} \frac{\partial F_a(U)}{\partial U} \right. \\ \left. + \frac{1}{m_t} \frac{\partial}{\partial c_\beta} \int \frac{d^3 \ell}{\ell^5} \frac{\ell_\alpha \ell_\beta}{|\epsilon_0^+(\ell, U)|^2} F_a(U) \right\} \end{aligned} \quad (3-15)$$

Eq. (3-13) is written

$$\frac{\partial \delta(\vec{c}-\vec{c}')}{\partial t} = \frac{-\partial}{\partial c_\alpha} \left\{ \mathcal{F}_\alpha(\vec{c}) \delta(\vec{c}-\vec{c}') \right\} + \frac{1}{2} \frac{\partial^2}{\partial c_\alpha \partial c_\beta} \left\{ T_{\alpha\beta}(\vec{c}) \delta(\vec{c}-\vec{c}') \right\} \quad (3-16)$$

The above equation for the evolution of the "test particle" stream distribution function is observed to display the mathematical form of a Fokker-Planck equation. As usual $\mathcal{F}_\alpha(\vec{c})$ is identified as the coefficient of dynamic friction and $T_{\alpha\beta}(\vec{c})$ is identified as the coefficient of dispersion.

Two wave-number integrations are carried out in Eqs. (3-9) and (3-15) and the integrals reduced to a single integration over the variable U.

$$\begin{aligned} \mathcal{F}_\alpha(\vec{c}) = \sum_{a=e,i} \frac{4\pi e^4}{m_t} \frac{c_\alpha}{c^3} \left\{ \frac{1}{m_a} \int_{-c}^c dU U \frac{\partial F_a(U)}{\partial U} I(U) \right. \\ \left. - \frac{1}{m_t} \int_{-c}^c F_a(U) I(U) dU + \frac{2c}{m_t} F_a(U) I(U) \right\} \end{aligned} \quad (3-17)$$

$$\begin{aligned} T_{\alpha\beta}(\vec{c}) = \frac{4\pi e^4}{m_t^2} \left\{ \sum_{a=e,i} \frac{c^2 \delta_{\alpha\beta} - c_\alpha c_\beta}{3} \int_{-c}^c dU (1-U^2/c^2) F_a(U) I(U) \right. \\ \left. + \frac{2c}{c^3} \int_{-c}^c dU U^2/c^2 I(U) F_a(U) \right\} \end{aligned} \quad (3-18)$$

where

$$I(U) = \int_0^{l_0} \frac{dl}{l |\epsilon_0^+(l, U)|^2} \quad (3-19)$$

If these relations are substituted into Eq. (3-16) and the transfer frequencies are computed by taking the appropriate velocity moments of the resulting equation, then it is found that¹

$$v_{Mta} = \left[\frac{1}{m_t^2 C^3} \right] \left\{ 8\pi e^4 \int_{-C}^C dU (1-U^2/C^2) F_a(U) I(U) \right\} \quad (3-20)$$

$$v_{Eta} = \left| - \left[\frac{1}{m_t^2 C^3} \right] \left[\frac{8\pi m_t e^4}{m_a} \right] \left\{ \int_{-C}^C dU U \frac{\partial F_a(U)}{\partial(U)} I(U) + 2C F_a(C) I(C) \right\} \right| \quad (3-21)$$

The integral $I(U)$, Eq. (3-19), is evaluated by Tidman and Eviatar [26] and is found to be

$$I(U) = \frac{-A(U)}{2|B(U)|} \left\{ \tan^{-1} \left(\frac{l_0^2/l_D^2 + A(U)}{|B(U)|} \right) - \tan^{-1} \left(\frac{A(U)}{|B(U)|} \right) \right\} + \frac{1}{4} \ln \left\{ \frac{[l_0^2/l_D^2 + A(U)]^2 + B^2(U)}{A^2(U) + B^2(U)} \right\} \quad (3-22)$$

For most plasmas of interest $\Lambda \equiv l_0/l_D \gg 1$ and $\Lambda \gg |A(U)|, |B(U)|$. Using these inequalities, Eq. (3-22) may be approximated to a high degree of accuracy as

$$I(U) = \ln \Lambda + G(U) \quad (3-23)$$

¹Expressions for τ_{Mta} and τ_{Eta} appear in the paper of Tidman and Eviatar [26]; however, there is an apparent error of a factor of two in the expression for τ_{Mta} .

where

$$G(U) = \frac{-A(U)}{2|B(U)|} \left\{ \frac{\pi}{2} - \tan^{-1} \left(\frac{A(U)}{|B(U)|} \right) \right\} - \frac{1}{4} \ln [A^2(U) + B^2(U)]$$

The $\ln \Lambda$ term in Eq. (3-23) is independent of velocity and leads to the usual collisional contribution to $\mathcal{F}_\alpha(\vec{c})$, $T_{\alpha\beta}(\vec{c})$, and the transfer frequencies exhibited by Montgomery and Tidman [6]. It is evident that the use of the Fourier transform of the dynamically screened potential, $eV_\ell/|\epsilon_0^+(\ell, U)|$, in place of the Fourier transform of the coulomb potential, eV_ℓ , yields a result which is the sum of the usual collisional term plus an additional velocity dependent term, $G(U)$, which arises due to the correct treatment of the long-range collective interactions.

Substituting $I(U)$ as given in Eq. (3-23) into Eqs. (3-20) and (3-21), it is apparent that the frequencies may be considered the sum of two terms. The term containing $\ln \Lambda$ is the usual collisional contribution to the frequency while the term containing $G(U)$ may be interpreted as being due to long-range collective effects. Using the symbol $()_c$ to denote the collisional contribution to the frequency, and the symbol $()_{coll.}$ to denote the collective contribution to the frequency, the collisional and collective contributions to the energy transfer and momentum transfer frequencies may be written

$$(\nu_{Mta})_c = \left| \left[\frac{\ln \Lambda}{m_t^2 C^3} \right] \left\{ 8\pi e^4 \int_{-C}^C dU (1-U^2/C^2) F_a(U) \right\} \right| \quad (3-24)$$

$$(\nu_{Mta})_{coll.} = \left| - \left[\frac{1}{m_t^2 C^3} \right] \left\{ 8\pi e^4 \int_{-C}^C dU (1-U^2/C^2) F_a(U) G(U) \right\} \right| \quad (3-25)$$

$$(\nu_{\text{Eta}})_c = \left| - \left[\frac{\ln \Lambda}{m_t^2 C^3} \right] \left[\frac{8\pi m_t e^4}{m_a} \right] \left\{ \int_{-C}^C dU \frac{U \partial F_a(U)}{\partial U} + 2C F_a(C) \right\} \right| \quad (3-26)$$

$$(\nu_{\text{Eta}})_{\text{coll.}} = \left| - \left[\frac{1}{m_t^2 C^3} \right] \left[\frac{8\pi m_t e^4}{m_a} \right] \left\{ \int_{-C}^C dU \frac{U \partial F_a(U)}{\partial U} G(U) + 2C F_a(C) G(C) \right\} \right| \quad (3-27)$$

Before closing this section, it should be pointed out that these results could have been obtained in another manner. Prior to its linearization, the Lenard-Balescu interaction integral could have been written

$$J_{\text{LB}} = J_L + J_{\text{coll.}} \quad (3-28)$$

where

J_L denotes the Landau interaction integral and the interaction integral $J_{\text{coll.}}$ includes only the long-range interactions due to collective effects. The form of $J_{\text{coll.}}$ is exhibited in Appendix B where this division of the Lenard-Balescu interaction integral is derived. After making this division, one could then linearize the equation, take the necessary velocity moments, and it would be found that all terms in the transfer frequency containing $\ln \Lambda$ could be attributed to J_L and all terms in the transfer frequency containing $G(U)$ could be attributed to $J_{\text{coll.}}$. Therefore, it is evident that the frequencies $(\nu_{\text{Mta}})_c$ and $(\nu_{\text{Eta}})_c$ are obtained from velocity moments of J_L , and the frequencies $(\nu_{\text{Mta}})_{\text{coll.}}$ and $(\nu_{\text{Eta}})_{\text{coll.}}$ are obtained from velocity moments of $J_{\text{coll.}}$.

In order to avoid confusion, expressions for the various frequencies are listed below.

$$\nu_{\text{Eta}} = \left| \frac{1}{(\tau_{\text{Eta}})_c} + \frac{1}{(\tau_{\text{Eta}})_{\text{coll.}}} \right| \quad (3-29)$$

$$\nu_{\text{Mta}} = \left| \frac{1}{(\tau_{\text{Mta}})_c} + \frac{1}{(\tau_{\text{Mta}})_{\text{coll.}}} \right| \quad (3-30)$$

$$(\nu_{\text{Eta}})_c = \left| \frac{1}{(\tau_{\text{Eta}})_c} \right|$$

$$(\nu_{\text{Eta}})_{\text{coll.}} = \left| \frac{1}{(\tau_{\text{Eta}})_{\text{coll.}}} \right|$$

$$(\nu_{\text{Mta}})_c = \left| \frac{1}{(\tau_{\text{Mta}})_c} \right|$$

$$(\nu_{\text{Mta}})_{\text{coll.}} = \left| \frac{1}{(\tau_{\text{Mta}})_{\text{coll.}}} \right|$$

Results and Discussion

Due to the fact that $\ln \Lambda$ is a velocity independent term, it is possible to obtain both the collisional momentum and collisional energy transfer frequencies from Eqs. (3-24) and (3-26) by performing the necessary integrations analytically. The collisional transfer frequencies that are obtained agree, as they must, with those computed utilizing the Fokker-Planck interaction integral with a Debye cut-off coulomb potential [6].

The various collisional transfer frequencies are as follows:

$$(\nu_{\text{Mee}})_c = 5.59 \left[\frac{n_e e^4}{m_e^{1/2} (k T_e)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{\text{Mei}})_c = 8.89 \left[\frac{n_i e^4}{m_e^{1/2} (k T_e)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Mie})_c = 11.70 \left[\frac{n_e e^4 m_e^{1/2}}{m_i (k T_i)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Mii})_c = 5.59 \left[\frac{n_i e^4}{m_i^{1/2} (k T_i)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Eee})_c = 0.0111 \left[\frac{n_e e^4}{m_e^{1/2} (k T_e)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Eei})_c = 8.89 \left[\frac{n_i e^4 m_e^{1/2}}{m_i (k T_e)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Eii})_c = 0.0111 \left[\frac{n_i e^4}{m_i^{1/2} (k T_i)^{3/2}} \right] \ln \Lambda$$

$$(\nu_{Eie})_c = \left| -13.37 \left[\frac{n_e e^4 m_e^{1/2}}{m_i (k T_i)^{3/2}} \right] (T_i/T_e)^{1/2} \right| \ln \Lambda$$

The absolute value need be taken in the case of $(\nu_{Eie})_c$ only. The negative sign occurs here because $(\partial E/\partial t)_{t=0}$ is positive in this case; that is, the ion "test particle" stream gains energy due to collisions with electrons.

Due to the fact that $G(U)$ is a rather complicated velocity dependent term, it was found necessary to compute numerically the integrals which are needed to obtain values for both the collective momentum and energy transfer frequencies. This numerical integration was performed on the University of New Mexico IBM 360-40 computer using a thirty-two point Gaussian quadrature formula to evaluate the integrals.

In order to simplify the presentation of collective transfer frequencies the following relations are introduced.

$$(\nu_{\text{Eta}})_{\text{coll.}} = |A_{\text{Eta}}/\ln\Lambda| (\nu_{\text{Eta}})_c$$

$$(\nu_{\text{Mta}})_{\text{coll.}} = |A_{\text{Mta}}/\ln\Lambda| (\nu_{\text{Mta}})_c$$

Table I contains the values of A_{Mta} and A_{Eta} ¹ for the various collective transfer frequencies for a range of temperature ratios from 1 to 100. In the case of collective momentum transfer, an agreement between the sign of A_{Mta} and the sign of $(\tau_{\text{Mta}})_c$ indicates that the collective interactions enhance momentum transfer. If the sign of A_{Mta} is not the same as that of $(\tau_{\text{Mta}})_c$ then the collective interactions impede momentum transfer.

In the case of collective energy transfer, a positive sign for A_{Eta} indicates that the "test particle" gains energy due to collective interactions, and a negative sign indicates an energy loss due to collective interactions. Agreement of the signs of A_{Eta} and $(\tau_{\text{Eta}})_c$ indicates an enhancement of energy transfer; if the signs do not agree, this indicates an impedance of energy transfer.

Consider the ratios

$$(\nu_{\text{Eta}})_{\text{coll.}}/(\nu_{\text{Eta}})_c = |A_{\text{Eta}}/\ln\Lambda| \quad (3-31)$$

and

$$(\nu_{\text{Mta}})_{\text{coll.}}/(\nu_{\text{Mta}})_c = |A_{\text{Mta}}/\ln\Lambda| \quad (3-32)$$

¹Note that the signs of A_{Mta} and A_{Eta} are the same as those of $(\tau_{\text{Mta}})_{\text{coll.}}$ and $(\tau_{\text{Eta}})_{\text{coll.}}$ respectively.

TABLE I - Values of A_{Mta} and A_{Eta} as Functions of T_e/T_i

	T_e/T_i										
	1	10	20	30	40	50	60	70	80	90	100
A_{Mee}	-0.03	0.86	1.29	1.67	2.04	2.39	2.75	3.09	3.43	3.77	4.11
A_{Mei}	-0.349	-0.122	-0.088	-0.068	-0.055	-0.049	-0.045	-0.042	-0.040	-0.033	-0.036
A_{Mie}	-0.24	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20
A_{Mii}	-0.44	-0.39	-0.38	-0.38	-0.38	-0.38	-0.38	-0.38	-0.38	-0.38	-0.38
A_{Eee}	-2.31	-0.923	-0.397	-0.080	0.148	0.326	0.472	0.596	0.708	0.799	0.884
A_{Eei}	-0.156	0.689	0.957	1.09	1.17	1.22	1.27	1.29	1.31	1.33	1.35
A_{Eie}	0.22	-0.12	-0.14	-0.15	-0.16	-0.16	-0.16	-0.16	-0.16	-0.16	-0.16
A_{Eii}	-1.51	-2.64	-2.76	-2.80	-2.82	-2.83	-2.84	-2.84	-2.85	-2.85	-2.85

The magnitude of the appropriate ratio is an excellent measure of the relative importance of the collective contribution in the transfer frequency of interest. Assuming that $\ln\Lambda=10^*$, one finds, by referring to Table I, that the collective term is a substantial fraction of the collisional term for the transfer frequencies ν_{Mee} , ν_{Eii} , ν_{Eee} , and ν_{Eei} . Plots of the ratio of the collective transfer frequency to the collisional transfer frequency as a function of the temperature ratio T_e/T_i for the frequencies noted above are included as Figs. 1-4.

Figure 1. Ratio of the collective electron-electron momentum transfer frequency to the collisional electron-electron momentum transfer frequency.

*This is an average value which is representative of the plasma systems of interest herein.

Figure 2. Ratio of the collective ion-ion energy transfer frequency to the collisional ion-ion energy transfer frequency

Figure 3. Ratio of the collective electron-ion energy transfer frequency to the collisional electron-ion energy transfer frequency

Figure 4. Ratio of the collective electron-electron energy transfer frequency to the collisional electron-electron energy transfer frequency

Except for the ratio $T_e/T_i=1$, where A_{Mee} is a very small negative number, the value of A_{Mee} is positive for all values of T_e/T_i given in Table I. $(\tau_{Mee})_c$ is positive for all values of T_e/T_i given in Table I. Therefore, one may make either of the following equivalent statements which apply for all values of T_e/T_i found in Table I except $T_e/T_i=1$. (1) The electrons are deflected more strongly than predicted by the collisional theory. (2) An initially anisotropic electron distribution will relax to an isotropic distribution more rapidly than predicted by the collisional theory.

Since A_{Eii} is always negative and $(\tau_{Eii})_c$ is always positive for the temperature ratio range of 1 to 100, it is evident that in this range an initially anisotropic ion distribution will relax to a Maxwellian ion distribution more slowly than predicted

by the collisional theory. Now A_{Eee} assumes both positive and negative values for the temperature ratio range of 1 to 100 while $(\tau_{Eee})_c$ is always positive for that range; therefore, whether an initially anisotropic electron distribution relaxes more rapidly or more slowly to a Maxwellian electron distribution than predicted by the collisional theory depends on the temperature ratio (T_e/T_i) for $(1 \leq T_e/T_i \leq 100)$.

For the temperature ratios shown in Table I, A_{Eei} assumes both positive and negative values while (τ_{Eei}) is positive in all cases. Therefore, it may be said that the characteristic time associated with the relaxation of ion and electron Maxwellians having temperatures T_i and T_e initially may be either longer or shorter than that predicted by the collisional theory for $(1 \leq T_e/T_i \leq 100)$.

Earlier in this chapter, several remarks were made pertaining to the "resonance approximation" which has been used to simplify the integral in Eq. (3-25). This approximation is obtained from Eq. (3-23) by assuming that the important contribution of the velocity dependent term occurs in the region where $A(U)$ is negative and $|A(U)| \gg |B(U)|$. In this approximation $G(U)$ is given by

$$G(U) = -\pi A(U) / 2 |B(U)|$$

To support the claim that the use of this approximation yields a prohibitively inexact value of the integral in question for temperature ratios between 1 and 100, Fig. 5, is included. This plot exhibits values of the integral which were computed numerically using the more exact relation Eq. (3-23) for $G(U)$, and the

values which were computed numerically using the "resonance approximation" for $G(U)$. Specifically, this plot refers to the frequency $(\nu_{\text{Mee}})_{\text{coll.}}$. This plot makes it quite obvious that the values obtained using the "resonance approximation" for $G(U)$ are indeed poor approximations to the values obtained using the more exact representation for $G(U)$ for $(1 \leq T_e/T_i \leq 100)$. It should be noted, however, that as the ratio T_e/T_i increases, the accuracy of the "resonance approximation" increases.

Figure 5. Comparison of the relative magnitudes of the integral in Eq. (3-25) for the frequency $(\nu_{\text{Mee}})_{\text{coll.}}$.

CHAPTER IV

DEVELOPMENT OF THE MODIFIED CHAPMAN-ENSKOG METHOD OF SOLUTION

Introduction

It is the purpose of this chapter to derive a modified form of the "classical" Chapman-Enskog method of solution of the Lenard-Balescu kinetic equation and to exhibit formal solutions for a fully ionized hydrogen plasma. As pointed out in Chapter II, two facts must be considered when treating two-temperature plasma systems. These two facts are (1) the Lenard-Balescu kinetic equation must be used since collective effects are important in the two-temperature plasma, and (2) the "classical" Chapman-Enskog method of solution, which is capable of describing only those systems near thermodynamic equilibrium, must be modified in order to obtain a meaningful solution for the two-temperature plasma.

Silin, et al [3] use the Lenard-Balescu kinetic equation for the determination of plasma transport properties of a two-temperature plasma. However, the "classical" Chapman-Enskog method is utilized to obtain the solution of the kinetic equation and the transport properties. These authors simply replace the temperature T by T_e and T_i in order to obtain the zeroth order approximations to the electron and ion one-particle distribution functions. These zeroth order approximations, however, are not solutions to the zeroth order equation. In addition, the electron equation only is considered; therefore, ion contributions to the transport properties are not included.

Pearson [5] also uses the Lenard-Balescu kinetic equation for the determination of plasma transport properties of a two-temperature plasma. He computes these transport properties using a numerical technique instead of employing a Chapman-Enskog type solution. This paper [5] does not include spatial variations of the one-particle distribution function on the left-hand side of the kinetic equation. The ion equation is not examined and, as a result, the ion contributions to the transport properties are not included. The "resonance approximation" for $G(U)$ is used, and, as has been shown in Chapter III, this is a poor approximation.

Previous attempts have been made to obtain a modified Chapman-Enskog method for the two-temperature plasma. A modified method is developed by Braginskii [1] for the Landau kinetic equation, and a similar modification is obtained by Chmielewski [4] for the Boltzmann kinetic equation. The development in neither case is applicable to a system described by the Lenard-Balescu kinetic equation, which must be used to correctly describe the two-temperature plasma. Further, neither of the works cited rigorously justifies the development of the modified ordering of terms in the kinetic equation.

In this chapter it will be assumed that the "test particle" transfer frequencies which were calculated and exhibited in Chapter III are good representations of the frequencies associated with interaction processes in the plasma of interest. These frequencies will ultimately be used to justify the ordering of terms which leads to the modified Chapman-Enskog method obtained herein.

Method of Ordering the Kinetic Equation

This section is intended to acquaint the reader with the ordering technique employed in sections to follow. Since this is the basic objective of this section, the treatment and the discussion given here are general and somewhat simplified.

For a two-constituent plasma, the equation describing the one-particle distribution function of one of the constituents may be written

$$\frac{\partial f_b}{\partial t} + \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \vec{A} \cdot \frac{\partial f_b}{\partial \vec{c}} = J(f_b, f_a) + J(f_b, f_b) \quad (4-1)$$

where $b=e,i$ and $a \neq b$ in the case of a fully ionized plasma. In order to explicitly exhibit the order of magnitude of each term in the equation, the variables on the left-hand side of Eq. (4-1) are expressed in terms of a characteristic time, a characteristic velocity, a characteristic length, and a characteristic acceleration.

That is

$$t = t_o t_1 \quad (4-2a)$$

$$\vec{r} = r_o \vec{r}_1 \quad (4-2b)$$

$$\vec{c} = c_o \vec{c}_1 \quad (4-2c)$$

$$\vec{A} = A_o \vec{A}_1 \quad (4-2d)$$

where, for example, c_o can be chosen as the average velocity of the constituent of type b .

On substitution of these relations, the left-hand side of Eq. (4-1) becomes

$$\left(\frac{1}{t_o}\right) \frac{\partial f_b}{\partial t_1} + \left(\frac{c_o}{r_o}\right) \vec{c}_1 \cdot \frac{\partial f_b}{\partial \vec{r}_1} + \left(\frac{A_o}{c_o}\right) \vec{A}_1 \cdot \frac{\partial f_b}{\partial \vec{c}_1} = J(f_b, f_b) + J(f_b, f_a) \quad (4-3)$$

It is convenient to introduce the frequencies

$$\omega_t = \frac{1}{t_0} \quad (4-4a)$$

$$\omega_r = \frac{c_0}{r_0} \quad (4-4b)$$

$$\omega_c = \frac{A_0}{c_0} \quad (4-4c)$$

and rewrite Eq. (4-3) as

$$\omega_t \frac{\partial f_b}{\partial t_1} + \omega_r \vec{c}_1 \cdot \frac{\partial f_b}{\partial \vec{r}_1} + \omega_c \vec{A}_1 \cdot \frac{\partial f_b}{\partial \vec{c}_1} = J(f_b, f_b) + J(f_b, f_a) \quad (4-5)$$

The order of magnitude of the terms on the left side are now characterized by ω_t , ω_r , and ω_c respectively.

The above technique is also applied to the interaction integral terms on the right-hand side of Eq. (4-5). That is

$$J_{(b,b)} = \omega_{bb} J_1(b,b)$$

$$J_{(b,a)} = \omega_{ba} J_1(b,a) ,$$

where ω_{bb} and ω_{ba} are frequencies which characterize the order of magnitude of the interaction terms. Eq. (4-5) now takes the form¹

$$\omega_t \frac{\partial f_b}{\partial t} + \omega_r \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \omega_c \vec{A} \cdot \frac{\partial f_b}{\partial \vec{c}} = \omega_{ba} J(f_b, f_a) + \omega_{bb} J(f_b, f_b) \quad (4-6)$$

¹At this point, the use of the subscript 1 will be terminated.

Having written the constituent equation in this form, the next step is to obtain a solution. The most straightforward method is to use a classical perturbation approach. To utilize this method, it is first necessary to identify an ordering parameter that (1) generates a set of equations which are easily soluble, and (2) is appropriate for the physical system under consideration.

The existence of a two-temperature plasma is the result of the fact that the energy exchange coupling between constituents is relatively weak, or stated in another way, like-particle interactions are dominant. It is therefore expected that $\omega_{ba}/\omega_{bb} \ll 1$. If ω_{ba}/ω_{bb} is chosen as the ordering parameter ϵ , then $J(f_b, f_b)$ appears in the $O(\epsilon^{-1})$ equation while $J(f_b, f_a)$ appears in the $O(\epsilon^0)$ equation. If Eq. (4-6) is divided by ω_{ba} , the following result is obtained:

$$\left(\frac{\omega_t}{\omega_{ba}} \right) \frac{\partial f_b}{\partial t} + \left(\frac{\omega_r}{\omega_{ba}} \right) \vec{c} \cdot \frac{\partial f_b}{\partial \vec{r}} + \left(\frac{\omega_c}{\omega_{ba}} \right) \vec{A} \cdot \frac{\partial f_b}{\partial \vec{c}} = \frac{1}{\epsilon} J(f_b, f_b) + J(f_b, f_a) \quad (4-7)$$

Since the validity of all the kinetic equations is dependent upon certain limitations on the maximum rates of change allowed for the terms on the left-hand side of the kinetic equation, it is always possible to obtain an upper limit for the terms ω_t/ω_{ba} , ω_r/ω_{ba} , and ω_c/ω_{ba} .

It has been shown in this section that the kinetic equation can be put into a form such that classical perturbation techniques can be utilized. Also, the feasibility of selecting an ordering parameter consistent with physical reality of a two-temperature system has been demonstrated. A slightly more

complicated version of this ordering technique is used in ordering the electron and ion constituent equations in the following section.

Ordering of the Ion and Electron Equations

The ion equation may be written

$$\frac{\partial f_i}{\partial t} + \vec{c} \cdot \frac{\partial f_i}{\partial \vec{r}} + \vec{A} \cdot \frac{\partial f_i}{\partial \vec{c}} = J_{LB}(f_i, f_i) + J_{LB}(f_i, f_e) \quad (4-8)$$

As shown in Appendix B and discussed in Chapter III, the Lenard-Balescu interaction integral, J_{LB} can be written

$$J_{LB} = J_L + J_{coll.}$$

Using this relationship for J_{LB} and employing the technique of the previous section, Eq. (4-8) may be written in the form

$$\begin{aligned} \omega_t \frac{\partial f_i}{\partial t} + \omega_r \vec{c} \cdot \frac{\partial f_i}{\partial \vec{r}} + \omega_c \vec{A} \cdot \frac{\partial f_i}{\partial \vec{c}} &= (\omega_{ii})_c J_L(f_i, f_i) \\ &+ (\omega_{ie})_c J_L(f_i, f_e) + (\omega_{ii})_{coll.} J_{coll.}(f_i, f_i) \\ &+ (\omega_{ie})_{coll.} J_{coll.}(f_i, f_e) \end{aligned} \quad (4-9)$$

At this point, it is assumed that the "test particle" frequencies, computed in Chapter III, can be used to represent the characteristic interaction frequencies in the above equation.

For example, $(\omega_{ii})_c$ is replaced by the larger of $(\nu_{Mii})_c$ and $(\nu_{Eii})_c$. Since the momentum frequencies are in all cases greater than or equal to the energy transfer frequencies, Eq. (4-9) can be written

$$\begin{aligned}
\omega_t \frac{\partial f_i}{\partial t} + \omega_r \vec{c} \cdot \frac{\partial f_i}{\partial \vec{r}} + \omega_c \vec{A} \cdot \frac{\partial f_i}{\partial \vec{c}} &= (v_{Mii})_c J_L(f_i, f_i) \\
+ (v_{Mie})_c J_L(f_i, f_e) + (v_{Mii})_{coll.} J_{coll.}(f_i, f_i) \\
+ (v_{Mie})_{coll.} J_{coll.}(f_i, f_e) &\quad (4-10)
\end{aligned}$$

For the reasons pointed out in the previous section both sides of the above equation are divided by $(v_{Mii})_c$ and the ordering parameter ϵ is identified as

$$\epsilon = (v_{Mie})_c / (v_{Mii})_c = (m_e/m_i)^{1/2}$$

Employing the values of the "test particle" transfer frequencies, exhibited in Chapter III, the resulting equation can be written

$$\begin{aligned}
\frac{\omega_t}{(v_{Mie})_c} \frac{\partial f_i}{\partial t} + \frac{\omega_r}{(v_{Mie})_c} \vec{c} \cdot \frac{\partial f_i}{\partial \vec{r}} + \frac{\omega_c}{(v_{Mie})_c} \vec{A} \cdot \frac{\partial f_i}{\partial \vec{c}} \\
= \frac{1}{\epsilon} J_L(f_i, f_i) + J_L(f_i, f_e) + J_{coll.}(f_i, f_i) \\
+ \epsilon J_{coll.}(f_i, f_e) &\quad (4-11)
\end{aligned}$$

Since the Lenard-Balescu equation is valid for only the class of plasma systems satisfying the requirements listed in Chapter II, limitations must be placed on the ratios appearing on the left-hand side of the above equation.

The usual hydrodynamic conditions require that

$$\frac{\omega_t}{(v_{Mii})_c}, \frac{\omega_r}{(v_{Mii})_c}, \frac{\omega_c}{(v_{Mii})_c} \lesssim 0(\epsilon^0)$$

or

$$\frac{\omega_t}{(v_{Mie})_c}, \frac{\omega_r}{(v_{Mie})_c}, \frac{\omega_c}{(v_{Mie})_c} \lesssim 0(\epsilon^0)$$

If it is assumed that the latter ratios are all $0(\epsilon^0)$, then Eq. (4-11) can be written

$$\begin{aligned} \frac{\partial f_i}{\partial t} + \vec{c} \cdot \frac{\partial f_i}{\partial \vec{r}} + \vec{\Lambda} \cdot \frac{\partial f_i}{\partial \vec{c}} &= \frac{1}{\epsilon} J_L(f_i, f_i) + J_L(f_i, f_e) \\ &+ J_{\text{coll.}}(f_i, f_i) + \epsilon J_{\text{coll.}}(f_i, f_e) \end{aligned} \quad (4-12)$$

Now if f_i is written

$$f_i = f_i^0 + \epsilon f_i^1 = f_i^0(1 + \epsilon \phi_i)$$

the zeroth order and first-order ion equations take the form

$$J_L(f_i^0, f_i^0) = 0 \quad (4-13)$$

$$\begin{aligned} Df_i^0 - J_L(f_i^0, f_e^0) &= J_L(f_i^0 \phi_i, f_i^0) + J_L(f_i^0, f_i^0 \phi_i) \\ &+ J_{\text{coll.}}(f_i^0, f_i^0) \end{aligned} \quad (4-14)$$

These equations show clearly that (1) the ordering parameter chosen has resulted in zeroth and first order equations that are simple and, in principle at least, easily soluble and (2) the parameter is appropriate for a two-temperature plasma because like-particle interactions are indeed dominant.

Following the same procedure that is used to order the ion equation, it is possible to obtain an equation which has the same form as Eq. (4-10) by operating on the electron equation. This equation is

$$\begin{aligned}
 \omega_t \frac{\partial f_e}{\partial t} + \omega_r \vec{c}' \cdot \frac{\partial f_e}{\partial \vec{r}} + \omega_c \vec{A}' \cdot \frac{\partial f_e}{\partial \vec{c}'} &= (\nu_{Mee})_c J_L(f_e, f_e) \\
 &+ (\nu_{Mei})_c J_L(f_e, f_i) + (\nu_{Mee})_{coll.} J_{coll.}(f_e, f_e) \\
 &+ (\nu_{Mei})_{coll.} J_{coll.}(f_e, f_i)
 \end{aligned} \tag{4-15}$$

The primes are used to denote electron velocities and accelerations in this chapter. From this point on, the steps used in ordering the electron equation are more complex than those used in ordering the ion equation.

There are, as mentioned previously, two "test particle" transfer frequencies associated with each interaction integral. Specifically, the interaction integral $J_L(f_e, f_e)$ in the above equation has associated with it, the transfer frequencies $(\nu_{Mei})_c$ and $(\nu_{Eei})_c$. Since the order of magnitude of $J_L(f_e, f_i)$ is determined by the larger of these two frequencies and since $(\nu_{Mei})_c$ is a factor (m_i/m_e) larger than $(\nu_{Eei})_c$ the frequency $(\nu_{Mei})_c$ appears in Eq. (4-15) as an indication of the order of magnitude of $J_L(f_e, f_i)$.

A more precise ordering is obtained if it is possible to find interaction integrals $J_M(f_e, f_i)$ and $J_E(f_e, f_i)$ such that $\nu_{Mei} J_L = \nu_{Mei} J_M(f_e, f_i) + \nu_{Eei} J_E(f_e, f_i)$, where the interaction integral $J_M(f_e, f_i)$ accounts for all momentum transfer and the

interaction integral $J_E(f_e, f_i)$ accounts for all energy transfer.

The interaction integral

$$J_M(f_e, f_i) = \frac{2e^4\pi}{m_e^2} \ln\Lambda \frac{\partial}{\partial u'_\alpha} \frac{(u')^2 \delta_{\alpha\beta} - u'_\alpha u'_\beta}{(u')^3} \frac{\partial f_e}{\partial u'_\beta} \int d^3u f_i(\vec{u}') \quad (4-16)$$

is obtained in Appendix B by inserting the following approximation into the Landau interaction integral.

$$\delta(\mu' - u\mu/u') \approx \delta(\mu') \quad (4-17)$$

where

$$\mu = \vec{\ell} \cdot \vec{u} / u$$

$$\mu' = \vec{\ell} \cdot \vec{u}' / u'$$

$\vec{\ell}$ is the Fourier transform variable (wave vector)

$\vec{u} = \vec{c} - \vec{w}$ is the ion peculiar velocity

$\vec{u}' = \vec{c}' - \vec{w}$ is the electron peculiar velocity

$\vec{w}$ is the hydrodynamic or mass flow velocity

The approximation above is clearly quite good since the ratio of ion velocity to electron velocity is on the average a small value.

It has been verified that when $J_M(f_e, f_i)$ is used and $(v_{Mei})_c$ computed, the value that is obtained agrees to $O(m_e/m_i)$ with that obtained in Chapter III using $J_L(f_e, f_i)$. On the other hand, the use of $J_M(f_e, f_i)$ in place of $J_L(f_e, f_i)$ yields a value of $(v_{Eei})_c$ that is identically equal to zero. Hence, $J_M(f_e, f_i)$ does indeed account for momentum transfer, but does not account for energy transfer. It follows, of course, that $J_E(f_e, f_i)$ accounts for energy transfer and not momentum transfer. The above results should be intuitively obvious since the approximation, Eq. (4-16), is equivalent to assuming that the ions are

infinitely massive.

Substituting

$$v_{Mei} J_L(f_e, f_i) = v_{Mei} J_M(f_e, f_i) + v_{Eei} J_E(f_e, f_i)$$

into Eq. (4-15) yields

$$\begin{aligned} \omega_t \frac{\partial f_e}{\partial t} + \omega_r \vec{c}' \cdot \frac{\partial f_e}{\partial \vec{r}} + \omega_c \vec{A}' \cdot \frac{\partial f_e}{\partial \vec{c}'} &= (v_{Mee})_c J_L(f_e, f_e) \\ &+ (v_{Mei})_c J_M(f_e, f_i) + (v_{Eei})_c J_E(f_e, f_i) \\ &+ (v_{Mee})_{coll.} J_{coll.}(f_e, f_e) + (v_{Mei})_{coll.} J_{coll.}(f_e, f_i) \end{aligned} \quad (4-18)$$

Once again both sides of this equation must be divided by the appropriate frequency and the ordering parameter ϵ must be identified. In this case, the equation is divided by (v_{Mei}) and the ordering parameter ϵ is identified as the ratio

$$\frac{(v_{Mei})_{coll.}}{(v_{Mee})_c} = \epsilon = \left(\frac{m_e}{m_i} \right)^{\frac{1}{2}}$$

Again, using the values of the "test particle" transfer frequencies exhibited in Chapter III, Eq. (14-8) becomes

$$\begin{aligned} \frac{\omega_t}{v_{Mei}} \frac{\partial f_e}{\partial t} + \frac{\omega_r}{v_{Mei}} \vec{c}' \cdot \frac{\partial f_e}{\partial \vec{r}} + \frac{\omega_c}{v_{Mei}} \vec{A}' \cdot \frac{\partial f_e}{\partial \vec{c}'} &= \frac{J_L(f_e, f_e)}{\epsilon} \\ &+ \frac{J_M(f_e, f_i)}{\epsilon} + \frac{J_{coll.}(f_e, f_e)}{\epsilon} + J_{coll.}(f_e, f_i) + \epsilon J_E(f_e, f_i) \end{aligned} \quad (4-19)$$

The following requirements must be satisfied for the usual hydrodynamic variation of the distribution function

$$\frac{\omega_t}{(v_{Mee})_c}, \frac{\omega_r}{(v_{Mee})_c}, \frac{\omega_c}{(v_{Mee})_c} \sim 0(\epsilon)$$

or

$$\frac{\omega_t}{(v_{Eei})_c}, \frac{\omega_r}{(v_{Eei})_c}, \frac{\omega_c}{(v_{Eei})_c} \lesssim 0(1)$$

If it is assumed that the latter ratios are all $0(1)$, the Eq. (4-19) can be written

$$\begin{aligned} \frac{\partial f_i}{\partial t} + \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{A} \cdot \frac{\partial f}{\partial \vec{c}} &= \frac{1}{\epsilon} J_L(f_e, f_e) + \frac{1}{\epsilon} J_M(f_e, f_i) \\ &+ \frac{1}{\epsilon} J_{\text{coll.}}(f_e, f_e) + J_{\text{coll.}}(f_e, f_i) + \epsilon J_E(f_e, f_i) \end{aligned} \quad (4-20)$$

Now, if f_e is written $f_e = f_e^0 + f_e^1 = f_e^0(1 + \epsilon \phi_e)$, then the zeroth-order electron equation is

$$J_L(f_e^0, f_e^0) + J_{\text{coll.}}(f_e^0, f_e^0) + J_M(f_e^0, f_i^0) = 0$$

or

$$J_{LB}(f_e^0, f_e^0) + J_M(f_e^0, f_i^0) = 0 \quad (4-21)$$

The first-order electron equation is

$$\begin{aligned} Df_e^0 - J_{\text{coll.}}(f_e^0, f_i^0) &= J_{LB}(f_e^0, f_e^0 \phi_e) + J_{LB}(f_e^0 \phi_e, f_e^0) \\ &+ J_M(f_e^0, f_i^0 \phi_i) + J_M(f_e^0 \phi_e, f_i^0) \end{aligned} \quad (4-22)$$

The forms of the zeroth-order electron and zeroth-order ion equations indicate that in this order, there is no energy coupling between constituents; however, there is a momentum coupling between constituents which appears in the zeroth-order electron equation.

Due to the large difference in the masses of the constituents, a purely intuitive argument could have been used to arrive at the same qualitative behavior indicated by the equation

exhibited here. However, it is impossible, using purely intuitive reasoning, to generate the set of ordered equations that has been derived.

It should be noted that using the ordering exhibited in this section, the simpler Landau form of the interaction integral is adequate for many of the terms.

Solutions of the Zeroth- and First-Order Equations For Ions and For Electrons

In this section, formal solutions are obtained for both the zeroth-order and first-order one-particle distribution functions for both electrons and ions. The mathematical analysis, employed to determine the solutions of f_a^0 and ϕ_a is similar to that of the "classical" Chapman-Enskog approach as will become evident in the following pages.

The solutions f_a^0 of the zeroth-order ion equation

$$J_L(f_i^0, f_i^0) = 0 \quad (4-23)$$

and of the zeroth-order electron equation

$$J_{LB}(f_e^0, f_e^0) + J_M(f_e^0, f_i^0) = 0 \quad (4-24)$$

are

$$f_a^0(\vec{c}) = n_a \left(\frac{m_a}{2\pi k T_a} \right)^{3/2} \exp \left[- \frac{m_a}{2k T_a} (\vec{c} - \vec{w})^2 \right] \quad (4-25)$$

where n_a , m_a , and T_a are the density, mass and temperature of the constituent of type a ($a=i, e$). The velocity $\vec{w}$ is the hydrodynamic or mass flow velocity. These solutions are Maxwellians about the hydrodynamic velocity $\vec{w}$ with a temperature T_a . In contrast to the zeroth-order "classical" solutions, the solutions exhibited above do not require that the constituents be at the

same temperature in zeroth order. In other words, these zeroth order solutions allow the treatment of two-temperature plasmas.

It should be mentioned that the solution f_e^0 is not a unique solution of the zeroth-order electron equation. It is the solution for which each term in the zeroth-order electron equation is zero. It is felt that this solution, although not unique, is indeed an appropriate solution for the physical system under consideration.

Before proceeding to solve the first-order electron equation in which the Lenard-Balescu interaction integral appears, a remark about the dielectric function $\epsilon^+(\vec{\mathcal{L}}, U)$ is necessary. As shown by Braun [32] and as discussed in Appendix A, it is possible to write

$$\frac{1}{|\epsilon^+(\vec{\mathcal{L}}, U)|^2} = \frac{1}{|\epsilon_0^+(\vec{\mathcal{L}}, U)|^2} [1 + \epsilon Q(f_e^0 \phi_e) + \epsilon Q(f_i^0 \phi_i)]$$

where Q is an integral operator.

In order to obtain this relation, it is assumed that

$$\begin{aligned} \epsilon^+(f_e, f_i, \vec{\mathcal{L}}, U) &= \epsilon_0^+(f_e^0, f_i^0, \vec{\mathcal{L}}, U) + \epsilon \epsilon_1^+(f_e^0 \phi_e, \vec{\mathcal{L}}, U) \\ &+ \epsilon \epsilon_1^+(f_i^0 \phi_i, \vec{\mathcal{L}}, U) \end{aligned}$$

Since $1/|\epsilon^+(\vec{\mathcal{L}}, U)|^2$ can be written in this manner, it is clear that $\epsilon_0^+(\vec{\mathcal{L}}, U)$ can be used in the Lenard-Balescu interaction integrals appearing in the first-order electron equation. The terms Q which are $O(\epsilon)$ will appear in the second order electron equation. For the remainder of this chapter it is to be understood that the dielectric function contained in J_{LB} is $\epsilon_0^+(\vec{\mathcal{L}}, U)$.

The first-order ion equation is

$$Df_i^0 - J_L(f_i^0, f_e^0) = J_L(f_i^0, \phi_i f_i^0) + J_L(f_i^0 \phi_i, f_i^0) \quad (4-26)$$

and the first-order electron equation is

$$Df_e^0 - J_{coll.}(f_e^0, f_i^0) = J_{LB}(f_e^0, f_e^0 \phi_e) + J_{LB}(f_e^0, \phi_e, f_e^0) + J_M(f_e^0 \phi_e, f_i^0) \quad (4-27)$$

It may be noted that the term $J_M(f_e^0, f_i^0 \phi_i)$ does not appear in the first-order electron equation since $J_M(f_e, f_i) = 0$ is satisfied for any isotropic f_e as may be easily verified.

Before proceeding, it is convenient to change the independent velocity variable from $\vec{c}$ to the peculiar velocity $\vec{u} = \vec{c} - \vec{w}$.

With this change of variable

$$Df_a(\vec{c}) = \frac{\partial f_a}{\partial t} + \vec{c} \cdot \frac{\partial f_a}{\partial \vec{r}} + \vec{A} \cdot \frac{\partial f_a}{\partial \vec{c}} \quad (4-28)$$

becomes

$$Df_a(\vec{u}) = \frac{Df_a}{Dt} + u_\alpha \frac{\partial f_a}{\partial r_\alpha} - \frac{\partial f_a}{\partial u_\alpha} u_\beta \frac{\partial w_\beta}{\partial r_\alpha} + \left[A_\alpha - \frac{Dw_\alpha}{Dt} \right] \frac{\partial f_a}{\partial u_\alpha} \quad (4-29)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{w} \cdot \frac{\partial}{\partial \vec{w}}$$

The first-order equation for each constituent is of the same form. Both equations are inhomogeneous linear integral equations for ϕ_a . The conditions for the existence of a solution of an equation of this type are that the solutions of the homogeneous equation be orthogonal to the inhomogeneity. These conditions require that

$$\int d^3u \left[Df_i^0 - J_L(f_i^0, f_e^0) \right] \phi_i^H = 0 \quad (4-30)$$

for the ion equation and

$$\int d^3u' \left[Df_e^O - J_{\text{coll.}}(f_i^O, f_e^O) \right] \phi_e^H = 0 \quad (4-31)$$

for the electron equation.

The solutions ϕ_i^H are the three "collisional invariants"

$$\phi_i^H = 1, \vec{u}, u^2$$

while the solutions ϕ_e^H are two of the "collisional invariants"

$$\phi_e^H = 1, (u')^2$$

The velocity or momentum is not a solution of the homogeneous electron equation. This is due to the fact that $J_M(f_e^{O\vec{u}}, f_i^O) \neq 0$ (i.e., momentum is not conserved in electron-ion interactions when J_M is used, since J_M is obtained by assuming that the ions are infinitely massive).

Since there are three solutions for ϕ_i^H , then three equations are generated using Eq. (4-30). These equations are referred to as ion Euler equations. There are only two solutions ϕ_e^H ; therefore, only two electron Euler equations are obtained. Before exhibiting the set of Euler equations for ions and for electrons, it is necessary to introduce the following relations.

$$\int f_a^O d^3u = n_a \quad (4-32a)$$

$$\int f_a^O u^2 d^3u = \frac{3n_a kT_a}{m_a} \quad (4-32b)$$

$$\int f_a^O \vec{u} d^3u = 0 \quad (4-32c)$$

$$\int J_L(f_i^O, f_e^O) d^3u = 0 \quad (4-33a)$$

$$\int J_{\text{coll.}}(f_e^O, f_i^O) d^3u' = 0 \quad (4-33b)$$

$$\int \vec{u} J_L(f_i^O, f_e^O) d^3u = 0 \quad (4-34a)$$

$$\int \vec{u}' J_{\text{coll.}}(f_e^O, f_i^O) d^3u' = 0 \quad (4-34b)$$

$$\int u^2 J_L(f_i^O, f_e^O) d^3u = \frac{2E_{ie}^O}{m_i} \quad (4-35a)$$

$$\int u'^2 J_{\text{coll.}}(f_e^O, f_i^O) d^3u' = \frac{2E_{ei}^O}{m_e} \quad (4-35b)$$

where Eqs. (4-32) apply to both electrons and ions; these relations may be easily verified by substituting f_e^O or f_i^O for f_a^O . Eqs. (4-33) are a result of the conservation of number density. Eqs. (4-34) occur since both zeroth order distributions are Maxwellian about the same velocity $\vec{w}$. E_{ie}^O is the collisional energy transfer from ions to electrons per unit volume, per unit time, and E_{ei}^O is the collective energy transfer from electrons to ions per unit volume per unit time.

Using the above relations, the two following Euler equations, which hold for both electrons and ions, result.

$$\frac{Dn_a}{Dt} + n_a \frac{\partial}{\partial \vec{r}} \cdot \vec{w} = 0 \quad (4-36)$$

$$\frac{D}{Dt} \left(\frac{3}{2} n_a kT_a \right) + \frac{5}{2} n_a kT_a \frac{\partial}{\partial \vec{r}} \cdot \vec{w} = E_{ab}^O \quad (4-37)$$

$$(a, b = e, i; a \neq b)$$

In addition, for ions there is a third Euler equation corresponding to $\vec{u}$. This Euler equation is

$$\frac{D\vec{w}}{Dt} - \frac{\vec{F}_i}{m_i} + \frac{1}{n_i m_i} \frac{\partial}{\partial \vec{r}} (n_i kT_i) = 0 \quad (4-38)$$

The three ion Euler equations are used to eliminate the time derivatives on the left-hand side of the first-order ion equation, Eq. (4-26). Having done this, the first-order ion equation becomes

$$\begin{aligned}
& f_i^0 \left\{ (W^2 - \frac{5}{2}) \frac{u_\alpha}{T_i} \frac{\partial T_i}{\partial r_\alpha} + 2 \left(W_\alpha W_\beta - \frac{W^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial W_\beta}{\partial r_\alpha} \right. \\
& \left. + \frac{2}{3n_i k T_i} (W^2 - \frac{3}{2}) E_{ie}^0 \right\} - J_L(f_i^0, f_e^0) \\
& = J_L(f_i^0, f_i^0 \phi_i) + J_L(f_i^0 \phi_i, f_i^0)
\end{aligned} \tag{4-39}$$

In a similar fashion the two electron Euler equations and the third ion Euler equation, Eq. (4-38), are used to cast the first-order electron equation, Eq. (4-27), into the following form.

$$\begin{aligned}
& f_e^0 \left\{ (W'^2 - \frac{5}{2}) \frac{u'_\alpha}{T_e} \frac{\partial T_e}{\partial r_\alpha} + 2 \left(W'_\alpha W'_\beta - \frac{W'^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial W'_\beta}{\partial r_\alpha} \right. \\
& \left. + \frac{2}{3n_e k T_e} (W'^2 - \frac{3}{2}) E_{ei}^0 + u'_\alpha d_\alpha^e \right\} - J_{coll.}(f_e^0, f_i^0) \\
& = J_{LB}(f_e^0, f_e^0 \phi_e) + J_{LB}(f_e^0 \phi_e, f_e^0) + J_M(f_e^0 \phi_e, f_i^0)
\end{aligned} \tag{4-40}$$

In the above equations¹

$$W = u(m_i/2kT_i)^{1/2}$$

$$W' = u'(m_e/2kT_e)^{1/2}$$

¹It will be noted that, in the relations which follow, the unfortunate but widely accepted practice of using both W's and u's in a given equation is followed. This is done to facilitate comparison of the results herein with the work of others.

In Eq. (4-40)

$$\begin{aligned} \vec{d}^e = & \frac{1}{n_e} \frac{\partial n_e}{\partial \vec{r}} - \left(\frac{T_i m_e}{T_e m_i} \right) \frac{1}{n_i} \frac{\partial n_i}{\partial \vec{r}} - \frac{1}{kT_e} \left\{ \vec{F}_e - \left(\frac{m_e}{m_i} \right) \vec{F}_i \right\} \\ & + \frac{1}{T_e} \left\{ \frac{\partial T_e}{\partial \vec{r}} - \left(\frac{m_e}{m_i} \right) \frac{\partial T_i}{\partial \vec{r}} \right\} \end{aligned}$$

Ignoring terms of $O(\epsilon^2)$

$$\vec{d}^e = \left(\frac{1}{n_e} \frac{\partial n_e}{\partial \vec{r}} - \frac{\vec{F}_e}{kT_e} + \frac{1}{T_e} \frac{\partial T_e}{\partial \vec{r}} \right) \quad (4-41)$$

This simplified form of $\vec{d}^e$ is used throughout the remainder of this report

Since E_{ab}^0 are scalar and are not functions of velocity and since $J_L(f_i^0, f_e^0)$ and $J_{coll.}(f_e^0, f_i^0)$ are isotropic functions of the velocity, the formal solutions of the first-order ion and electron equations, in terms of the undetermined coefficients A_i , B_i , ψ_i , h_1 , h_2 , and h_3 , are

$$\begin{aligned} \phi_i(W) = & -A_i(W^2) W_\alpha \frac{\partial}{\partial r_\alpha} (\ln T_i) - B_i(W^2) \left(W_\alpha W_\beta - \frac{W^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial W_\beta}{\partial r_\alpha} \\ & + h_1 + h_2 W_\alpha + h_3 W^2 + \psi_i(W^2) \end{aligned} \quad (4-42)$$

$$\begin{aligned} \phi_e = & -A_e(W'^2) W'_\alpha \frac{\partial}{\partial r_\alpha} (\ln T_e) - B_e(W'^2) \left(W'_\alpha W'_\beta - \frac{W'^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial W_\beta}{\partial r_\alpha} \\ & - D_e(W'^2) W'_\alpha d_\alpha^e + h'_1 + h'_3 u'^2 + \psi_e(W'^2) \end{aligned} \quad (4-43)$$

Each of these solutions is the sum of the homogeneous solution plus the particular integral. The particular solution in the electron case, for example, has been obtained by noting that ϕ_e must be a scalar which is a linear combination of $\frac{\partial \ln T_e}{\partial \vec{r}}$, $\frac{\partial W_\beta}{\partial r_\alpha}$, $\vec{d}^e$ and an arbitrary scalar.

Following the "classical" Chapman-Enskog development, the following conditions are imposed to insure a unique solution of ϕ_i

(recall $f_i^1 = f_i^0 \phi_i$)

$$\int f_i^0 \phi_i d^3u = 0$$

$$\int f_i^0 \phi_i \vec{u} d^3u = 0$$

$$\int f_i^0 \phi_i u^2 d^3u = 0$$

Substituting ϕ_i as given by Eq. (4-43) into the above relations and neglecting vanishing integrals, the equivalent conditions

$$\int f_i^0 [h_1 + h_3 u^2 + \psi_i(W^2)] d^3u = 0$$

$$\int f_i^0 [-A_i(W^2) \vec{W} \cdot \frac{\partial T_i}{\partial \vec{r}} + \vec{h}_2 \cdot \vec{u}] u d^3u = 0$$

$$\int f_i^0 [h_1 + h_3 u^2 + \psi_i(W^2)] u^2 d^3u = 0$$

are obtained.

Again, following the "classical" Chapman-Enskog procedure the conditions selected which satisfy these equations are

$$h_1 = h_3 = 0; \vec{h}_2 = 0$$

and

$$\int f_i^0 A_i(W^2) W_\alpha W_\beta d^3u = 0$$

$$\int f_i^0 \psi_i(W^2) d^3u = 0$$

$$\int f_i^0 \psi_i(W^2) W^2 d^3u = 0$$

Therefore, ϕ_i becomes

$$\begin{aligned} \phi_i = & -A_i(W^2) W_\alpha \frac{\partial (\ln T_i)}{\partial r_\alpha} - B_i(W^2) \left(W_\alpha W_\beta - \frac{W^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial W_\beta}{\partial r_\alpha} \\ & + \psi_i(W^2) \end{aligned} \quad (4-44)$$

This ϕ_i is now a unique solution so long as the restrictions on $A_i(W^2)$ and $\Psi_i(W^2)$ are satisfied.

Since there are only two solutions to the homogeneous first-order electron equation, only two conditions are required to determine a unique solution ϕ_e . The conditions are

$$\begin{aligned}\int f_e^0 \phi_e d^3 u' &= 0 \\ \int f_e^0 \phi_e u'^2 d^3 u' &= 0\end{aligned}$$

These relations are equivalent to

$$\begin{aligned}\int f_e^0 [h'_1 + h'_3 u'^2 + \Psi_e(W'^2)] d^3 u' &= 0 \\ \int f_e^0 [h'_1 + h'_3 u'^2 + \Psi_e(W'^2)] u'^2 d^3 u' &= 0\end{aligned}$$

These relations may be satisfied by choosing $h'_1 = h'_3 = 0$ and by choosing

$$\begin{aligned}\int f_e^0 \Psi_e(W'^2) d^3 u' &= 0 \\ \int f_e^0 \Psi_e(W'^2) W'^2 d^3 u' &= 0\end{aligned}$$

Therefore, ϕ_e becomes

$$\begin{aligned}\phi_e = & -A_e(W'^2) W'_\alpha \frac{\partial (\ln T_e)}{\partial r_\alpha} - B_e(W'^2) \left(W'_\alpha W'_\beta - \frac{W'^2 \delta_{\alpha\beta}}{3} \right) \frac{\partial w_\beta}{\partial r_\alpha} \\ & - D_e(W'^2) W'_\alpha d_\alpha^e + \Psi_e(W'^2)\end{aligned}\quad (4-45)$$

This solution ϕ_e is now a unique solution if the restrictions on $\Psi_e(W'^2)$ are satisfied.

Having determined unique solutions ϕ_i and ϕ_e to the first-order equations, it is possible to calculate the first-order constituent heat flux $\vec{q}_a$, the first-order constituent traceless pressure tensor $(P_{\alpha\beta}^a)^0$, and the first-order constituent electrical current $\vec{j}_a$. In a reference system moving with a velocity $\vec{w}$ with respect to the laboratory, these quantities are defined by

$$\vec{q}_a = \frac{1}{2} m_a \int f_a^0 \phi_a u^2 \vec{u} d^3u$$

$$(P_{\alpha\beta}^a)^0 = m_a \int f_a^0 \phi_a (u_\alpha u_\beta - \frac{u^2 \delta_{\alpha\beta}}{3}) d^3u$$

$$\vec{j}_a = eZ_a \int f_a^0 \phi_a \vec{u} d^3u$$

It may be noted that since the f_a^0 are isotropic, the zeroth-order contributions to the above quantities are identically zero.

The first-order ion solution ϕ_i is, with the exception of the scalar term $\psi_i (W^2)^*$, exactly the same form as the "classical" Chapman-Enskog solution for a simple gas having a temperature T_i . Therefore, all ion transport coefficients and transport properties are identical in form to those of a simple one component gas. Specifically, the first-order ion current is zero, and the ion traceless pressure tensor and ion heat flux have a form identical to those of a simple gas. The simple solution

*It should be noted that since the terms ψ_e and ψ_i are scalars, they make no contribution to the transport coefficients of interest.

ϕ_i is obtained due to the lack of both energy and momentum coupling between electrons and ions in the first-order ion equation.

Except for the term containing $\tilde{d}^e$, the electron solution ϕ_e has the same form as the ion solution ϕ_i . The term containing $\tilde{d}^e$ occurs since there is a momentum coupling between electrons and ions in the first-order electron equation. This momentum coupling results in a first order electron current.

To point out clearly the major difference between the "classical" Chapman-Enskog method of solution and the modified method of solution just developed, the following brief comments are included. In the "classical" Chapman-Enskog method of solution, a very small ordering parameter ϵ is chosen, so small in fact that both like and unlike particle interaction integrals appear in the zeroth-order equation in spite of the fact that these terms may differ in magnitude by as much as a factor $(m_e/m_i)^{1/2}$. Although this choice allows the determination of a very exact solution, the solution is valid for only the class of plasma systems near thermodynamic equilibrium. In this chapter it has been shown that at a small expense in accuracy (terms $O(\epsilon^2)$ are neglected; $\epsilon^2 = m_e/m_i$), it is possible to develop, in a systematic manner, a modified Chapman-Enskog method of solving the Lenard-Balescu kinetic equation for two-temperature fully ionized plasmas.

All the transport coefficients of interest (i.e., electrical conductivity, thermal conductivity, and viscosity) can be computed for a fully ionized hydrogen plasma in which $1 \leq T_e/T_i \leq 100$

by using the modified solutions obtained in this chapter.¹

Transport coefficients obtained by using the modified solution will agree with those obtained by using the "classical" solution only when $T_e/T_i=1$.

In order to obtain the numerical values associated with the transport coefficients of interest, the standard Sonine polynomial expansion can be used to represent $A_a(W^2)$, $B_a(W^2)$, and $D_e(W^2)$. These expansions are

$$A_a(W^2) = \sum_{n=0}^{\infty} a_n^a S_{3/2}^n(W^2)$$

$$B_a(W^2) = \sum_{n=0}^{\infty} \mathcal{B}_n^a S_{5/2}^n(W^2)$$

$$D_e(W^2) = \sum_{n=0}^{\infty} \mathcal{D}_n^e S_{3/2}^n(W^2)$$

This means of determining numerical results is discussed in detail in several references [17,18,24].

¹It is a simple matter to extend the results of this chapter to any fully ionized plasma in which the mass ratio of the constituents is a small number.

CHAPTER V

SUMMARY

A detailed study, of momentum and energy transfer frequencies for a tenuous "test particle" beam interacting with a two-temperature Maxwellian plasma, has been carried out for the range of T_e/T_i from 1 to 100. The Lenard-Balescu equation was employed; therefore, long-range collective effects were included in the study. In fact, each frequency has been exhibited as the sum of two terms. The first term takes account of the collisional interactions within a Debye sphere and hence agrees with the results of Montgomery and Tidman [6] who compute some of these times using the Fokker-Planck equation with a Debye cutoff potential. The second term takes account of long-range collective effects as had been anticipated since the Lenard-Balescu equation differs from the Fokker-Planck equation in that long-range collective effects are accounted for through the inclusion of the dielectric function $[\epsilon^+(\mathcal{L}, U)]$. The long-range collective term was found to range in magnitude up to 40 percent of the usual collisional term (see Figs. 1 thru 4).

In Chapter IV a systematic development of a modified Chapman-Enskog method of solving the Lenard-Balescu equation has been presented. The modification is such that the method is appropriate for the description of the behavior of two-temperature fully ionized plasmas. The systematic ordering of the interaction integrals in the Lenard-Balescu kinetic equation has been carried out by accepting for ordering purposes,

the corresponding "test particle" transfer frequencies computed in Chapter III. An expansion parameter, $\varepsilon = (m_e/m_i)^{1/2}$, has been identified as appropriate for two-temperature plasmas. The Lenard-Balescu equation has been ordered in ε and the resulting zeroth and first-order equations solved by a mathematical procedure similar to the "classical" Chapman-Enskog approach. The formal solutions exhibit a simple form. This simplicity is not too surprising since the ordering, being consistent with physical reality, is such that energy coupling does not occur in zeroth or first order.

Although attention has been directed to a hydrogen plasma to facilitate comparison with the results of other investigators, it should be noted that the method is applicable to any plasma system within which the constituent mass ratio is small.

As indicated in the first section of Chapter IV, this development constitutes the first systematic development of a method of solution of the Lenard-Balescu kinetic equation consistent with the physical realities of two-temperature plasmas. In addition, the employment of the Lenard-Balescu equation has made it possible to include the influence of long-range collective interactions which are important in two-temperature plasmas.

Preliminary investigation of the magnitudes of the coefficients of the terms in the formal solutions indicate that corrections¹ resulting from the inclusion of long-range

¹Here it should be understood that these are corrections to the widely accepted results of Spitzer and Harm [21].

collective interactions will amount to tens of percent, and corrections resulting from the use of the modified Chapman-Enskog method may be even more significant.

APPENDIX A

In this appendix the dielectric function for a fully ionized hydrogen plasma will be derived. The derivation used is simple and direct and yields the same result as the more rigorous, and more complicated derivations.

The Maxwell equations for a material medium are

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = - \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{j}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

where $\vec{D}$ obeys the equation

$$\frac{\partial \vec{D}}{\partial t} = \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j}_o \quad (\vec{j}_o \rightarrow \text{induced current}) \quad (\text{A-1})$$

This system of equations must be supplemented by a material equation giving an explicit expression for the displacement vector $\vec{D}$. If the electric field, $\vec{E}$, is constant and if the medium is homogeneous and isotropic, one assumes a linear relation $\vec{D} = \epsilon \vec{E}$ holds. The dielectric constant ϵ is a characteristic property of the material medium. This parameter ϵ describes the polarization produced in the medium by the electric field.

If rapidly varying fields are considered, the polarization cannot follow the variation of the electric field because of the inertia of the plasma constituents. Therefore, there is a temporal and spatial lag between cause and response. In the case of time varying $\vec{E}$, the relation between $\vec{D}$ and $\vec{E}$ takes

the form

$$\vec{D}(\vec{r}, t) = \int_{-\infty}^{\infty} d^3r \int_0^t dt' \epsilon(\vec{r}-\vec{r}'; t-t') \vec{E}(\vec{r}'; t')$$

A Fourier-Laplace transform of this relation yields

$$\vec{D}(\vec{\ell}, \omega) = \epsilon(\vec{\ell}, \omega) \vec{E}(\vec{\ell}, \omega)$$

Here, the Fourier and Laplace convolution theorems have been applied.

The response of the plasma to any small perturbation which in turn generates a small perturbation, f_a^1 , to the initial one-particle distribution function, f_a^0 , is now studied. It is assumed that in the initial state the plasma system is stationary and homogeneous (i.e., f_a^0 is a function of the velocity $\vec{c}$ only).

The evolution of f_a is assumed to be governed by the equation

$$\frac{\partial f_a}{\partial t} + \vec{c} \cdot \frac{\partial f_a}{\partial \vec{r}} + \frac{Z_a e_a}{m_a} \vec{E} \cdot \frac{\partial f_a}{\partial \vec{c}} = - \frac{1}{\tau} (f_a - f_a^0) \quad (A-2)$$

where

$$f_a = f_a^0 + f_a^1$$

$$f_a^0 \gg f_a^1$$

Here $\tau > 0$ is the effective interaction time, for which the limit $\tau \rightarrow \infty$ will be taken later in the derivation, (note that Eq. (A-2) becomes the Vlasov equation if the limit $\tau \rightarrow \infty$ is taken).

The electric field is a self-consistent field, and therefore,

$$E \sim f_a^1$$

By linearizing Eq. (A-2) in f_a^1 the following equation is obtained

$$\frac{\partial f_a^1}{\partial t} + \vec{c} \cdot \frac{\partial f_a^1}{\partial \vec{r}} + \frac{Z_a e \vec{E}}{m_a} \cdot \frac{\partial f_a^0}{\partial \vec{c}} = - \frac{f_a^1}{\tau} \quad (A-3)$$

If solutions of the form

$$f_a^1(\vec{c}, \vec{r}, t) = f_a^1(\vec{c}, \vec{\mathcal{L}}, \omega) \exp(i\vec{\mathcal{L}} \cdot \vec{r} - i\omega t)$$

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{\mathcal{L}}, \omega) \exp(i\vec{\mathcal{L}} \cdot \vec{r} - i\omega t)$$

are assumed then Eq. (A-3) becomes

$$-i\omega f_a^1 + i\vec{\mathcal{L}} \cdot \vec{c} f_a^1 + \frac{Z_a e a}{m_a} \vec{E} \cdot \frac{\partial f_a^0}{\partial \vec{c}} = - \frac{1}{\tau} f_a^1$$

The solution f_a^1 is, therefore

$$f_a^1 = -i \frac{Z_a e a}{m_a} \vec{E} \cdot \frac{\partial f_a^0}{\partial \vec{c}} \left[\omega - \vec{\mathcal{L}} \cdot \vec{c} + \frac{i}{\tau} \right]^{-1}$$

Since

$$\vec{j} = \sum_a Z_a e \int \vec{c}' f_a^1 d^3 c'$$

It is possible to write, using Eq. (A-1)

$$\epsilon \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{E}}{\partial t} + 4\pi \sum_{a=e,i} Z_a e \int \vec{c}' f_a^1 d^3 c'$$

Substituting for f_a^1 and solving for ϵ , it is found that

$$\epsilon(\vec{\mathcal{L}}, \omega) = 1 + \lim_{\tau \rightarrow \infty} \frac{4\pi}{\omega} \sum_{a=e,i} \frac{Z_a^2 e^2}{m_a} \int \frac{\vec{c}' \cdot \partial f_a^0 / \partial \vec{c}'}{\omega - \vec{\mathcal{L}} \cdot \vec{c}' + \frac{i}{\tau}}$$

Using the Plemelj formula derived in Muskhelishvili [33]

$$\lim_{\Delta \rightarrow 0} \frac{1}{y+i\Delta} \text{Pr} \left(\frac{1}{y} \right) - i\pi\delta(y) = i\delta_-(y)$$

the dielectric $\epsilon(\vec{\mathcal{L}}, \omega)$ can be cast in the form shown following Eq. (2-2). That is,¹

$$\epsilon^+ = 1 + i \sum_{a=e,i} \frac{4\pi^2 e^2}{m_a \mathcal{L}^2} \int \delta_-(\vec{\mathcal{L}} \cdot \vec{g}) \vec{\mathcal{L}} \cdot \partial f_a^0 / \partial \vec{c}' d^3 c' \quad (\text{A-4})$$

where the identification $\omega = \vec{\mathcal{L}} \cdot \vec{c}$ is made. This corresponds to a phase velocity $\frac{\omega}{\mathcal{L}} = U$.

If Maxwellian distributions are substituted for the f_a^0 , the result is the form $\epsilon_o^+(\mathcal{L}, U)$ exhibited in Eq. (3-6). If f_a^0 is Maxwellian and if it is assumed that

$$f_a = f_a^0 + \epsilon f_a^0 \phi_a$$

then

$$\epsilon^+(\vec{\mathcal{L}}, U) = \epsilon_o^+(\mathcal{L}, U) + \epsilon Q' (f_a^0 \phi_a)$$

where

$$Q' = i \sum_a \frac{4\pi^2 e^2}{m_a \mathcal{L}^2} \int d^3 c' \delta_-(\vec{\mathcal{L}} \cdot \vec{g})^+ \cdot \frac{\partial}{\partial \vec{c}'}$$

With this form for $\epsilon^+(\vec{\mathcal{L}}, U)$ then

$$\frac{1}{|\epsilon^+(\vec{\mathcal{L}}, U)|^2} = \frac{1}{|\epsilon_o^+(\mathcal{L}, U)|^2} \left(1 - \frac{\epsilon \text{Re}(\epsilon_o^+ Q'^*)}{\text{Re}(\epsilon_o^+)} + \dots \right)$$

where Q'^* is the complex conjugate of Q' . If Q is defined as

$$Q = \frac{\text{Re}(\epsilon_o^+ Q'^*)}{\text{Re}(\epsilon_o^+)}$$

then the form of $1/\epsilon^+(\vec{\mathcal{L}}, U)$ found on page 49 is obtained.

¹The plus superscript on ϵ indicates that $\frac{1}{\tau} \rightarrow 0$ through positive values.

APPENDIX B

In Chapters III and IV the following form of the Lenard-Balescu interaction integral is used.

$$J_{LB} = J_L + J_{coll}. \quad (B-1)$$

The derivation of this relation is exhibited in this appendix and the interaction integral J_{coll} is identified. Also, in this appendix the method of obtaining the interaction integration $J_M(f_e, f_i)$ is demonstrated. It will be recalled that $J_M(f_e, f_i)$ is an approximation to $J_L(f_e, f_i)$.

The Lenard-Balescu interaction integral in tensor form is

$$J_{LB}(f_a, f_b) = \frac{2e^4}{m_a} \int d^3c' \int d^3\ell \frac{\ell_\alpha \ell_\beta}{\ell^4} \frac{\partial}{\partial c_\alpha} \frac{\delta(\vec{\ell} \cdot \vec{g})}{|\epsilon(\vec{\ell}, U)|^2} \left\{ \frac{1}{m_a} \frac{\partial}{\partial c_\beta} - \frac{1}{m_b} \frac{\partial}{\partial c'_\beta} \right\} f_a(\vec{c}) f_b(\vec{c}') \quad (B-2)$$

To facilitate the derivation of Eq. (B-1), it is convenient to introduce the following coordinate system

Using this coordinate system, the Lenard-Balescu equation becomes

$$J_{LB}(f_a, f_b) = \frac{2e^4}{m_a} \int d^3c' \int_0^{2\pi} d\phi \int_{-1}^1 d \cos\theta \delta(lg \cos\theta) \begin{Bmatrix} \sin\theta & \cos\phi \\ \sin\theta & \sin\phi \\ \cos\theta \end{Bmatrix} \begin{Bmatrix} \sin\theta & \cos\phi \\ \sin\theta & \sin\phi \\ \cos\theta \end{Bmatrix} \int_0^{l_0} \frac{dl l^4}{|l^2 + \psi(\cos\theta, \phi)|^2} \quad (B-3)$$

where the dielectric function given by Eq. (A-4) has been written $\epsilon^+ = \frac{l^2 + \psi(\cos\theta, \phi)}{l^2}$.

Since

$$\delta(lg \cos\theta) = \frac{\delta(\cos\theta)}{lg}$$

the integration over $\cos\theta$ in the above equation yields

$$J_{LB}(f_a, f_b) = \frac{2e^4}{m_a g} \int d^3c' \int_0^{2\pi} d\phi \begin{Bmatrix} \cos\phi & \cos\phi \\ \sin\phi & \sin\phi \\ 0 & 0 \end{Bmatrix} I(0, \phi) \quad (B-4)$$

where

$$I(0, \phi) = \int_0^{l_0} \frac{dl l^3}{|l^2 + \psi(0, \phi)|} \quad (B-5)$$

For Maxwellian distributions, this $I(0, \phi)$ reduces to the $I(U)$ in Eq. (3-23). The integration of $I(0, \phi)$ yields

$$I(0, \phi) = \ln\Lambda + G(0, \phi) \quad (B-6)$$

It is clear that the integration over ϕ yields components (xy, xz, yz, yx, zx, zy) that are zero. The zz component is identically zero, and the xx and yy components are equal.

Therefore,

$$\int_0^{2\pi} d\phi \begin{Bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{Bmatrix} \begin{Bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{Bmatrix} = \frac{\pi}{g^2} (g^2 \delta_{\alpha\beta} - g_\alpha g_\beta) \quad (B-7)$$

where it has been recalled that the vector $\vec{g}$ is along the z axis.

Using Eqs. (B-6) and (B-7), the Lenard-Balescu interaction integral becomes

$$J_{LB}(f_a, f_b) = J_L(f_a, f_b) + J_{coll.}(f_a, f_b)$$

where J_L is exactly Landau's [12] original form of the interaction integral

$$J_L(f_a, f_b) = \frac{2e^4 \pi}{m_a} \ln \Lambda \int d^3 c' \frac{\partial}{\partial c_\alpha} \left(\frac{g^2 \delta_{\alpha\beta} - g_\alpha g_\beta}{g^3} \right) \left(\frac{1}{m_a} \frac{\partial}{\partial c_\beta} - \frac{1}{m_b} \frac{\partial}{\partial c'_\beta} \right) f_a(\vec{c}) f_b(\vec{c}') \quad (B-8)$$

and where $J_{coll.}$ arises due to the term $G(O, \phi)$ in $I(O, \phi)$ and is

$$J_{coll.}(f_a, f_b) = \frac{2e^4}{m_a g} \int d^3 c' \int_0^{2\pi} d\phi \begin{Bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{Bmatrix} \begin{Bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{Bmatrix} G(O, \phi) \left(\frac{1}{m_a} \frac{\partial}{\partial c_\beta} - \frac{1}{m_b} \frac{\partial}{\partial c'_\beta} \right) f_a(\vec{c}) f_b(\vec{c}') \quad (B-9)$$

As stated in the body of this thesis, the interaction integral $J_L(f_a, f_b)$ includes collisional interactions only while $J_{coll.}(f_a, f_b)$ includes long-range collective interactions only.

To facilitate the derivation of the interaction integral $J_M(f_e, f_i)$ the following coordinate system is introduced.

The Landau interaction integral $J_L(f_e, f_i)$ which is simply the Lenard-Balescu interaction integral, Eq. (B-2), with $\epsilon^+(\vec{l}, U) = 1$, can be written using the coordinate system above.

$$J_L(f_e, f_i) = \frac{2e^4}{m_e} \int d^3c \int_0^{2\pi} d\phi \int d\cos\theta \int_{l_D}^{l_0} dl$$

$$\left\{ \begin{matrix} \sin\theta' \cos\phi' \\ \sin\theta' \sin\phi' \\ \cos\theta' \end{matrix} \right\} \left\{ \begin{matrix} \sin\theta' \cos\phi' \\ \sin\theta' \sin\phi' \\ \cos\theta' \end{matrix} \right\} \frac{\partial}{\partial c'_\alpha} \delta(lc' \cos\theta' - lc \cos\theta)$$

$$\left\{ \frac{1}{m_e} \frac{\partial}{\partial c'_\beta} - \frac{1}{m_i} \frac{\partial}{\partial c_\beta} \right\} f_e(\vec{c}') f_i(\vec{c}) \quad (B-10)$$

Since

$$\delta(lc' \cos\theta' - lc \cos\theta) = \frac{\delta\left(\cos\theta' - \frac{c \cos\theta}{c'}\right)}{lc'}$$

and since

$$\frac{\delta\left(\cos\theta' - \frac{c \cos\theta}{c'}\right)}{lc'} \approx \frac{\delta(\cos\theta')}{lc'} *$$

*This approximation is quite good since on the average, the ratio of ion velocity to electron velocity is a small number.

The integration over ϕ' and θ' in Eq. (B-10) may be carried out in precisely the same manner as in the previous derivation. These integrations and the integration over ℓ yield

$$J_L(f_e, f_i) = \frac{2e^4\pi}{m_e^2} \ln\Lambda \frac{\partial}{\partial c'_\alpha} \frac{(c')^2 \delta_{\alpha\beta} - c'_\alpha c'_\beta}{(c')^3}$$

$$\frac{\partial f_e}{\partial c'_\beta} \int d^3c f_i \quad (B-11)$$

where

$$\ln\Lambda = \int_{l_D}^{l_0} \frac{d\ell}{\ell}$$

If the velocity variables

$$\vec{u}' = \vec{c}' - \vec{w}$$

$$\vec{u} = \vec{c} - \vec{w}$$

are introduced then Eq. (4-16) is obtained.

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