COO-3533-13
SU-4205-13
January 1973

$\eta \rightarrow 3\pi$ In a Renormalizable SU(3) Sigma Model

by

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* Work supported by the U. S. Atomic Energy Commission

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ABSTRACT

The problem of predicting the $\eta \rightarrow 3\pi$ decay rate in terms of the $K^+ - K^0$ mass difference is discussed in the framework of a renormalizable SU(3) sigma model. Order of magnitude agreement for the process is finally achieved. In addition, some connections with unified weak-electromagnetic gauge schemes are explored.

I. INTRODUCTION

Although the current-algebra or phenomenological Lagrangian technique adequately predicts the energy spectrum of $\eta \rightarrow \pi^+ \pi^- \pi^0$ to linear order it has generally given a rate which is an order of magnitude too small. This result follows from relating the π^0 - η mixing angle to the "tadpole" part of the K^+K^0 mass shift. In a rather general form⁽¹⁾ of the SU(3) sigma model it was previously shown⁽²⁾ that the energy spectrum comes out right but that the overall magnitude involves some constants which can not be fixed merely by requiring chiral symmetry for the Lagrangian. Thus, although the correct rate could in principle be obtained, more information is needed.

In the present note it is shown that requiring the general SU(3) sigma model to be renormalizable fixes these previously unknown constants and, in fact, leads to the correct order of magnitude for the decay rate. What is essentially happening is that in addition to the π^0 - η mixing term there is another term resulting from the electromagnetic mixing of the analogous scalar mesons which enter the process virtually. This latter term actually dominates the former and together they build up the rate.

We shall continue to use the same notation and formalism as in refs. (1) and (2) but shall briefly define some relevant quantities again for the sake of coherence. The Lagrangian of the general sigma model is:

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2}\text{Tr}(\partial_\mu S \partial_\mu S) - V_0 - V_{SB} \quad (1.1)$$

In (1.1) ϕ and S are the 3×3 matrices of pseudoscalar and scalar fields. The members of the scalar nonet are denoted as $\{(\kappa^+, \bar{\kappa}^0), (\epsilon^+, \bar{\epsilon}^0, \epsilon^-), (\bar{\kappa}^+, \bar{\kappa}^0), (\sigma, \sigma')\}$. V_0 is the most general function of the following chiral $SU(3) \times SU(3)$ invariants:

$$\begin{aligned} I_1 &= \text{Tr}[(S+i\phi)(S-i\phi)] \\ I_2 &= \text{Tr}\{[(S+i\phi)(S-i\phi)]^2\} \\ I_3 &= \text{Tr}\{[(S+i\phi)(S-i\phi)]^3\} \\ I_4 &= 6[\det(S+i\phi) + \det(S-i\phi)] \end{aligned} \quad (1.2)$$

It is convenient to use the abbreviations

$$V_i = \left\langle \frac{\partial V_0}{\partial I_i} \right\rangle_0, \quad V_{ij} = \left\langle \frac{\partial^2 V_0}{\partial I_i \partial I_j} \right\rangle_0 \quad (1.3)$$

The symmetry breaking term V_{SB} will be taken to have the following $[(3, 3^*) + (3^*, 3)]$ form:

$$V_{SB} = -2(A_1 S_1^1 + A_2 S_2^2 + A_3 S_3^3) \dots, \quad (1.4)$$

where the A_i are three constants analogous to the three quark masses. A possible origin for these terms will be discussed later. It is crucial to remark that there is not only ordinary symmetry breaking in $\mathcal{L}$ but also "spontaneous" breaking in the ground state. This is described by the three quantities

$$\alpha_i = \langle S_i^i \rangle_0 \quad (1.5)$$

Thus, in addition to the V_i , V_{ij} , etc., the system is described by the six parameters $\{A_i, \alpha_i\}$. In the isospin invariant limit, which we shall take as a first approximation:

$$\begin{aligned} \alpha_1 &= \alpha_2 \equiv \alpha, & A_1 &= A_2 \\ \alpha_3 &\equiv w\alpha \end{aligned} \quad (\text{isospin limit}) \quad (1.6)$$

A remarkable feature of the dynamics, discussed in ref. (1), is that a large number of interesting results can be found without specifying the explicit form of V_0 (i.e., the $V_i, V_{ij}, \dots$). This follows because one can derive two generating equations [(2.18) and (2.21) of ref. (1)] which on differentiation yield a chain of Ward-like identities⁽³⁾ between n and $(n-1)$ point functions of the theory. In addition, one uses the extremum equations

$$\left\langle \frac{\partial V_0}{\partial S} \right\rangle_0 + \left\langle \frac{\partial V_{SB}}{\partial S} \right\rangle_0 = 0 \quad (1.7)$$

These take the explicit form:

$$\begin{aligned} \alpha_1 [V_1 + 2V_2(\alpha_1)^2 + 3V_3(\alpha_1)^4] + 6\alpha_2\alpha_3 V_4 &= A_1 \\ \alpha_2 [V_1 + 2V_2(\alpha_2)^2 + 3V_3(\alpha_2)^4] + 6\alpha_1\alpha_3 V_4 &= A_2 \\ \alpha_3 [V_1 + 2V_2(\alpha_3)^2 + 3V_3(\alpha_3)^4] + 6\alpha_1\alpha_2 V_4 &= A_3 \end{aligned} \quad (1.8)$$

From (1.8) we see that if $\alpha_1 = \alpha_2 = \alpha_3$ [SU(3) invariant vacuum] we must have $A_1 = A_2 = A_3$ [SU(3) invariant Lagrangian] and similarly, if $\alpha_1 = \alpha_2$ [isospin invariant vacuum] we must have $A_1 = A_2$ [isospin invariant Lagrangian]. Thus the symmetry properties of the vacuum and the Lagrangian are linked in this model. Besides the extremum conditions (1.7) there are stability conditions on the

second derivatives of V_0 which lead to certain allowed ranges for the parameters A_i and α_i [see footnote 15 of ref. 1].

As explained in ref. 1, it is possible to determine the fundamental parameters of the system in terms of a limited set of input quantities. We initially work in the isospin invariant limit and choose as input:

$$\begin{aligned} \pi^2 &= 1, \quad K^2 = 13.6\pi^2, \quad \eta^2 = 16.5\pi^2, \quad \eta'^2 = 50.3\pi^2 \\ F_\pi &= 1.01\pi \end{aligned} \quad (1.9)$$

where the particle symbol denotes its mass [π^0 mass equals unity] and F_π is the pion decay constant. This choice then enables us to find the basic parameters:

$$\begin{aligned} \frac{1}{2}(A_1 + A_2) &= 0.25\pi^3, \quad A_3 = 9.05\pi^3, \quad \alpha = 0.5\pi \\ w &= 1.71, \quad V_4 = -1.85\pi \end{aligned} \quad (1.10)$$

One then has the predictions

$$\theta_p = 0.48^\circ, \quad F_k = 1.37\pi, \quad \kappa^2 = 50.5\pi^2 = (959.2 \text{ MeV})^2 \quad (1.11)$$

where θ_p is the η - η' mixing angle and F_k is the kaon decay constant. In addition there are many predictions on the 3,4,... point vertices of the theory given in ref. 1. An interesting feature is that the scalar meson masses (except for the κ) are free parameters in the general sigma model. Placing restrictions other than chiral $SU(3) \times SU(3)$ invariance on V_0 will then relate some of these scalar masses. The additional restriction of scale invariance was investigated in ref. 1. Here we shall impose the requirement of renormalizability.

II. THE RENORMALIZABLE SU(3) SIGMA MODEL

The renormalizability criterion is that V_0 contain no terms of order higher than 4 in ϕ and S . This gives, in the notation used in the preceding section:

$$V_0 = [V_1 - V_{11}\alpha^2(2+w^2)]I_1 + \frac{1}{2}V_{11}(I_1)^2 + V_2I_2 + V_4I_4 \quad (2.1)$$

(In (2.1) the quantities α and w are considered to be fixed at the values given in (1.10)). Thus V_0 is described by the four constants V_1 , V_{11} , V_2 , and V_4 . While the motivation for requiring a phenomenological Lagrangian to be renormalizable is certainly not as strong as the corresponding motivation for a "fundamental" Lagrangian it is, nevertheless, an attractive idea. It is interesting to note that Chan and Haymaker⁽⁴⁾ have calculated the one loop corrections to the tree approximation for the one and two point functions in such a model and found them to be only of the order of 5%.

We must now specify the four constants appearing in V_0 . These will be determined in the isospin invariant limit. V_4 occupies a somewhat special position in that it has already been determined from the general sigma model [eq. (1.10)]. V_1 and V_2 can be found from the extremum conditions (1.8). Noting that $V_3 = 0$ here and that we are working in the isospin invariant limit gives:

$$\begin{aligned} V_2 &= \frac{A_3/w-1}{2\alpha^3(w^2-1)} + \frac{3V_4}{\alpha w} = 3.75 \\ V_1 &= \frac{A_1}{\alpha} - 2V_2\alpha^2 - 6V_4\alpha w = 8.17\pi^2 \end{aligned} \quad (2.2)$$

Knowing only V_1 , V_2 and V_4 is sufficient to give the squared mass of the scalar-isovector particle

$$\begin{aligned} \epsilon^2 &= \left\langle \frac{\partial^2 V_0}{\partial S_1^2 \partial S_2^2} \right\rangle_0 = 2V_1 + 12V_2\alpha^2 - 12V_4\alpha w \\ &= 46.98\pi^2 = (925 \text{ MeV})^2. \end{aligned} \quad (2.3)$$

This result might tempt one to identify the ϵ with the $\mathbb{S}(960)$.

The remaining constant V_{11} may be determined if the mass of the σ -meson is specified. This, however, involves one in the intricacies of the σ - σ' mixing problem. A painless way to proceed is as follows. First define the objects:

$$m_{ab} = \left\langle \frac{\partial^2 V_0}{\partial S_a^a \partial S_b^b} \right\rangle_0 \quad (2.4)$$

One then notes that the m_{ab} can either be expressed in terms of the squared masses and the scalar mixing angle, θ_S , or in terms of the parameters in V_0 . Explicitly, in the isospin invariant limit,

$$m_{11} = \frac{1}{2} \epsilon^2 + b'^2 \sigma^2 + a'^2 \sigma'^2 = 2V_1 + 12V_2\alpha^2 + 4\alpha^2 V_{11} \quad (2.4a)$$

$$m_{12} = -\frac{1}{2} \epsilon^2 + b'^2 \sigma^2 + a'^2 \sigma'^2 = 12V_4\alpha w + 4\alpha^2 V_{11} \quad (2.4b)$$

$$m_{13} = \sqrt{2} a' b' (\sigma'^2 - \sigma^2) = 12V_4\alpha + 4\alpha w V_{11} \quad (2.4c)$$

$$m_{33} = 2a'^2 \sigma^2 + 2b'^2 \sigma'^2 = 2V_1 + 12V_2\alpha^2 w^2 + 4\alpha^2 w^2 V_{11} \quad (2.4d)$$

where a' and b' are certain useful combinations of θ_S :

$$a' = \frac{1}{\sqrt{6}} (\sin\theta_S + \sqrt{2}\cos\theta_S), \quad b' = \frac{1}{\sqrt{6}} (\cos\theta_S - \sqrt{2}\sin\theta_S) \quad (2.5)$$

$$a'^2 + b'^2 = \frac{1}{2}$$

We will also need the analogous combinations of the pseudo-scalar mixing angle θ_p :

$$a = \frac{1}{\sqrt{6}} (\sin\theta_p + \sqrt{2}\cos\theta_p) , \quad b = \frac{1}{\sqrt{6}} (\cos\theta_p - \sqrt{2}\sin\theta_p) \quad (2.6)$$

From the set of eqs. (2.4a) - (2.4d) we may solve for V_{11} in terms of σ^2 and known quantities

$$V_{11} = \frac{72(V_4)^2 - \frac{1}{4\alpha^2} (2V_1 + 12V_2\alpha^2 + 12V_4\alpha w - \sigma^2) (2V_1 + 12V_2\alpha^2 w^2 - \sigma^2)}{(4+2w^2)V_1 + 36\alpha^2 w^2 V_2 + 12\alpha w(w^2 - 4)V_4 - (2+w^2)\sigma^2} \quad (2.7)$$

Thus, choosing σ^2 gives us V_{11} from (2.7) and completes the specification of V_0 . From (2.4) we may then find the σ' squared mass

$$\sigma'^2 = m_{11} + m_{12} + m_{33} - \sigma^2 , \quad (2.8)$$

and the σ - σ' mixing angle

$$\tan 2\theta_S = \frac{-2\sqrt{2}(m_{11} + m_{12} - m_{13} - m_{33})}{m_{33} - m_{11} - m_{12} - 8m_{13}} \quad (2.9)$$

In solving (2.9) for θ_S a spurious solution will be found in addition to the correct one. The spurious solution may be eliminated by checking to see if it satisfies (2.4c). The actual choice of σ^2 used to be considered controversial but now π - π phase shift analyses⁽⁵⁾ give an extremely broad object centered around 850 MeV. To be on the safe side we shall investigate this system for a range of σ masses in the correct region. A listing of V_{11} , σ'^2 and θ_S for various values of σ^2 is given in Table I.

We note that the positivity of the σ, σ' masses does not allow all choices of σ^2 in eqn.(2.8). For m_σ lying between 0-863 MeV,

m_{σ} , lies between 993- ∞ MeV; choosing m_{σ} between 863-993 MeV gives an imaginary m_{σ} , therefore, ruling out this region. Choosing m_{σ} between 993- ∞ MeV, m_{σ} , lies between 0-863 MeV.

III. ISOSPIN VIOLATIONS

Setting $A_1 \neq A_2$ will introduce a violation of isospin into our Lagrangian. By the extremum eqs. (1.8) this will induce an isospin violation into the vacuum, i.e., $\alpha_1 \neq \alpha_2$. In fact, subtracting the first of eqs. (1.8) from the second and combining with (2.3) gives:

$$\epsilon^2 = \frac{2(A_1 - A_2)}{\alpha_1 - \alpha_2} \quad (3.1)$$

Since ϵ^2 has already been fixed in terms of the system parameters this means that the two new isospin violating quantities $(A_1 - A_2)$ and $(\alpha_1 - \alpha_2)$ are linked. It is thus sufficient to find $(\alpha_1 - \alpha_2)$ to specify the system in the presence of a small isospin violation. In ref. 2 it was shown that this quantity is related to the $K^+ - K^0$ mass difference. From (3.5) and (3.6) of ref. 2, we find

$$(K^+)^2 - (K^0)^2 = d_K + \frac{\alpha_1 - \alpha_2}{\alpha(1+w)} (\epsilon^2 - K^2), \quad (3.2)$$

where d_K is the contribution from the electromagnetic self-energy diagrams. Conventionally⁽⁶⁾ this is taken to be

$$d_K \approx 0.15 \pi^2. \quad (3.3)$$

Adopting (3.3) and using (3.1) and (3.2) then gives:⁽⁷⁾

$$\alpha_1 - \alpha_2 = -.0149\pi, \quad A_1 - A_2 = -.352 \pi^3 \quad (3.4)$$

Now that the system is completely specified we can go on to make predictions about $\eta \rightarrow 3\pi$ without introducing any arbitrary parameters. Because of the isospin violation there will be $\sigma - \epsilon^0$ and $\sigma' - \epsilon^0$ mixing in addition to the $\sigma - \sigma'$ mixing which results from SU(3) violation. The physical ϵ^0 , σ , and σ' are related to the mathematical quantities S_1^1 , S_2^2 , and S_3^3 by :

$$\begin{pmatrix} \epsilon^0 \\ \sigma \\ \sigma' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \chi_1 b' + \chi_3 a', & -\frac{1}{\sqrt{2}} \chi_1 b' + \chi_3 a', & \sqrt{2}(-\chi_1 a' + \chi_3 b') \\ -\frac{1}{\sqrt{2}} \chi_1 + b', & \frac{1}{\sqrt{2}} \chi_1 + b', & -\sqrt{2} a' \\ -\frac{1}{\sqrt{2}} \chi_3 + a', & \frac{1}{\sqrt{2}} \chi_3 + a', & \sqrt{2} b' \end{pmatrix} \begin{pmatrix} S_1^1 \\ S_2^2 \\ S_3^3 \end{pmatrix} \quad (3.5)$$

In (3.5) χ_1 is the $\epsilon^0 - \sigma$ mixing angle and χ_3 is the $\epsilon^0 - \sigma'$ mixing angle.⁽⁸⁾ A similar formula [eq. (3.1) of ref. 2] holds for mixing in the $\pi^0 - \eta - \eta'$ complex. There, ψ_1 is the $\pi^0 - \eta$ mixing angle and ψ_3 is the $\pi^0 - \eta'$ mixing angle. The pseudoscalar meson mixing angles have already been determined in the general SU(3) sigma model. Taking over (3.13) of ref. 2 gives

$$\begin{aligned} \psi_1 &= \frac{b(\alpha_1 - \alpha_2)}{\sqrt{2}\alpha(\eta'^2 - \pi^2)} [\pi^2 - \epsilon^2 + 2a(\sqrt{2}bw - a)(\eta'^2 - \eta^2)] \\ &= .0166 = -(1.109)(\alpha_1 - \alpha_2) \\ \psi_3 &= \frac{a(\alpha_1 - \alpha_2)}{\sqrt{2}\alpha(\eta'^2 - \pi^2)} [\pi^2 - \epsilon^2 + 2b(\sqrt{2}aw + b)(\eta'^2 - \eta^2)] \\ &= -8 \times 10^{-4} = (.0537)(\alpha_1 - \alpha_2) \end{aligned} \quad (3.6)$$

A computation analogous to the one leading to (3.6) gives for the ε^0 - σ and ε^0 - σ' mixing angles:

$$\begin{aligned} \chi_1 &= \frac{4(\alpha_1 - \alpha_2)}{\varepsilon^2 - \sigma^2} [\sqrt{2}\alpha b' (3V_2 + V_{11}) + a' (3V_4 - \alpha w V_{11})] \\ \chi_3 &= \frac{4(\alpha_1 - \alpha_2)}{\varepsilon^2 - \sigma'^2} [\sqrt{2} \alpha a' (3V_2 + V_{11}) - b' (3V_4 - \alpha w V_{11})] \end{aligned} \quad (3.7)$$

Due to the presence of V_{11} in (3.7), χ_1 and χ_3 depend on our choice of σ^2 . Thus, their numerical values are listed in Table I. Besides the isospin violating two point functions (mixing angles) we will also need certain isospin violating three point vertices for computing $\eta \rightarrow 3\pi$ decay. A straightforward differentiation of V_0 [eq. (2.1)] gives us the needed objects:

$$\begin{aligned} f_{\sigma\pi\eta} &= \left\langle \frac{\partial^3 V_0}{\partial \sigma \partial \pi^0 \partial \eta} \right\rangle_0 = \sum_{i,j,k} \frac{\partial \phi_i^i}{\partial \eta} \frac{\partial \phi_j^j}{\partial \pi^0} \frac{\partial S_k^k}{\partial \sigma} \left\langle \frac{\partial^3 V_0}{\partial S_k^k \partial \phi_j^j \partial \phi_i^i} \right\rangle_0 \\ &= \frac{8}{\sqrt{2}} V_2 b b' (\alpha_1 - \alpha_2) + \frac{8\chi_1}{\sqrt{2}} (3aV_4 - \sqrt{2}\alpha b V_2) \\ &\quad + 8b' \{ 2\Psi_1 (-\alpha a^2 V_2 + 3\sqrt{2}\alpha b V_4) + \Psi_3 [2\alpha a b V_2 - 3\sqrt{2}(b^2 - a^2)V_4] \} \\ &\quad - 4\sqrt{2}a' \{ \Psi_1 [4\alpha w a^2 V_2 - 3(1+2b^2)V_4] - 2ab\Psi_3 [2\alpha w V_2 + 3V_4] \} \end{aligned} \quad (3.8a)$$

$$\begin{aligned}
 f_{\sigma', \pi\eta} = & \frac{8}{\sqrt{2}} V_2 b a' (\alpha_1 - \alpha_2) + \frac{8}{\sqrt{2}} \chi_3 (3aV_4 - \sqrt{2}\alpha bV_2) \\
 & + 8a' \{ 2\Psi_1 (-\alpha a^2 V_2 + 3\sqrt{2}abV_4) + \Psi_3 [2\alpha abV_2 - 3\sqrt{2}(b^2 - a^2)V_4] \} \\
 & + 4\sqrt{2}b' \{ \Psi_1 [4\alpha wa^2 V_2 - 3(1+2b^2)V_4] - 2ab\Psi_3 [2\alpha wV_2 + 3V_4] \} \quad (3.8b)
 \end{aligned}$$

The numerical values of $f_{\sigma\pi\eta}$ and $f_{\sigma', \pi\eta}$ are also listed in Table I for various choices of σ mass.

IV. $\eta \rightarrow 3\pi$ DECAY

The details of the calculation of this process for the general σ -model are explained in sect. IV of ref. 2, the final formula being eq. (4.6). In its derivation, non-trivial use was made of the Ward-like identities referred to after (1.6). This formula involves the quantities χ_1 , χ_3 , $f_{\sigma\pi\eta}$, $f_{\sigma', \pi\eta}$, σ'^2 , and θ_S which were not fixed in the general σ -model but which are given in the last two sections. For simplicity we write this formula in the (fairly reasonable) approximation where the relevant pseudoscalar squared masses are considered small compared to the scalar squared masses. Then:

$$\begin{aligned}
 T(\eta \rightarrow \pi^+ \pi^- \pi^0) & \approx \frac{-\eta^2}{\alpha} \left(1 - \frac{2\omega_0}{\eta}\right) \left\{ \frac{b' f_{\sigma\pi\eta}}{\sigma^2} + \frac{a' f_{\sigma', \pi\eta}}{\sigma'^2} \right. \\
 & \quad \left. + \frac{b}{\alpha} [(b'\chi_1 + a'\chi_3) - (b\Psi_1 + a\Psi_3)] \right\} \\
 & \equiv \bar{X} \left(1 - \frac{2\omega_0}{\eta}\right) \quad (4.1)
 \end{aligned}$$

where ω_0 is the energy of the π^0 in the η rest frame. The spectrum shape⁽⁸⁾ represented by the $(1 - 2\omega_0/\eta)$ factor has also been obtained by the "current algebra" technique and is very close to the experimental one. For our present purposes we note that

the "experimental" amplitude to linear order looks like⁽⁸⁾

$$T_{\text{exp}} \approx \pm (0.994 \pm .112) \left(1 - \frac{2\omega_0}{\eta}\right) \quad (4.2)$$

The early work using the current-algebra approach essentially neglected all terms in (4.1) except for the ψ_1 term (π^0 - η mixing term). Using (3.4) would give in this "current algebra" limit

$$T_{\text{CA}} \approx .163 \left(1 - \frac{2\omega_0}{\eta}\right) \quad (4.3)$$

Thus the predicted width would be about 1/25 of the experimental one.⁽⁹⁾ Now, if all the terms of (4.1) are taken into account we get a much more satisfactory result. This is illustrated in Fig. 1 where a plot is made of the factor $\bar{X}$ in (4.1) as a function of the assumed (input) mass of the σ -meson. Note that the predicted value of $\bar{X}$ is relatively constant for the interesting range of σ -masses; at the favored value of 850 MeV we see $\bar{X} \approx 0.64$. Since the experimental value may be as low as $\bar{X} \approx 0.89$ the agreement is reasonable, considering the simplicity of our model. Reference to Table I shows that the dominant contribution to (4.1) comes from the χ_1 (ϵ^0 - σ mixing angle) term. This term is about three times larger than the ψ_1 (π^0 - η mixing angle) term, which represents the contribution which originally had been thought to be the most important one. Actually a similar situation to the present one had already been conjectured in ref. 2.

How can the agreement with experiment be improved further? One possibility is that the conventional electromagnetic self-energy contribution to the K^+-K^0 mass shift (d_K) should be modi-

fied from the value given in (3.3). In fact a value

$$d_K = (.35 \pm .07) \pi^2 \quad (4.4)$$

is sufficient to give experimental agreement for $m_\sigma = 850$ MeV. Eq. (4.4) is easily found by first noting that each term in (4.1) is proportional to $(\alpha_1 - \alpha_2)$ and then substituting the required value of $(\alpha_1 - \alpha_2)$ into (3.2). Another possibility for improving the agreement is to take more particles into account in our model; including vector and axial vector mesons⁽¹⁰⁾ would seem to be a logical though complicated first step in this direction.

V. POSSIBLE ORIGIN FOR ISOSPIN VIOLATION

We have correlated the "tadpole" part of the $(K^+ - K^0)$ mass shift with the $\eta \rightarrow 3\pi$ decay rate by using the $SU(3) \times SU(3)$ symmetry breaking interaction (1.4). The chiral $SU(3)$ is broken because not all of the A_i are zero, the ordinary $SU(3)$ is broken because the three A_i are not equal, and finally the isospin symmetry is broken because $A_1 \neq A_2$. This type of situation is entirely analogous to the type of symmetry breaking in the quark model. There one expects the symmetry breaking term to be:

$$\mathcal{L}_{SB} = -m_1 \bar{q}_1 q_1 - m_2 \bar{q}_2 q_2 - m_3 \bar{q}_3 q_3 \quad (5.1)$$

where q_1, q_2, q_3 are the three quarks and m_1, m_2 , and m_3 are their "masses". Eq. (5.1) has the same $SU(3) \times SU(3)$ transformation properties as (1.4) so we may identify A_1, A_2, A_3 as being analogous to m_1, m_2, m_3 .

What is the origin of this symmetry breaking? One possibility^(11,12) is to consider that the strong interaction is exactly

SU(3) x SU(3) invariant and that the symmetry breaking "masses" arise from a unified weak-electromagnetic gauge theory in much the same way that the electron and muon masses would arise in such a theory. Then the total Lagrangian of the strong, electromagnetic, and weak interaction would be invariant with respect to the weak-electromagnetic symmetry group. However, this symmetry would be spontaneously broken giving rise to (5.1) (or a similar expression) in the process. The precise implementation of such a mechanism depends on a particular choice of a theory of weak-electromagnetic interactions including the hadrons. Probably all the present theories are at best "steps in the right direction". Thus we shall content ourselves with illustrating this mechanism in two simple models. Similar approaches can of course be carried out for other models. In a simple theory⁽¹³⁾ which contains only three quarks the transformation properties of the quarks with respect to the weak electromagnetic SU(2) are:

$$\begin{aligned} \text{doublet: } \begin{pmatrix} Q_{1L} \\ Q_{2L} \end{pmatrix} &= \frac{1}{2}(1+\gamma_5) \begin{pmatrix} q_1 \\ q_2 \cos \theta + q_3 \sin \theta \end{pmatrix} \\ \text{singlets: } Q_{1R} &= \frac{1}{2}(1-\gamma_5)q_1, \quad Q_{2R} = \frac{1}{2}(1-\gamma_5)(q_2 \cos \theta + q_3 \sin \theta) \\ Q_{3L,R} &= \frac{1}{2}(1 \pm \gamma_5)(-q_2 \sin \theta + q_3 \cos \theta) \end{aligned} \quad (5.2)$$

In (5.2) θ is the Cabbibo angle. Then an SU(2) x U(1) invariant form⁽¹¹⁾ involving a scalar doublet $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ is:

$$\begin{aligned}
 & (\bar{Q}_{1L} \bar{Q}_{2L}) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} (f_1 Q_{2R} + f_2 Q_{3R}) + f_3 \bar{Q}_{3L} Q_{2R} \\
 & + f_4 \bar{Q}_{3L} Q_{3R} + i f_5 (\bar{Q}_{1L} \bar{Q}_{2L}) \tau_2 \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix} Q_{1R} + \text{h.c.} \quad (5.3)
 \end{aligned}$$

where the f_i are some real constants. Eq. (5.3) can be identified with (5.1) having arbitrary m_1 , m_2 , and m_3 when we replace $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ by its vacuum expectation value $\begin{pmatrix} \theta \\ \lambda \end{pmatrix}$ and impose⁽¹¹⁾

$$\lambda f_2 = f_3 = \frac{1}{2}(f_4 - \lambda f_1) \tan 2\theta.$$

In such a three quark model the unwanted semi-leptonic decays can be eliminated by introducing⁽¹³⁾ another U(1) gauge field, thereby extending the weak-electromagnetic group from SU(2) x U(1) to SU(2) x U(1) x U(1). The unwanted non-leptonic processes can be disregarded if we are willing⁽¹³⁾ to postulate an exact dynamical suppression of non-octet components in the effective Hamiltonian.

Another way to eliminate unwanted weak processes is to introduce⁽¹⁴⁾ a fourth quark, q_4 . (q_4 has the same charge as q_1 but is distinguished from the other three by having a different "charm" quantum number.) The transformation properties with respect to the weak electromagnetic SU(2) are then:

$$\begin{aligned}
 \text{doublets: } \begin{pmatrix} Q_{1L} \\ Q_{2L} \end{pmatrix} &= \frac{1}{2}(1+\gamma_5) \begin{pmatrix} q_1 \\ q_2 \cos \theta + q_3 \sin \theta \end{pmatrix} \\
 \begin{pmatrix} Q'_{1L} \\ Q'_{2L} \end{pmatrix} &= \frac{1}{2}(1+\gamma_5) \begin{pmatrix} q_4 \\ -q_2 \sin \theta + q_3 \cos \theta \end{pmatrix} \quad (5.4)
 \end{aligned}$$

$$\begin{aligned}
 \text{singlets: } Q_{1R} &= \frac{1}{2}(1-\gamma_5) q_1, \quad Q_{2R} = \frac{1}{2}(1-\gamma_5) (q_2 \cos \theta + q_3 \sin \theta) \\
 Q'_{1R} &= \frac{1}{2}(1-\gamma_5) q_4, \quad Q'_{2R} = \frac{1}{2}(1-\gamma_5) (-q_2 \sin \theta + q_3 \cos \theta)
 \end{aligned}$$

An SU(2) invariant form is similarly

$$\begin{aligned}
 & (\bar{Q}_{1L} \bar{Q}_{2L}) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} (h_1 Q_{2R} + h_2 Q'_{2R}) + (\bar{Q}'_{1L} \bar{Q}'_{2L}) \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} (h_3 Q_{2R} + h_4 Q'_{2R}) \\
 & + (\bar{Q}_{1L} \bar{Q}_{2L}) (i\tau_2) \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix} (h_5 Q_{1R} + h_6 Q'_{1R}) \\
 & + (\bar{Q}'_{1L} \bar{Q}'_{2L}) (i\tau_2) \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix} (h_7 Q_{1R} + h_8 Q'_{1R})
 \end{aligned} \tag{5.5}$$

where the h_i are some real constants. Eq. (5.5) will become identical to

$$-m_1 \bar{q}_1 q_1 - m_2 \bar{q}_2 q_2 - m_3 \bar{q}_3 q_3 - m_4 \bar{q}_4 q_4 \tag{5.6}$$

with arbitrary m_i if we replace $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ by $\begin{pmatrix} 0 \\ \lambda \end{pmatrix}$ and in addition impose:

$$\begin{aligned}
 h_1 &= \frac{-1}{2\lambda} [(m_2 + m_3) + (m_2 - m_3) \cos 2\theta] \\
 h_2 &= h_3 = \frac{1}{2\lambda} (m_2 - m_3) \sin 2\theta \\
 h_4 &= -\frac{1}{2\lambda} [(m_2 + m_3) - (m_2 - m_3) \cos 2\theta] \\
 h_5 &= -m_1/\lambda \\
 h_6 &= h_7 = 0 \\
 h_8 &= -m_4/\lambda
 \end{aligned} \tag{5.7}$$

Although both the above models apparently do not give us any information on the values of the m_i , they do suggest a heuristic

ansatz. Noting that the electron doublet is $\frac{1}{2}(1+\gamma_5) \begin{pmatrix} \nu_e \\ e \end{pmatrix}$ we are tempted to imagine a similarity between the neutrino and q_1 and to therefore postulate⁽¹¹⁾ $m_1 = 0$. In the case of the σ -model this would be $A_1 = 0$. Using (1.10) and (3.4) shows that (with the choice (3.3)) we have from fitting the spin zero mass spectrum to experiment:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} .08 \\ .43 \\ 9.05 \end{pmatrix} \pi^3 \quad (5.8)$$

Thus, $A_1 = 0$ does not seem too unreasonable. What would happen if we were to impose this to determine $\alpha_1 - \alpha_2$ instead of relying on (3.2) and (3.3)? Then we would have⁽¹¹⁾ from the π^2 and ϵ^2 mass formulas of the general sigma model and using the value for ϵ given in (2.3)

$$(\alpha_1 - \alpha_2) = -2\alpha \left(\frac{\pi}{\epsilon} \right)^2 = -.0215\pi \quad (5.9)$$

This is 1.44 times larger than the value given in (3.3), precisely large enough to bring (4.1) into numerical agreement with experiment [Remember that (4.1) is proportional to $(\alpha_1 - \alpha_2)$]. Perhaps this is more than a coincidence.

Another question⁽¹⁵⁾ arises in connection with $\eta \rightarrow 3\pi$ in the presence of a unified weak-electromagnetic gauge scheme. When we calculated $\eta \rightarrow 3\pi$ above, we neglected the contribution from the emission and absorption of a virtual photon. Some justification for doing this is provided by Sutherland's theorem⁽¹⁶⁾ which states that in the current-algebra approximation this contribution

vanishes. However, in a unified weak-electromagnetic scheme one should treat the contribution from virtual intermediate vector meson exchange on the same footing as the virtual photon contribution. Even though these effects are expected to be small because of the large masses of the intermediate bosons it is perhaps of some formal interest to see if Sutherland's theorem holds for these contributions too.

For definiteness we consider the question in the four quark model. The interaction of the gauge fields with the hadronic currents is then given by:

$$\begin{aligned}
 \mathcal{L} = & e a_\mu [K(V_{1\mu}^1 + V_{4\mu}^4) + (K-1)(V_{2\mu}^2 + V_{3\mu}^3)] \\
 & + \frac{e}{2} Z_\mu \left\{ \left(\frac{1}{\sin 2\phi} + 2K \tan \phi \right) (V_{1\mu}^1 + V_{4\mu}^4) - \frac{1}{\sin 2\phi} (A_{1\mu}^1 + A_{4\mu}^4) \right. \\
 & + \left[\left(2K - \frac{3}{2} \right) \tan \phi + \frac{1}{2} \cot \phi \right] (V_{2\mu}^2 + V_{3\mu}^3) + \frac{1}{\sin 2\phi} (A_{2\mu}^2 + A_{3\mu}^3) \Big\} \\
 & - \frac{e}{2\sqrt{2}\sin\phi} \{ W_\mu^+ [\cos\theta (V_2^1 + A_2^1) + \sin\theta (V_{3\mu}^1 + A_{3\mu}^1) \\
 & - \sin\theta (V_{2\mu}^4 + A_{2\mu}^4) + \cos\theta (V_{3\mu}^4 + A_{3\mu}^4)] + \text{h.c.} \} \quad (5.10)
 \end{aligned}$$

where K is the charge of q_1 , a_μ is the photon field, Z_μ is a heavy neutral intermediate boson field, W_μ^+ is the charged intermediate boson, and ϕ is a mixing angle. The vector and axial vector currents are

$$V_{a\mu}^b = i \bar{q}_b \gamma_\mu q_a, \quad A_{a\mu}^b = i \bar{q}_b \gamma_\mu \gamma_5 q_a. \quad (5.11)$$

The proof of Sutherland's theorem rests on the equal time commutation relations between the axial charges

$$B_a^b(t) = i \int d^3x A_{a4}^b(x,t) \quad (5.12)$$

and the currents. These commutation relations are

$$\begin{aligned} [B_b^a, A_{d\mu}^c] &= \delta_d^a V_{b\mu}^c - \delta_b^c V_{d\mu}^a \\ [B_b^a, V_{d\mu}^c] &= \delta_d^a A_{b\mu}^c - \delta_b^c A_{d\mu}^a \end{aligned} \quad (5.13)$$

Now the matrix element $\langle \pi^+ \pi^- \pi^0 | H_{\text{eff}} | \eta \rangle$ is related by the current algebra technique to $\langle \pi^+ \pi^- | [(B_1^1 - B_2^2), H_{\text{eff}}] | \eta \rangle$ in the limit when the π^0 4-momentum goes to zero and to $\langle \pi^- \pi^0 | [B_1^2, H_{\text{eff}}] | \eta \rangle$ in the limit when the π^+ 4-momentum goes to zero. If the original matrix element is then expanded to linear order in the squared momenta of the particles one can see⁽¹⁷⁾ that it will vanish identically (taking into account momentum conservation and charge conjugation invariance) provided that

$$\text{i) } [(B_1^1 - B_2^2), H_{\text{eff}}] = 0 \text{ or an isovector,} \quad (5.14)$$

$$\text{ii) } [B_1^2, H_{\text{eff}}] = 0 \text{ or an isovector}$$

The reason an isovector is sufficient on the right hand sides of (5.14) is that $\langle \pi\pi | (\text{Isovector}) | \eta \rangle = 0$ by isospin covariance *plus* Bose statistics. Thus to check Sutherland's theorem it is sufficient to see if (5.14) is satisfied by the effective current-current products resulting from (5.9). For the photon exchange contribution (5.14) holds as in the original proof. For the Z

exchange contribution the effective current-current product looks like (suppressing inessentials)

$$H_{\text{eff}}(Z) \sim (V_1^1 - V_2^2) [r_1 (V_1^1 + V_2^2 + V_3^3) + r_2 (V_1^1 + V_2^2 - 2V_3^3) + r_3 V_4^4] \\ + (A_1^1 - A_2^2) [r'_1 (A_1^1 + A_2^2 + A_3^3) + r'_2 (A_1^1 + A_2^2 - 2A_3^3) + r'_3 A_4^4] \quad (5.15)$$

where the r_i and the r'_i are some constants. Note that we have extracted the parity conserving, $G = -1$ part of the current-current product. Using (5.13) we readily verify

$$[(B_1^1 - B_2^2), H_{\text{eff}}(Z)] = 0 \quad (5.16)$$

$$[B_1^2, H_{\text{eff}}(Z)] = \text{isovector}$$

so that the Z-exchange contribution vanishes in the current-algebra limit. Finally, the $\Delta S = 0$, parity conserving, charm conserving, and $G = -1$ part of the W-exchange current-current product looks like

$$H_{\text{eff}}(W) \sim (V_1^3 V_3^1 - V_2^3 V_3^2) + (A_1^3 A_3^1 - A_2^3 A_3^2) \\ + (V_2^4 V_4^2 - V_1^4 V_4^1) + (A_2^4 A_4^2 - A_1^4 A_4^1) \quad (5.17)$$

Using (5.13) we see that

$$[(B_1^1 - B_2^2), H_{\text{eff}}(W)] = 0 \quad (5.18)$$

and also

$$[B_1^2, H_{\text{eff}}(W)] \sim -(V_1^3 A_3^2 + A_1^3 V_3^2) + (V_1^4 A_4^2 + A_1^4 V_4^2) \quad (5.19)$$

Each term on the right hand side of (5.19) is evidently the product of two isospinors and each product bears a net electric charge. Thus it is an isovector and a generalized Sutherland's theorem holds.

ACKNOWLEDGEMENTS

We would like to thank Professor Y. Ueda for helpful discussions on the subject matter of this paper.

FOOTNOTES AND REFERENCES

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TABLE AND FIGURE CAPTIONS

Table I: Values for the σ -dependent parameters of the renormalizable sigma model for various choices of σ . [σ in MeV; V_{11}, χ_1, χ_3 unitless; θ_s in degrees; $\sigma'^2, f_{\sigma\pi\eta}, f_{\sigma'\pi\eta}$ in $\pi=1$ units].

Figure 1: A graph of the parameter $\bar{X}$ defined in equation (4.1). The experimental band given by equation (4.2) and the current algebra result given in reference 2 are also shown for comparison.

TABLE I

σ	V_{11}	σ'^2	θ_s°	χ_1	χ_3	$f_{\sigma\pi\eta}$	$f_{\sigma'\pi\eta}$
600 MeV	5.50	$66.42\pi^2$	-57.54	-0.0194	0.0211	-0.0434π	0.292π
650	7.16	71.32	-52.57	-0.0226	0.0212	-0.0327	0.241
700	9.49	79.31	-46.36	-0.0267	0.0206	-0.0123	0.189
750	13.29	94.43	-38.63	-0.0318	0.0189	0.0271	0.135
800	21.89	133.4	-29.26	-0.0388	0.0158	0.108	0.0812
825	33.69	190.4	-23.98	-0.0434	0.0136	0.182	0.0553
845	64.72	344.4	-19.52	-0.0484	0.0116	0.276	0.0357
850	86.21	451.9	-18.38	-0.0499	0.0110	0.307	0.0310
855	131.6	679.4	-17.23	-0.0516	0.0105	0.343	0.0263
860	291.4	1482.	-16.07	-0.0536	0.0099	0.384	0.0218
863	1186.	5978.	-15.37	-0.0548	0.0096	0.412	0.0191

FIG. 1