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DEFORMATION ANALYSIS OF A CIRCULAR COUPLED-PLATE STRUCTURE

by

H. Halle

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H. Halle

Reactor Engineering Division

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<u>Symbol</u>	<u>Definition</u>
a	Outside radius of plates, in.
A_g	Cross-sectional area of individual grid member, in. ²
A_s	Plate area associated with individual grid member, in. ²
c	Distance to neutral axis of bending, in.
C	Bending-moment intensity coefficient for beam (refer to Eq. 41), lb
C_1, C_2	Constants of integration
D_p	Flexural rigidity of individual plate, $D_p = E_p h^3 / 12(1 - \nu^2)$, in.-lb
D_t	Flexural rigidity of two parallel plates about their common neutral axis (refer to Eq. 1), in.-lb
E_g	Modulus of elasticity of grid-member material, lb/in. ²
E_p	Modulus of elasticity of plate or beam-flange material, lb/in. ²
f	Constant indicating degree of edge restraint
F	Constant (refer to Eq. 9)
F_g	End-reaction force of grid member, lb
G_g	Shear modulus of grid member, lb/in. ²
h	Plate or beam-flange thickness, in.
I_g	Moment of inertia of grid member, in. ⁴
I_p	Moment of inertia of individual beam flange, in. ⁴
I_t	Moment of inertia of two parallel beam flanges about their common neutral axis, in. ⁴
I_0, I_1, K_0, K_1	Modified Bessel functions of first and second kinds of orders zero and one
k	Constant, $k^2 = (K_a + K_b) / (D_{pa} + D_{pb})$, in. ⁻¹
K	Bending-moment intensity coefficient for circular plate, lb/in.
l	Distance between plates or flanges, in.
L	Span length of beam, in.
M_g	End moment of grid member, in.-lb

NOMENCLATURE

<u>Symbol</u>	<u>Definition</u>
N	Constant, $N = K_a + K_b$, lb/in.
P	Central, concentrated load, lb
q	Normal pressure load on plate structure, lb/in. ²
q'	Uniformly distributed load on beam, lb/in.
Q	Shear force per unit length on radial boundaries of plate element, lb/in.
r	Radial coordinate, in.
s	Distance between adjacent grid members, in.
S	Constant, $S = C_a + C_b$, lb
u	Constant (refer to Eq. 30)
w	Plate deflection, in.
x	Distance coordinate, in.
y	Beam deflection, in.
α	Ratio of maximum to average shear stress
β	Ratio M_{gb} / M_{ga} (refer to Eq. 45)
γ	Euler's constant = 0.5772 ...
ϵ	Strain
ν	Poisson's ratio of plate material
σ	Stress, lb/in. ²

Subscripts

Pertaining to
(unless otherwise defined above)

a, b	Top and bottom plates, respectively
e, i	Plate or flange surface external and internal, respectively, with respect to grid structure
r, t	Radial and tangential components, respectively
1, 2	Rigid-grid and flexible-grid components, respectively
2-1, etc.	Refer to text, Section IV

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ABSTRACT

A structure consisting of two circular, parallel plates, interconnected by a grid work of equally spaced members, subjected to normal axisymmetric loading, is analyzed for its overall deformation behavior. Approximate equations presented for the evaluation of deflections and stresses are shown to be sufficiently accurate for engineering estimates and permit examination of the relative influence of the various plate and grid parameters on the deformation behavior of the structure. Model tests provided verification of the analysis.

The analytical method is essentially based on the superposing of two deformation components which account for (1) the bending of the plates, when the connecting grid members are considered to be ideally rigid, and (2) the additional effects resulting from the actual deformation of these members. These two components represent bending of the plates (1) about a common intermediate, neutral axis, and (2) about their individual internal neutral axes, respectively. The analytical and experimental study was also extended to coupled-beam structures.

I. INTRODUCTION AND STATEMENT OF PROBLEM

This report develops simple design equations for the approximate analysis of the overall, axially symmetric deformation of a circular, coupled-plate structure. The structure analyzed consists of two flat, parallel plates, interconnected by a grid work of equally spaced structural members as shown on Fig. 1. Its deformation performance is of particular pertinence when the structure is used to support the core of a nuclear reactor, since associated fuel configuration changes may influence reactor performance. Arrangements of this type, called at times coupled tube sheets, grid-support structures, etc., have been previously analyzed.¹⁻³ This paper presents deflection and stress equations which are suitable to provide an

engineering-design estimate with slide-rule calculations. In addition, the closed-form solutions of the equations permit a general examination of the influence of the various plate and grid parameters on the structural behavior, an important feature for optimization of a design.

Fig. 1. Circular Coupled-plate Structure

The suggested analytical method is also applied to determine the deformation of coupled-beam structures, extending an existing analysis of Vierendeel girders.⁴

Experiments were performed with both beam and circular plate model structures to verify the analysis.

II. ANALYSIS

A. Basic Assumptions

The analysis is subjected to the following assumptions:

1. Loads and edge restraints are axisymmetric.
2. The deformation of the plates is based on conditions customarily taken for elastic, thin-plate theory.
3. Both plates have the same outside diameter and edge restraint.
4. The normal deflection curves of both plates are identical, determining the overall deformation of the structure. This implies that the grid members do not deform in the axial direction.
5. The elastic grid members are of uniform material and size and have a symmetrical cross section. They are spaced in a uniform pattern, which covers the entire plate surface.

6. The effects of the reaction forces imposed upon the plates by the grid members at the mounting joints are considered negligible. This implies that the bending reactions exerted by the grid members result in the application of pure couples at the plate joints.

7. The overall restraining effect of the grid structure is represented by a continuous moment-distribution function. This implies that the distance between adjacent grid members is small compared to the plate radius.

8. The two-component deformation principle discussed in Section B below is valid and applicable.

B. Formulation of Governing Differential Equations

The principal concept of this approximate deformation analysis is the assumption that upon application of a normal, axisymmetrical load the structure deforms as a result of two separate phenomena. These two superposed components account (1) for the deformation of the plates when the connecting grid members are considered to be ideally rigid, and (2) for the additional effects resulting from the deformation of the grid structure. This type of approach is commonly used for the analysis of thick beams, plates, and sandwich cores, where the additional deformation term represents the correction resulting from the shear deformation.

For the problem at hand, these components will be designated the "rigid-grid" and "flexible-grid" deformation components, identified by the subscripts 1 and 2, respectively. The rigid-grid deformation represents bending of the plates about a common neutral axis somewhere between them, and the flexible-grid deformation accounts for bending of the plates about their individual internal neutral axes.

The determination of the rigid-grid deformation component is straightforward. It requires the computation of the location of the common neutral axis and the total flexural rigidity (D_t) of the two plates of the structure. These computations are based on the condition that the strains are proportional to their distance from this common neutral axis. The connecting grid members are considered to be rigid to provide the above condition, but do not contribute in any other manner to the flexural rigidity. The commonly used plate equations determine deflections (w_1) and bending moments as a function of radius.

For two identical plates, the total flexural rigidity is

$$D_t = \frac{E_p}{12(1 - \nu^2)} [(\ell + 2h)^3 - \ell^3] = \frac{E_p h(\ell + h)^2}{2(1 - \nu^2)} + 2D_p. \quad (1)$$

The flexible-grid component, representing additional bending of the plates about their individual neutral axes, will now be considered. As this deformation proceeds, the rigidly mounted ends of the grid members are rotated through an angle corresponding to the slope of the additional plate deflection. The strained grid members resist deformation by exerting bending moments, proportional to the slope, at each mounting joint (Fig. 2). The method of analysis developed here for circular plates was analogous to one presented by M. Hetényi⁴ for Vierendeel girders.

Fig. 2. Deformed Grid Member

Let the magnitude of the bending moment applied to each plate at the joints be denoted by M_g , and let the plate area associated with each individual grid member be A_g . For example, a triangular array of pitch s

will have $A_s = 0.866s^2$. The bending-moment intensity in terms of moment applied per unit area may now be expressed as

$$\frac{M_g}{A_s} = K \frac{dw_2}{dr}, \quad (2)$$

where the constant K represents the parameters of the grid structure. Section IV will consider the determination of K values to be used in the calculations. At this point it suffices to state that $K \approx 6E_g I_g / A_s \ell$. When this moment application is considered in conjunction with the other forces acting on a plate element (e.g., Ref. 5, p. 52, Fig. 28), the analysis of the equilibrium conditions results in the differential equation

$$\frac{d^3 w_{2a}}{dr^3} + \frac{1}{r} \frac{d^2 w_{2a}}{dr^2} - \left(\frac{1}{r^2} + \frac{K_a}{D_{pa}} \right) \frac{dw_{2a}}{dr} = \frac{Q_a}{D_{pa}}, \quad (3)$$

where the subscript a identifies the top plate under consideration here. When written with subscript b , Eq. 3 represents the deformation of the bottom plate. Following Hetényi's derivation, this approximation will make no distinction between the deflections of the two plates. Setting $w_{2a} = w_{2b} = w_2$, we obtain the following basic differential equation:

$$\frac{d^3 w_2}{dr^3} + \frac{1}{r} \frac{d^2 w_2}{dr^2} - \left(\frac{1}{r^2} + \frac{K_a + K_b}{D_{pa} + D_{pb}} \right) \frac{dw_2}{dr} = \frac{Q_a + Q_b}{D_{pa} + D_{pb}} \quad (4)$$

Analogous to Hetényi's findings for the Vierendeel girder, Eq. 4 is identical with the deformation equation of a single, solid plate having a flexural rigidity $D_{pa} + D_{pb}$, subjected to transverse, axisymmetric loading represented by $Q = Q_a + Q_b$ and, simultaneously, to a uniform tension $N = K_a + K_b$ applied radially around the periphery in the middle plane of the plate (Ref. 5, p. 391). Defining a constant $k^2 = (K_a + K_b) / (D_{pa} + D_{pb})$, we rewrite Eq. 4 as

$$\frac{d^3 w_2}{dr^3} + \frac{1}{r} \frac{d^2 w_2}{dr^2} - \left(\frac{1}{r^2} + k^2 \right) \frac{dw_2}{dr} = \frac{Q}{N} \quad (5)$$

C. Deformation Resulting from a Uniform Pressure Load

For a uniform-pressure load q , the shear force $Q = qr/2$. The rigid-grid deflection, as obtained from Eq. 1 of Ref. 1 is represented by

$$w_1 = \frac{q(a^2 - r^2)}{64D_t} \left(\frac{5 + \nu - 4f}{1 + \nu} a^2 - r^2 \right), \quad (6)$$

where f represents the edge restraint defined as the ratio of radial moment at the plate edge to the corresponding radial-edge moment of a clamped plate having $f = 1$.

Determination of the flexible-grid deformation requires the general solution of Eq. 5 given by

$$\frac{dw_2}{dr} = C_1 I_1(kr) - \frac{qr}{2N}, \quad (7)$$

which may be expressed as

$$\frac{dw_2}{dr} = \frac{qa}{2N} \left[F \frac{I_1(kr)}{I_1(ka)} - \frac{r}{a} \right]. \quad (8)$$

The edge restraint of the structure determines the constant

$$F = \frac{fka I_0(ka) + (1 + \nu - 2f) I_1(ka)}{ka I_0(ka) - (1 - \nu) I_1(ka)}. \quad (9)$$

Integration of Eq. 8 yields the flexible-grid deflection,

$$w_2 = \frac{qa^2}{4N} \left[\frac{a^2 - r^2}{a^2} + F \frac{2[I_0(kr) - I_0(ka)]}{ka I_1(ka)} \right]. \quad (10)$$

Finally, the total deflection of the structure $w = w_1 + w_2$ is given by the sum of Eqs. 6 and 10.

At the center of a clamped plate, the maximum deflection is given by

$$w_{\max} = \frac{qa^4}{64D_t} + \frac{qa^2}{4N} \left\{ 1 - \frac{2[I_0(ka) - 1]}{ka I_1(ka)} \right\}. \quad (11)$$

For a plate structure with simply supported edges ($f = 0$), the central deflection is

$$w_{\max} = \frac{qa^4}{64D_t} \left(\frac{5 + \nu}{1 + \nu} \right) + \frac{qa^2}{4N} \left\{ 1 - \frac{2(1 + \nu)[I_0(ka) - 1]}{ka[ka I_0(ka) - (1 - \nu) I_1(ka)]} \right\}. \quad (12)$$

One factor required for the stress analysis is

$$\frac{d^2 w_2}{dr^2} = \frac{q}{2N} \left[F \frac{ka I_0(kr) - (a/r) I_1(kr)}{I_1(ka)} - 1 \right]. \quad (13)$$

At the center of the plate, this reduces to

$$\left(\frac{d^2 w_2}{dr^2} \right)_{r=0} = \left(\frac{1}{r} \frac{dw_2}{dr} \right)_{r=0} = \frac{q}{2N} \left[F \frac{ka}{2I_1(ka)} - 1 \right] \quad (14)$$

D. Deformation Resulting from a Central Concentrated Load

For a central concentrated load P , the shear force $Q = P/2\pi r$, and the differential equation for the rigid-grid deformation is

$$\frac{d^3 w_1}{dr^3} + \frac{1}{r} \frac{d^2 w_1}{dr^2} - \frac{1}{r^2} \frac{dw_1}{dr} = \frac{P}{2\pi r D_t} \quad (15)$$

The solution of Eq. 15 for the rigid-grid deflection is

$$w_1 = \frac{P}{16\pi D_t} \left\{ 2r^2 \ln(r/a) + (a^2 - r^2) \left(\frac{3 + \nu - 2f}{1 + \nu} \right) \right\} \quad (16)$$

The general solution of Eq. 5 for the flexible-grid deformation component is

$$\frac{dw_2}{dr} = C_1 I_1(kr) + C_2 K_1(kr) - \frac{P}{2\pi r N} \quad (17)$$

The boundary condition $dw_2/dr = 0$ at $r = 0$ determines the constant of integration C_2 . As r approaches zero, $I_1(kr)$ vanishes and $K_1(kr)$ approaches the value $1/k$; consequently, $C_2 = Pk/2\pi N$.

The constant C_1 is dependent on the edge restraint of the plates of the structure. The general expression is

$$C_1 = \frac{P}{2\pi a N I_1(ka)} \left\{ \frac{[fka I_0(ka) - (1 - \nu) I_1(ka)][1 - ka K_1(ka)] + (1 - f)(ka)^2 I_1(ka) K_0(ka)}{ka I_0(ka) - (1 - \nu) I_1(ka)} \right\} \quad (18)$$

For clamped edges,

$$C_1 = \frac{P}{2\pi a N I_1(ka)} [1 - ka K_1(ka)], \quad (19)$$

and for simply supported edges,

$$C_1 = \frac{P}{2\pi a N} \left\{ \frac{ka[ka K_0(ka) + (1 - \nu) K_1(ka)] - (1 - \nu)}{ka I_0(ka) - (1 - \nu) I_1(ka)} \right\} \quad (20)$$

Integrating Eq. 17 results in

$$w_2 = -\frac{P \ln(r/a)}{2\pi N} + \frac{C_1}{k} [I_0(kr) - I_0(ka)] - \frac{P}{2\pi N} [K_0(kr) - K_0(ka)]. \quad (21)$$

The total deflection is then given by $w = w_1 + w_2$, the sum of Eqs. 16 and 21. When r approaches zero, the function $K_0(kr)$ approaches the value

$$-\ln(ka/2) - \ln(r/a) - \gamma.$$

Thus, at $r = 0$, the maximum deflection is

$$w_{\max} = \frac{Pa^2(3+\nu-2f)}{16\pi D_t(1+\nu)} + \frac{C_1}{k} [1 - I_0(ka)] + \frac{P}{2\pi N} [K_0(ka) + \ln(ka/2) + \gamma]. \quad (22)$$

Stress analysis requires the evaluation of

$$\frac{d^2 w_2}{dr^2} = C_1[k I_0(kr) - (1/r) I_1(kr)] - \frac{Pk}{2\pi N} [k K_0(kr) + (1/r) K_1(kr)] + \frac{P}{2\pi r^2 N}. \quad (23)$$

Since the pertinent equations approach infinity when r approaches zero, these are not suitable for computing stresses at or near the plate center as discussed in Ref. 5, p. 69.

E. Additional Comments

Up to this point in the analysis, no restriction has been placed on the material or thickness properties of the two flat plates of the structure. For clarity, brevity, and also expediency in evaluating the experimental work, the remainder of the report will generally be limited to the special case of a structure having identical top and bottom plates ($K_a = K_b = K = N/2$). The general condition of beam flanges having different thicknesses is treated in detail by M. Hetényi.⁴ This method is applied in Section IV to the problem at hand. A cursory investigation based on a comparison of the coupled-beam and circular-plate analyses permits replacement of the quantity

$$\frac{E_p I_p}{s} \text{ by } D_p. \quad (24)$$

Investigation of Eq. 11 permits some interesting observations. As the value of K , representing the strength of the grid members, increases, the $w_{2\max}$ term decreases. On the other hand, when K vanishes, the expected value of $w_{2\max} = qa^4/128 D_p$.

Note that ℓ , the distance between plates, affects both the w_1 and w_2 terms. At the extreme values of $\ell = 0$ and $\ell = \infty$, the magnitudes of the deflections $w_{\max}$ approach the ratio 1 to 4, representing, respectively, the deflection of a plate of double thickness and two stacked plates of single thicknesses each (Fig. 3). A minimum total deflection may occur for an intermediate value of ℓ . In case of very long grid members, their axial deformation ($< qA_s \ell / A_g E_g$) may not be negligible. Comparison with w_2 will establish a limiting criterion.

Fig. 3. Qualitative Relationship of Central Deflection vs. Grid-member Length

This deflection, $qa^2/4N$, is multiplied by a term, which approaches unity, for larger values of the ka parameter, as indicated in Fig. 4. Thus, in this region, the flexible-grid deflection of the structure is that of a membrane, having the following essential features: (1) The deflection increases with the square of the radius, rather than with the fourth power as for plates; and (2) the deflection depends on the parameter N , which is principally a measure of the grid member strength and is only slightly influenced by the strength of the plates. With increasing ka parameters, the effect of the type of edge restraint becomes less significant as the membrane condition is approached. The corresponding conclusions applied to the Vierendeel girder have been discussed by M. Hetényi,⁴ who mentions additional features that are likely to be pertinent for the design and optimization of a circular, coupled-plate structure also.

F. Strain and Stress Analysis

To compute strains and stresses resulting from the overall deformation, it is expedient to determine the radial and tangential strains from the relationships

$$\epsilon_r = +c \frac{d^2 w}{dr^2},$$

and

$$\epsilon_t = +\frac{c}{r} \frac{dw}{dr},$$

(25)

Fig. 4: Pertinent Functions of Parameter ka

where c represents the distance to the neutral axis appropriate to the type of bending considered. A positive ϵ denotes tension in the top surface of the plate facing the load. For example, for a structure having identical plates and a common neutral axis midway between them, the strain components at the top external plate surface of the structure are

$$\epsilon_{re} = \left(\frac{\ell}{2} + h \right) \frac{d^2 w_1}{dr^2} + \frac{h}{2} \frac{d^2 w_2}{dr^2},$$

and

$$\epsilon_{te} = \left(\frac{\ell}{2} + h \right) \frac{1}{r} \frac{dw_1}{dr} + \frac{h}{2r} \frac{dw_2}{dr}.$$

(26)

At the internal surface of the top plate, the strain components are given by

$$\epsilon_{ri} = \frac{\ell}{2} \frac{d^2 w_1}{dr^2} - \frac{h}{2} \frac{d^2 w_2}{dr^2},$$

and

$$\epsilon_{ti} = \frac{\ell}{2r} \frac{dw_1}{dr} - \frac{h}{2r} \frac{dw_2}{dr}.$$

(27)

The stresses are determined by the relationships

$$\sigma_r = \frac{E_p}{1 - \nu^2} (\epsilon_r + \nu \epsilon_t),$$

and

$$\sigma_t = \frac{E_p}{1 - \nu^2} (\epsilon_t + \nu \epsilon_r).$$

(28)

Not included in the above relationships are stresses resulting from localized load and support applications. These will have to be considered separately.

For this approximate analysis, each grid member is considered to be a beam of length $\ell + h$ subjected to end moments M_g and plate reactions $F_g = 2M_g/(\ell + h)$ in the case of equal plate thicknesses (Fig. 2). The magnitude of these end moments is given by Eq. 2. The maximum bending moment is applied to those members located where the plate experiences the maximum slope. At this radial position,

$$\frac{d^2 w_2}{dr^2} = 0.$$

For every individual member, the bending moment varies linearly with the distance from the neutral axis midway between plates and determines the bending stresses at a particular location on the member. The end reactions subject the member to a constant shear F_g , and the resulting shear stresses can be of considerable magnitude, particularly if the distance between plates is small. In addition, the member may be subjected to axial loads that transfer part of the load from one plate to the other. No attempt has been made to compute these loads in this study.

III. ANALYSIS OF COUPLED-BEAM STRUCTURE

The deflection equations for coupled-beam structures are based on the same two-component concept applied to the circular-coupled plate structures. The rigid-grid deformation is based on ordinary beam theory, using a stiffness value of $E_p I_t$ that is analogous to the plate flexural rigidity D_t . The flexible-grid component is exactly the Vierendeel girder deformation developed by Hetényi.⁴ As an example, take a simply supported beam of span length L , subjected to a uniformly distributed load q' ; the deflection is given by $y = y_1 + y_2$, where

$$y_1 = \frac{q'x}{24E_p I_t} (L^3 + x^3 - 2Lx^2), \quad (29)$$

and⁴

$$y_2 = \frac{q'}{2S} x(L-x) + \frac{q'L^2}{4Su^2} \left\{ \frac{\cosh u(1-2x/L)}{\cosh u} - 1 \right\}, \quad (30)$$

where

$$u = \frac{L}{2} \left[\frac{C_a + C_b}{E_{pa} I_{pa} + E_{pb} I_{pb}} \right]^{1/2}$$

and $S = C_a + C_b$ is the fictitious axial tensile force applied to the beam structure.

For a beam having identical flanges, the strains on the external and internal flange surfaces, respectively, are

$$\epsilon_e = \left(\frac{\ell}{2} + h \right) \frac{d^2 y_1}{dx^2} + \frac{h}{2} \frac{d^2 y_2}{dx^2},$$

and

$$\epsilon_i = \frac{\ell}{2} \frac{d^2 y_1}{dx^2} - \frac{h}{2} \frac{d^2 y_2}{dx^2}$$

(31)

The end moments imposed on the grid members are given by

$$M_g = Cs \frac{dy_2}{dx}. \quad (32)$$

The discussion presented for the stress analysis of the circular structure is also applicable to the beam.

IV. DETERMINATION OF BENDING-MOMENT INTENSITY COEFFICIENTS

The coefficients K and C relate the plate slope to the restraining grid moments in Eqs. 2 and 32. The flexible-grid slope dw_2/dr (or dy_2/dx) can be considered to be the sum of as many as four components (Fig. 5) which result from

1. bending of the grid member dw_{2-1}/dr ,
2. local bending of the plate dw_{2-2}/dr ,
3. shear deflection of grid members dw_{2-3}/dr ,
4. the additional deflection due to the elasticity of the support as described in Ref. 6, but not explicitly considered here.

The value obtained for dw_{2-1}/dr depends on the assumed boundary and loading conditions at the mounting joints. Only identical plates are considered here. Using the conditions shown on Fig. 2, and assuming perfectly rigid mounting in the plates, one obtains

$$\frac{dw_{2-1}}{dr} = \frac{M_g \ell^3}{6E_g I_g (\ell + h)^2}. \quad (33)$$

Locating the forces F_g at the internal plate surfaces, with rigid mounting, Ref. 1 derives

$$\frac{dw_{2-1}}{dr} = \frac{M_g \ell^2}{6E_g I_g (\ell + h)}. \quad (34)$$

Considering nil effective rigidity within the plate mounts results in

$$\frac{dw_{2-1}}{dr} = \frac{M_g (\ell + h)}{6E_g I_g}. \quad (35)$$

Fig. 5. Plate-slope Components

This writer does not presume to have found the most representative relationship. The geometric mean between Eqs. 34 and 35 was selected to correlate the work reported herein; thus,

$$\frac{dw_{2-1}}{dr} = \frac{M_g \ell}{6E_g I_g} \quad (36)$$

The slope component for local bending of the plate was developed in Ref. 4, where it is implicitly presented as

$$\frac{dy_{2-2}}{dx} = \frac{M_g s}{12E_p I_p} \quad (37)$$

For the circular plate, the substitution of relationship 24 in Eq. 37 yields

$$\frac{dw_{2-2}}{dr} = \frac{M_g}{12D_p} \quad (38)$$

The deflection resulting from the shear of the grid member subjected to the shear force F_g is

$$\frac{dw_{2-3}}{dr} = \frac{\alpha F_g}{A_g G_g} = \frac{2\alpha M_g}{A_g G_g (\ell + h)} \quad (39)$$

Since

$$\frac{dw_2}{dr} = \frac{dw_{2-1}}{dr} + \frac{dw_{2-2}}{dr} + \frac{dw_{2-3}}{dr}, \quad (40)$$

one obtains, by substituting Eq. 40 in Eq. 2,

$$K = \frac{6E_g I_g}{A_s \left(\frac{E_g I_g}{2D_p} + \ell + \frac{12\alpha E_g I_g}{A_g G_g (\ell + h)} \right)} \quad (41)$$

Similarly, using Eq. 32, one obtains

$$C = \frac{6E_g I_g}{s \left(\frac{s E_g I_g}{2E_p I_p} + \ell + \frac{12\alpha E_g I_g}{A_g G_g (\ell + h)} \right)} \quad (42)$$

The above derivation contains some doubtful factors, which influence the flexible-grid deformation component when the grid length is short. In that region, however, the relative importance of the flexible-grid effect diminishes as the rigid-grid effects become dominant. Thus, fortunately, the uncertainty is less likely to influence the overall computational results significantly.

At this point we will return to the consideration of the structure having plates of different sizes and materials. It is assumed that an analysis presented in Ref. 4 for the beam can be extended to the case of the circular plate also. Applying relationship 24, in conjunction with Eq. 41

and work presented in Ref. 4, results in the following unverified expressions for the bending-moment intensity coefficients of the top and bottom plates, respectively:

$$K_a = \frac{6E_g I_g}{A_s \left[\frac{E_g I_g}{2D_{pa}} + \ell(2 - \beta) + \frac{12\alpha E_g I_g(1 + \beta)}{A_g G_g(2\ell + h_a + h_b)} \right]}, \quad (43)$$

and

$$K_b = \frac{6E_g I_g}{A_s \left[\frac{E_g I_g}{2D_{pb}} + \frac{\ell(2\beta - 1)}{\beta} + \frac{12\alpha E_g I_g(1 + \beta)}{A_g G_g(2\ell + h_a + h_b) \beta} \right]}, \quad (44)$$

where

$$\beta = \frac{M_{gb}}{M_{ga}} = \frac{D_{pb}(E_g I_g + 6\ell D_{pa})}{D_{pa}(E_g I_g + 6\ell D_{pb})}. \quad (45)$$

V. EXPERIMENTAL WORK

A. Beam Tests

Since a beam structure involves a much simpler experimental technique than a plate, coupled-beam models were tested initially (Fig. 6). The

Fig. 6
Coupled-beam Model

beams were constructed of two identical 0.245-in.-thick, 2-in.-wide, acrylic plastic flanges, interconnected by a grid work of 0.26-in.-diam rounds. Nine different configurations having different grid-member lengths, grid pitches, and beam span lengths were tested. Three load conditions were applied to each simply supported configuration: (a) a uniformly distributed load, (b) a concentrated midspan load, and (c) two concentrated loads applied at quarter-point span-length locations. The following data were taken for all 27 tests:

1. Midspan (maximum) deflection.
2. Longitudinal strain in flanges at midspan, measured externally and internally with respect to the grid structure.
3. Strain measurement to determine bending of grid members.

The theoretical data were computed by the methods presented above. Comparison of the midspan deflection data indicated agreement of theoretical and experimental values within 10%. As far as the longitudinal flange strain was concerned, agreement was usually within 20%; however, the experimental data clearly reflected the significant effects of loading conditions and configuration parameters predicted by theory.

Representative analytical and experimental data are shown in Figs. 7 and 8 for a uniform load. To reduce local loading effects for the concentrated and uniform load tests, the strain-gage location was protected with a metal block ("zeroed in" before any experiment) as shown in Fig. 6. The quarter-point loading condition, involving no load application near the strain gages, provided the best correlation for the midspan strain measurements. The measurements of bending strain made on one of the grid members agreed well with theory.

B. Circular-plate Structure Tests

The subsequent test phase involved two circular-plate models. Constructed of acrylic plastic, each model had a pair of 22-in.-diam plates, 0.17 in. thick, interconnected by 0.26-in.-diam grid members (Figs. 9 and 10). The lengths of these members were 0.375 and 2 in., respectively.

Initially, these structures were fitted with 109 grid members on a 2-in., equilateral, triangular pitch. Additional test configurations were provided by removing selected members to obtain a 31-member grid having a 4-in. pitch. The models were supported on a 21-in.-diam ring, determining the plate radius $a = 10.5$ in. Two loading conditions were used, a central concentrated load and a uniformly distributed pressure load, the

Fig. 7. Coupled-beam Strain vs. Grid-member Length for Uniform Load ($q' = 0.2$ lb/in.)

Fig. 8. Midspan Beam Deflection vs. Grid-member Length for Uniform Load ($q' = 0.2 \text{ lb/in.}$)

Fig. 9. Coupled-plate Model

Fig. 10. Plate-model Construction

latter being applied with a vacuum underneath the structure. The following data were taken for the four configurations tested:

1. Central (maximum) deflection.
2. Radial and tangential strains at a 1-in. radial distance from the plate center, measured externally and internally with respect to the grid structure.
3. Strain measurement to determine bending of a grid member mounted 8 in. from the plate center.

Comparison of the central deflection data (Table I) indicated the experimental deflections to be greater than predicted by theory. For the central

concentrated load, the data range was confined within plus 7 and 11%; for the uniform load, it was between plus 14 and 28%. It is felt that the higher experimental results are, in part, the result of increased creep deformation in the structure, particularly at the epoxy-glued grid member-plate joints.

TABLE I. Comparison of Circular Coupled-plate Data

Load* Cond.	Grid Length l , in.	Spacing s , in.	Central Deflection,** 10^{-3} in.					Bending Strain on Grid Member, [†] 10^{-6}		
			Theory			Exp., w	Ratio, Exp./Th., w	Theory, ϵ	Exp., ϵ	Ratio, Exp./Th., ϵ
			w_1	w_2	w					
Unif.	2.004	2	0.37	17.3	17.6	20.2	1.15	525	530	1.01
Unif.	2.004	4	0.37	54.1	54.4	62.0	1.14	1610	1445	0.897
Unif.	0.374	2	5.72	7.39	13.1	16.8	1.28	243	235	0.967
Unif.	0.374	4	5.72	26.7	32.5	39.0	1.20	869	1047	1.20
Conc.	2.004	2	0.027	1.86	1.89	2.04	1.08	28.0	27.0	0.965
Conc.	2.004	4	0.027	4.61	4.64	5.14	1.11	93.3	84.4	0.904
Conc.	0.374	2	0.415	0.960	1.38	1.51	1.10	12.4	11.6	0.933
Conc.	0.374	4	0.415	2.64	3.06	3.36	1.10	47.0	53.6	1.14

*Unif. = uniform pressure load, 0.1-lb/in.² basis; Conc. = central concentrated load, 1-lb basis.

** w_1 , w_2 = rigid- and flexible-grid deformation components, respectively; w = total deflection.

[†]Corrected for location of gage and strengthening effect of gage installation. Models constructed of acrylic plastic having $E_p = E_g = 440,000$ lb/in.²; $\nu = 0.35$.

The radial and tangential strains of this complex structure did not always correlate as well as had been the case with the beam. Apparently, the strain gages mounted on the biaxially stressed plates were more sensitive to local effects than was the case with the beam. The bending strains experienced by an individual grid member correlated reasonably well, generally within 15% of theoretical predictions.

VI. CONCLUDING REMARKS

It would be less than candid to omit stating that the experimental work was subjected to shortcomings. Most of these shortcomings resulted from the use of plastic rather than metal for the model material. However, the ease of fabrication and pressure-load application within the limited effort to be spent were decisive. The problems encountered included strain-gage heating effects, creep, dimensional tolerances, and uncertainties involving material properties. These problems were partially counteracted by conducting all tests under uniform procedures to particularly enhance the relative accuracy required to compare results.

Notwithstanding the above remarks, it is felt that the experiments performed with plastic models of coupled beams and circular-plate structures, both of which had various dimensional parameters, verified the analytical method. Good agreement between theoretical and experimental results of the coupled-beam tests indicate that for the purpose of design

calculations, the deformation of the coupled structure can be represented by two superposed components. Even though a less accurate correlation could be expected for the subsequently performed circular-plate deformation experiments, the results obtained here, considered in conjunction with those obtained from the beam tests, are felt to provide confidence in the presented application.

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