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LAWRENCE LIVERMORE LABORATORY
University of California, Livermore, California 94550

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**SOME OBSERVATIONS RELATED TO MODELING
EXPLOSIVE CRATERING PHENOMENA**

John W. White
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ABSTRACT

One of several applications studied in the Plowshare Program (Peaceful Uses of Nuclear Explosives) involves the formation of large craters on the surface of the earth. For nuclear cratering to become an efficient engineering tool, it is necessary to predict the results of a cratering shot. The two techniques most frequently used to predict the dimensions of a crater involve empirical formulas and computer calculations. This thesis presents improved methods for both of these approaches.

The empirical formulas have consisted of statistical correlations of crater dimensions as a function of yield (explosive size) and depth of burial. There has not been a successful attempt to provide a quantitative theoretical model for the cratering process. A quantitative theoretical model of the cratering process is proposed, and it results in a formula predicting the crater size. Experimental data are used to determine empirically the value of the terms which are obtained from the derivation of the formula.

An improved method for describing the failure surface of a material is presented. This description is incorporated in a failure model used in computer calculations for modelling rock mechanics. Other aspects of modelling rock failure are also incorporated in this model. They include stress relaxation and the manner in which failure surfaces change. Finally, an improved method for treating shocks has been developed for continuum mechanics simulation on a computer. All of these computational developments contribute to an improved capability for simulating rock mechanics phenomena which in turn will aid in improving the tools for predicting the results of explosive cratering projects.

I. OPENING REMARKS

There are a number of applications of nuclear explosive cratering which are of potential benefit to man. The digging of a sea level canal is one frequently mentioned example; there are many others such as cratering a deep water harbor, constructing a crater-lip dam, and digging a reservoir. The main attractions for nuclear cratering are the efficiency, speed, and low cost of the mechanical process. However, there are some difficulties attendant to the use of nuclear explosives. Specifically, the release of radioactivity and the generation of strong seismic waves. For this reason, it is necessary to be able to predict crater dimensions accurately as a function of explosive yield, depth of burial, and rock-mechanical properties. Only when this is achieved, will it be possible to consider large engineering projects with precision and confidence.

The two major approaches used to predict crater dimensions have been

- Empirical formulas derived from experimental observations.
- Computer solution of the equations of motion appropriate for the rock-mechanical behavior of the cratering medium.

Both of these approaches evidence need for considerable improvement and it is the purpose of this thesis to provide some of this improvement.

Previous empirical formulas had a number of shortcomings. However, the most serious of these was that none of these formulas was based on a quantitative theoretical model. Chapter II proposes a model that is used to derive a cratering formula. The free parameters in the formula are then determined from the data observed in experiments. The correlation between this semiempirical approach and experimental evidence is excellent.

Chapter III introduces a new method of describing the yield surface of a material. This method is quite general and should be capable of describing any geologic material. Chapter IV combines this yield surface description with physical models for describing the manner in which yield surfaces change and it contains models for stress relaxation appropriate for yielding materials. This combination provides a method for simulating material failure in a computer calculation. Chapter V contains a new

method for implementing the artificial viscosity technique.* This new formulation greatly reduces the difficulties encountered with previous formulations. The use of these methods (from Chapters III, IV and V) in computational rock mechanics calculations results in considerable improvement when compared to the capability of previous computational schemes.

* Artificial viscosity is the name given to a numerical method which permits the simulation of a discontinuous phenomena (shocks) with a finite difference scheme.

II. AN EMPIRICALLY DERIVED CRATERING FORMULA

INTRODUCTION

A crude model of the cratering process is proposed from which a cratering formula is derived. The key feature of the model is that the hydrodynamic effects are separated from the ballistic effects. The free parameters of the formula are determined from experimental observations. The resulting formula is quite simple and should be applicable to engineering problems.

Previous empirical formulas were obtained by arithmetical procedures. Certain mathematical forms were fitted to experimental observations by statistical means. Frequently, the form of the mathematical formulas implied that the physical processes involved in cratering were scalable. This is not correct, and this writer has published a note¹ describing the shortcomings of previous efforts in the cratering formula area. Briefly, that note states that cratering is not a scalable phenomena because the gravitational constant cannot be scaled on earth when going from a large crater to a small crater, i.e., the value of g is a constant. None of these formulas was based on a quantitative theoretical physical model. Nonetheless, it is a fact that a simple power law on yield does correlate experimental data reasonably well.

Before a model is proposed, we will recall some of the pertinent features of the so-called "scaling" curves. Figure 1 presents a rough sketch of the two types of scaling curves most frequently encountered (for hard rock and alluvial material).

Different curves are required for different materials and for different kinds of explosive energy sources (e.g., nuclear vs chemical). Every time a new cratering medium is encountered, a new "scaling curve" must be obtained experimentally. A cratering formula would be much more useful if it were applicable, in general, to the variety of materials encountered geologically.

Fig. 1. "Scaled" crater volume (apparent) vs "scaled" DOB. W is the energy of the explosive (yield).

Fig. 2. Outline of crater shape. R_L is the lip radius, R_a is the apparent radius, Y_a is the apparent depth, and Y_d is the depth of burial.

From Fig. 1 we note that hard rock is stronger and denser than the alluvial material. Despite the fact that these differences might seem considerable, the volumes ejected at comparable conditions are not greatly different. Thus, we note that the cratering results are a weak function of material properties of the cratering medium. This fact suggests that a model might be proposed in which various material properties provide slight perturbations about some reference system. Such an approach may, in fact, be simple and direct.

Before proceeding with a discussion of a cratering model, we will also review some of the standard terminology associated with the characteristic parameters of cratering. Figure 2 contains an outline of a crater's shape with accompanying definitions.

A Cratering Model

The model will be restricted to explosive charges buried at or near their optimum (in the sense of maximum ejecta) depth of burst (DOB) in a homogeneous medium. It will be assumed that the material of interest is confined to a right circular cone whose apex is at the explosive detonation point (Fig. 3).

Note that any cratered material which falls back inside R_a (Fig. 2) will be returned to the initial cone of interest. However, any material that falls back beyond R_L will be ejected. Between R_a and R_L some of the material is ejected and some material slides back into the cone; the ejected material becomes part of the lip.

Next, let R define a radius between R_a and R_L such that (on the average) if a material element falls back inside R it returns to within the initial cone and if it falls beyond R it is ejected. Then also assume that $R_c = R$.

Assume that each differential volume element moves along a straight line passing through the apex and the element until it reaches the ground surface, at which point it goes into a ballistic trajectory. The object then is roughly to determine whether or not r_f is greater than R for each element and to integrate this result over the entire cone volume.

Fig. 3. Geometry of a cratering model. The notation r is the perpendicular distance of a differential volume element from the axis, y is the distance of the same element from the shot point, along the axis, R_c is the radius of the cone, y_d is the depth of burial of the explosive, r_i is the initial perpendicular distance from the axis at the ground surface for the ballistic trajectory, and r_f is the final perpendicular distance.

Let P be an approximation of the probability that an element will be ejected. It will be assumed that

$$P = C \left(\frac{r_f - r_i}{R} \right) \left(\frac{r_i}{R} \right). \quad (\text{II-1})$$

This weighting function assumes that the likelihood of an element being able to escape is a product of the perpendicular distance from the axis to the point where the ballistic trajectory begins times the horizontal distance the element moves from the point. The function is dimensionless (as it must be) since each of the above mentioned dimensions is divided by R . A dimensionless constant C is then available for the purpose of fitting the weighting function to experimental data. We will assume, near the optimum DOB for a given yield, that R is slowly varying and we will treat it as a constant. Appendix IIA examines the weighting function (and alternative choices) in more detail. Thus, the ejected volume is to be obtained from the equation

$$V_{ej} = \int_{\text{cone}} P \, dV \quad (\text{II-2a})$$

$$V_{ej} = \int_{\text{cone}} C \left(\frac{r_f - r_i}{R} \right) \left(\frac{r_i}{R} \right) dV. \quad (\text{II-2b})$$

However, since we assume straight line motion for an element before it reaches the surface, we obtain

$$r_i = \frac{r}{y} y_d. \quad (\text{II-3})$$

If we can express $r_f - r_i$ in simple terms, the integral can be performed directly. Fortunately $r_f - r_i$ is easily determined from ballistic analysis if the initial velocity is known at r_i .

The element starts and finishes at the surface. Thus, $y_f - y_i = 1/2 g t^2 - v_{0y} t = 0$, where v_{0y} is the y component of velocity at time zero when the element is at r_i , g is the acceleration of gravity, and t is the time spent in going from r_i to r_f . Therefore

$$t = \frac{2 v_{0y}}{g} . \quad (\text{II-4})$$

But

$$r_f - r_i = v_{0r} t = \frac{2 v_{0y} v_{0r}}{g} . \quad (\text{II-5})$$

Also

$$v_{0y} = \frac{v y}{\sqrt{y^2 + r^2}} \text{ and } v_{0r} = \frac{v r}{\sqrt{y^2 + r^2}} , \quad (\text{II-6})$$

where v is the "speed" or magnitude of the velocity of the element at $t = 0$. Combining expressions we obtain

$$r_f - r_i = \frac{2 v^2 y r}{g (y^2 + r^2)} . \quad (\text{II-7})$$

Thus, the problem reduces to a determination of the speed of the element as it crosses the original ground surface.

The speed of a particle and the kinetic energy are simply related.

$$v^2 = \frac{2}{\rho} \epsilon , \quad (\text{II-8})$$

where ϵ is the kinetic energy per unit volume and ρ is the material density. Combining and simplifying terms yields

$$r_f - r_i = \frac{4 y r}{(y^2 + r^2)} \frac{\epsilon}{\rho g} . \quad (\text{II-9})$$

The energy of the particle at the surface will be ϵ_k (the energy imparted to it by the hydrodynamic response of the earth to the explosive) minus ϵ_g (the energy lost moving through the gravitational potential). Thus, $\epsilon_g = \rho g (y_d - y)$. We have assumed that the energy can be described by a single term. In actual cratering, the accelerations are affected in several stages: 1) the shock 2) the rarefaction and 3) the gas acceleration phase. The last of these dominates the cratering process and is the major contribution represented by ϵ_k .

$$\epsilon = \epsilon_k - \rho g (y_d - y) . \quad (II-10)$$

Assume further that the material in the cone acts as if it were rigid and that all the kinetic energy instantaneously appears in the differential volume elements with each element receiving the same energy per unit volume. Let W denote the energy potential of the explosive. Only a fraction (f) of this energy will appear in the kinetic energy of the cratering material.

Then we have

$$\epsilon_k = f W/V_c , \quad (II-11)$$

where V_c is cone volume. Thus, f indicates the efficiency with which the explosive energy is converted to kinetic energy of the cratering material. Note that f could be a function of material properties such as strength and water content. It might also be a function of the energy of the explosive since the crater forming process is not scalable. These effects will be assumed separable so that

$$f = f_w f_s f_\omega \text{ etc.}, \quad (II-12)$$

where f_w denotes the factor of f which results from explosive size, f_s denotes the cratering material strength factor, and f_ω denotes the water content factor.

The terms which make up f would then be determined empirically.

Thus,

$$v_{ej} = \int_{\text{con}} \frac{C r^2 y_d^4}{\rho g R^2 (y^2 + r^2)} \left[\frac{3 f W}{\pi R^2 y_d} - \rho g (y_d - y) \right] dV \quad (II-13)$$

where

$$dV = 2\pi r dr dy.$$

This becomes

$$v_{ej} = \int_0^{y_d} \int_0^{\frac{R}{y_d}} C \left[\frac{24 f W}{\rho g R^4} - \frac{8\pi y_d (y_d - y)}{R^2} \right] \frac{r^3}{y^2 + r^2} dr dy. \quad (II-14)$$

The $\int dr$ can be performed immediately.

$$V_{ej} = \int_0^{y_d} C \left[\frac{24 \pi W}{\rho g R^4} - \frac{8\pi y_d (y_d - y)}{R^2} \right] \frac{y^2}{2} \left(\left(\frac{R}{y_d} \right)^2 - \ln \left[\left(\frac{R}{y_d} \right)^2 + 1 \right] \right) dy \quad (II-15)$$

The integral contains only y dependent variables and can now be evaluated easily.

$$V_{ej} = C \left[\frac{4 \pi W}{\rho g R^2} y_d - \frac{\pi}{3} y_d^3 \right] \left(\frac{y_d}{R} \right)^2 \left(\left(\frac{R}{y_d} \right)^2 - \ln \left[\left(\frac{R}{y_d} \right)^2 + 1 \right] \right) \quad (II-16)$$

This expression can be simplified even further by assuming that $R = y_d$ (i.e., $R/y_d = 1$), and by expanding the logarithmic term with a series.

Use

$$\ln x = \frac{x-1}{x} + 1/2 \left(\frac{x-1}{x} \right)^2 + 1/3 \left(\frac{x-1}{x} \right)^3 + \dots \quad (II-17)$$

and factor out a $(R/y_d)^2$ to obtain

$$\ln \left[\left(\frac{R}{y_d} \right)^2 + 1 \right] = \left(\frac{R}{y_d} \right)^2 \left[\frac{1}{1 + (R/y_d)^2} + 1/2 \frac{(R/y_d)^2}{[1 + (R/y_d)^2]^2} + \dots \right] \quad (II-18)$$

Within the parentheses apply the approximation $R = y_d$.

$$\ln \left[\left(\frac{R}{y_d} \right)^2 + 1 \right] = \left(\frac{R}{y_d} \right)^2 \left[1/2 + 1/8 + 1/24 + \dots \right] \quad (II-19a)$$

$$= 0.7 \left(\frac{R}{y_d} \right)^2. \quad (II-19b)$$

Table 1 provides some simple comparisons of values for which the assumption $R = y_d$ is reasonably good.

The formula for the ejected volume is thus approximated by the following equation:

$$V_{ej} = 0.3 C \left[\frac{4 \pi W}{\rho g R^2} y_d - \frac{\pi}{3} y_d^3 \right] \quad (II-20)$$

Table 1. Examination of the approximation.

R/y_d	$0.7 (R/y_d)^2$	$\ln[(R/y_d)^2 + 1]$
0.9	0.57	0.59
1.0	0.70	0.69
1.1	0.85	0.79

This equation provides two terms which can be easily interpreted. The positive contribution to the ejection process results from the energy available in the explosive (W). The negative term opposing the explosive energy results from the gravitational potential. From an engineering point of view, we are particularly interested in obtaining the optimum DOB. By optimum we mean the volume of material ejected. Designate y_0 to stand for the optimum DOB for a given material (f) and explosive (W).

R is a constant and it will be assumed that $R = 1.1 y_0$. V_{ej} is then given by

$$V_{ej} = 0.3 C \left[\frac{4 fW}{\rho_B (1.1)^2 y_0^2} y_d - \frac{\pi}{3} y_d^3 \right] \quad (II-21)$$

To obtain y_0 , we calculate (dV_{ej}/dy_d) and set it equal to zero. The y_0 is just the value of y_d for which the equation is valid. Note that the derivative can be easily obtained and that it is a very simple expression because we used an approximation to evaluate the logarithmic term.

$$\frac{4 fW}{\rho_B 1.21 y_0^2} - \pi y_0^2 = 0 \quad (II-22)$$

or

$$y_0^4 = 1.05 \frac{fW}{\rho_B} \quad (II-23)$$

or

$$y_0 = \left(\frac{fW}{\rho_B} \right)^{1/4} \quad (II-24)$$

Note that we have derived a formula that predicts the yield exponent to be $1/4$ rather than the $1/3.5$ or $1/3.4$ value which is observed experimentally. To eliminate this difficulty we must carefully examine f , the efficiency coefficient.

Energy Considerations

If a cratering shot involved only hydrodynamic forces the volume would be proportional to the explosive energy. Likewise, any linear dimension would be proportional to the cube root of the explosive energy.

$$y_0 = K_1 W^{1/3} \text{ or } W = K_1^3 y_0^3 . \quad (\text{II-25})$$

On the other hand, if only gravitational forces were involved, the explosive energy would be proportional to the volume of material times the distance it was lifted. Thus, any linear dimension would be proportional to the fourth root of the explosive energy.

$$y_0 = \bar{K}_2 E_g^{1/4} = K_2 W^{1/4} \text{ or } W = K_2^4 y_0^4 , \quad (\text{II-26})$$

where E_g is the gravitational energy.

In fact, the cratering process involves a balance between these two extremes. We will assume that the following equation is approximately correct.

$$y_0 = K_3 W^{1/3.5} \text{ or } W = K_3^3 y_0^{3.5} . \quad (\text{II-27})$$

If we then assume that the gravitational energy and initial kinetic energy are the same, we can conclude that the efficiency f is given by

$$f_w = \frac{E_g}{W} \propto \frac{y_0^4}{y_0^{3.5}} = y_0^{0.5} = K_4 W^{1/7} . \quad (\text{II-28})$$

Thus, this equation indicates that the efficiency increases as the explosive energy gets larger. However, the rate of increase is a slow one; W would have to increase by a factor greater than 100 in order to double the efficiency.

The above analysis has an obvious limitation. If it were universally true, the efficiency could become greater than 100% if the yield were increased indefinitely. The above formula might be expected to hold as long as the efficiency is very small, i.e., as long as the hydrodynamic dissipative mechanisms absorb most of the energy. Since the efficiency of a 1-kt Plowshare crater is only a few percent, the above equation might be valid over many more orders of magnitude of explosive yield. Let

$$f = f_0 \left(\frac{W}{W_0} \right)^{1/7} f_m \quad (\text{II-29})$$

i.e., let

$$K_4 = f_0 (W_0)^{-1/7} f_m, \quad (\text{II-30})$$

where f_0 is the "reference" efficiency of a material with no significant energy degrading or enhancing characteristics (such as strength and water content) and f_m is a multiplier which accounts for the change in f due to the presence of such energy degrading or enhancing characteristics in cratering materials. Assume $f_m = f_s f_w$, where f_s is the strength factor and f_w is the water factor. Since f must be dimensionless, $(W_0)^{-1/7}$ is required. This constant reflects the yield at which f_0 is determined. Returning to our cratering formulas for the optimum DOB, we obtain

$$y_0 = \left(\frac{f_0 f_m}{g} \right)^{1/4} \left(\frac{W}{W_0} \right)^{1/28} W^{1/4}, \quad (\text{II-31})$$

which is consistent with our assumption $y_0 = K_3 W^{1/3.5}$. The ejected volume is expressed by

$$v_{ej} = \frac{C(f_0 f_m)^{1/2} y^{1/2}}{(\rho g)^{1/2}} \left(\frac{W}{W_0} \right)^{1/4} y_d - \frac{\pi C}{10} y_d^3. \quad (\text{II-32})$$

Note that the sensitivity of these functions to the accuracy of the material properties is very weak. Thus, only a crude estimate of f_0 and f_m is required. If f_m were 46% too large, an error of only 10% in y_0 would result.

The discussion of the model is essentially complete at this point. For the formulas to become practical engineering tools, it will be necessary to determine the values of C , f_0 , f_w , and f_s . They will be determined empirically by fitting the formulas to experimental observations.

Experimental Data

Table 2 contains a summary of the U.S. nuclear cratering experiments. The sources of this information are given in Refs. 2-7.

Table 2. U.S. Cratering Data

	Sedan	Danny Boy	Cabriolet	Schooner
ρ gr/cc density	1.9	2.6	2.3	2.2
ω vol. fraction water near shot point	0.15	0	0	0.2
τ psi unconf. comp. strength	~0	30,000	5,000	7,000
W (kt) yield of explosive	100	0.42	2.3	35
y_d (10^3 cm) DOB	19.3	3.35	5.2	10.8
y_a ($\times 10^3$ cm) apparent depth	9.9	1.9	3.6	6.3
R_L (10^3 cm) lip radius	20.7 ^a	4.0 ^b	6.4	14.7
R_a (10^3 cm) apparent radius	18.5	3.3	5.5	13.0
V_a ($\times 10^{12}$ cc) apparent volume	5.1	0.027	0.138	1.74

^aEstimated from Fig. 2.9 of Ref. 5.

^bEstimated from Fig. 9 of Ref. 4.

Only nuclear data are presented because the geologic properties were carefully measured, the geometry of the explosive was well defined, and the explosive was carefully sealed. Most of the cratering with high explosives is not satisfactory from the point of view of the precision with which the cratering conditions were known.

The volume fraction of water represents an average value near the explosive. The other material properties data represent average values for the cone of integration. If we compare the apparent radius to the depth of burial we find that they are roughly comparable. However, we see from Table 2 that sometimes y_d is greater than R_a and sometimes the reverse is true. There is a systematic effect evidenced by this data. For example, the Sedan⁵ burial point was predicted by using "scaling" laws which were determined for high explosive (HE). High explosive is a more efficient energy source for cratering than is a nuclear explosive. Thus, the predicted optimum burial point was too low. Fortunately, the burial was conservatively selected, with the result that Sedan was buried only slightly below optimum DOB and the crater was satisfactory. Both Danny^{4,7} and Cabriolet^{6,7} performed basically as had been predicted by computer calculations and were both near optimum DOB. Schooner^{7,8} on the other hand was not buried deeply enough. Preshot computer calculations had not accounted for the increased working fluid effectiveness resulting from the presence of water around the shot. Unfortunately, had the explosive been buried deeper, it would have been too deep because the deeper rock layers did not have large water components. This serves as a warning that cratering in a layered medium may be quite complicated.

The derivation of the cratering formula involved the assumption that the radius was not a strong function of DOB provided that DOB was not too far from the optimum. The fact that R or R_a should be a weak function of DOB can be qualitatively understood by noting that a certain tensile force normal to the surface is required to spall the material. As the DOB is made larger than optimum, the shock reaching the surface at R is weaker but its normal component carries a larger fraction of the energy. Thus, the normal tensile force at R changes very slowly with respect to DOB. The same argument applies as burial is more shallow than the optimum.

Experimental data shown in Table 2 are in good agreement with this assumption. However, if R_a is not a strong function of DOB, then Table 2 indicates further that R_a and y_0 (optimum DOB) are approximately equal. We will conclude they are equal for the remainder of the analysis. Remember also the assumption $R = 1.1 y_0$, where R is the radius used to determine whether or not ejecta escapes. Thus, we now have $R = 1.1 R_a$, which places R beyond R_a but short of R_L in Table 2. This also confirms that these assumptions of the model are not seriously at odds with experimental observations.

Fitting the Formula

The formula for y_0 contains the efficiency factor f which must be determined by comparison with the experimental data. But we have

$$f = f_0 f_\omega f_s . \quad (\text{II-33})$$

To obtain f_0 , we would like to use an experiment for which $f_\omega = f_s = 1.0$. Unfortunately, Sedan is the only experiment for which $f_s = 1$, but $f_\omega \neq 1$. Note that $f_s = 1$ if the medium has no significant strength, and that $f_\omega = 1$ for a dry rock. Each of these factors will be written in the following perturbation-like form:

$$f_i = (1 + \psi_i) . \quad (\text{II-34})$$

Thus, the change in f_i is measured by ψ_i and f_i is unchanged if $\psi_i = 0$. ψ_ω will be positive and ψ_s will be negative. The Schooner experiment gave a much larger crater than expected because the effect of the water was not accounted for in preshot calculations. One of the reasons that water enhances the effectiveness of the explosive is that less energy is wasted in the latent heat of vaporization, and it does not condense as readily as rock. The effect of water in Schooner could be estimated by noting that computer calculations indicated that the explosive source acted as if the yield were increased by approximately 40% in a comparable dry medium. The water content in the cone near the shot point was 20% by volume, and if we assume the effect is proportional to the volume fraction filled with water,

we obtain

$$f_w = (1 + 2w) , \quad (\text{II-35})$$

where w is the volume fraction of water.

For W_0 we can arbitrarily select 1 kt and note that 1 kt = 0.42×10^{20} ergs. We can now determine f_0 from the Sedan cratering data in Table 2 by using the equation for y_0 :

$$y_0 = \left(\frac{f_0 f_w f_s}{\rho g} \right)^{1/4} \left(\frac{W}{W_0} \right)^{1/28} w^{1/4} \quad (\text{II-36})$$

For Sedan we have:

$$1.85 \times 10^4 = \left[\frac{f_0 (1 + 0.3)}{1.9 (980)} \right]^{1/4} (100)^{1/28} (42 \times 10^{20})^{1/4} \quad (\text{II-37})$$

Thus,

$$f_0 = 0.021. \quad (\text{II-38})$$

This says that only a small fraction of an optimally buried nuclear explosive yield goes into kinetic energy for a 1-kt shot in a material with low strength and no water. Actually, even less of the energy does "useful" work, since the earth material, on the average, is lifted considerably above the ground surface before falling back. The equation for optimum DOB is now given as follows:

$$y_0 = \left[\frac{0.021 (1 + 2w) f_s}{\rho g} \right]^{1/4} \left(\frac{W}{W_0} \right)^{1/28} w^{1/4} \quad (\text{II-39})$$

The only remaining term is f_s . We can use the formula for y_0 and the Danny Boy data of Table 2 to obtain f_s for the Danny Boy medium. The same process can be repeated for Schooner thereby obtaining the results shown in Table 3.

Table 3. Strength factor.

	Danny Boy	Schooner
f_s	0.73	0.88

It would be desirable to express f_s as a continuous function of τ , the unconfined compressive strength. Such a formula would have to fit the above data; it would have to be such that f_s approaches 1 as the strength approached zero; and f_s should always be greater than zero. The following formula achieves the above numerical constraints, but it is devoid of any physical interpretation.

$$f_s = 1 - (0.27)^{3\sqrt{\tau_0/\tau}} \quad (\text{II-40})$$

where τ_0 is the unconfined compressive strength for Danny Boy and τ is the same quantity for the experiment whose f_s value is being calculated. Note that this formula has a weak dependence on the ratio τ_0/τ . This provides some hope that this crude estimate of f_s may be sufficient for most materials.

The formula for y_0 has now been fit to experimental data. As a check, it should now be possible to calculate y_0 for Cabriolet.

$$y_0 = \left[\frac{(0.021)(0.91)}{(2.3)(980)} \right]^{1/4} (2.3)^{1/28} [2.3 \times 0.42 \times 10^{20}]^{1/4} \quad (\text{II-41a})$$

$$y_0 = 5.4 \times 10^3 \text{ cm} \quad (\text{II-41b})$$

This agrees well with the Cabriolet value for R_a (5.5×10^3 cm) found in Table 2. Thus, we obtain the first check on the formula for y_0 . The only other U.S. cratering shot fired near optimum DOB was Euggy, a row charge experiment for which the analysis of this model is inappropriate. The experience of the Soviet program has not been included in the central text because it is very difficult to know the conditions of their experiments precisely from their published data. For instance, they describe shot 1004 as "a charge of over 100 kt placed in a hole at depth of around 200 meters." Their experience is examined in Appendix B and the results provide further confirmation of this approach. Unfortunately, the above mentioned lack of precision in reporting their experiments minimizes the value of the confirmation.

The equation for y_0 has been used to evaluate f_0 , f_w , and f_s . They are given as follows along with the equation for y_0

$$f_0 = 0.021 \quad (II-42a)$$

$$f_w = 1 + 2w \quad (w = \text{volume \% water near } y_0) \quad (II-42b)$$

$$f = 1 - (0.27) \sqrt[3]{\tau_0/\tau} \quad (\tau_0 \text{ is unconfined compression strength of Danny Boy rock [30,000 psi] and } \tau \text{ is the strength of the rock of interest.}) \quad (II-42c)$$

$$y_0 = \left(\frac{f_0 f_w f_s}{\rho g} \right)^{1/4} \left(\frac{W(kt)}{I(kt)} \right)^{1/28} W^{1/4} \quad (II-42d)$$

The equation for volume ejected is given as follows (rewriting Eq. 32)

$$v_{ej} = C \left(\frac{f_0 f_w f_s}{\rho g} \right)^{1/2} \left(\frac{W(kt)}{I(kt)} \right)^{1/14} W^{1/2} y_d - \frac{C\pi}{10} y_d^3 \quad (II-43)$$

In Appendix IIA we show that one might expect the value of C to be on the order of 1.5. Having determined f_i values we can now determine the value of C by comparing the calculated volumes to the apparent volumes of Table 2. This should be a reasonable comparison because any compaction or bulking which takes place for the fallback material applies only to a small fraction of the material which started out in the cone of integration. The results are listed in Table 4.

Table 4. Volume comparisons.

	Sedan	Danny Boy	Cabriolet	Schooner
$v_{ej} (\times 10^{12} \text{ cm}^3)$	5.3	0.025	0.136	1.73
$v_a (\times 10^{12} \text{ cm}^3)$	5.1	0.027	0.138	1.74

(The value of C has been set equal to 1.2 and this value is consistent with the considerations of Appendix IIA). The agreement is excellent.

For engineering purposes, it would be desirable to predict R_a and y_a . We have already seen that $R_a = y_0$. From Table 2 we note that the following relationship is approximately valid.

$$y_a = 0.575 R_a \quad (\text{II-44})$$

However, there is noticeable scatter in the data of Table 2 (about $\pm 10\%$ for a relative standard deviation). This reflects a number of factors which complicate the determination of y_a . Slope stability may vary from one material to another, and some craters (shallow burial) may have slopes well below the critical slope. Also, y_a strongly depends on the volume of fallback because the apex of the cone fills up first. This same factor makes y_a more sensitive to bulking or compaction of material. Nonetheless it does not seem unreasonable to estimate that the above prediction (II-44) for y_a should be accurate to within 15%. This subjective statement assumes that R_a can be predicted quite reliably (on the order of 5%). The formulas for predicting crater dimensions are summarized in Appendix IIC.

Discussion

The volume equation reflects the basic principle of the model, that being the separation of the hydrodynamic processes and the gravitational process. Perhaps the most interesting aspect of the formula becomes evident when the equation for V_{ej} is divided by $W^{3/3.5}$. We then obtain:

$$\left(\frac{V_{ej}}{W^{3/3.5}} \right) = A \left(\frac{y_d}{W^{1/3.5}} \right) - B \left(\frac{y_d}{W^{1/3.5}} \right)^3 \quad (\text{II-45})$$

where A and B are constant for any given material. But this form is similar to the so called "scaling curves." At the risk of repetition, the above is not legitimate scaling but a formula using a yield exponent form. It seems appropriate to point out that this equation has the right general functional behavior as the DOB is increased. In particular, the abrupt

failure of an explosive to produce a crater (when the DOB is too large) is evidenced by the cubic term.

The advantage of the formulas for y_0 and V_{ej} in the practical sense is that they can accommodate a continuum of material properties. The model also provides a thinking tool and a more fundamental understanding of the experimental data as plotted on the yield exponents curves. Because y_0 is a weak function of material properties, it is reasonable to expect good results when they are applied in engineering situations.

As examined herein, the material is assumed to be dominated by silicates and typical silicate failure does not have a strongly dissipative plastic flow mechanism. In carbonate rocks one might expect an enhancement as a result of CO_2 vaporization. Furthermore, the nonsilicate rock component has a different latent heat of vaporization which must also be accounted for. It might be unlikely that a cratering shot would be fired in salt, but such a shot would be an example of a need to account for dissipative plastic flow. It would be possible to embellish the formula to take into consideration such factors as bulking/compaction of the fallback and the volume effect of the vaporized rock. However, the model is sufficiently crude at this time that such an attempt is hard to justify. In the future a more detailed model may be practical, particularly as computer simulation becomes more accurate and equation of state input is known better. In particular, one would like to have a more physically meaningful form for f_s .

It might also be possible to use y_0 to estimate the DOB for an optimum retarc (i.e., $y_{\text{retarc}} = C y_0$ where C is a constant greater than 1). (Note that retarc is crater spelled backwards and simply means a mound.) A retarc results when the DOB is a little too deep for cratering to occur but may result when a rubble cone extends from the DOB to above the ground surface. It should also be possible to extend the formula to include large HE craters. We might anticipate that HE is some factor G times as efficient for cratering as a nuclear explosive. However, the initial cavity pressures are lower and no significant water vaporization can occur. Thus, one might set f_0 for high explosive craters equal to G times the value of f_0 for nuclear craters and f_m would be restricted to f_s .

Future investigation might lead to a better determination of f_s and also to the correlation of HE data. However, experimental conditions for HE cratering are poorly documented and considerable care would have to be exercised. It may also be profitable to extend the model to account for inhomogeneities such as the presence of layers of different geologic materials. One might also attempt to account for row charge effects. Although the problem would be considerably more complex (varied angles of incidence and explosive no longer at a point source), it might also be possible to extend this approach to calculate meteorite craters. Finally, it must be recognized that the data on the nuclear cratering experiments is crude even though it is the best data available. The accuracy of various measurements is crude in itself, but of greater importance is the fact that many factors have not been measured at all. Thus, measurements were taken at discrete locations and the geologic inhomogeneities remain unknown (but their existence is fairly well assured). The effects of large scale cracks and preexisting planes of weakness are unknown, as are the effects of anisotropies. The measurement of explosive yields also involves some uncertainties. Furthermore it has been necessary to treat the data (averaging procedures for material property and, estimates of optimum DOB) in a somewhat subjective way. All this has been done within the framework of a model which contains a large number of assumptions. Thus, the reader must consider the results of this analysis somewhat cautiously, in spite of the rather close agreement with the experiment. In this light, it seems appropriate to suggest that a significant amount of testing and refinement of this approach may be necessary.

Appendix IIA

The Weighting Function

We can examine the weighting function $P = 1.5 \left(\frac{r_f - r_i}{R} \right) \left(\frac{r_i}{R} \right)$

(i.e., let $C = 1.5$) by noting that r_i/R is a constant along "subcones" within the cone. We designed the model with the assumption that elements move along these constant angle lines until reaching the surface.

Let $r_i/R = \alpha$ and examine the behavior of W for different values of α . Thus

$$P = 1.5 \left(\frac{r_f}{R} - \alpha \right) \alpha \quad (\text{IIA-1})$$

Note that when $\alpha = 0$ (elements on the cone axis) P must equal zero. Presumably $P = 0$ for the maximum value of α , i.e., $\alpha = r_i/R = 1$. Thus, the elements along the cone edge don't get enough energy to escape. This is reported in Fig. 2 of UCRL-6438 for the Scooter experiment.⁹ In Fig. 4 we see a comparison of P (for different constant values of α) and P' (an estimate of what a precise probability function looks like).

Again referring to the Scooter experiment we know that some throw-out is found at a distance of $2R$. Thus, Fig. 4 indicates that P is not a bad approximation to P' . This is especially true if we note that the P curve for $\alpha = 3/4$ must be given more weight since the volume of material in the cone at large α is greater. The straight lines for P at constant α are terminated at each end because of energy considerations. Thus, for any constant α , there is a range of energies available for all the volume elements. For example, no element has an infinite amount of energy. At the low energy end, P probably should not become negative since most elements will have enough energy to rise above the original ground surface.

In any event, P is really a weighting function. As long as the shape of the cone and the relative position of the throw-out doesn't change drastically, P should prove satisfactory. Of course, the normalizing constant could be changed by a significant amount (perhaps as much as 50%) and Fig. 4 would still give a reasonable match for

Fig. 4. The weighting function.

P and P'. Several such graphs were drawn and $C = 1.5$ was arbitrarily selected.

In fact there are an infinite number of choices for a weighting function expressed as a function of $(f/R)^\alpha$ and α . For example, we could have used

$$W = C_2 (f/R - \alpha)^\gamma (\alpha)^\beta \quad (\text{IIA-2})$$

where γ and β are any real numbers. However, for $\gamma \neq 1$, we are unable to separate the hydrodynamic energy from the gravitational energy in a simple manner. For $\beta = 1$, we obtain a qualitatively reasonable ratio of the relative effect of the gravitational term to the hydro term.

Appendix IIB

The Soviet Cratering Program

The following includes calculations for y_0 and V_{ej} for three Soviet cratering shots.¹⁰ For some reason, the Soviet descriptions of their cratering shots are imprecise and obscure. I have calculated their experiments using values which have been "corrected" insofar as I'm capable. I'm fairly confident these calculations are realistic although no guarantee can be given.

Shot T1

$$\rho = 1.74$$

$$\omega = 0.1 + 0.2 \text{ (for gasable material)} f_w = 1.6$$

$$Y = 0.2 \text{ kt}$$

$$y_d = 3.1 \times 10^3 \text{ cm}$$

$$f_s = 1$$

$$R_A = 3.6 \times 10^3 \text{ cm}$$

Then using the formulas in equations (II-42d) and (II-43) gives

$$y_0 = 3.4 \times 10^3 \text{ cm}$$

and

$$V_{ej} = 3.1 \times 10^4 \text{ m}^3$$

The shot is reported to have not been buried deeply enough because of the effect of the gasable materials which I have arbitrarily accounted for in the same way as water. The experimentally observed apparent volume was about $3.2 \times 10^4 \text{ m}^3$. (The apparent volume was not reported but it can be crudely calculated from the reported dimensions.)

Shot 1003

$$\rho = 1.8$$

$$\omega = 0.2$$

$$Y = 1.1 \text{ kt}$$

$$y_d = 4.8 \times 10^3 \text{ cm}$$

$$f_s = 1$$

$$R_A = 5.5 \times 10^3 \text{ cm}$$

The values published for the crater dimensions are sloughed values two years after the experiment. I have estimated the unsloughed radius in the above list. Then

$$y_0 = 5.3 \times 10^3 \text{ cm}$$

and

$$V_{ej} = 1.2 \times 10^5 \text{ m}^3$$

The burial was again shallow (compare y_d to R_a) and the reported apparent volume is $1.11 \times 10^5 \text{ m}^3$.

Shot 1004

$$\rho = 1.9$$

$$\omega = 0.2$$

$$Y = 125 \text{ kt}$$

$$y_d = 1.78 \times 10^4 \text{ cm}$$

$$f_s = 1$$

$$R_A = 2.1 \times 10^4 \text{ cm}$$

Then we obtain

$$y_0 = 2.0 \times 10^4 \text{ cm}$$

and

$$V_{ej} = 6.6 \times 10^6 \text{ m}^3$$

The reported value for the apparent volume is $6.4 \times 10^6 \text{ m}^3$ in good agreement with the above calculation. The radius of the apparent crater again indicates that the explosive was not buried deeply enough. The consistency with which the Soviet shots are buried too shallow might indicate a conservative engineering approach to ensure that the experiment produces a crater rather than a retarc. However, this has the disadvantage of permitting the maximum amount of radioactivity to escape.

Appendix IIC

Formula Summary

The following equations summarize the cratering formulas for engineering purposes.

$$y_0 = \left(\frac{f_0 f_w f_s}{\rho g} \right)^{1/4} \left(\frac{W(kt)}{1(kt)} \right)^{1/28} W^{1/4} \quad (\text{optimum DOB})$$

$$R_a = y_0 \quad (\text{apparent radius})$$

$$y_a = 0.575 y_0 \quad (\pm 10\%) \quad (\text{apparent depth})$$

$$V_{ej} = 1.2 \left(\frac{f_0 f_w f_s}{\rho g} \right)^{1/2} \left(\frac{W(kt)}{1(kt)} \right)^{1/4} \\ \times W^{1/2} y_d - \frac{1.2 \pi}{10} y_d^3 \quad (\text{volume ejected})$$

where

$$f_0 = 0.021 \text{ for nuclear explosives in silicate rocks,}$$

$$f_w = 1 + 2\omega \quad (\omega = \text{volume \% of water near } y_0),$$

and

$$f_s = 1 - (0.27)^{3\sqrt{\tau_0/\tau}}$$

[τ_0 is unconfined compression strength of Danny Boy rock (30,000 psi) and τ is the strength of the rock of interest.]

III. AN INVARIANT DESCRIPTION OF THE FAILURE SURFACE OF AN ISOTROPIC MEDIUM

INTRODUCTION

"Because of a lack of a generally accepted means of presenting these data (on rock failure), the results of the experiments are given in three forms."¹¹ This quotation exemplifies the fact that a systematic and illuminating method of displaying experimental rock failure data would be very useful. For convenience, a relatively simple description of the failure surface would be best. In rock-mechanics computer codes, it is also desirable to be able to describe failure surfaces simply and efficiently; this chapter examines one promising approach.

The analysis presented here assumes that material stresses are given in the principal coordinate system. Only isotropic materials are considered. We assume that a failure condition that depends only on the stress state of the material, and not on the path by which the stress state was arrived at, can be clearly defined (this is not precisely true, but it seems to be a reasonable first approximation for geologic materials). Of course, failure in rocks also depends on strain rate, temperature, and other parameters, but these are considered to be held constant and thus not included in the failure-surface analysis.

THEORY

The development of invariant quantities is well-known; the following brief discussion is presented to establish the sign conventions.

There are three principal stresses σ_i ($i = 1, 2, 3$) for any infinitesimal element. Assume that σ_i is positive for tension, and define the stress deviators by the following:

$$s_i = \sigma_i - \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3). \quad (\text{III-1})$$

The invariants of interest are defined by the following equations:

$$I_1 = (\sigma_1 + \sigma_2 + \sigma_3), \quad (\text{III-2a})$$

where pressure is just $-1/3 I_1$.

$$I_{2D}^{1/2} = \sqrt{\frac{1}{2} (S_1^2 + S_2^2 + S_3^2)}. \quad (\text{III-2b})$$

$$I_{3D}^{1/3} = (S_1 S_2 S_3)^{1/3}. \quad (\text{III-2c})$$

Many standard references¹²⁻¹⁴ state that, if a material's failure depends only on the stress state, then an equation of the failure surface exists of the functional form

$$F_1(\sigma_1, \sigma_2, \sigma_3) = 0 \quad (\text{III-3})$$

or

$$F_1(I_1, I_{2D}, I_{3D}) = 0.$$

The specific form of the relationship between the failure condition and the principal stresses (or some combination of invariants) may, however, be quite complicated, and a closed form of the solution may be difficult to obtain. We can examine the specific relationship

$$F_2(I_1, I_{2D}^{1/2}, I_{3D}^{1/3}) = 0 \quad (\text{III-4})$$

to see what properties the $I_1, I_{2D}^{1/2}, I_{3D}^{1/3}$ space has.

When any two of the principal stresses are equal, the material is either in a state of compression or extension: extension if the unique third stress is more tensile than the others, and compression if it is less tensile. Then $I_{2D}^{1/2}$ and $I_{3D}^{1/3}$ take on the following special forms for compression tests or extension tests:

$$I_{2D}^{1/2} = (1/3)^{1/2} (\sigma_1 - \sigma_2) \quad (\text{III-5})$$

$$I_{3D}^{1/3} = (2/27)^{1/3} (\sigma_1 - \sigma_2), \quad (\text{III-6})$$

where σ_1 and σ_2 are the two different values of principal stress. Hence,

$$I_{3D}^{1/2} = \frac{3^{1/2}}{2^{1/3}} I_{3D}^{1/3}, \quad \left(\frac{3^{1/2}}{2^{1/3}} \approx 1.4 \right) \quad (\text{III-7})$$

for compression and extension. Pure shear is defined as those stress states for which $I_{3D}^{1/3} = 0$; this occurs whenever σ_1 is greater (or less) than σ_2 by the same amount that σ_3 is less (or greater) than σ_2 (i.e., whenever $S_2 = 0$). Note that when all three stresses are equal, $I_{2D}^{1/2} = I_{3D}^{1/3} = 0$ and the state of the material lies on the I_1 axis.

For a constant value of I_1 , we can examine a cross-section of the $I_1, I_{2D}^{1/2}, I_{3D}^{1/3}$ space. Figure 5 shows a hypothetical plot of $I_{2D}^{1/2}$ vs $I_{3D}^{1/3}$ at constant I_1 . The straight lines OA and OB give the locus of points for compression and extension, respectively. The line $I_{3D}^{1/3} = 0$ is the locus for torsion test points. For points P_1, P_2 , and P_3 , the material fails in compression, extension, and torsion, respectively. All possible states of the material lie between the straight lines OA and OB. The points below the curve $P_1 P_2 P_3$ contain all the states of the material experiencing no failure. Thus we observe that the geometrical properties of this space are quite simple and attractive. It should also be noted that all the coordinates of this space have dimensions of stress. The standard Von Mises yield criterion would appear as a horizontal straight line such as the line from P_1 to P_3 (Fig. 5).

Figure 6 shows an artist's conception of what the three-dimensional stress space might look like. At high pressure (large I_1), the $I_{2D}^{1/2}$ (at failure) flattens out and loses its dependence on $I_{3D}^{1/3}$ for ductile failure. However, at lower pressure, the failure becomes brittle and the dependence on $I_{3D}^{1/3}$ becomes important. Finally, the surface closes onto the I_1 axis for the hydrostatic tension condition that induces failure.

Significantly, experimental evidence indicates that $I_{2D}^{1/2}$ (at failure) is a single-valued function of I_1 and $I_{3D}^{1/3}$. Thus we might consider replacing Eq. (III-4) by the following equation:

$$I_{2D}^{1/2} (f) = I_{2D}^{1/2} \left(I_1, I_{3D}^{1/3} \right), \quad (\text{III-8})$$

Fig. 5. $I_{1/2}^{2D}$ vs $I_{1/3}^{3D}$ for constant I_1 .

Fig. 6. Projected view of the failure surface.

where (f) stands for "at failure."

If the surface is such that $I_{2D}^{1/2}(f)$ is a slowly varying function of I_1 and $I_{3D}^{1/3}$, one might consider expanding this dependence in a power (Taylor) series and expect to have only a few terms in the expansion. Rather than generate multiterm power series, we examine a cut in the space at constant I_1 (as in Fig. 5). Assume data are available for compression, extension, and some state between them (torsion would appear to be convenient, from the experimental standpoint). Then a quadratic curve can be drawn through these three points, and it will be a good approximation to the failure surface if the surface is slowly varying, smooth, continuous, and in general mathematically well-behaved. Thus, we have

$$I_{2D}^{1/2}(f) = A_1(I_1) + A_2(I_1) I_{3D}^{1/3} + A_3(I_1) I_{3D}^{2/3}. \quad (\text{III-9})$$

where A_1 , A_2 , and A_3 are functions of I_1 . If A_1 , A_2 , and A_3 were expanded in a power series, we would have a truncated Taylor series expansion for the failure-surface description. The fact that the stress states of the material are "confined" between the straight lines OA and OB of Fig. 5 makes the power series form of Eq. (III-9) very attractive. However, the pressure states available to a material are not limited mathematically, and expansion of A_1 , A_2 , and A_3 in powers of I_1 would require a higher-order series for most rocks. If a graphical plot of A_1 , A_2 , and A_3 as a function of I_1 is obtained, then the complete failure surface is determined to a very good approximation.

EXPERIMENTAL FIT

To use Eq. (III-9) to generate a complete failure surface, it is necessary to have (for many different pressures) data for failure in compression, extension, and somewhere between compression and extension. The data reported on Solenhofen limestone by Handin et al.¹⁵ is the only satisfactory set of data from this point of view.

For example, the glass and dolomite data reported by Handin et al.¹⁵ were not useful for mapping the failure surface, because the three tests

(compression, extension, and torsion) did not cover the same pressure regions. Some data, such as that presented by Mogi,¹⁶ suffer from the limitation of covering a small range of pressures and having insufficient torsion data. The need for more data is obvious.

The limestone data of Handin et al.¹⁵ that are treated here have been limited to the tests performed on Blocks 3 and 5 because they are quite similar in their mechanical characteristics. Where two tests were performed at the same conditions, the average stresses in the two tests were used. A few test points were rejected as being premature failures (these judgments were concurred in by Hugh Heard). To generate failure curves of $I_{2D}^{1/2}$ vs $I_{3D}^{1/3}$, it was necessary to plot $I_{2D}^{1/2}$ vs I_1 as a smooth curve for the three different tests (compression, extension, and torsion) and to take values of $I_{2D}^{1/2}$ for the three tests at the same I_1 values. Figure 7 shows the limestone failure curves for $I_{2D}^{1/2}$ vs $I_{3D}^{1/3}$ for several constant values of I_1 . Note the inadequacy of the Von Mises criterion.

Equation (III-9) can be solved directly by linear algebra for the values of $A_1(I_1)$, $A_2(I_1)$, and $A_3(I_1)$. Figure 8 provides a plot of $A_1(I_1)$, $A_2(I_1)$, and $A_3(I_1)$ as a function of I_1 for the limestone. Note that for I_1 greater than 12 kbar, A_2 and A_3 go to zero as the failure becomes more ductile. These values can now be used as input for a computer rock-mechanics code in a compact and efficient form with A_1 , A_2 , and A_3 in a table as a function of I_1 , using linear interpolation between points.

This approach represents a greater capability for describing a failure surface for rock mechanics computations than that reported by Cherry et al.¹⁷ Cherry's model included the assumption that the I_{3D} effect did not vanish as pressures increased. Another assumption was that the torsion failure curve was intermediate between the compression curve and the extension curve when $I_{2D}^{1/2}$ was plotted against pressure. As we have seen in the limestone data [Handin et al., 1962], these assumptions are not in agreement with experimental observation.

Fig. 7. $I^{1/2}$ vs $I^{1/3}$ for limestone at various values of I_1 .

Fig. 8. A_1 , $-A_2$, and A_3 as functions of I_1 . A_1 is in kbar; $-A_2$ is dimensionless; A_3 is in kbar^{-1} .

DISCUSSION

At this point it seems appropriate to emphasize that while this chapter provides a mathematical mapping technique for a material failure surface, it does not contain any physics information. It does, however, provide a systematic way of classifying material failure. Hopefully, the bulk of experimental data can be organized into units of comprehensible proportions, so that the underlying physics can appear. In addition, it provides an efficient means of defining the failure surface in a numerical modelling code. Before either of these areas can be widely exploited, it is necessary to have more experimental data.

Much of the data reported in the literature does not contain information about failure between compression or extension such as torsion. In Fig. 8, we noted that A_3 can be quite significant for some values of I_1 . Since A_3 is a measure of the curvature in a cut at constant I_1 for the $I_{2D}^{1/2}$ vs $I_{3D}^{1/3}$ curve, it is important to have torsion data. Conclusions reached without such data (as has been done by Mogi)¹⁶, would seem to be very hazardous.

So far, there has been no attempt at physical interpretation for the limestone data. It would be simple to speculate that the continual increase of $A_1(I_1)$ as I_1 increases represents some kind of frictional effect that increases the strength of the rock. One might also speculate that decreasing the magnitude of $A_2(I_1)$ as I_1 increases indicates that the opportunity for statistically random tensile locations to exist in the material decreases as the material becomes more ductile. Consequently, a large value of $A_3(I_1)$ might indicate a preference for a shearing mode of failure.

However, putting speculation aside, the plot of Fig. 8 indicates clearly that A_3 "turns on" over a rather limited range of I_1 . Thus, the mapping process has revealed that something special is happening physically. $A_1(I_1)$ would be expected to become constant for large I_1 as the failure becomes perfectly ductile. The behavior of A_2 is inconclusive in Fig. 8. It would be interesting to see more data on Solenhafen limestone to see what the behavior of $A_2(I_1)$ and $A_3(I_1)$ are at low pressure. Such experiments are difficult, since they would require confining pressures (which would also include negative pressures) with tensions and torsion.

IV. A NEW MODEL FOR SIMULATING STRESS RELAXATION AND BRITTLE-DUCTILE MATERIAL FAILURE

INTRODUCTION

The SOC and TENSOR computer codes and their physical models are described in Refs. 18-20. They have been critically reviewed in Refs. 21 and 22. The previous chapter was written because the SOC and TENSOR codes included a failure surface description which was not general, and in fact, it forced the scientist using the code to use a failure surface which was not always consistent with experimental data. The scheme used in SOC and TENSOR to relax stresses was in error because numerically the difference equation did not properly represent a legitimate differential equation. This introduced nonphysical discontinuities and noise in the solution. The most distressing result was that the solution failed to converge as the zone size was allowed to approach zero! Many additional shortcomings have been documented by Shatz.²¹ The UKO²³ computer code contains improved calculational procedures which can be of benefit to the Plowshare Program.

THE FAILURE MODEL

This section describes the essential new failure modelling techniques which are the basis for the UKO⁶ computer code. The determination of a failure surface is simply the mathematical determination of material failure. It does not specify how the material fails. We must first determine how to specify the conditions required for failure. Then if failure has occurred, we must determine how the stresses are to be relaxed.

We will provide two failure surfaces as outlined in the previous chapter, one for consolidated material and one for "rubble." These are intended to represent limiting cases for the material. As discussed in the previous chapter each of these surfaces (consolidated and rubble) is

most easily constructed from several curves, each for a certain test type. A multidimensional code requires three such curves for reasonably complete surface definition. The most efficient and convenient set might provide one each for compression, torsion, and extension. For a 1-D code only the compression and extension curves are required because two of the principal stresses are always equal.

Initially, the consolidated surface will be appropriate for describing a material. Whenever the material fails (passes through the failure surface as I_{2D} gets larger), an index is set which determines how to mix the consolidated surface with the rubble surface to obtain the new failure surface. At low pressures the rubble curve will dominate and at high pressures, the consolidated curve will dominate (Fig. 9). Suppose the state of an element of material in Fig. 9 has gone from A to B. It has penetrated the failure surface and must fail. However, because it was in a state of tension, the new equilibrium surface is probably well described by the rubble curve since tensile failure might be expected to break it into many pieces. Thus, the element will continue to fail until the conditions of the element are within the rubble failure surface. We must allow for mechanisms which will allow the stresses to relax in order to complete the description of failure. The fraction of the failure surface apportioned to the rubble curve can be made to vary smoothly as a function of pressure. At high pressures this fraction might eventually become zero.

For relaxation associated with brittle failure, we can use a dislocation model and/or a modified Maxwell solid model similar to those reported by Wilkins.²⁴ A dislocation model just says that the stress is reduced at a constant rate. The modified Maxwell solid model says that the stress relaxation rate is proportional to the stress change the material will undergo in returning to the failure surface. For ductile failure, we can use a Von Mises-like relaxation, which simply limits the $I_{2D}^{1/2}$ value so that the stress state cannot leave the failure surface. This is appropriate at the high pressure end of Fig. 9 where the two surfaces have become the same, i.e., they are flat. We allow the model to mix these two relaxation schemes in different proportions as we go from brittle failure at negative and low pressures to ductile failure

Fig. 9. Consolidated-rubble curve.

at high pressure. Since we define the failure surface to be that locus of material states ($1^{1/2}_{2D}$ values) below which failure will not occur, we limit the brittle relaxation process so that the stress state stops relaxing when the failure surface is reached.

The shear modulus also is part of the equation of state (EOS) description and must be consistently described in relation to the other EOS considerations. It is expressed as a function of stress and energy. In particular, as the melt condition is reached, the shear modulus and shear stresses become zero. This is accomplished by multiplying the shear modulus and the failure surface by a fraction which goes linearly from one to zero as the energy goes from a specified fraction of the melt energy to the melt energy.

When material spalls, the pressures and stresses both relax completely and the shear modulus is set to zero. As material recompacts and positive pressures are achieved, the degree of ductile behavior is used as a criterion to allow the shear modulus to heal so that its full value is regained when the material is completely ductile. The failure surface is used to calculate the minimum pressure which the material can withstand; this pressure is assumed to correlate with spall. The P-V data is used to calculate the corresponding volume for spall. This spall volume is then also used as a criterion to test against, in case the material recompacts and it becomes necessary to use the P-V data again. Note that the spalling phenomenon involves discontinuous behavior when spalled material is recompacted. We have found that the noise in the solution for recompaction of spalled material is reduced by the use of artificial viscosity. Typically, only a linear q is used with a 0.5 multiplier. Artificial viscosity is discussed in the next chapter.

Of course, the failure model is only a part of the overall equation of state description, and it is essential that all the parts be consistent. For example, the P-V description must change after the material fails and a new failure surface is established. Thus, the material should not be allowed to develop negative pressures comparable to point A in Fig. 9 after the rubble curve has replaced the consolidated curve. Also the P-V relationship must include the capability of describing porosity. Thus, if a material has a high porosity, its P-V relationship for loading will

be different than that for unloading after the voids have been eliminated. This in turn may affect the failure surface.

DISCUSSION

UKO calculations²³ of the Lawrence Livermore Lab cratering experiment Pre-Schooner II have been performed. Since the code UKO is one-dimensional, the calculation is only valid for obtaining the spall (or jump-off) velocity of the earth. The measured velocity was 3.9 cm/m sec and the calculated velocity was 4.0 cm/m sec. This agreement is satisfactory when one considers the inhomogeneous nature of earth materials and the limited information available regarding their equations of state. The linear q (artificial viscosity discussed in a later chapter) also introduces a small error by converting too much kinetic energy into internal energy.

The code provides smooth and continuous material behavior with its solution converging as the zone size is reduced. The scale of the failure description is macroscopic rather than microscopic and it assumes a continuum mechanics model. Thus, one should not expect to model the propagation of a single crack using UKO. In fact, the input format allows the code user to input EOS information in a physically inconsistent way and to include discontinuities in these equations of state. Thus, as is true with almost all computer programs, the code user must appreciate the limits of the model before he can use it profitably. In order to simulate a real shock and the subsequent relaxation, one would need to model the real viscosity (rather than artificial viscosity) and have the zone size small compared to the shock rise thickness. Then one could also use physically measured values for relaxation rate constants. Unless this is done, the relaxation mechanisms must be considered as numerical artifacts and a close cousin of artificial viscosity.

V. A NEW FORM OF ARTIFICIAL VISCOSITY

INTRODUCTION

The classical paper by von Neumann and Richtmyer²⁵ (1950) discussed a numerical method for shocks. Von Neumann and Richtmyer introduced an artificial damping term (q) to spread the shock so that the hydrodynamic equations would not have discontinuities in solutions for variables. Thus, for conservation of energy, one obtains (in Lagrangian coordinates)

$$\frac{\partial \epsilon}{\partial t} = -(P + q) \frac{\partial V}{\partial t}, \quad (V-1)$$

and for conservation of momentum one obtains

$$\frac{\partial U}{\partial t} = - \frac{1}{\rho_0} \frac{\partial}{\partial x} (P + q). \quad (V-2)$$

Both equations are modified by the presence of the damping term q . The continuity equation does not contain the q , and is unchanged.

The standard approach is to write an equation for q that implies a viscous property, because physical viscosity will always spread a shock wave. Since Ref. 25 was published, it has become apparent that a number of difficulties occur when the standard method is used. One serious difficulty is that the q heating obtained at reflecting boundaries may not be consistent with the Rankine-Hugoniot conditions. Furthermore, the quadratic q for plane waves given by

$$q = \rho (c \Delta x)^2 \left(\frac{\Delta u}{\Delta x} \right)^2 = \rho c^2 |\Delta u|^2 \quad (V-3)$$

for zones undergoing compression (otherwise, $q = 0$) does not provide sufficient damping, and the solution contains numerical noise. Frequently a linear q is used to smooth the solution. A linear q lowers the order of the accuracy of the solution, and numerical experiments show that the Rankine-Hugoniot conditions are not satisfied if the linear q contribution

becomes large. Furthermore, a linear q frequently introduces oscillations, at material interfaces, which persist after the shock has passed. (Interface noise can also result from the use of empirical q forms such as those that use $\partial^2 u / \partial x^2$.)

Cameron²⁶ has proposed a method for reducing these interface errors, but his technique is insufficient. To this date, there remains a need for improving the behavior of solutions at boundaries. Thorne and Dahlgren²⁷ compiled a comparison of different computational techniques, which underlines this need for improving artificial-viscosity calculations.

THEORY AND DISCUSSION

For the quadratic q term, let us use

$$q = \rho (c \Delta x)^2 \left| \frac{\Delta u}{\Delta x} \sqrt{-\frac{\Delta P}{\Delta x} \frac{\Delta \frac{1}{\rho}}{\Delta x}} \right| \quad (V-4)$$

for zones being compressed; $q = 0$, otherwise. Integrating over a shock, this expression is approximately equivalent to that in Ref. 25 because, for a shock, the second Rankine-Hugoniot equation applies:

$$(\Delta u)^2 = -\Delta P \Delta \frac{1}{\rho}. \quad (V-5)$$

However, in differential form, the q of Eq. (V-4) is not identical to that in Ref. 25. (See Appendix VA for elaboration of this point.)

Note that away from shocks the q of Eq. (V-4) is quite different, when compared to the q of Eq. (V-3). The q of Eq. (V-4) is in a form more uniquely associated with shocks because shocks are identified by large velocity changes, large pressure changes, and abrupt density changes. All these effects are present in a shock and, conversely, when all these effects are observed, a shock is present. It is possible to have any one of the effects alone and not have a shock. For example, large velocity gradients can exist because of geometrical effects. Equation (V-4) would intrinsically treat an adiabatically squeezed sphere

correctly. No significant q would develop because the pressure gradient would be small, even though a large velocity gradient might be present. Codes frequently avoid q 's in these situations by testing on velocity gradients or other quantities. However, it seems preferable to have a q whose very form avoids these problems. Another example (as shown in Figs. 10 and 11) would be the description of a spherical charge of high explosive detonated in the earth. After the shock has passed the earth/high-explosive interface, a steep velocity gradient develops (proportional to $1/r^2$ in the earth), but the pressure gradient is small.

We are now using in Eq. (V-4) a mix of gradients of zone quantities (pressure and density⁻¹) and grid quantities (velocity). When a perfectly reflecting boundary is reached, a mirror zone is required. Boundary conditions are imposed on both zonal and grid quantities in the calculation of q . Example calculations are presented here that show that the Rankine-Hugoniot conditions are satisfied better at reflecting walls.

Within the shock, perturbations will be damped out in proportion to $e^{-\alpha t}$ (in which α is a real number) for the quadratic q . However, the value of α may be quite small near the edges of the shock wave, and additional damping may be required. This result can be obtained by using a lower-order q such as a linear one. Let us try a $3/2$ power q given by

$$q_{3/2} = \rho c^{3/2} W^{1/2} \left| \Delta u \sqrt{-\Delta P \frac{1}{\rho}} \right|^{3/4}, \quad (V-6)$$

where W is a function with dimensions of velocity. (Appendix VB examines the stability of a $3/2$ power q .) The stability analysis of q requires that the determinant resulting from a particular set of equations equals zero. This determinantal equation should change continuously when going from the shocked region to the unshocked region. This is equivalent to requiring that the damping term α should be continuous in going from the shocked to the unshocked region. The quadratic q and $3/2$ power q (see Appendix VB) both satisfy this condition; but the linear q does not, unless it is used in the shocked and unshocked regions. This, however, has the undesirable consequence of introducing more artificial dissipation.

Although a linear q has a low-order accuracy and its stability analysis leads to a discontinuity in the determinantal equation, there

Fig. 10. Velocity vs position.

Fig. 11. Pressure vs position. The dotted line represents the noise that can be introduced by a linear term.

are other arguments that make it attractive. For an elastic solid, the ratio q/P goes to zero for a quadratic q as the pressure approaches zero, whereas it approaches a constant multiplied by $\Delta P/P$ for a linear q (see Appendix VC). UNEC,²⁸ the first code applied to nuclear explosion interactions with earth materials, used a linear q in addition to the quadratic q ²⁹ to avoid this problem. However, a small multiplier is desirable for the best accuracy.

Note that in the limit of infinitely weak shocks (sound waves), the relation

$$\Delta u = \sqrt{-\Delta P \Delta \frac{1}{\rho}} \quad (V-7)$$

becomes

$$\Delta u = \frac{\Delta P}{\rho} \left(\frac{\Delta \rho}{\Delta P} \right)^{1/2} = \frac{\Delta P}{\rho SS}, \quad (V-8)$$

where SS is the speed of sound, which is just the solution of the characteristic equation for sound waves. The writer has tried this relation in q equations, and it works quite well for weak shocks. Weak shocks in this case had pressure changes of more than 10 times the pre-shock pressure. The form in Eq. (V-8) is simpler to program and could be of value in an Eulerian grid (an application that comes to mind is the simulation of air-frame dynamics) or in a multidimensional Lagrangian grid. Appendix VD examines Eq. (V-8) as to the ways in which it may be used in a formulation for q . Time derivatives might also be used for complicated meshes; they are also discussed in Appendix VD.

The use of the second Rankine-Hugoniot condition naturally suggests the use of the third Rankine-Hugoniot condition:

$$\Delta E = \bar{P} \Delta \frac{1}{\rho}, \quad (V-9)$$

where $\bar{P}$ is the average of the pressure behind the shock and the pressure in front of the shock. Thus, one might try using

$$\Delta U = \sqrt{\frac{\Delta P}{P}} \Delta E \quad (V-10)$$

in the formulation for q or some combination of Eqs. (V-7) and (V-10).

PROGRAMMING

For zones at material interfaces, the term $\sqrt{-\Delta P \Delta(1/\rho)}$ is obtained by a one-sided extrapolation.

For a reflecting boundary, an imaginary zone must be added, and its physical state must be identical to that of the boundary zone. These modifications in the coding have not increased the running time of test problems by less than 1%. Problems have been run using $\sqrt{-\Delta P \frac{\Delta t}{\rho}}$ to completely replace Δu in the q equation, and they are stable but quite noisy. All problems were run on UKO, a modified version of the elastic-plastic KO code.³⁰

KO currently uses $q_L = \rho c \Delta x (\Delta u / \Delta x) \sqrt{P/\rho}$ for a linear term; that is, it uses $\sqrt{P/\rho}$ to get a velocity term rather than using a constant or the sound speed. UKO has not followed this example for the linear term but instead uses the sound speed. This ensures the damping of shocks in an elastic medium.

A number of "time-selection" schemes have been used to investigate the importance of achieving centering for the viscosity in the energy and momentum equations. The velocity, pressure, and density were used from the previous cycle (and half cycle) and also from two previous cycles in a variety of combinations. None of these schemes evidenced any significant differences. The simplest scheme was selected, i.e., to use whatever values of velocity, pressure, and density that exist in the code at the time the viscosity is calculated. There was no attempt to iterate on the pressure at the current time step.

Let us consider one approach that might be applicable for an Eulerian formulation. Assume that material A is the material taking up the largest fraction of a zone volume, but that there is none in a neighboring zone. One might use the pressure in the neighboring zone to calculate what the density of A would be if it were present. Then this imaginary density would be used to calculate $\Delta(1/\rho)$. Of course, if neighboring zones contain a material common to both, $\Delta(1/\rho)$ can be obtained directly.

PROBLEM EXAMPLES

A number of problem results from UKO calculations are presented here. The first group of examples were run with two different q 's:

$$q_u = \rho c_1^2 |\Delta u|^2 + \rho c_2^{3/2} \left(\frac{p}{\rho}\right)^{1/4} |\Delta u|^{3/2} \quad (V-11)$$

and

$$q_p = \rho c_1^2 |\Delta u| \left| \sqrt{-\Delta P \left(\Delta \frac{1}{\rho}\right)} \right| + \rho c_2^{3/2} \left(\frac{p}{\rho}\right)^{1/4} \times \left[|\Delta u| \left| \sqrt{-\Delta P \left(\Delta \frac{1}{\rho}\right)} \right| \right]^{3/4}. \quad (V-12)$$

The same coefficients are used to spread the shock over the same number of zones in each case, thereby allowing direct comparisons. For convenience and brevity, the first will be called the velocity q and the second the pressure q . Otherwise, comparisons were made between identical problems. It is worth noting that q_u and q_p responded experimentally to the same stability condition.

The first examples will be for the case of a plane wave shock. Consider a 30-cm length of ideal gas with $\gamma = 1.4$ and one end against an immovable wall. At the other end apply a constant pressure of 1.0 Mbar. Let the gas have an initial density of 0.008 g/cm^3 and a pressure of 0.00004 Mbar. This is a good approximation to an infinitely strong shock. Let $c_1^2 = 4$ and $c_2^{3/2} = 0.4$ with zone size $\Delta x = 0.5 \text{ cm}$ (60 zones). After 2.72 μsec , the shock has bounced off the wall. Table 5 shows the pressure profile of the shock as calculated by the two different q 's.

We know analytically that, for an infinitely strong shock in an ideal gas,

$$\frac{P_r}{P_i} = \frac{3\gamma - 1}{\gamma - 1} \quad (= 8 \text{ for } \gamma = 1.4), \quad (V-13)$$

where P_r is the reflected shock pressure and P_i is the incident shock pressure. We note that the q_p solution is both smoother and more accurate.

Table 5. Pressure profile.

J (zone)	P (Mbar) for q_u	P (Mbar) for q_p
40	7.91	7.99
41	7.99	7.99
42	7.91	7.98
43	7.96	7.99
44	7.96	7.98
45	7.45	7.83
46	5.13	6.57
47	3.03	4.44
48	1.74	2.49
49	1.20	1.33
50	1.03	1.03
51	0.99	1.00
52	1.00	1.00

Table 6. Plane shock reflection.

J	For q_u			For q_p		
	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η
1	7.94	1.55	12.8	7.99	0.91	21.8
2	7.96	1.16	17.2	7.99	1.10	18.2
3	7.96	0.88	22.4	7.99	0.92	21.7
4	7.96	0.79	25.1	7.99	0.95	21.1
5	7.95	0.79	25.1	7.99	0.93	21.6
6	7.92	0.80	24.7	7.98	0.94	21.1
7	7.96	0.82	24.3	7.99	0.94	21.1
8	7.94	0.83	23.8	7.99	0.95	21.0
9	7.91	0.85	23.2	7.99	0.95	21.0
10	7.96	0.88	22.6	7.99	0.95	21.0

The shock seems to be slightly sharper, although this effect is not particularly significant. The zone values near the wall are given by Table 6. Zone J = 1 is next to the reflecting wall.

In Table 6 we observe the anomalous heating introduced by the q , but we also observe that the effect is much worse in the q_u case. For the q_p problem, the largest error for E (E should = 0.95) is about 15%, but in the q_u problem the largest error is about 60%. Thus, we see that the Rankine-Hugoniot conditions are preserved better by the use of the pressure q . The q_p calculation comes to equilibrium much quicker than q_u does; that is, the E and η values stabilize very near the reflecting wall. The values for E and η for the q_u solution stabilize and approach those of the q_p solution, but at a much greater distance from the wall (larger J value). Again we note that the q_p solution is both smoother and more accurate. We ran a weak-shock calculation, and the results were similar.

For our next example, let us investigate a spherically converging shock of infinite strength in an ideal gas of $\gamma = 1.4$. Use the same initial fluid conditions as in the previous problem, with a 1-Mbar pressure at the outside. Let the outer radius be 30 cm with a constant $\Delta x (= 0.3 \text{ cm})$ from the outside into a 6-cm radius. Let $\Delta x = 0.01 \text{ cm}$ at the origin, and let the size of zones increase by a constant percentage ($\sim 10\%$) until the zone at 6 cm radius is 0.3 cm thick. We will examine the problem solution near the origin, where the approximation of an infinitely strong shock is valid. Let $c_1^2 = 4$ and $c_2^{3/2} = 0.4$ again.

The analytic solution is due to Guderley.³¹ For $\gamma = 1.4$ we expect $P_r/P_i = 26^{3/2}$ for a fixed point in space; we also expect that $P t^2/x^2$ will be a constant,³³ where P is the pressure of the shock when the shock is at position x at time t . The time is negative for convergence, zero when the shock is at the center, and positive for divergence. The pressure ratio (reflected to incident values) is not a particularly sensitive function; both q_u and q_p give approximately the same answers ($P_r/P_i \approx 23$), even though the answers to the two problems are noticeably different in the amount of heating done to the zones in the center.

Before the results of problems with the different q 's are compared, it should be pointed out that care must be taken in evaluating the pressure following a converging shock in spherical coordinates, because there is an adiabatic pressure increase following the shock (due to convergence).

Table 7. Spherical shock reflection.

J	For q_u			For q_p		
	P (Mbar)	$E \left(\frac{\text{Mbar-cm}^3}{\text{cm}^3} \right)$	η	P (Mbar)	$E \left(\frac{\text{Mbar-cm}^3}{\text{cm}^3} \right)$	η
1	1048	242	10.8	891	127	17.5
2	1058	175	15.1	893	143	15.5
3	1077	120	22.5	896	116	19.4
4	1104	92	30.0	903	98	23.0
5	1133	85	33.5	914	85	26.9
6	1160	77	37.8	926	77	30.0
7	1170	69	42.4	932	70	33.5
8	1158	64	45.1	930	63	37.0
9	1130	59	47.6	921	57	40.0
10	1110	54	51.2	911	53	47.2

In Table 7 we can examine zonal values near the center of the sphere after the shock has been reflected (at a time when the reflected shock is at a radius of 0.17 cm). The two solutions are quite different, and we might surmise that the anomalous q heating is much improved by the pressure q . The flatter pressure distribution of the q_p solution also appears more believable, but such observations are not always conclusive.

However, we know that Pt^2/r^2 must be a constant for the shock, and in Fig. 12 this quantity is plotted for both the converging and diverging shocks. Note that the scales for the Pt^2/r^2 axes are different for the converging shock and the diverging shock. The solution is clearly much more favorable for the q_p case.

Some spherical problems were also run with weak shocks generated at the outer surface, and they displayed qualitatively similar properties.

For a final example, consider a 10-cm-radius sphere of high explosive at the center of a 100-cm-radius sphere of iron; let the explosive be ignited throughout its volume at the start of the problem, with $\Delta x = 1$ cm. Table 8 presents the pressure profile of the shock when it has nearly reached the outer boundary of the iron sphere. Also shown in

Fig. 12. Characteristics for the Guderley problem.

Table 8. Spherically detonated explosive.

J	X(cm)	P (kbar) for q_u	P (kbar) for q_p	
94	94.0	3.95	3.49	Iron
93	93.0	6.13	5.72	
92	92.0	8.32	8.15	
91	91.0	10.1	10.2	
90	90.0	11.2	11.5	
89	89.0	11.6	11.9	
88	88.0	11.5	11.7	
87	87.1	11.0	11.1	
86	86.1	10.5	10.4	
85	85.1	10.0	9.9	
.	.	.	.	
.	.	.	.	
.	.	.	.	
15	19.0	5.6	5.8	
14	18.4	5.4	5.7	
13	17.8	5.5	5.6	
12	17.3	5.1	5.5	
11	16.5	4.8	5.3	
Material Interface				
10	14.8	5.1	5.1	High Explosive
9	13.1	5.0	5.1	
8	11.4	5.1	5.1	
7	9.7	5.1	5.1	
6	8.0	5.1	5.1	

Table 8 is the region of the explosive-iron interface at the same time (time = 178 μ sec). Here $c_1^2 = 5.5$ and $c_2^{3/2} = 3.0$ (more damping is usually required for spherically diverging waves in solids). The q_p solution is better because the shock is sharper and the pressure profile is smoother at the interface.

A wide variety of other problems (and materials) were examined, all of which showed improvement through the use of q_p . A plane-wave problem was run with a pressure profile on each end; and, after the shocks collided, the result was identical to the result of the rigid-wall problem. Planar problems with many materials (gases and solids) were run to examine the anomalous heating at interfaces; in every case, the q_p result was a considerable improvement over the q_u result. Note that all the above problems avoided interfaces with large differences in zone size. Problems were run with large zone-size changes at interfaces, and the anomalous heating was a serious problem for both forms of the q (although slightly less troublesome for q_p).

The previous problem examples were run basically with a quadratic term, while the $3/2$ power term could be considered as a perturbation term (admittedly large in the last example) to obtain smoothing. This meant that the analysis in Ref. 25 applied reasonably well.

We have also seen that the $3/2$ power term has desirable damping properties by itself.

We will now examine some problems that have only one q term, in order to isolate specific effects and to demonstrate that the $3/2$ power q , alone, is sufficient for many problems. The problem example used here is the same as the first: a plane strong shock in gas with $\gamma = 1.4$. Table 9 gives the results of using several different q 's. The constants were chosen to provide shock widths of the same size. The results can also be compared to those of Table 6.

First we note that the $3/2$ power term in $|\Delta u|$ has provided considerable smoothness and slightly more accuracy than the quadratic term provides. This results from the fact that the $3/2$ power term provides more damping near the trailing edge of the shock. However, going to $3/2$ power in Δu has not helped the heating at the wall.

The heating at the wall has been improved by the Δu mixed with $\sqrt{-\Delta P \Delta(1/\rho)}$ (and the accuracy in the pressure column has also been slightly improved). One might anticipate that the three-point difference scheme for ΔP or $\Delta(1/\rho)$ is a reason for the big improvement and replace Δu by $\sqrt{-\Delta P \Delta(1/\rho)}$ entirely. However, the results of such a calculation are similar to the pure $|\Delta u|$ examples in Table 9, although the calculation is noisier. This supports the idea that it is the mix of zone and grid quantities that provides the improvement.

Table 9. Plane shock reflection.

J (wall)	$q = 4\rho \Delta u ^2$		$q = 3\rho\left(\frac{P}{\rho}\right)^{1/4} \Delta u ^{3/2}$		$q = 3\rho\left(\frac{P}{\rho}\right)^{1/4} \Delta u ^{3/4} \left \sqrt{-\Delta P \Delta \frac{1}{\rho}} \right ^{3/4}$	
	P	E	P	E	P	E
1	7.959	1.54	7.974	1.59	7.994	0.96
2	7.953	1.13	7.974	1.07	7.993	1.13
3	7.954	0.86	7.974	0.96	7.993	0.95
4	7.938	0.78	7.975	0.83	7.992	0.96
5	7.938	0.79	7.974	0.84	7.992	0.94
6	7.935	0.80	7.974	0.86	7.992	0.95
7	7.925	0.82	7.975	0.87	7.991	0.95
8	7.930	0.84	7.974	0.89	7.992	0.95
9	7.937	0.86	7.974	0.90	7.992	0.95
10	7.932	0.87	7.974	0.92	7.993	0.95

Plane-wave ideal-gas ($\gamma = 1.4$) problems have been run with a small q coefficient, specifically using

$$q = 2\rho\left(\frac{P}{\rho}\right)^{1/4} |\Delta u|^{3/4} \left| \sqrt{-\Delta P \Delta \frac{1}{\rho}} \right|^{3/4}, \quad (V-14)$$

and the results are extremely smooth and accurate (almost as good as the right-hand column of Table 5. However, the pressure values at the shock give a much sharper shock (by a factor of approximately 2/3). The accuracy of the values near the shock is still excellent and virtually noise-free. Cutting the multiplier in half (from 2 to 1) does, however, lead to a noisy and less satisfactory solution. A smaller multiplier means savings in machine time.

This chapter has not examined the linear q formulation. However, problems have been run using it, and their performance is improved by mixing zone and grid quantities via the second Rankine-Hugoniot condition.

USE OF THE THIRD RANKINE-HUGONIOT EQUATION

Up to this point, we have seen that shock shapes are relatively unaffected by use of the second Rankine-Hugoniot equation. This might have been anticipated, since Δu and $\sqrt{-\Delta P \Delta (1/\rho)}$ could be expected to show

similar functional behavior in a shock. However, the third Rankine-Hugoniot equation is

$$\Delta E = - \frac{P_1 + P_2}{2} \Delta \frac{1}{\rho}, \quad (V-15)$$

where P_1 and P_2 are the pressure values of the regions before and after the shock. These values (P_1 and P_2) are really unknowns, and the replacement of $(1/2)(P_1 + P_2)$ by some quantity such as P will introduce an uncertain functional behavior into a q . In fact, it will be seen that the shape of the shock can be significantly affected.

A wide variety of problem types have been run, and they indicate that the third Rankine-Hugoniot equation provides improvement in the anomalous heating at boundaries and interfaces. Again, the best choice for q is indicated experimentally to be an equal mix of zone quantities and grid quantities.

Then a $3/2$ power q would have the following form:

$$q = \rho c^{3/2} W^{1/4} |\Delta u|^{3/4} \left| \sqrt{\Delta P \left(- \frac{\Delta(1/\rho) \Delta E}{P} \right)^{1/2}} \right|^{3/4}. \quad (V-16)$$

The results of this q in a plane strong shock for $\gamma = 1.4$ (like the results of Table 9) are given in Table 10 and are compared to the best q of Table 9.

The accuracy is considerably improved at the wall; the size of the worst error in the calculation of E has been reduced by a factor of $2\frac{1}{2}$. At the same time, we note that the shock is sharper and of a considerably different shape for the reflected shock shown in Table 10. However, the incident shock did not have a significantly different shape. Weak shock problems also did not display significantly different shock shapes for the different q forms.

Approximating $\frac{1}{2}(P_1 + P_2)$ by some quantity such as P might be found to have adverse effects for some material equations of state, although this has not yet been observed. Equation (V-16) also provides less damping at the trailing edge of the shock and results in a noisier solution, which can be smoothed by adding more damping (such as a lower-order of q).

Table 10. Plane shock reflection.

$q = 3\rho\left(\frac{p}{\rho}\right)^{1/4} \Delta u ^{3/4} \left \sqrt{-\Delta P \Delta \frac{1}{\rho}} \right $ $q = 3\rho\left(\frac{p}{\rho}\right)^{1/4} \Delta u ^{3/4} \left \sqrt{\Delta P - \Delta \frac{1}{\rho} \frac{\Delta E}{P}} \right ^{3/4}$				
J	P	E	P	E
51	1.000	0.417	1.000	0.417
50	1.005	0.420	1.004	0.417
49	1.065	0.427	1.058	0.424
48	1.456	0.484	1.565	0.497
47	2.802	0.652	4.040	0.756
Shock 46	4.728	0.804	6.555	0.900
45	6.506	0.896	7.750	0.947
44	7.571	0.938	7.981	0.955
43	7.945	0.951	7.957	0.954
42	7.992	0.952	7.985	0.954
41	7.994	0.952	7.976	0.953
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
10	7.993	0.951	7.998	0.950
9	7.992	0.950	7.997	0.949
8	7.992	0.950	7.997	0.947
7	7.991	0.945	7.996	0.944
6	7.992	0.946	7.996	0.939
5	7.992	0.940	7.997	0.938
4	7.992	0.964	7.997	0.937
3	7.993	0.955	7.998	0.948
2	7.993	1.140	7.998	1.03
Wall 1	7.994	0.963	7.998	0.929

CONCLUSIONS

Empirically derived forms of artificial viscosity have been described that are uniquely associated with shocks; and, for the problems investigated, they provide smoother and more accurate answers, in comparison with methods given in Ref. 25. They improve solution behavior at interfaces and, in particular, they improve anomalous heating by a considerable factor at walls. Whether a quadratic term, 3/2 power term, linear term, some other term, or a combination of terms should be used

in a calculation depends on the types of materials being modeled and the pressure regime of interest. In Appendix VE are presented some conclusions I've reached regarding the use of these artificial viscosity forms.

A weak-shock version could be used easily in some Eulerian codes, but clever programming would be required for an Eulerian code capable of describing strong-shock systems; still, one suspects it can be done, just as one might also expect benefits in multidimensional Lagrangian codes. The use of Rankine-Hugoniot relations in the difference equations for shock-sensitive forms may be widely useful, for example, in conservative schemes.

Appendix VA

Equation Equivalence

The q Eqs. (V-3) and (V-4) would be identical if

$$\frac{\partial U}{\partial x}^2 = - \frac{\partial P}{\partial x} \frac{\partial (1/\rho)}{\partial x} . \quad (\text{VA-1})$$

Consider a coordinate system moving with the shock; then ω is the distance from the shock where

$$\omega = x - st, \quad (\text{VA-2})$$

where s is the shock speed. Then we would have identical forms if

$$\left(\frac{dU}{d\omega} \right)^2 = - \frac{dP}{d\omega} \frac{dV}{d\omega}, \quad (\text{VA-3})$$

where $V = 1/\rho$. But we know from the continuity equation that

$$-M \frac{dV}{d\omega} = \frac{dU}{d\omega}, \quad (\text{VA-4})$$

where $M = \rho_0 s$ (ρ_0 is the density in undisturbed regions). Thus the forms are the same if

$$-M^2 \frac{dV}{d\omega} = \frac{dp}{d\omega}. \quad (\text{VA-5})$$

In contradiction, however, the equation of motion [Eqs. (V-2)] and (VA-4) gives the result

$$-M^2 \frac{dV}{d\omega} = \frac{d}{d\omega} (p + q). \quad (\text{VA-6})$$

For the solution by von Neumann and Richtmeyer,²⁵ $\int \frac{dq}{d\omega} d\omega = 0$ when evaluated over the shock. This will be true for any q that equals zero at the shock edges. Thus, one is not surprised that the new q gives results like those of Ref. 25. In fact, numerical experiments with the pressure q produce shock-wave shapes (pressure vs ω , density vs ω , etc.) that are virtually indistinguishable from the traditional q given by von Neumann and Richtmeyer.

Appendix VB

The 3/2 Power Damping Analysis

Consider perturbations δU , δV , δP , and δq on the desired solutions for a γ -law gas. Assume these perturbations are of the form

$$\delta U = \delta U_0 e^{ikx + \alpha t}, \quad \delta V = \delta V_0 e^{ikx + \alpha t}, \text{ etc.}, \quad (\text{VB-1})$$

where δU_0 , δV_0 , δP_0 , δq_0 , k , and α are constant and k is real. We then obtain from the equations of continuity, momentum, and energy and the equation for q the following set of linear equations:

$$\alpha \rho_0 \delta V_0 - ik \delta U_0 = 0 \quad (\text{VB-2})$$

$$ik \delta q_0 + \alpha \rho_0 \delta U_0 + ik \delta P_0 = 0 \quad (\text{VB-3})$$

$$(\gamma - 1) \frac{\partial V}{\partial t} \delta q_0 + \left[\frac{\partial P}{\partial t} + \alpha \gamma P + \alpha (\gamma - 1) \frac{(c \Delta x)^{3/2}}{V} \right. \\ \left. \times \left| \frac{\partial U}{\partial x} \right|^{1/2} \frac{\partial U}{\partial x} \right] \delta V_0 + \delta P_0 \left(V \alpha + \gamma \frac{\partial V}{\partial t} \right) = 0 \quad (\text{VB-4})$$

$$-\delta q_0 + \sqrt{W} \frac{(c \Delta x)^{3/2}}{V^2} \frac{\partial U}{\partial x} \left| \frac{\partial U}{\partial x} \right|^{1/2} \delta V_0 - \frac{3}{2} ik \frac{(c \Delta x)^{3/2}}{V} \sqrt{W} \left| \frac{\partial U}{\partial x} \right|^{1/2} \delta U_0 = 0, \quad (\text{VB-5})$$

where we have used

$$q = -\frac{(c \Delta x)^{3/2}}{V} W^{1/2} \left| \frac{\partial U}{\partial x} \right|^{1/2} \frac{\partial U}{\partial x} \quad (\text{VB-6})$$

for a shock traveling in the positive x direction. (Thus q is positive, since $\partial U / \partial x$ is negative.) The determinant of the above must equal zero in order to have a solution. We then obtain the determinantal equation:

$$\begin{aligned} 0 = & -(\alpha \rho_0)^2 \left(V \alpha + \gamma \frac{\partial V}{\partial t} \right) + (\gamma - 1) \alpha \rho_0 \frac{3}{2} \frac{(c \Delta x)^{3/2}}{V} \left| \frac{\partial U}{\partial x} \right|^{1/2} \sqrt{W} k^2 \frac{\partial V}{\partial t} \\ & - \alpha \rho_0 k^2 \frac{3}{2} \frac{(c \Delta x)^{3/2}}{V} \left| \frac{\partial U}{\partial x} \right|^{1/2} - k^2 (\gamma - 1) \frac{\partial V}{\partial t} \frac{(c \Delta x)^{3/2}}{V^2} \sqrt{W} \frac{\partial U}{\partial x} \left| \frac{\partial U}{\partial x} \right|^{1/2} \\ & + k^2 \left(V \alpha + \gamma \frac{\partial V}{\partial t} \right) \frac{(c \Delta x)^{3/2}}{V^2} \frac{\partial U}{\partial x} \left| \frac{\partial U}{\partial x} \right|^{1/2} \\ & - k^2 \left[\frac{\partial P}{\partial t} + \alpha P + \alpha (\gamma - 1) \frac{(c \Delta x)^{3/2}}{V} \left| \frac{\partial U}{\partial x} \right|^{1/2} \frac{\partial U}{\partial x} \right]. \quad (\text{VB-7}) \end{aligned}$$

If we restrict ourselves to the dominant terms (highest order in k and α), we get

$$\alpha = -\frac{3}{2} k^2 \frac{(c\Delta x)^{3/2}}{\rho_0 V} \sqrt{W} \left| \frac{\partial U}{\partial x} \right|^{1/2} \text{ in shocked regions} \quad (\text{VB-8})$$

and

$$\alpha^2 = -\frac{k^2 \gamma P}{2 \rho_0 V} \text{ in normal regions.} \quad (\text{VB-9})$$

Thus, perturbations are damped out in the shock, but they are propagated without change in normal regions. This result is like von Neumann and Richtmeyer's²⁵ for the quadratic q . A linear q would have obtained a value for α in the shock region proportional to $|\partial U/\partial x|$, which does not go to zero at the shock boundaries. (At this point, it is appropriate to restate that the purpose of the q is to eliminate the discontinuities in the solutions of the differential equations, so that difference equations can be applied.) It would seem that the decay constant α is one solution variable (even though a numerical artifact) that should be continuous. Thus, we would prefer to avoid a linear q if at all possible. The same kind of analysis can be performed for

$$q = \frac{(c\Delta x)^{3/2}}{V} W^{1/2} \left(\frac{\partial U}{\partial x} \sqrt{\frac{\partial P}{\partial x} \frac{\partial (1/\rho)}{\partial x}} \right)^{3/2},$$

with similar results.

Appendix VC

Elastic Solid EOS

An elastic solid is characterized by an equation of state (EOS) for which the pressure is a function of only the volume. The following equation defining the bulk modulus (K) is a standard way to express the EOS:

$$K = -V \frac{dP}{dV}, \quad (\text{VC-1})$$

where V is the specific volume ($1/\rho$) and P is the pressure. The EOS is complete if K is given as a function of volume.

Then, for the quadratic q we obtain

$$\frac{q}{P} = \frac{\rho c^2 |\Delta P \Delta V|}{P}. \quad (\text{VC-2})$$

But $\Delta V = \frac{-V \Delta P}{K}$ from Eq. (V-C1), hence

$$\frac{q}{P} = \frac{V \rho c^2}{K} \left(\frac{\Delta P}{P} \right) \Delta P, \quad (\text{VC-3})$$

which approaches zero as ΔP approaches zero.

For the linear q we have

$$\frac{q_L}{P} = \frac{(SS) \rho c \Delta u}{P}, \quad (\text{VC-4})$$

where SS is the sound speed. Then, using Eq. (VC-1) we get

$$\frac{q}{P} = \frac{V^{1/2} SS \rho c}{K^{1/2}} \left(\frac{\Delta P}{P} \right). \quad (\text{VC-5})$$

This function is well-behaved, in the sense that it is only the relative value of ΔP compared to P that matters.

Appendix VD

Eulerian and Multidimensional Forms

Introduction

Previously, we introduced a new form of artificial viscosity (q) which performs better in a one-dimensional Lagrangian code by substituting the Rankine-Hugoniot (R-H) relations directly into the form originally suggested by Von Neumann and Richtmeyer.²⁵ However, these new forms are difficult to program in multidimensional or Eulerian programs because of the presence of interfaces between different materials. Across interfaces, density (or energy) gradients cannot be used in the R-H relations. This appendix will suggest a form that will not have such difficulties.

Discussion

From the first R-H equation, we have

$$\rho_0 \dot{\xi} = \sqrt{-\Delta p \left(\Delta \frac{1}{\rho}\right)^{-1}}, \quad (\text{VD-1})$$

where $\dot{\xi}$ is the shock speed, ρ_0 is the unshocked material density, and Δp and $\Delta \frac{1}{\rho}$ are the changes in pressure and inverse density across the shock. From the second R-H equation we have

$$\Delta U = \sqrt{-\Delta p \Delta \frac{1}{\rho}}, \quad (\text{VD-2})$$

where ΔU is the change in material velocity across the shock. Combining (VD-1) and (VD-2) yields

$$\Delta U = \frac{\Delta p}{\rho_0 \dot{\xi}}. \quad (\text{VD-3})$$

Since pressure is a continuous variable across interfaces, this would be a convenient relation to program into the formulation for q . Unfortunately,

ξ is usually an unknown. Instead, let us examine the replacement of ΔU in a q as follows:

$$\Delta U \rightarrow \frac{\Delta p}{\rho c_s} \quad (\text{VD-4})$$

For physical systems involving gravitational potentials in astrophysical systems, it might be necessary to use only that portion of Δp which can give rise to velocity changes (i.e., that portion not balanced by a potential gradient). The sun is an example of a system possessing a large pressure gradient in balance with a gravitational potential — yet, there is no velocity change.

For infinitely weak shocks (sound waves), $\Delta U = \Delta p / (\rho c_s)$. However, for strong shocks, ρc_s will be smaller than $\rho_0 \xi$ ahead of the shock, but the reverse will be true behind the shock. Averaged over a shock of finite width, one might anticipate that using Eq. VD-4 in a q formulation will not change the overall behavior too much. However, it would produce increased damping at the leading edge of a shock and less at the trailing edge. Numerical experiments indicate that a good form of q is

$$q = \rho c^2 |\Delta U|^{3/2} \left| \frac{\Delta p}{\rho c_s} \right|^{1/2} \quad (\text{VD-5})$$

A larger power in the exponent of $(\Delta p / \rho c_s)$ results in a noisy solution, consistent with the fact that Eq. VD-4 results in less damping at the shock's trailing edge. The q in Eq. VD-5 is sensitive to gradients in zone quantities (pressure) and grid quantities (velocities). Calculations using VD-5 give more accurate results than the standard q ($q = \rho c^2 |\Delta U|^2$) at interfaces and reflecting walls; anomalous heating is essentially eliminated for an adiabatically compressed sphere (which has no pressure gradients).

K. Trigger³⁴ has suggested that time derivatives might also be programmed conveniently in various computer codes. Thus, one might take advantage of the characteristic solution nature of shock wave propagation. Then for plane wave shocks we can use

$$\left| \frac{\Delta f}{\Delta x} \right| = \left| \frac{1}{\xi} \frac{\Delta f}{\Delta t} \right| \quad (\text{VD-6})$$

since $f = f(x - \xi t)$, where f is velocity, pressure, density, etc. Again, it might be necessary to use such a substitution to a low order if ξ is approximated by c_s .

Example Calculation

In Table 11 below we present the result of a $\gamma = 1.4$ gas ($\rho_0 = 0.008$ g/cc, $p_0 = 1.6 \times 10^{-5}$ Mbar) after an infinite pressure (1 Mbar) has been applied to one end of a one-dimensional slab and bounced off a

Table 11. Shock reflection at a rigid wall.

$$q = 2.0 \rho |\Delta U|^{3/2} \left| \frac{\Delta p}{\rho c_s} \right|^{1/2} + 0.2 \rho c_s |\Delta U|^{3/4} \left| \frac{\Delta p}{\rho c_s} \right|^{1/4}$$

Zone	Pressure (Mbar)	Energy (Mbar cc/cc)
1 ^a	7.93	1.10
2	7.94	0.95
3	7.92	0.87
4	7.94	0.88
5	7.95	0.91
6	7.93	0.93
7	7.96	0.94
8	7.94	0.95
9	7.96	0.96
10	7.96	0.96
11	7.95	0.95

^aZone 1 is next to the reflecting wall.

perfectly reflecting wall. The analytic solution for the reflected pressure is 8.0 Mbar and for the reflected energy, 20/21 (approximately = 0.95) Mbar - cc/cc. Thus, the worst error in the energy is approximately 15%. The same problem run with the standard q (using the same constants to

multiply the quadratic and linear components) resulted in an error of 60%. The improved q reported earlier

$$\left(q_p = pc^2 \Delta U \sqrt{-\Delta p \Delta \frac{1}{\rho}} \right)$$

resulted in an error of 10%. Thus, the thermodynamics are almost as good as that obtained with q_p and the programming is much simpler. For the one-dimensional Lagrangian code used here, the programming is only important for one-zone regions. But for other codes (Eulerian and multi-dimensional) the significance of simpler programming may be quite important. There are many kinds of such computer programs and I have not attempted any investigations using Eq. VD-5 with any of them. A wide variety of tests were run with a one-dimensional Lagrangian code in addition to the one presented here in the example; in every case the results were like those of this example. That is, the new q of Eq. VD-5 was almost as good as q_p and much better than q_u .

Appendix VE

The Linear Q.

Viecelli³⁵ has recently shown how a linear q determines the form of the solution for spherically decaying shocks in a solid. This work gives ample evidence of the need to minimize the linear q coefficient. The linear q introduces a serious problem for solids in that too much kinetic energy is converted into internal energy. Thus calculated velocities are too small. In Viecelli's analysis, the linear q is calculated for positive pressures, regardless of the sign of $\partial u / \partial x$. (Thus, q_L is negative for expanding zones.) I have found it convenient to implement the linear q in the same fashion; but q is set equal to zero if its absolute value is less than a given fraction (typically set = 0.001) of the pressure. For most problems, $c^2 = 2$ (for the quadratic multiplier) and $c_L = 0.2$ proves quite adequate. This provides sharp shocks (a smaller c^2 than was used in the earlier examples) that are still smooth, and yet the c_L is not too large for most calculations. Then we have

$$\left. \begin{array}{l} \text{I. for } P \leq 0.0, \\ \quad q = 0.0 \\ \text{II. for } P > 0.0, \\ \quad \text{for } dV < 0 \left\{ \begin{array}{l} q = 2.0 \rho \Delta u (-\Delta P \Delta 1 / \rho)^{1/2} \\ \quad + 0.2(SS)\rho |\Delta u|^{1/2} (-\Delta P \Delta 1 / \rho)^{1/4} \\ \text{for } dV \geq 0 \left\{ \begin{array}{l} q = -0.2(SS)\rho |\Delta u|^{1/2} (-\Delta P \Delta 1 / \rho)^{1/4} \end{array} \right. \end{array} \right. \\ \text{If } |q| < \text{fraction (typically } 0.001)P, \\ \quad q = 0.0 \end{array} \right\} \quad (\text{VE-1})$$

Furthermore, I have observed that using a large Δt improves the anomalous heating slightly, provided that the stability condition is not violated. The form recommended here is quite similar to the one published in Ref. 30.

Let

$$\Delta t = \frac{2}{3} \frac{\Delta x}{\sqrt{(SS)^2 + b^2}}, \quad (\text{VE-2})$$

where $b = 2c^2 \Delta x \frac{\dot{V}}{V}$, and SS is the sound speed.

Using this Δt control and the q multipliers given in Eq. (VE-1), the largest error in the anomalous heating effect for the problem described in Tables 6, 7, 9 and 10 was only 8% (similar to the best result in Table 10).

These recommendations are obviously subjective and reflect my set of calculations and my bias regarding shock sharpness, smoothness, accuracy, computer time, etc.

VI. SUMMARY AND DISCUSSION

The field of explosive cratering technology has not received much attention from the technical community until very recently. Hence, it is not surprising that the methods are primitive or based on erroneous assumptions. The purpose of this thesis has been to advance the technology associated with predicting cratering phenomena. This attempt has been made along two separate avenues; one being a rule-of-thumb formula approach and the other being numerical simulation.

The effort to replace the "scaling law curves" with a general semi-empirical formula founded on a physical model has been quite successful. One major improvement lies in the formulas' applicability to a broad continuum of rock types. One might expect improvements in the formula by making refinements. For example, it is likely that a better method can be obtained for determining the effect of strength. This might come about by computer simulation. But if significant improvements are to be obtained, it will be necessary to avoid making such assumptions as that of the material being isotropic and homogeneous. However, avoiding such assumptions makes the model much more complicated. In fact, the method starts to approach that of computer simulation!

Advances in computer simulation are also quite significant and they should eventually allow for an improvement in accuracy for predicting optimum depth of burial. These improvements include a new and general way of describing a material's failure surface efficiently in a computer calculation. A numerical model for relaxing stresses and for changing the failure envelope as a result of failure has also been developed. The computer codes used previously in the Plowshare Program have the unacceptable feature of containing zone size dependent physics in the failure model. This new failure model does not. Finally, a new form of artificial viscosity has been developed which provides smoother (less noisy) solutions and which also provides more accurate thermodynamics when shocks cross interfaces, reflecting walls, or boundaries. Unfortunately, it will be hard to take full advantage of these improvements. This is because the experimental techniques for measuring the rock properties economically are not satisfactory. Thus the computer

technique's major advantage of being able to simulate in detail is severely limited by a lack of data which will provide that detail.

In conclusion, it seems that the most urgent need for advancement in crater phenomenology prediction will involve the development of cheaper methods of measuring the material properties accurately throughout the region of interest.

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