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Energy Loss of a Relativistic Electron Beam In a Plasma

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by

R. A. Gerwin

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NOTATION

n_b = density of beam electrons in the laboratory frame

u = beam speed

c = light speed

$\beta = u/c$

$\gamma = (1 - \beta^2)^{-\frac{1}{2}}$

a = beam radius

D = beam length

$\tau = D/u$ = beam pulse time

m = electron rest mass

$v = (n_b \pi a^2) (e^2/mc^2) = \frac{1}{4} \omega_b^2 a^2/c^2$

n_p = density of plasma electrons

$\omega_p = (4\pi n_p e^2/m)^{\frac{1}{2}}$ = electron plasma frequency

$\kappa_u = \omega_p/u$

$\kappa_c = \omega_p/c$

ν_c = collision frequency of plasma electrons

$\ell_c = u/\nu_c$ = collisional decay length; $\nu_c \tau = D/\ell_c$

$\ell_j = (u/\nu_c) a^2 \kappa_c^2 = \ell_c a^2 \kappa_c^2$ = decay length of plasma reverse current

k_z = axial wave-number

$\epsilon_c = \nu_c/(u k_z)$

$\mathcal{E}$ = energy of a beam particle

ℓ_f = active length of beam involved in energy loss to longitudinal plasma waves

L_f = characteristic slowing-down length due to energy loss to longitudinal plasma waves

ℓ_j = active length of beam involved in energy loss to reverse current

L_j = characteristic slowing down length due to energy loss to reverse current

$|Z_-|$ = distance of an observation point behind the beam head

Z_+ = distance of an observation point ahead of the beam tail

P = expended power

P_b = initial beam power

ENERGY LOSS OF A RELATIVISTIC ELECTRON BEAM IN A PLASMA

by

R. A. Gerwin

ABSTRACT

The energy loss of a finite radius, finite length, low density, relativistic electron beam propagating through a resistive, infinite, field-free plasma is calculated. The presentation is both more general and more quantitative than the estimate of Lovelace and Sudan [Phys. Rev. Lett. 27 (1971) 1256] who neglected the excitation of plasma oscillations, assumed a beam of infinite extent behind the beam head, and did not utilize the exact solution of the relevant electromagnetic diffusion equation. Here, a unified calculation of the energy loss to both longitudinal and transverse plasma disturbances is presented wherein are incorporated the exact solutions of Maxwell's equations coupled to a linearized cold plasma model. In the appropriate parameter range, the results are in essential agreement with the work of Lovelace and Sudan but encompass a broader parameter range.

It is noted that the condition usually stated for the validity of the linearized plasma dynamics, beam density (n_b) $\ll$ plasma density (n_p), is insufficient. Instead, the much more restrictive condition, $n_b/n_p \ll \nu_c/\omega_p$, is required, where ν_c and ω_p are the plasma-electron collision frequency and plasma frequency, respectively.

I. INTRODUCTION

Recent developments in electron beam technology have allowed the production of very high-power beams ($\sim 10^{12}$ watts). Such beams may have a number of applications, such as the production and heating of nuclear-fusion plasmas.¹

A beam traveling in vacuum not only suffers from lack of space-charge neutralization, but at sufficiently high currents the beam electrons are unable to propagate across their self-magnetic field. This leads to a limiting maximum current, $I_A = 17000 \beta \gamma$ amps, where $\beta = u/c$, $\gamma = (1 - \beta^2)^{-1/2}$, with u = beam speed and c = light speed.²

It has been found that a plasma can current-neutralize as well as charge-neutralize an electron beam,³ thus allowing the above-mentioned current-limitation to be overcome. Lovelace and Sudan⁴ pointed out that appreciable beam drag and associated plasma heating are to be expected from excitation of the plasma reverse current by high-current

beams. As they point out, the importance of this mechanism is that the ions can be heated by the reverse-current drift velocity of plasma electrons via the ion-acoustic instability, whereas the beam-electron, plasma-electron two-stream instability primarily heats electrons.

Lovelace and Sudan⁴ estimated the beam energy loss due to excitation of the reverse current. They neglected the excitation of plasma oscillations, assumed an infinitely extended beam behind the beam head, and did not fully utilize the relevant electromagnetic diffusion equation. Here, more quantitative results are obtained, allowing both for the excitation of plasma oscillations and for finite length beams. The beam can easily be shorter than the reverse current decay length, so finite beam-length calculations are relevant. For very long beams, the present results are in essential agreement with those of Ref. (4).

II. GENERAL RESULTS

Let P_0 be the initial beam power, n_b the beam density as observed in the laboratory frame, a the beam radius, and define a beam density parameter

$$\nu = n_b \pi a^2 e^2 / (mc^2) = \frac{1}{4} (\omega_p^2 a^2 / c^2) \quad (1)$$

where m is the electron rest mass.

Then, the fractional power lost (hence the minus sign) by the beam has been found to be (as derived later),

$$P/P_0 = -C(f + g) \quad (2)$$

where

$$C = 8 \frac{\nu}{\gamma - 1} = \frac{240}{\beta} \frac{I'}{V'} \quad (3)$$

(I' is the current in amps, V' the voltage in volts), with the contribution due to radiation of longitudinal waves reading

$$f = \left[1 - e^{-\frac{1}{2} \frac{D}{\ell_c} \cos(\kappa_u D)} \right] \cdot \int_0^{\infty} x \, dx \frac{J_1^2(x)}{x^2} \frac{1}{x^2 + a^2 \kappa_u^2} \quad (4)$$

and the contribution due to excitation of the reverse current being

$$g = \beta^2 \int_0^{\infty} x \, dx \frac{J_1^2(x)}{x^2} \frac{a^2 \kappa_c^2}{x^2 + a^2 \kappa_c^2} \cdot \frac{1 - \exp\left(-\frac{D}{\ell_c} \frac{x^2}{x^2 + a^2 \kappa_c^2}\right)}{x^2} \quad (5)$$

Here, D is the beam length, $\kappa_u = \omega_p / u$ and $\kappa_c = \omega_p / c$ with ω_p the plasma-electron plasma frequency, and $\ell_c = u / \nu_c$ with ν_c the plasma-electron collision frequency. Also, $J_1(x)$ is a Bessel function of the first kind, of order unity. A basic assumption in the derivation of these expressions has been $\nu_c \ll \omega_p \beta$, and precludes the limit $\gamma \rightarrow 1$ ($u \rightarrow 0$) in Eq. (3).

III. EVALUATION OF THE LONGITUDINAL RADIATION LOSS

Since $\int_0^{\infty} x^{-1} J_1^2(x) dx = \frac{1}{2}$, Eq. (4) becomes, for large $a^2 \kappa_u^2$ which is usually the case,

$$f = \left[1 - e^{-\frac{1}{2} \frac{D}{\ell_c} \cos(\kappa_u D)} \right] \frac{1}{2 a^2 \kappa_u^2} ; a^2 \kappa_u^2 \gg 1 \quad (6)$$

and if $D \gg \ell_c$ as may realistically be expected, Eqs. (2) and (3) give the fractional power loss due to radiation of longitudinal plasma oscillations as

$$Cf = 4 \frac{\nu}{\gamma - 1} \frac{u^2}{\omega_p^2 a^2} = \frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1} ; a^2 \kappa_u^2 \gg 1 ; D \gg \ell_c, \quad (7)$$

where n_p is the plasma electron density.

Attempts have been made to produce millimeter beams with a Bennett cathode guide.⁵ In a plasma of density 10^{14} cm^{-3} , one would then have, when $u \sim c$, $u/\omega_p \sim a$ or $a\kappa_u \sim 1$. For general values of $a\kappa_u$, one then needs to evaluate the complete Bessel function integral of Eq. (4). This can be done approximately in terms of elementary functions, if one notes (see Fig. 1) that the dominant range of integration is determined by $J_1^2(x)/x^2$ which is fairly well represented in this range by $0.28 (1 - \frac{1}{3}x)$. (The integration is now cut off at $x = 3$). Then one finds, from Eq. (4),

$$f = 0.28 \left[1 - e^{-\frac{1}{2} \frac{D}{\ell_c} \cos(\kappa_u D)} \right] f\left(\frac{3}{a\kappa_u}\right), \quad (8)$$

where the function $f(\epsilon)$ is plotted in Fig. 2, and reads

$$f(\epsilon) = \frac{1}{2} \ln(1 + \epsilon^2) - (1 - \epsilon^{-1} \arctan \epsilon). \quad (9)$$

As $\epsilon \rightarrow 0$, $f(\epsilon) \approx \frac{1}{8} \epsilon^2$, so the approximate f of Eq. (8) becomes $f \approx [1 - \exp(-\frac{1}{2} D/\ell_c) \cos(\kappa_u D)] 0.42 (a\kappa_u)^{-2}$ for $a^2 \kappa_u^2 \gg 1$, which is only 16% smaller than the exact expression, Eq. (6).

Fig. 1. An approximation for $J_1^2(x)/x^2$.

Fig. 2. An estimate of the $(a\kappa_u)$ dependence of Eq. (4). Here, $\epsilon \equiv 3/(a\kappa_u)$.

Combining Eqs. (2), (3), and (8), one has the fractional power radiated to longitudinal oscillations for $a\kappa_u \geq 1$, namely

$$C_f \approx 2.24 \frac{v}{\gamma - 1} f\left(\frac{3}{a\kappa_u}\right). \quad (10)$$

For $\epsilon \approx 3$ or $a\kappa_u \approx 1$, Fig. 2 gives $f(\epsilon) \approx \frac{1}{2}$, and the fractional radiated power is then of order $v/(\gamma - 1)$.

As discussed later, the result for $a\kappa_u \approx 1$ is valid only for very small $-v$ beams.

IV. EVALUATION OF THE POWER EXPENDED IN DRIVING THE REVERSE CURRENT

A. Short Beams

Suppose $v_c = 3 \times 10^{10} \text{ sec}^{-1}$, $u = c$, $\omega_p = 10^{12} \text{ sec}^{-1}$, and $a = 1 \text{ cm}$. Then the reverse current exists within a length l_j of the beam head,^{3,4} where $l_j = \frac{u}{v_c} \left(\frac{\omega_p a}{c}\right)^2 \approx 10^3 \text{ cm}$ for this example. For beam pulse times $< 30 \text{ nsec}$, one would then have $D < l_j$. Thus, the short-beam case is realistic.

Now consider Eq. (5) for $\mathcal{J}$. (Note that in the absence of collisions, $l_c \rightarrow \infty$, $\mathcal{J} \rightarrow 0$, and there is no net energy loss to the reverse current. This is because the transverse waves associated with this term have phase velocities larger than c , and so cannot be driven resonantly by the beam). From Fig. 1, one sees that $J_1^2(x)/x^2$ effectively cuts off the integration at $x \approx 3$. For $a\kappa_c \geq 3$, the argument of the exponential in this range is then roughly $(-D/l_j)$, where $l_j = l_c(a^2\kappa_c^2)$. Thus, the exponential can be replaced by $\sim$ unity if $D \ll l_j$ which is the case we now consider. Thus, expanding the exponential, Eq. (5) becomes

$$\mathcal{J} = \beta^2 a^2 \kappa_c^2 \frac{D}{l_c} \int_0^\infty x dx \frac{J_1^2(x)}{x^2} \cdot \frac{1}{(x^2 + a^2 \kappa_c^2)^2}; \quad D \ll l_j. \quad (11)$$

When $a^2 \kappa_c^2 \gg 1$, this becomes, since $\int_0^\infty x^{-1} dx J_1^2(x) = \frac{1}{2}$,

$$\mathfrak{J} = \frac{1}{2} \frac{q^2}{a^2 \kappa_c^2} \frac{D}{\ell_c}; a^2 \kappa_c^2 \gg 1; D \ll \ell_j \quad (12)$$

and, from Eqs. (2) and (3), the fractional power expended in driving the reverse current reads

$$\mathfrak{C}\mathfrak{J} = 4 \frac{v}{\gamma - 1} \beta^2 \frac{D}{\ell_j} = \frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1} \frac{D}{\ell_c};$$

$$a^2 \kappa_c^2 \gg 1; D \ll \ell_j; \quad (13)$$

since $\ell_j = \ell_c a^2 \kappa_c^2$. Comparing with Eq. (7), one sees that the power expended by a short beam in driving the reverse current is just (D/ℓ_c) times that radiated to plasma oscillations.

To validate the assertion that $\mathfrak{C}\mathfrak{J}$ really is the fractional power expended in driving the reverse current, note that in a steady state, this must just be the (fractional) resistive power dissipated by the reverse current itself. Then, since the current neutralization involves equal magnitude beam (n_{be}) and plasma (j_{pl}) current densities,^{3,4} one can calculate the (fractional) resistive power dissipated by the induced plasma current to be (σ = plasma conductivity)

$$\frac{j_{pl}^2}{\sigma} \frac{\text{beam volume}}{\text{beam power}} = \frac{n_b^2 e^2 \beta^2 c^2}{n_p e^2 / (m v_c)} \frac{\pi a^2 D}{n_b \beta c n a^2 (\gamma - 1) m c^2} = \frac{n_b \beta^2 D}{n_p (\gamma - 1) \ell_c}$$

in exact agreement with Eq. (13).

For general values of $a \kappa_c$, the integral of Eq. (11) can be evaluated approximately, as in Section III. One notes that

$$\int_0^\infty x dx \frac{J_1^2(x)}{x^2} \frac{1}{(x^2 + a^2 \kappa_c^2)^2}$$

$$= - \frac{d}{d(a^2 \kappa_c^2)} \left[\int_0^\infty x dx \frac{J_1^2(x)}{x^2} \frac{1}{x^2 + a^2 \kappa_c^2} \right] = - \frac{d}{d(a^2 \kappa_c^2)} \left[0.28 f \left(\frac{3}{a \kappa_c} \right) \right] = \frac{0.14}{a^2 \kappa_c^2} \left[1 - \frac{a \kappa_c}{3} \arctan \left(\frac{3}{a \kappa_c} \right) \right]$$

where $f(\epsilon)$ is the function of Eq. (9).

Then Eq. (11) reads

$$\mathfrak{J} \approx 0.14 \beta^2 \frac{D}{\ell_c} \left[1 - \frac{a \kappa_c}{3} \arctan \left(\frac{3}{a \kappa_c} \right) \right]; D \ll \ell_j. \quad (14)$$

Since $[1 - \epsilon^{-1} \arctan \epsilon] \rightarrow \frac{1}{3} \epsilon^2$ as $\epsilon \rightarrow 0$, the approximate $\mathfrak{J}$ becomes $\mathfrak{J} \approx 0.42 \frac{\beta^2}{a^2 \kappa_c^2} \frac{D}{\ell_c}$ when $a^2 \kappa_c^2 \gg 1$, which is $\approx 16\%$ smaller than the exact expression, Eq. (12).

From Eqs. (2), (3), and (14), the power expended by short beams in driving the reverse current becomes

$$\mathfrak{C}\mathfrak{J} \approx 1.12 \frac{v}{\gamma - 1} \beta^2 \frac{D}{\ell_c} \left[1 - \frac{a \kappa_c}{3} \arctan \left(\frac{3}{a \kappa_c} \right) \right]; \quad D \ll \ell_j. \quad (15)$$

The function $[1 - \epsilon^{-1} \arctan \epsilon]$ is plotted in Fig. 3, which shows that, for $a \kappa_c \approx 1$ or $c/w_p \approx a$, corresponding to $\epsilon \approx 3$, the fractional power loss is of order $\frac{1}{3} \frac{v}{\gamma - 1} \beta^2 \frac{D}{\ell_c}$. This case corresponds now to $D \ll \ell_c$, since $\ell_c = \ell_j$ when $a \kappa_c = 1$, and thus involves ultra-short beam pulses.

Fig. 3. An estimate of the $(a\kappa_c)$ dependence of Eq. (11).

B. Long Beams

We now suppose that the beam length, D , far exceeds the length over which the plasma reverse current exists, l_j . It now proves convenient to rewrite Eq. (5) as

$$\mathfrak{I} = \mathfrak{I}_1 - \mathfrak{I}_2, \text{ where} \quad (16)$$

$$\mathfrak{I}_1 = \beta^2 a^2 \kappa_c^2 \left(\frac{\lim}{\delta \rightarrow 0} \right) \int_0^\infty x \, dx \frac{J_1^2(x)}{x^2} \frac{1}{x^2(x^2 + a^2 \kappa_c^2)}, \quad (17)$$

and

$$\mathfrak{I}_2 = \beta^2 a^2 \kappa_c^2 \left(\frac{\lim}{\delta \rightarrow 0} \right) \int_0^\infty x \, dx \frac{J_1^2}{x^2} \cdot$$

$$\cdot \exp \left\{ - \frac{D}{l_c} \frac{x^2}{x^2 + a^2 \kappa_c^2} \right\} \frac{1}{x^2(x^2 + a^2 \kappa_c^2)} \quad (18)$$

For $a^2 \kappa_c^2 \geq 1$, it is not difficult to show that the dominant term of $\mathfrak{I}_1$, as $\delta \rightarrow 0$, is given by [since $J_1^2(x)/x^2 \rightarrow \frac{1}{4}$ as $x \rightarrow 0$]

$$\mathfrak{I}_1 \approx - \frac{1}{4} \beta^2 \ln \delta + \text{terms of order unity.} \quad (19)$$

As for $\mathfrak{I}_2$, the argument of the exponential is

$$- \frac{D}{l_j} \frac{x^2}{1 + (x/a\kappa_c)^2}$$

so the integration is essentially completed when

$x \sim \sqrt{l_j/D} \ll 1$, since we now suppose $D^{\frac{1}{2}} \gg l_j^{\frac{1}{2}}$.

Therefore, in Eq. (18), $J_1^2(x)/x^2$ can be replaced by $\frac{1}{4}$, and x^2 can be neglected against $a^2 \kappa_c^2$ in the denominator since only $a^2 \kappa_c^2 \geq 1$ is of interest.

In the exponential also, x^2 can be neglected against $a^2 \kappa_c^2$ provided that $l_j/D \ll a^2 \kappa_c^2$, which is certainly true under the assumed conditions. Then Eq. (18) becomes

$$\mathfrak{I}_2 \approx \frac{1}{4} \beta^2 \frac{\lim}{\delta \rightarrow 0} \int_0^\infty \frac{x \, dx}{x^2} \exp \left(- \frac{D}{l_j} x^2 \right) = \frac{1}{8} \beta^2 \frac{\lim}{\delta \rightarrow 0} \int_0^\infty \frac{dy}{y} e^{-y} = \frac{1}{8} \beta^2 \frac{\lim}{\delta \rightarrow 0} E_1 \left(\frac{D}{l_j} \delta^2 \right) \quad (20)$$

where E_1 is a well-known form of the exponential integral. As $\epsilon \rightarrow 0$, the dominant term of $E_1(\epsilon)$ is $-\ln \epsilon$, so $\mathfrak{I}_2$ becomes

$$\mathfrak{I}_2 \approx - \frac{1}{8} \beta^2 \ln \left(\frac{D}{l_j} \delta^2 \right) + \text{terms of order unity.} \quad (21)$$

Combining Eqs. (16), (19), and (21), one has, to dominant order,

$$\mathfrak{I} \approx \frac{1}{4} \beta^2 \ln \left(\frac{D}{l_j} \right); \quad D \gg l_j; \quad a^2 \kappa_c^2 \geq 1. \quad (22)$$

Then, Eqs. (1) and (2) give the fractional power expended in driving the reverse current as

$$\Omega \approx \frac{v}{v-1} \beta^2 \ln \left(\frac{D}{\ell_j} \right); D \gg \ell_j; a^2 \kappa_c^2 \geq 1. \quad (23)$$

This result is in essential agreement with that of Lovelace and Sudan,⁴ differing only in the logarithmic factor. They have $\ln(b^2/a^2)$ where b is the outer radius of the plasma through which propagates a semi-infinite beam. The present model, however, consists of a finite-length beam in an unbounded plasma. It should also be noted that Eq. (23) was derived under the more general condition $a\kappa_c \geq 1$, whereas Ref. (4) used $a\kappa_c \gg 1$ in the form $B^p \approx -B^B$, at the beam head, i.e., complete magnetic neutralization at the beam head. (When $a\kappa_c \sim 1$, much of the reverse current flows outside the beam, and there is then little magnetic neutralization).³

V. DERIVATION OF THE GENERAL RESULTS OF SECTION II

All calculations are done in the laboratory frame, in which the midplane of the beam has coordinate ut . The beam head is at $ut + D/2$ and the tail at $ut - D/2$. In a steady-state, all fields, and charge and current distributions will be functions only of the combination $\vec{r}' = \vec{r} - \vec{u}t = (\vec{R}, z - ut)$, when the beam moves along the z -axis.

The power lost by the beam is then

$$P = u \int_{\infty} E_z(\vec{r}, t) \rho_b(\vec{r}, t) d^3r = u \int_{\infty} E_z(\vec{r}', t) \rho_b(\vec{r}', t) d^3r' \quad (24)$$

where ρ_b is the beam charge density, and E_z is the total axial electric field.

In terms of the Fourier transform

$$\psi(\vec{k}) = (2\pi)^{-3/2} \int_{\infty} \psi(\vec{r}') e^{-i\vec{k} \cdot \vec{r}'} d^3r',$$

Eq. (24) can be rewritten

$$P = u \int_{\infty} E_z(\vec{k}) \rho_b(-\vec{k}) d^3k. \quad (25)$$

If one notes that, for $\vec{k} = (\vec{R}, k_z)$, the Bessel functions obey the relations $2\pi J_0(KR) = \int_0^{2\pi} \exp(-iKR \cos \theta) d\theta$ and $(d/dx)[xJ_1(x)] = xJ_0(x)$, then

the uniform electron beam charge distribution of length D , radius, a , centered at $\vec{r}' = 0$, is easily found to transform to

$$\rho_b(\vec{k}) = -\sqrt{\frac{2}{\pi}} n_b e a \frac{J_1(Ka)}{K} \frac{\sin(\frac{1}{2}k_z D)}{k_z}; K = |\vec{k}|. \quad (26)$$

It is well known that a cold plasma model is sufficient for the study of disturbances whose characteristic lengths far exceed the Debye length, $\sim v_{th}/\omega_p$, where v_{th} is the plasma electron thermal speed. This is certainly the case in the present problem, wherein the shortest lengths of interest prove to be $\sim c/\omega_p$, and the plasma is non-relativistic ($v_{th} \ll c$) with $u \gg v_{th}$. Assuming that the beam only slightly perturbs the plasma, one then finds the transform of the axial electric field to be

$$E_z(\vec{k}) = -4\pi i k_z \rho_b(\vec{k}) \gamma^2 \left(\frac{k_z^2}{D_L D_T} - \frac{\beta^2}{D_T} \right) \quad (27)$$

where

$$D_L = k_z^2 - \kappa_u^2/(1 + i\epsilon_c) \quad (28)$$

$$D_T = k_z^2 + \gamma^2[k^2 + \kappa_c^2/(1 + i\epsilon_c)], \text{ and}$$

$$\epsilon_c = v_c/(uk_z). \quad (29)$$

Equation (27) is derived in the Appendix.

Substitution of Eqs. (26) and (27) into (25) produces

$$P = -16\pi i u \gamma^2 n_b^2 e^2 a^2 \int_0^{\infty} K dK \frac{J_1^2(Ka)}{K^2} \cdot \int_{-\infty}^{\infty} dk_z \frac{\sin^2(\frac{1}{2}k_z D)}{k_z^2} \left(\frac{k_z^2}{D_L D_T} - \frac{\beta^2}{D_T} \right). \quad (30)$$

In vacuum, $\omega_p = 0$, so $D_L = k_z^2$ and $D_T = k_z^2 + \gamma^2 k^2$. Then, the k_z -integral of Eq. (30) is

$$\int_{-\infty}^{\infty} dk_z \frac{\sin^2(\frac{1}{2}k_z D)}{k_z^2} \frac{1 - \beta^2}{k_z^2 + \gamma^2 k^2} = 0$$

as it is the integral of an odd function over an even interval. Thus, the self-field of the beam produces no net energy loss, as we expect.

In collisionless plasma, for a steady state, $w = uk_z$, the roots of D_L correspond to $w^2 = u_p^2$ and thus to excitation of longitudinal plasma oscillations. The roots of D_T correspond to $w^2 = u_p^2 + c^2 k^2$ and thus to (non-resonant) excitation of transverse plasma disturbances. In a slightly resistive plasma (meaning essentially $v_c \ll u_p$), D_T has an additional root corresponding to the resistive diffusion of the transverse disturbances.

If one writes $\sin^2(\frac{1}{2}k_z D) = \frac{1}{2}[1 - \frac{1}{2}(e^{ik_z D} + e^{-ik_z D})]$ then each of the various terms of the k_z integral of Eq. (30) may be integrated by contour integration. The roots of D_L are, approximately,

$k_{\pm}^L \approx \pm k_u - \frac{1}{2}i \frac{v_c}{u}$, provided $v_c^2 \ll u_p^2$. The roots of D_T are, approximately $k_{\pm}^T \approx \pm i\gamma\sqrt{K^2 + \kappa_c^2}$

$- \frac{1}{2}i \frac{v_c}{u} \frac{\kappa_c^2}{K^2 + \kappa_c^2}$, provided $v_c \ll \beta u_p$; and

$k_v \approx -i \frac{v_c}{u} \frac{K^2}{K^2 + \kappa_c^2}$, provided $\frac{v_c^2}{2} \ll \beta^2 \gamma^2 \frac{u_p^2}{c^2}$ or

$\ell_j^2 \gg \frac{c^2}{\gamma^2 u_p^2}$. The latter root has to do with the

resistive diffusion of the transverse waves.

The most stringent requirement on v_c is the second one, $v_c \ll \beta u_p$, or $\frac{c}{u_p} \ll \ell_c$. It precludes the limit $u \rightarrow 0$ or $\gamma \rightarrow 1$.

In addition it should be noted that the β^2 term of Eq. (30) has a pole at $k_z = 0$.

The various roots and integration paths are displayed in Fig. 4. (The original integrand has no pole at $k_z = 0$, and so the path can be trivially indented above the origin). The algebra is tedious but straightforward. When terms of order (v_c/u_p) are neglected everywhere except in exponents, the expended power, Eq. (30), can be written

Fig. 4. The integration path and poles for the K_z integration of the power equation, Eq. (30).

$$P = 8\pi^2 u n_b^2 e^2 a^2 \int_0^{\infty} K dK \frac{J_1^2(Ka)}{K^2}$$

$$\cdot \left\{ - \frac{1 - \exp(-\frac{1}{2} \frac{D}{\ell_c}) \cos(\kappa_u D)}{K^2 + \kappa_u^2} - \frac{\beta^2 \kappa_c^2}{K^2 (K^2 + \kappa_c^2)} \left[1 - \exp\left(-\frac{D}{\ell_c} \frac{K^2}{K^2 + \kappa_c^2}\right) \right] + \frac{(1 - \beta^2) K^2}{(K^2 + \kappa_u^2)(K^2 + \kappa_c^2)} \operatorname{sh}\left(\frac{1}{2} \frac{D}{\ell_c} \frac{\kappa_c^2}{K^2 + \kappa_c^2}\right) \cdot \exp\left(-\gamma D \sqrt{K^2 + \kappa_c^2}\right) \right\} \quad (31)$$

When the variable of integration is changed to $x = Ka$, and one divides by the beam power, $F_b = n_b u_m a^2 (\gamma - 1) m c^2$, then the first two terms of Eq. (31) are just those expressed by Eqs. (2) to (5). The last term (normalized to P_b) can be shown to be small of order less than $\frac{v}{\gamma - 1} \frac{1}{\gamma^2 a^2 \kappa_c^2} \exp\left(-\gamma \frac{P}{c}\right) \exp\left(\frac{1}{2} \frac{D}{\ell_c}\right)$, and will generally be negligible compared to the terms remaining, since $c/u_p \ll D$ and $v_c \ll u_p$. (Recall that $\ell_c = u/v_c$).

VI. CHARGE AND CURRENT NEUTRALIZATION

Although charge and current neutralization of the beam by the plasma have been discussed elsewhere,³ a brief account is presented here for completeness.

A. Charge Neutralization

The quantity $ik \cdot \vec{E}(k)$ is furnished by Eq. (A-6) of the Appendix. Taking the inverse transform, $\psi(\vec{r}') = (2\pi)^{-3} \int \psi(k) e^{ik \cdot \vec{r}'} d^3k$, and applying Eq. (26) for $\rho_b(\vec{k})$, one finds

$$\nabla \cdot \vec{E} = 4\pi(\rho_{\text{plasma}} + \rho_b) = -\frac{4n_b e a}{2i} \int_0^\infty dK J_0(KR) J_1(Ka) \cdot \int_{-\infty}^\infty dk_z \frac{k_z + i v_c/u}{k_z^2 (1 + i \epsilon_c) - \kappa_u^2} \left(e^{ik_z Z_+} - e^{ik_z Z_-} \right) \quad (32)$$

where $\epsilon_c = v_c/(u \kappa_z)$, and

$$Z_\pm = Z' \pm \frac{1}{2}D = (\text{signed}) \text{ distance of observation point from beam } \begin{cases} \text{tail} \\ \text{head} \end{cases}. \quad (33)$$

The k_z poles are in the lower half-plane, so the $e^{ik_z Z_+}$ term gives no contribution. Since $Z_- < 0$ within the beam, the integration of the $\exp(ik_z Z_-)$ term is closed in the lower half-plane. Then, using $\int_0^\infty J_0(KR) J_1(Ka) dK = \begin{cases} a^{-1} \\ 0 \end{cases}$ for $R < a$, and neglecting terms of order (v_c/u_p) except in exponents, one finds that Eq. (32) becomes

$$\rho_{\text{plasma}} + \rho_b = -n_b e \exp\left(-\frac{1}{2} \frac{|Z_-|}{\ell_c}\right) \cos(\kappa_u Z_-); \quad (34)$$

$R < a$;

with $\ell_c = u/v_c$, $\kappa_u = u_p/u$.

This equation shows that the beam only becomes charge-neutralized beyond a distance $2 \ell_c$ behind the beam head. Closer to the head, there exist plasma oscillations involving plasma density fluctuations as large as $2 n_b$.

B. Current Neutralization

The Fourier inversion of the axial electric field, Eq. (27), may be performed by the usual methods. The roots are as shown in Fig. 4, except that there is no longer a pole at $k_z = 0$. After substitution of Eq. (26) for $\rho_b(\vec{k})$ into Eq. (27), and applying contour integration to the k_z integral, one finds (note $Z_- < 0$)

$$E_z(\vec{r}') = 4\pi n_b e a \int_0^\infty dK J_0(KR) J_1(Ka) \cdot \left\{ -\frac{\kappa_u}{K^2 + \kappa_u^2} \exp\left(\frac{v_c}{2u} Z_- \right) \sin(\kappa_u Z_-) + \frac{\beta^2 (v_c/u) \kappa_c^2}{(K^2 + \kappa_c^2)^2} \exp\left(Z_- \frac{v_c}{u} \frac{K^2}{K^2 + \kappa_c^2}\right) + \frac{(1 - \beta^2) K^2}{(K^2 + \kappa_u^2)(K^2 + \kappa_c^2)} \frac{\gamma \sqrt{K^2 + \kappa_c^2}}{2} \cdot \left[\exp\left(-Z_+ \left(\gamma \sqrt{K^2 + \kappa_c^2} - \frac{1}{2} \frac{v_c}{u} \frac{\kappa_c^2}{K^2 + \kappa_c^2} \right) \right) - \exp\left(Z_- \left(\gamma \sqrt{K^2 + \kappa_c^2} + \frac{1}{2} \frac{v_c}{u} \frac{\kappa_c^2}{K^2 + \kappa_c^2} \right) \right) \right] \right\}, \quad (35)$$

provided terms of order (v_c/u_p) are neglected except in exponents.

The final term (in square brackets) may be shown to contribute an electric field small of order less than

$$\frac{4\pi n_b e}{\gamma \kappa_c} \exp\left(-|Z_\pm| \gamma \frac{u_p}{c}\right) \exp\left(\frac{1}{2} \frac{v_c}{u} Z_+\right)$$

in which the positive exponential applies only to the Z_+ contribution. Thus, this term is negligible more than an electromagnetic skin depth away from the ends of the beam.

In the limit $a^2 \kappa_u^2 \gg 1$, noting

$$\int_0^\infty dK J_0(KR) J_1(Ka) = a^{-1}$$

for $R < a$, the first term of Eq. (35) reads

$$E_{1Z} \approx -\frac{4\pi m_b e}{\kappa_u} \exp\left(\frac{v_c}{2u} Z_-\right) \sin\left(\kappa_u Z_-\right); R < a; \quad (36)$$

$$a^2 \kappa_u^2 \gg 1;$$

and is associated with the transient plasma oscillations near the beam head.

The argument of the exponential in the second term of Eq. (35) is of order $(-|Z_-|/\ell_j)$ where $\ell_j = \ell_c a^2 \kappa_c^2$ is the characteristic decay length of the plasma reverse current. Well within ℓ_j of the beam head, the exponential may be replaced by unity, and in the limit $a^2 \kappa_c^2 \gg 1$, one finds ($\ell_c = u/v_c$),

$$E_{2Z} \approx \frac{4\pi m_b e \beta^2}{\ell_c \kappa_c^2}; R < a; a^2 \kappa_c^2 \gg 1; \quad (37)$$

$$|Z_-| \ll \ell_j.$$

Far beyond the distance ℓ_j from the beam head, $|Z_-| \gg \ell_j$, the exponential becomes very small and the integration is essentially completed for $K \sim a^{-1}(\ell_j/|Z_-|)^{1/2}$, then, one can replace $J_0(KR)J_1(Ka)$ by $(\frac{1}{2}Ka)$ in Eq. (35), and can neglect K^2 against κ_c^2 for $a\kappa_c > 1$. Then the second term of Eq. (35) becomes

$$E_{2Z} \approx \left(\frac{4\pi m_b e \beta^2}{\ell_c \kappa_c^2}\right) \left(\frac{\ell_j}{4|Z_-|}\right) \quad (38)$$

$$= \frac{\pi m_b e \beta^2 a^2}{|Z_-|}; R \lesssim a; a\kappa_c \geq 1; |Z_-| \gg \ell_j.$$

Beyond $2\ell_c$ from the beam head, the transient plasma oscillations have died out, and the linearized plasma-electron equation of motion,

$$(\partial/\partial t + v_c)v_z = -\frac{e}{m} E_z, \quad (39)$$

reduces to

$$v_z = -\frac{e}{mv_c} E_z. \quad (40)$$

Substitution of Eq. (37) into Eq. (40) immediately yields the reverse current, $v_z = -(n_b/n_p)u$ for $|Z_-| \ll \ell_j$. For $|Z_-| \gg \ell_j$, substitution of Eq.

$$(38) \text{ into Eq. (40) yields } v_z = -\left(\frac{n_b}{n_p} u\right) \left(\frac{1}{4} \frac{\ell_j}{|Z_-|}\right),$$

that is, algebraic decay of the reverse current.

VII. DISCUSSION

A. The Energy Loss

General expressions [Eqs. (2) to (5)] were derived for the expended beam power in a resistive plasma, allowing for the excitation of plasma oscillations, finite beam length, finite beam radius, and allowing for the possibility of thin beams, $a \sim c/w_p$. These general expressions incorporate the exact solutions of Maxwell's equations coupled to the linearized dynamics of a cold plasma.

Specific evaluation of these expressions produced the following results for the expended power, when $a \gg c/w_p$ and $\gamma \gg 1$. The power radiated into plasma oscillations is $P_f = 4 \frac{\gamma}{v} \frac{c^2}{2a^2} P_b$, provided

$D \gg \ell_c$. When $D \ll \ell_j$, the power expended in

driving the reverse current is $P_3 = 4 \frac{\gamma}{v} \frac{D}{\ell_j} P_b = \frac{D}{\ell_c}$ times the power radiated into plasma oscillations. When $D \gg \ell_j$, the power expended in driving the reverse current is $P_3 = \frac{\gamma}{v} \ln\left(\frac{D}{\ell_j}\right) P_b$, in essential

agreement with the work of Lovelace and Sudan.⁴ In addition, it was checked that the expended power is just the ohmic power dissipated by the reverse current, which is to be expected in a steady state.

For very thin beams, $a \sim c/w_p$, $\ell_j \rightarrow \ell_c$, these results remain as stated above, but with somewhat different numerical factors [except for $P_3(D \gg \ell_j)$

which is valid as is]. However, due to the use of the linearized equation of motion for the plasma electrons, the thin-beam results apply only to very small v beams, as discussed below.

B. The Restrictions of Linearized Theory

In the Appendix, the plasma current density was approximated by $-en_p \vec{v}$. Thus, the excited plasma density fluctuations were assumed small against n_p . But, according to Eq. (34), these fluctuations are of order n_b . Therefore, the assertion that $\vec{J}_{\text{plasma}} = -en_p \vec{v}$ requires $n_b \ll n_p$. This restriction of the linearized theory has been noted elsewhere.³ It can also be written $v \ll \frac{1}{4} \omega_p^2 a^2 / c^2$, see Eq. (1).

Similar restrictions are obtained when the equation of motion (39) is considered. In the transient region, the dominant velocity term, $\partial v_z / \partial t$, is of the order $\omega_p v_z$, and the neglected velocity terms are $v_z \partial v_z / \partial z$, $v_R \partial v_z / \partial R$, and $\omega_c \text{net} v_R$, the last two occurring only in a thin tube of thickness $\sim c/\omega_p$ around $R = a$, wherein $v_R \sim v_z$.³ If one notes that $v_z \sim n_b c / n_p$, that the characteristic lengths are of order c/ω_p , and that $\omega_c \text{net} = eB \text{net} / mc$ is the cyclotron frequency in the magnetic field of the net current, $I_{\text{net}}^{(R=a)} \sim I_b / (\omega_p a / c)$,³ then one finds again that the neglected terms are of order (n_b/n_p) compared to the dominant terms. But, these neglected terms should also be small even against $v_c v_z$, and this leads to the more restrictive inequality $n_b \ll n_p \frac{v_c}{\omega_p}$, or $v \ll \frac{1}{4} \left(\frac{\omega_p a}{c} \right)^2 \left(\frac{v_c}{\omega_p} \right)$.

When the transients have died away, further than $2 \ell_c$ from the beam head, the $\partial_t v_z \sim \omega_p v_z$ term is effectively no longer present in Eq. (39). Then, the neglected terms must be compared to $v_c v_z$ only. In this region, the above-mentioned v_R terms have died away too,³ but one still has to compare $v_z \partial v_z / \partial z$ against $v_c v_z$. Now, however, the transients are absent, and so the characteristic length is $\sim \ell_j$, rather than c/ω_p . Recalling that $\ell_j = \frac{u}{v_c} \frac{\omega_p^2 a^2}{c^2}$, one finds that $|v_z \partial v_z / \partial z| \ll v_c v_z$ only requires $n_b \ll n_p (\omega_p a / c)^2$ in the post-transient region. Of course, this is superseded by the more restrictive inequality $n_b \ll n_p$, required everywhere, since the plasma density fluctuations were everywhere neglected against n_p as stated above.

To summarize, it has been found that the usually stated condition $n_b \ll n_p$, or $v \ll \frac{1}{4} (\omega_p a / c)^2$, is insufficient for the application of the linearized theory to the transient region. What appears to be required is the much more restrictive $n_b \ll n_p$ times (v_c / ω_p) , or $v \ll \frac{1}{4} (\omega_p a / c)^2 (v_c / \omega_p)$. (However, beyond the transient region, $n_b \ll n_p$ again appears to be sufficient). When $a \rightarrow c/\omega_p$, this requires a very small value of v indeed, namely $v \ll \frac{1}{4} (v_c / \omega_p)$.

C. Slowing-Down Ranges

It is natural to ask how far the beam travels before losing a significant fraction of its energy, i.e., what is the range of the beam. The fractional expended power is directly interpretable in terms of the range. Let ℓ be the length of beam involved in a given energy loss mechanism. Then, assuming all particles in the volume $\pi a^2 \ell$ lose energy at equal rates, one has (ϵ = single particle energy)

$$\frac{P}{P_b} \approx \frac{n_b \pi a^2 \ell (d\epsilon/dt)}{n_b \pi a^2 \epsilon} = \frac{\ell}{u \epsilon} u \frac{d\epsilon}{dz} = \frac{\ell}{L} \quad (41)$$

where the range L of an affected beam particle is defined as

$$L = \epsilon / (d\epsilon/dz) . \quad (42)$$

Since (P/P_b) is known, L can be found if one knows ℓ .

1. Range Due to Excitation of Plasma Oscillations.

Since alternate layers of beam of thickness $\sim u/\omega_p$ are losing and gaining energy, the active length here is effectively $\ell_f \sim u/\omega_p$. In fact, using Eq. (36), the net fractional radiated power may be estimated as

$$\begin{aligned} (P/P_b)_f &\sim \frac{n_b \pi a^2 \ell_f e E_{1z} u}{n_b \pi a^2 u (\gamma - 1) mc^2} \sim \frac{\ell_f e E_{1z}}{(\gamma - 1) mc^2} \\ &\sim \frac{(u/\omega_p) e (4\pi n_b e / \kappa u)}{(\gamma - 1) mc^2} \sim \frac{\omega_p^2 \beta^2}{(\gamma - 1) \omega_p^2} \end{aligned}$$

in agreement with Eq. (7). This shows that $\ell_f \sim u/\omega_p$ is indeed the effective active beam length for this particular loss mechanism. Then, from Eqs. (41) and (7) the range of the affected beam particles is

given by

$$L_f \approx \ell_j / (P/P_b)_f \approx \frac{(u/w_p)}{\left(\frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1}\right)} \approx \frac{(c/w_p)}{\left(\frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1}\right)} \quad (43)$$

for a highly relativistic beam in the last expression, or also

$$L_f \approx \frac{(u/w_p)}{(2n_b/n_p)} \quad (44)$$

in the non-relativistic limit. (Strictly speaking, this limit is forbidden by the assumptions).

2. Range Due to Driving the Reverse Current.

For a short beam ($D \ll \ell_j$), it is clear that the entire beam is involved in driving the reverse current so $\ell_j \approx D$. In fact, using Eq. (37),

$$\begin{aligned} (P/P_b)_j &\sim \frac{n_b \pi a^2 \ell_j e E_{2z} u}{n_b \pi a^2 u (\gamma - 1) m c^2} = \frac{\ell_j e E_{2z}}{(\gamma - 1) m c^2} \\ &= \frac{De(4\pi n_b e \beta^2 / \ell_c \kappa_c^2)}{(\gamma - 1) m c^2} = \frac{D}{\ell_c} \frac{\omega_b^2 \beta^2}{\omega_p^2 (\gamma - 1)} \end{aligned}$$

in agreement with Eq. (13). This verifies that, for a short beam the active beam length for this mechanism is indeed $\ell_j \approx D$. Then, using Eqs. (13) and (41), the range is given by

$$\ell_j \approx \ell_j / (P/P_b)_j = \frac{D}{\left(\frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1} \frac{D}{\ell_c}\right)} = \frac{\ell_c}{\left(\frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1}\right)} \quad (45)$$

$$\approx \frac{\ell_c}{\frac{n_b}{n_p} \frac{\beta^2}{\gamma - 1}} ; (D \ll \ell_j)$$

for a highly relativistic beam in the final expression; or also

$$L_j \approx \frac{\ell_c}{(2n_b/n_p)} ; (D \ll \ell_j) ; \quad (46)$$

in the non-relativistic limit.

For a long beam ($D \gg \ell_j$), it is clear that the affected beam length is $\ell_j \approx \ell_j$. In fact, again using Eq. (37), and recalling that $\ell_j = \ell_c (\omega_p a/c)^2$,

$$\begin{aligned} (P/P_b)_j &\sim \frac{\ell_j e E_{2z}}{(\gamma - 1) m c^2} = \frac{\ell_j e (4\pi n_b e \beta^2 / \ell_c \kappa_c^2)}{(\gamma - 1) m c^2} \\ &= \frac{\omega_b^2 \beta^2 (\omega_p a/c)^2}{(\gamma - 1) \omega_p^2} = \frac{4v\beta^2}{\gamma - 1} \end{aligned}$$

in essential agreement with Eq. (23) except for the logarithmic factor. [Here, we have used Eq. (1) for v]. This verifies that, for a long beam, the active beam length for this mechanism is indeed $\ell_j \approx \ell_j$. The two results will agree exactly if we take $\ell_j = \frac{1}{4} \ell_j \ln \left(\frac{D}{\ell_j} \right)$. Then, using Eqs. (23) and (41), the range is

$$L_j \approx \ell_j / (P/P_b)_j = \frac{\frac{1}{4} \ell_j \ln \left(\frac{D}{\ell_j} \right)}{\frac{v}{\gamma - 1} \beta^2 \ln \left(\frac{D}{\ell_j} \right)} \quad (47)$$

$$= \frac{\frac{1}{4} \ell_c (\omega_p a/c)^2}{\frac{(\omega_b^2 a^2/c^2)}{\gamma - 1} \beta^2} = \frac{\ell_c}{\left(\frac{n_b \beta^2}{(\gamma - 1) n_p} \right)} ; D \gg \ell_j ;$$

in agreement with the short beam result, Eq. (45).

It is also useful to note that Eqs. (45) and (47) can be written

$$L_j \approx \frac{\ell_j}{[4v/(\gamma - 1)]} , \quad (48)$$

which is the basis of the statement that high v/γ beams will be stopped in a distance short compared to ℓ_j .

3. Comparison with the Two-Stream Instability.

Consider first a non-relativistic beam. The

two-stream instability growth rate is ⁶ $\delta_1 \sim \omega_p \left(\frac{n_b}{n_p} \right)^{\frac{1}{3}}$

corresponding to a range $L_{t.s.}^{(1)} \sim \frac{(u/w_p)}{(n_b/n_p)^{\frac{1}{3}}}$ which is

much shorter than Eq. (44). However, after a few growth times, the beam is smeared in axial velocity, and the quasi-linear regime takes over, wherein⁷
 $\delta_2 \sim \omega_p (n_b/n_p)$. Then, the range is $L_{t.s.}^{(2)} \sim \frac{(u/w_p)}{(n_b/n_p)}$ which is of the same order as Eq. (44). In either case, the two-stream ranges, as well as the range given by Eq. (44), are much shorter than the range L_j for driving the reverse current, because of the condition $v_c \ll \omega_p$.

It should also be noted that the range $L_{t.s.}$ applies to all the beam electrons, whereas L_f applies to only those within $\sim \ell_c$ of the beam head, and L_j applies to those within ℓ_j of the beam head (for a short beam, this would involve all the beam electrons).

Consider next a relativistic beam. The initial growth rate is now⁸ $\delta_1 \sim \omega_p (n_b/\gamma n_p)^{\frac{1}{2}}$ corre-

sponding to a range $L_{t.s.}^{(1)} \sim \frac{(c/\omega_p)}{(n_b/\gamma n_p)^{\frac{1}{2}}}$ which is

again much shorter than Eq. (43). Because of the large longitudinal mass of the relativistic beam particles, axial velocity smearing is ineffective. Transverse velocity smearing leads to the two-stream growth rate in the quasi-linear regime⁸ $\delta_2 \sim$

$\frac{\omega_p}{\gamma} \left(\frac{n_b}{n_p} \right)^{\frac{1}{2}}$, which is not too different from δ_1 . This

gives a range $L_{t.s.}^{(2)} \sim \frac{\gamma c/\omega_p}{(n_b/n_p)^{\frac{1}{2}}}$ which is again much

shorter than Eq. (43). Reference (8) points out that the relativistic beam loses a significant fraction of its energy already during the δ_1 (hydrodynamic) phase.

In summary, the two-stream electron-electron process appears to be a significantly more effective energy loss mechanism than either of the processes represented by Eqs. (43) or (45), contrary to the assertion of Lovelace and Sudan.⁴ Nevertheless, the energy loss to the transverse current remains important because of the possibility of

heating the plasma ions when the reverse-current drift velocity exceeds the ion acoustic speed.

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APPENDIX

FOURIER TRANSFORM OF THE AXIAL ELECTRIC FIELD

If one eliminates the magnetic field from Maxwell's equations, one has, for the electric field,

$$\nabla \times \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = - \frac{4\pi}{c^2} \frac{\partial}{\partial t} \vec{j}_{\text{tot}} \quad (\text{A-1})$$

where $\vec{j}_{\text{tot}} \equiv \vec{j}_{\text{beam}} + \vec{j}_{\text{plasma}}$ is the total current density.

The linearized equation of motion for a plasma electron's velocity reads

$$\left(\frac{\partial}{\partial t} + v_c \right) \vec{v} = - \frac{e}{m} \vec{E} \quad (\text{A-2})$$

(where v_c is the plasma-electron collision frequency) which is equivalent to the following equation for the plasma current density,

$$\left(\frac{\partial}{\partial t} + v_c \right) \vec{j}_{\text{pl.}} = \frac{\omega_p^2}{4\pi} \vec{E}. \quad (\text{A-3})$$

In a steady state, wherein all quantities depend only on the combination $\vec{r}' = \vec{r} - \vec{ut} = (\vec{R}, z - ut)$, one can replace $\nabla \equiv \nabla_{\vec{r}}$ by $\nabla' \equiv \nabla_{\vec{r}'}$, and further

$\frac{\partial}{\partial t} = -u \frac{\partial}{\partial z'}$. Then, one Fourier-transforms Eqs. (A-

1) and (A-3) with respect to $\vec{r}'$, i.e.,

$$\psi(\vec{k}) \equiv (2\pi)^{-3/2} \int d^3 r' e^{-i\vec{k} \cdot \vec{r}'} \psi(\vec{r}'). \text{ Then they read}$$

$$k^2 \vec{E} - \vec{k} (\vec{k} \cdot \vec{E}) - \beta^2 k_z^2 \vec{E} = \frac{4\pi u}{c^2} i k_z (\vec{j}_{\text{pl.}} + \vec{j}_b) \quad (\text{A-4})$$

$$(-u i k_z + v_c) (4\pi \vec{j}_{\text{pl.}}) = \omega_p^2 \vec{E}, \quad (\text{A-5})$$

where $\beta \equiv u/c$.

If one takes $\vec{k}$ of Eqs. (A-4) and (A-5), the following expression for $\vec{k} \cdot \vec{E}$ can be obtained,

$$\vec{k} \cdot \vec{E} = - \frac{4\pi i k_z^2 \rho_b(\vec{k})}{k_z^2 - \kappa_u^2 / (1 + i\epsilon_c)} \equiv - \frac{4\pi i k_z^2 \rho_b(\vec{k})}{D_L} \quad (\text{A-6})$$

where $\kappa_u \equiv \omega_p/u$, $\epsilon_c \equiv v_c/(u k_z)$, and we have used

$$\vec{j}_b = 2 \vec{j}_{bz} = 2 u \rho_b.$$

One can now take the axial components of Eqs. (A-4) and (A-5), and, with the help of Eq. (A-6) can find $E_z(\vec{k})$. The result is

$$E_z(\vec{k}) = - 4\pi i k_z^2 \gamma^2 \rho_b(\vec{k}) \left\{ \frac{k_z^2}{D_L} - \beta^2 \right\} \frac{1}{D_T} \quad (\text{A-7})$$

where

$$D_T \equiv k_z^2 + \gamma^2 \left(k^2 + \kappa_c^2 / (1 + i\epsilon_c) \right),$$

and $\gamma \equiv (1 - \beta^2)^{-1/2}$, with $\beta \equiv u/c$.