FUNDAMENTAL INVESTIGATION OF LIQUID-METAL
LUBRICATED JOURNAL BEARINGS

Quarterly Technical Report No. 6
Contract No. AT(11-1)-1228
Project Agreement No. 8
Modification No. 7

R. A. Burton
Y. C. Hsu

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

to

U.S. ATOMIC ENERGY COMMISSION
Washington 25, D. C.

March 1, 1966

Report No. RS-482

SOUTHWEST RESEARCH INSTITUTE
SAN ANTONIO HOUSTON

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APPROVED:

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Report No. RS-482

FOREWORD

This report was prepared by Southwest Research Institute under Contract No. AT(11-1)-1228, Project Agreement No. 8, Modification No. 7. The work was administered by the Engineering Development Branch, Division of Reactor Development, U. S. Atomic Energy Commission, Washington 25, D. C., with Mr. Nicholas Grossman as Project Engineer.

This report covers work performed from November 1, 1965 through March 1, 1966.

ABSTRACT

The calculated performance of finite tilting pad bearings is reported in terms of mean pressure and center-of-pressure location, as functions of length/width ratio, and tilt angle.

The performance of a stepped pad is calculated by an iterative numerical program, and the pressure distribution is shown to resemble closely that measured experimentally.

Interactions between a shear flow and pressure flows of different directions and magnitudes are investigated by application of the mixing length approach.

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LIST OF SYMBOLS

b	Local half-thickness of film
C_f	Friction coefficient, $C_f = 2T/\rho q^2$
C_{xc}	Friction coefficient, $C_{xc} = 2T_{yx}/\rho(u_{bc})^2$
H	Dimensionless film thickness
h	Local film thickness
L_x	Pad bearing length in tangential direction
L_z	Pad bearing length in axial direction
m	Velocity profile exponent for axial flow
n	Velocity profile exponent for tangential flow
P	Dimensionless pressure, $\frac{p}{\rho(u_s/2)^2}$
p	Gage pressure
p_a	Ambient pressure
q	Velocity vector, $\sqrt{u^2 + w^2}$
R	Reynolds number based on local film thickness
U	$\Delta u_b/u_{bc}$
U^+	Dimensionless velocity in tangential direction
u	Velocity in tangential direction
V^+	Dimensionless velocity vector, $\sqrt{(U^+)^2 + (W^+)^2}$
v	Velocity across film
W^+	Dimensionless velocity in axial direction
W	Dimensionless velocity w/u_{bc}

LIST OF SYMBOLS (Cont'd)

w	Velocity in axial direction
X	Dimensionless distance
x	Coordinate in tangential direction
Y^+	Dimensionless wall distance
y	Coordinate across film
Z	Dimensionless axial coordinate
z	Axial coordinate
α	Tilted pad bearing slope
δ^*	Displacement thickness
ζ	Dimensionless coefficient in integrated turbulent film equation
μ	Absolute viscosity
ν	Kinematic viscosity
ρ	Fluid density
T	Wall shear stress vector, $\sqrt{(T_{yx})^2 + (T_{yz})^2}$
T_{yx}	Wall shear stress in tangential direction
T_{yz}	Wall shear stress in axial direction
τ_{yx}	Local shear stress in tangential direction
τ_{yz}	Local shear stress in axial direction
$()_b$	Refers to midchannel
$()_{bc}$	Refers to Couette flow at midchannel
$()_c$	Refers to Couette flow
$()_s$	Refers to surface velocity of moving wall

I. INTRODUCTION

This report summarizes progress in three areas: (1) Estimation of performance of finite tilted pad-type bearings with turbulent flow, (2) Extension of the numerical calculation procedure to short stepped pads, and (3) Investigation of the interaction between turbulent Couette and pressure flows, using mixing length techniques.

The tilted finite pad studies indicate the predominant effect of entering pressure on the pressure distribution. For example, this causes the center of pressure to be far ahead of the bearing center in the cases examined.

The short stepped-pad study shows that the unexpected drop in pressure at the step can be predicted by the numerical calculation technique, and is a consequence of the momentum or inertia effects at the step.

The study of bearing flows by the mixing length technique shows the predictions to be consistent with experiment, insofar as this can be checked. It also provides comparisons with the power-law profile approach discussed in earlier reports. More important it provides a systematic set of predictions of the local behavior of a flow subject to both boundary sliding and pressure gradients. These will be useful for comparison with the predictions by other techniques in later reports.

II. COMPUTER STUDIES OF FINITE BEARINGS

In Progress Report No. 5, an iterative procedure was outlined for finite bearing calculations. Here this has been applied to computation of the performance of tilted pads and a short stepped pad. In improved form, the method is outlined below, and the results are presented in the form of plots.

A. Iterative Procedure for Finite-Length Turbulent-Film Bearing Calculations

The simplified turbulent-film equations are

$$\begin{aligned} \frac{\zeta_1}{\zeta_2} \left[2 (1 + U) \frac{\partial U}{\partial X} + \frac{(dH/dX)}{H} (1 + U)^2 \right] + W \frac{\partial U}{\partial Z} + (1 + U) \frac{\partial W}{\partial Z} = \\ = - \frac{1}{\zeta_2} \frac{\partial P}{\partial X} - \frac{2\zeta_5}{H\zeta_2} C_{xc} U \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{2\zeta_3}{\zeta_2} W \frac{\partial W}{\partial Z} + (1 + U) \frac{\partial W}{\partial X} + W \frac{\partial U}{\partial X} + \frac{(dH/dX)}{H} W (1 + U) \\ = - \frac{1}{\zeta_2} \frac{\partial P}{\partial Z} - \frac{\zeta_6}{\zeta_2} C_{xc} \frac{W}{H} \end{aligned} \quad (2)$$

$$\frac{(dH/dX)}{H} (1 + U) + \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} = 0 \quad (3)$$

The boundary conditions for the present analysis are

$$\text{at } X = 0, \quad P = P_0 \quad U = U_0$$

$$\text{at } X = L, \quad P = 0$$

$$\text{at } Z = Z_{\max}, \quad P = 0$$

$$\text{at } Z = 0, \quad W = 0$$

This means: that the shearing motion is in the X-direction; and Z-flow is leakage flow; that the line $Z = 0$ corresponds to the midline of a symmetrical bearing, where there is no flow across this line. It is assumed that there is some leading-edge pressure in the X-direction, P_0 , and some inlet velocity, U_0 . In the analysis which follows, the contour (variation of H) will be assumed to be a function of X alone, though the general approach is not necessarily subject to this restriction.

To produce a first approximation, which may be improved by iteration, the inertial terms in Equations (1) and (2) may be neglected and the quantities ζ_i and C_{xc} may be treated as constants in the (X, Z) space. After pressures and velocities are computed on this trial basis, the velocity magnitude may be used to compute approximate magnitudes for the inertial term; and, with tables given in prior reports, may be used to compute improved values of ζ_i , as functions of X and Z . This is possible because in cases of real interest, the inertial terms are of secondary importance, and the variation of ζ_i is not great. Even with this simplification, there remains a significant problem in computing the pressures for a given set of edge conditions and a given variation of H .

Without the inertia terms, Equations (1), (2), and (3) become, respectively,

$$F + \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} = 0 \quad (4)$$

$$\frac{\partial P}{\partial X} = - 2\zeta_5 C_{xc} \frac{U}{H} = - A \frac{U}{H} \quad (5)$$

$$\frac{\partial P}{\partial Z} = - \zeta_6 C_{xc} \frac{W}{H} = - B \frac{W}{H} \quad (6)$$

Since a number of terms have been relegated to the second iteration, linearity of (2) is achieved by similarly dropping U in this equation in the first iteration, and then supplying the approximate value of this variable at the same time as the inertial terms and ζ_i terms for the second iteration. Letting $2\zeta_5 C_{xc}$ be defined as A, $\zeta_6 C_{xc}$ be defined as B, and letting $(1 + U) \frac{(dH/dX)}{H}$ be defined as F, the equations may be written in reverse order to become

$$- F = \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} \quad (7)$$

$$\frac{\partial P}{\partial X} = - \frac{AU}{H} \quad (8)$$

$$\frac{\partial P}{\partial Z} = - \frac{BW}{H} \quad (9)$$

Solutions may be obtained that satisfy (7) and that may or may not satisfy (8) and (9); however, if a systematic group of solutions can be obtained, it is permissible to sum these to achieve a combined solution which satisfies all three equations. The problem is to devise a system of generation and summing such solutions, which is economical of computer time. The method chosen here is to begin with a primitive

solution of Equation (7) and allow the discrepancy between the pressures computed from (8) and (9) from this solution to serve as a guide in generation of successive solutions. For example, the following represents a valid solution for (7)

$$\frac{\partial U}{\partial X} = - \frac{F}{2} \quad (10)$$

with the complementary statement

$$\zeta_7 \frac{\partial W}{\partial Z} = - \frac{F}{2} \quad (11)$$

It follows from (10) that

$$U = U_o - \frac{1}{2} \int_0^X F dX \quad (12)$$

where U_o is an initial value at $X = 0$. It also follows from (8) that

$$P_x = P_o - \int_0^X \frac{A}{H} U dX \quad (13)$$

where P_o is an initial value, and P_x is P as computed by the X integration.

Combining (12) and (13)

$$P_x = P_o - \int_0^X \frac{A}{H} \left[U_o - \frac{1}{2} \int_0^X F dX \right] dX \quad (14)$$

but U_o is determined by the requirement that the trailing edge pressure be zero; hence

$$U_o = \left\{ P_o + \int_0^L \left[\frac{A}{2H} \int_0^X F dX \right] dX \right\} / \left\{ \int_0^L \frac{A}{H} dX \right\} \quad (15)$$

From Equation (11)

$$\frac{\partial W}{\partial Z} = - \frac{F}{2\zeta_7} \quad (16)$$

or

$$W = - \frac{1}{2} \int_0^Z \frac{F}{\zeta_7} dZ \quad (17)$$

because $W = 0$ at $Z = 0$.

Since $P = 0$ at $Z_{\max}$ and P_z is computed by integration in the Z -direction

$$P_z = - \frac{1}{2} \int_{Z_{\max}}^Z \frac{B}{H} \int_0^Z \frac{F}{\zeta_7} dZ dZ \quad (18)$$

it would only be fortuitous that P_z equal P_x throughout the entire field.

A solution can be obtained, however, which, when added to the prior solution, tends to produce values of P_z closer to those of P_x , than in the primitive solution. This is obtained by letting, at each point

$$\frac{\partial U}{\partial X} = a(P_z - P_x) \quad (19)$$

and noting that to satisfy Equation (7) there must be a complementary solution

$$\zeta_7 \frac{\partial W}{\partial Z} = -a(P_z - P_x) \quad (20)$$

In (19) and (20) $a \ll 1$, say 0.001, and is called the relaxation coefficient.

For the next approximate solution the boundary conditions become

$$W = 0 \text{ at } Z = 0$$

$$P = 0 \text{ at } Z = Z_{\max}$$

$$P = 0 \text{ at } X = 0$$

$$P = 0 \text{ at } X = L$$

A solution which satisfies these boundary conditions may be added to the previous solution, and thus the sum will satisfy all of the previous boundary conditions as stated. For the incremental solution this is equivalent to substituting $-a(P_z - P_x)$ for $F/2$, in Equation (10), (12), (14) and $a(P_z - P_x)$ for $F/2$ in (16), (17), (18), and letting $P_o = 0$. By this means, the new P_z and P_x magnitudes may be produced for all (X, Z) points and may be added to the previously obtained matrices of P_x and P_z . The new matrices of P_x and P_z may be used in Equations (19) and (20) for additional repeats of the process of successive approximation.

B. Performance of Finite Tilted Pad Bearings

A series of computer solutions were carried out for bearings having $L_z/L_x = 1, 1/2, 1/3$, to provide estimates of the performance of real

bearings. A range of pad angles, as determined by the slope α , and a range of bearing lengths were investigated. Here the length is dimensionless and is expressed in multiples of entry film thickness (or maximum film thickness). Though the calculation procedure gives complete pressure data over the pad, it has been found advantageous here to summarize the results in terms of mean dimensionless pressure $\bar{P}$ and position of center of pressure, measured back from the leading edge. Throughout the calculation a leading edge pressure of $P = 0.25$ was assumed on the basis of earlier reported experiments.

To illustrate the type of pressure variation encountered, Figure 1 shows the variation of pressure in the direction of flow along the center-line of the bearing, for $\alpha = 0.0005$, $L_x = 750$, and for the three length/width ratios, and for $\zeta_1 = 1$.

Load-Carrying Capacity. The curves of $\bar{P}$ versus L_x are plotted in Figures 2, 3, and 4 for the cases of $L_z/L_x = 1$, $1/2$, and $1/3$, respectively. The mean dimensionless pressure, $\bar{P}$ is seen to be significantly affected by the values of L_z/L_x and α . Increasing the quantities of L_x and α , the value of $\bar{P}$ is shown to increase. It is reasonable that the narrower the bearing, the lower the value of $\bar{P}$ because of larger side leakage. For the case of narrower and flatter pad bearing, the inertia effect on the value of $\bar{P}$ is of very small order, except for the leading edge pressure rise which is present in all cases.

FIGURE 1. PRESSURE SOLUTIONS FOR SELECTED LENGTH WIDTH RATIOS FOR TILTED PADS

FIGURE 2. LOAD SUPPORT CHARACTERISTICS OF TILTED PAD FOR $L_z/L_x = 1$

FIGURE 3. LOAD SUPPORT CHARACTERISTICS OF TILTED PAD FOR $L_z/L_x = 1/2$

FIGURE 4. LOAD SUPPORT CHARACTERISTICS OF TILTED PAD FOR $L_z/L_x = 1/3$

Location of Center of Pressure. In Figures 5, 6, and 7, the symbol $\bar{x}/L_x$ denotes the distance back from the leading edge to the center of pressure, expressed as a fraction of pad bearing length L_x . These figures show three plots of $\bar{x}/L_x$ versus L_x , showing the effects of L_z/L_x and α . For the case of $L_z/L_x = 1$, Figure 5, the center of pressure is typically well ahead of the maximum value of $\bar{x}/L_x = 0.525$. It moves backward for larger values of bearing length, L_x , and for increasing pad angle, α . Figures 6 and 7, for the cases of $L_z/L_x = 1/2$ and $L_z/L_x = 1/3$, show similar relations for different pad angles. For the corresponding L_x and α , the narrower the bearing, the more forward the center of pressure. For the narrower and flatter pad bearing, the inertia effect on the center of pressure can be neglected. The backward movement of the center of pressure with increasing α insures stability. Given a pivot position of a pad bearing, there will be a stable condition in the sense that pad angle will increase until the center of pressure moves back to the pivot location.

C. Short Stepped Pad

In the large scale turbulence apparatus, a stepped pad was operated at Reynolds numbers in the turbulent range, and pressures were measured at points along its length. Typical results were reported in Quarterly Progress Report No. 5 and are reproduced in Figure 8. The pad was approximately 3 ft long, and 20 in. wide. Entering film thickness was 0.54 in., and the film thickness stepped down at midlength to 0.28 inch.

FIGURE 5. LOCATION OF CENTER OF PRESSURE FOR
 TURBULENT FLOW UNDER A TILTED
 PAD FOR $L_z/L_x = 1$

FIGURE 6. LOCATION OF CENTER OF PRESSURE FOR
 TURBULENT FLOW UNDER A TILTED
 PAD FOR $L_z/L_x = 1/2$

FIGURE 7. LOCATION OF CENTER OF PRESSURE FOR
 TURBULENT FLOW UNDER A TILTED
 PAD FOR $L_z/L_x = 1/3$

FIGURE 8. PRESSURE DISTRIBUTION FOR
 STEPPED PAD AT $R = 2400$ WHERE
 $R = bu_b/\nu$

The most interesting feature of the pressure profile is the drop at the step, and the depressed pressure over the remainder of the length. This drop is attributable to the momentum changes at the step and serves to illustrate a case where the momentum effects dominate the behavior of the bearing.

To determine whether or not the short bearing calculation procedure was adequate for such a severely distorted pressure distribution, a calculation was undertaken for the following conditions: $L_x = 74$, $L_z = 40$, $h_2/h_1 = 0.5$, which would correspond to a bearing 37 in. long and 20 in. wide if the entering film thickness were 0.5 inch. Figure 9 shows the computed results. These differ from the measured results of Figure 8 in the magnitudes indicated, but show the same general trends, including negative pressures in the thin-film portion of the clearance.

The solution was for 57 grid spaces in the x-direction, and this grid coarseness appeared to spread the midlength pressure drop over several segments. It is possible that the pressure curve for the forward half of the pad could be extrapolated to give a maximum considerably above that shown, and more nearly comparable to the experimental maximum. In any event the relative success of the computation is encouraging, and it is expected that it can be improved in detail by straightforward modifications.

FIGURE 9. COMPUTED PRESSURE FOR STEPPED
PAD AT $R = 2400$ WHERE $R = bu_b/\nu$

III. A STUDY OF THE INTERACTIONS BETWEEN TURBULENT SHEAR FLOW AND DISPLACEMENT FLOW BETWEEN PARALLEL WALLS

Tao⁽¹⁾ and Constantinescu⁽²⁾ have solved the turbulent film bearing problem by neglecting the coupling effects between orthogonal shear and pressure flows. Prandtl⁽³⁾, however, has derived a mixing length equation, taking account of the coupling effect in such flows where the shear stress varies in direction with distance from a wall. Burton⁽⁴⁾ has applied this equation to the short bearing problem for the special case of a weak pressure (or displacement) flow at right angles to a strong Couette (or shear) flow.

This is extended here to the interactions between a fully turbulent shear flow and displacement flow of varying strength acting both along and across the shear flow. Using the digital computer, solutions are obtained for the set of differential equations recommended by Prandtl. To provide the necessary expression for mixing length variation required to complete these equations, an empirical expression is used, which gives a continuous statement for mixing length in the turbulent case and through the sublayer up to the boundary wall.

A. Basic Equations

In the presence of coupling effects, the mixing length equations⁽³⁾ for shear stress variation in Cartesian coordinates are:

$$\tau_{yx} = (\rho l^2 |J| + \mu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (21)$$

$$\tau_{yz} = (\rho l^2 |J| + \mu) \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \quad (22)$$

where

$$|J| = \left| \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right]^2 + \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]^2 + \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]^2 \right\}^{1/2} \right| \quad (23)$$

For the case of parallel flow, it is assumed that $\frac{\partial u}{\partial x} = \frac{\partial w}{\partial z} = \frac{\partial u}{\partial z} =$

$\frac{\partial w}{\partial x} = v = 0$. Using this assumption in Equations (21) to (23), Equations (21)

and (22) reduce to

$$\tau_{yx} = \left\{ \mu + \rho l^2 u_y \left| \left[1 + \left(\frac{w_y}{u_y} \right)^2 \right]^{1/2} \right| \right\} u_y \quad (24)$$

$$\tau_{yz} = \left\{ \mu + \rho l^2 u_y \left| \left[1 + \left(\frac{w_y}{u_y} \right)^2 \right]^{1/2} \right| \right\} w_y \quad (25)$$

where

$$u_y = \frac{du}{dy} \quad (26)$$

$$w_y = \frac{dw}{dy} \quad (27)$$

Here x- and z-coordinates are parallel to the boundary surfaces, and the y-coordinate is perpendicular to those. The velocity of u and w represents the flow in x- and z-directions, respectively. The viscosity, fluid density and mixing length are denoted by μ , ρ , and l , respectively. The components

in x- and z-directions of shear stresses variation across the fluid film are denoted by τ_{yx} and τ_{yz} , respectively.

Burton⁽⁵⁾ has proposed the one-dimensioned empirical dimensionless mixing length equation based on a good direct fit of the whole range of empirical velocity profile. For the very small Y^+ $\left[Y^+ = \frac{y}{\nu} \left(\frac{\tau_{yx}}{\rho} \right)^{1/2} \right]$, this equation follows a cubic relation for which Emmons⁽⁶⁾ has provided some measurements in terms of eddy dimension. As for the higher values of Y^+ , this becomes nearly linear and corresponds to the ordinary assumption in mixing length theory. Hence, this equation will be employed and also extended to the two-dimensional case as follows:

$$(L^+)^2 = 0.801 \left(\frac{V^+}{8.74} \right)^6 \left[1 + 0.801 \left(\frac{V^+}{8.74} \right)^6 \right] \quad (28)$$

where

$$V^+ = [(U^+)^2 + (W^+)^2]^{1/2} \quad (29)$$

the above-mentioned dimensionless quantities are defined as follows:

$$(L^+)^2 = Tl^2 / \rho \nu^2 \quad (30)$$

$$U^+ = u \sqrt{\rho / T} \quad (31)$$

$$W^+ = w \sqrt{\rho / T} \quad (32)$$

$$T = \sqrt{(\tau_{yx})^2 + (\tau_{yz})^2} \quad (33)$$

The momentum and continuity equations for a thin incompressible fluid under a steady-state condition are [see Equations (9) and (11) in Reference 7]:

$$\rho \left[u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right] = - \frac{\partial p}{\partial x} + \frac{\partial T_{yx}}{\partial y} \quad (34)$$

$$\rho \left[u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right] = - \frac{\partial p}{\partial z} + \frac{\partial T_{yz}}{\partial y} \quad (35)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0 \quad (36)$$

Introduction of the parallel flow assumptions do satisfy Equation (36)

and can reduce Equations (34) and (35) to

$$\frac{\partial p}{\partial x} = \frac{\partial T_{yx}}{\partial y} \quad (37)$$

$$\frac{\partial p}{\partial z} = \frac{\partial T_{yz}}{\partial y} \quad (38)$$

Given local shear stresses (τ_{yx} and τ_{yz}) and mixing length in Equation (28), Equations (24), (25), (37), and (38) constitute the basic equations to solve for u , w , and p .

B. Solutions for u , w , and p

Dividing Equation (25) by (24), w_y is

$$w_y = Gu_y \quad (39)$$

where

$$G = \tau_{yz}/\tau_{yx} \quad (40)$$

Substituting Equation (39) into (24), the governing differential equation for u_y becomes

$$(u_y)^2 + A(u_y) - \frac{B}{4} = 0 \quad (41)$$

where

$$A = \mu / \left\{ \rho \ell^2 \left| (1 + G^2)^{1/2} \right| \right\} \quad (42)$$

$$B = 4\tau_{yx} / \left\{ \rho \ell^2 \left| (1 + G^2)^{1/2} \right| \right\} \quad (43)$$

Solving Equation (41) for u_y as

$$u_y = \frac{1}{2} [-A + (A^2 + B)^{1/2}] \quad (44)$$

Because of positive value of u_y across fluid film, the value of τ_{yx} is, hence, positive too. Thus B is a definitely positive quantity. Owing to $(A^2 + B)^{1/2} > A$ and positive value of u_y , a "plus" sign in front of $(A^2 + B)^{1/2}$ is chosen in Equation (44) as

$$u_y = \frac{1}{2} [-A + (A^2 + B)^{1/2}] \quad (45)$$

Integrating u_y in Equation (45) with respect to y and using conditions of $u = 0$ at $y = 0$, u is obtained in terms of integral form

$$u = \frac{\mu}{2\rho} \int_0^y \frac{-1 + \left[1 + \frac{4\rho \ell^2 \tau_{yx}}{\mu^2} \left| (1 + G^2)^{1/2} \right| \right]^{1/2}}{\ell^2 \left| (1 + G^2)^{1/2} \right|} dy \quad (46)$$

Substituting Equation (45) into (39) and integrating it with respect to y with $w = 0$ at $y = 0$, w in integral form is

$$w = \frac{\mu}{2\rho} \int_0^y G \frac{-1 + \left[1 + \frac{4\rho \ell^2 \tau_{yx}}{\mu^2} \left| (1 + G^2)^{1/2} \right| \right]^{1/2}}{\ell^2 \left| (1 + G^2)^{1/2} \right|} dy \quad (47)$$

Using the quantities in Equations (30) to (33) and defining $q = \sqrt{u^2 + w^2}$,

$R = \frac{bq}{\nu}$ and $C_f = \frac{2}{\rho} \frac{T}{q^2}$, Equations (46) and (47), in dimensionless form

are, respectively,

$$U^+ = \beta \int_0^{y/b} Kd\left(\frac{y}{b}\right) \quad \left(0 \leq \frac{y}{b} \leq 1\right) \quad (48)$$

$$W^+ = \beta \int_0^{y/b} GKd\left(\frac{y}{b}\right) \quad \left(0 \leq \frac{y}{b} \leq 1\right) \quad (49)$$

where

$$\beta = \frac{R\sqrt{C_f}}{2\sqrt{2}} \quad (50)$$

$$K = \frac{-1 + \left[1 + \frac{4\tau_{yx}}{T} (L^+)^2 \left|(1 + G^2)^{1/2}\right|\right]^{1/2}}{(L^+)^2 \left|(1 + G^2)^{1/2}\right|} \quad (51)$$

and L^+ and G are identical to the same forms as in Equations (28) and (40), respectively.

Equations (48) and (49) are the general solutions in integral form for U^+ and W^+ . Of interest here are two cases: (1) a Couette flow with orthogonal superposed pressure flows, and (2) a Couette flow with co-directional pressure flow and orthogonal superposed pressure flows.

C. Case 1. Orthogonal Couette Flow and Pressure Flow

If only Couette flow acts in x-direction, τ_{yx} across the fluid film may be assumed to be constant. Hence

$$\tau_{yx} = T_{yx} \quad \left(0 \leq \frac{y}{b} \leq 2\right) \quad (52)$$

Because of symmetry of w about $y = b$, τ_{yz} is an odd linear function about $y = b$. Hence,

$$\tau_{yz} = T_{yz} \left(1 - \frac{y}{b} \right) \quad \left(0 \leq \frac{y}{b} \leq 2 \right) \quad (53)$$

where T_{yx} and T_{yz} are the wall shear stresses.

By means of Equations (52) and (53), τ_{yx}/T and G are, respectively

$$\frac{\tau_{yx}}{T} = \frac{1}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \quad (54)$$

$$G = \frac{T_{yz}}{T_{yx}} \left(1 - \frac{y}{b} \right) \quad (55)$$

Equations (48) and (49) with (54) and (55) are the solutions of U^+ and W^+ for Case 1. The values of U^+ and W^+ near the wall are

$$U^+ = \frac{2\beta}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \cdot \frac{y}{b} \quad (56)$$

$$W^+ = \frac{2\beta(T_{yz}/T_{yx})}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \cdot \frac{y}{b} \quad (57)$$

Using Equations (52) and (53), Equations (37) and (38) reduce to

$$p = - \frac{2T_{yz}}{h} z \quad (58)$$

With condition of $p = p_{\max}$ at $z = 0$, Δp is

$$\Delta p = 2 \frac{T_{yz}}{h} z \quad (59)$$

The pressure drop in z -direction in Equation (59) is a linear function in z .

At corresponding values of T_{yz} and z , Δp increases with decreasing thickness of fluid film.

D. Case 2. Couette Flow with Both Co-direction and Orthogonal Pressure Flows

For a Couette flow with co-directional pressure flow in x-direction,

τ_{yx} is

$$\tau_{yx} = T_{yx} \left(1 - \frac{y}{2b} \right) \quad \left(0 \leq \frac{y}{b} \leq 2 \right) \quad (60)$$

Identical to Case 1, τ_{yz} also reads

$$\tau_{yz} = T_{yz} \left(1 - \frac{y}{b} \right) \quad \left(0 \leq \frac{y}{b} \leq 2 \right) \quad (61)$$

By means of Equations (60) and (61), τ_{yx}/T and G are, respectively,

$$\frac{\tau_{yx}}{T} = \frac{\left(1 - \frac{y}{2b} \right)}{[1 + (T_{yz}/T_{yx})^2]^{1/2}} \quad (62)$$

$$G = \frac{T_{yz}}{T_{yx}} \frac{\left(1 - \frac{y}{b} \right)}{\left(1 - \frac{y}{2b} \right)} \quad (63)$$

Equations (48) and (49) with (62) and (63) are the solutions of U^+ and W^+ for Case 2. As for the values of U^+ and W^+ near the wall, they are in the same forms as in Equations (56) and (57).

Using Equations (60) and (61), Equations (37) and (38) are, respectively,

$$p = - \frac{T_{yx}}{2b} x + f(z) \quad (64)$$

$$p = - \frac{T_{yz}}{b} z + g(x) \quad (65)$$

From Equations (64) and (65), p is

$$p = - \frac{1}{h} (T_{yx}x + 2T_{yz}z) \quad (66)$$

The pressure in Equation (66) is a function of x and z . This means that co-directional pressure flow with Couette flow in x -direction and orthogonal superposed pressure flow in z -direction are represented by Equation (66).

For Case 1, Equation (66) reduces to

$$p = - \frac{2T_{yx}}{h} z \quad (67)$$

where Equation (67) is identical to the same form as in Equation (58).

E. Discussion of Case 1

Numerical evaluation of u and w . Given a parametric study of

$R = 3 \times 10^3$, $C_f = 0.0057$ and $T_{yz}/T_{yx} = 0.2, 1.0$ and 1.8 , the dimensionless velocity profiles were computed. These are plotted in Figures 10 and 11.

In the range of $0 \leq \frac{Y}{b} \leq 0.2$, the stronger the pressure flow in z -direction,

the higher the quantities of u/u_{bc} and w/w_b . The opposite is true for

u/u_{bc} in the range of $0.2 \leq \frac{Y}{b} \leq 1$. The ratio of w/w_b is independent with

T_{yz}/T_{yx} in the range of $0.2 \leq \frac{Y}{b} \leq 1$.

Dimensionless displacement thickness. The dimensionless displacement thicknesses in the half fluid film in x - and z -directions are, respectively:

FIGURE 10. VARIATION OF DIMENSIONLESS VELOCITY PROFILES FOR u/u_{bc}

FIGURE 11. VARIATION OF DIMENSIONLESS VELOCITY PROFILES FOR w/w_b

$$\frac{\delta_x^*}{b} = 1 - \int_0^1 \left(\frac{u}{u_{bc}} \right) d\left(\frac{y}{b} \right) \quad (68)$$

$$\frac{\delta_z^*}{b} = 1 - \int_0^1 \left(\frac{w}{w_b} \right) d\left(\frac{y}{b} \right) \quad (69)$$

Using the same parameters in the last section and performing numerical integration of the second term on the right side in Equations (68) and (69), the values of δ_x^*/b and δ_z^*/b are listed in Table 1.

TABLE 1. CALCULATION OF DIMENSIONLESS
DISPLACEMENT THICKNESSES

T_{yz}/T_{yx}	δ_x^*/b	δ_z^*/b
0.2	0.2	0.11
0.2	0.1991	0.1300
1.0	0.1991	0.1211
1.8	0.2082	0.1211

Ref. 3

For the orthogonal weak pressure flows in the range of $0.2 \leq \frac{T_{yz}}{T_{yx}} \leq 1$,

δ_x^*/b remains unchanged and δ_z^*/b decreases with increasing intensity of orthogonal pressure flow. In the range of orthogonal strong pressure

flows such that $1 \leq \frac{T_{yz}}{T_{yx}} \leq 1.8$, δ_z^*/b remains unchanged and δ_x^*/b increases

with increasing intensity of pressure flow.

Evaluation of ζ_1 , ζ_2 , and ζ_3 . From Equations (44) to (46) in

Reference 7, the quantities of ζ_1 , ζ_2 , and ζ_3 are

$$\zeta_1 = \frac{\int_0^h u^2 dy}{h(u_{bc})^2} \quad (70)$$

$$\zeta_2 = \frac{\int_0^h u w dy}{h u_{bc} w_b} \quad (71)$$

and

$$\zeta_3 = \frac{\int_0^h w^2 dy}{h w_b^2} \quad (72)$$

Because of the symmetries inherent in the Couette and pressure flow relative to $y/b = 1$, Equations (70) to (72) are reduced to

$$\zeta_1 = 2 + \int_0^1 \left(\frac{u}{u_{bc}} \right)^2 d \left(\frac{y}{b} \right) - 2 \int_0^1 \left(\frac{u}{u_{bc}} \right) d \left(\frac{y}{b} \right) \quad (73)$$

$$\zeta_2 = \int_0^1 \left(\frac{w}{w_b} \right) d \left(\frac{y}{b} \right) \quad (74)$$

and

$$\zeta_3 = \int_0^1 \left(\frac{w}{w_b} \right)^2 d \left(\frac{y}{b} \right) \quad (75)$$

The quantities of ζ_1 to ζ_3 , which were numerically evaluated directly from Equations (73) to (75), are listed in Table 3.

Instead of using Equations (73) to (75) and the actual velocity profiles in Figures 10 and 11, a simple integrated form for ζ_1 to ζ_3 is obtained by approximating the actual velocity profiles by means of power-law velocity distributions. The turbulent power laws in Equations (71) to (74) in Reference 7 are:

u- profile:

$$\frac{u}{u_{bc}} = \left(\frac{y_o}{b} \right)^n \quad \left(0 \leq \frac{y_o}{b} \leq 1 \right) \quad (76)$$

$$\frac{u}{u_{bc}} = 2 - \left(\frac{y_h}{b} \right)^n \quad \left(0 \leq \frac{y_h}{b} \leq 1 \right) \quad (77)$$

w- profile:

$$\frac{w}{w_b} = \left(\frac{y_o}{b} \right)^m \quad \left(0 \leq \frac{y_o}{b} \leq 1 \right) \quad (78)$$

$$\frac{w}{w_b} = \left(\frac{y_h}{b} \right)^m \quad \left(0 \leq \frac{y_h}{b} \leq 1 \right) \quad (79)$$

Denote y_o and y_h as the reference position across the film with respect to the lower and upper surfaces of the wall, respectively. The values of n and m refer to the power-law velocity profiles in x - and z -directions, respectively. Substituting Equations (76) to (79) into (70) to (72) and integrating them, the quantities of ζ_1 to ζ_3 are

$$\zeta_1 = \frac{1}{1 + 2n} + \frac{2n}{1 + n} \quad (80)$$

$$\zeta_2 = \frac{1}{1 + m} \quad (81)$$

and

$$\zeta_3 = \frac{1}{1 + 2m} \quad (82)$$

Substituting Equations (76) and (77) into (68), the relation between n and δ_x/b is

$$n = \frac{\delta_x^*/b}{1 - (\delta_x^*/b)} \quad (83)$$

Similarly, from Equations (69), (78) and (79), m is

$$m = \frac{\delta_z^*/b}{1 - (\delta_z^*/b)} \quad (84)$$

Substituting the values of δ_x^*/b and δ_z^*/b in Table 1 into Equations (83) and (84), the values of n and m are calculated and listed in Table 2.

TABLE 2. POWER-LAW VELOCITY PROFILE EXPONENTS

T_{yz}/T_{yx}	n	m
0.2	0.2486	0.1494
1.0	0.2486	0.1378
1.8	0.2630	0.1378

Using the values of n and m in Table 2 and Equations (80) to (82), the values of ζ_1 to ζ_3 are calculated and listed in Table 3.

The calculated values of ζ_1 to ζ_3 by means of simple power-law forms are shown to agree well with those by means of numerical evaluation from Equations (73) to (75).

TABLE 3. VALUES OF ζ_1 , ζ_2 AND ζ_3

T_{yz}/T_{yx}	ζ_1		ζ_2		ζ_3	
	D. E. *	S. F. **	D. E.	S. F.	D. E.	S. F.
0.2	1.0798	1.0661	0.8736	0.8700	0.8028	0.7699
1.0	1.0786	1.0661	0.8765	0.8774	0.8046	0.7839
1.8	1.0773	1.0718	0.8801	0.8774	0.8069	0.7839

*D. E. denotes the direct numerical evaluation of ζ_1 to ζ_3 from the actual velocity profiles in Equations (73) to (75).

**S. F. denotes the calculated values of ζ_1 to ζ_3 from the simple forms in Equations (80) to (82) obtained by approximating the actual velocity profiles by power-law distributions.

F. Discussion of Case 2

To provide a similar parametric study of Case 1, the dimensionless velocity profiles were computed and plotted in Figures 12 and 13. The character of u/u_b and w/w_b for Case 2 is qualitatively similar to that for Case 1. Experimental measurements are reported for a single combination of flows produced by inserting a blockage into the gap of a concentric-cylinder, Couette-flow apparatus. The flow so produced was of Case 2. Hot wire anemometer measurements serve to provide local magnitude of the velocity components in the profiles so produced. Measured results are shown to agree well with theory, within the experimental scatter.

FIGURE 12. VARIATION OF DIMENSIONLESS VELOCITY PROFILES FOR u/u_b

FIGURE 13. VARIATION OF DIMENSIONLESS VELOCITY PROFILES FOR w/w_b

IV. CONCLUSIONS

The results of the finite tilted pad calculations stand as a set of predicted performance plots which cover a practical range of pad tilt angles and film thicknesses. They show the predominant effect of the leading edge pressure rise. They also show that, other than at the leading edge, inertia effects are of secondary importance.

The short stepped-pad study demonstrates that the numerical calculation procedure can be applied to this abrupt type of boundary change. Some further development is called for, but the essential features of the experimental measurements are represented by the computed results.

The study of Couette-flow interactions with pressure flows provides estimates of the velocity profile shapes in flows when the velocity varies in direction with distance from the walls. These calculations serve also to give numerical values to the ξ coefficients in the bearing film equations.

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