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CONJUGACY TO UNITARY GROUPS
WITHIN THE
INFINITE-DIMENSIONAL SYMPLECTIC GROUP

by

Irving E. Segal*

Applied Mathematics Division

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*Consultant, Massachusetts Institute of Technology

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ABSTRACT

The characterization of Cook for stable, "amenable," symplectic groups in an infinite-dimensional space is sharpened and reformulated so as to be applicable in part to arbitrary groups.

I. INTRODUCTION

In a recent manuscript, Cook¹ extended to infinite-dimensional spaces a fundamental classical characterization of stable linear motions in phase space: any such motion is similar to one determined by a one-parameter unitary group. Actually, the result applies to any group having the "fixed-point property," i.e., with the property that its action on a compact convex set necessarily admits a fixed point. An example is given to show that non-trivial topological assumptions are required. An assumption is made, which is satisfied in important cases, but which is stronger than is desirable for certain applications, and the manuscript indicates that the case of "uniform nondegeneracy" of the fundamental skew-symmetric form is being extended.

The present report provides an extension to a case in which uniform nondegeneracy in a certain sense may be lacking. The hypotheses are reduced chiefly to inequalities that are readily verified in practice. The proof is based on a reduction to a case similar to that treated by Cook and on the use of known properties of "amenable" groups due to Day. However, for the convenience of the reader, a self-contained account is given.

The main result may be formulated roughly as the statement that any uniformly bounded, amenable subgroup of the symplectic group in an infinite number of dimensions is similar to a unitary group. This is reminiscent of the classical theorem that any compact subgroup of the symplectic group in a finite number of dimensions is similar to a unitary subgroup (a special case of the more general theorem that all maximal compact subgroups of a semi-simple Lie group are conjugate). It is not known, and doubtful, whether an arbitrary uniformly bounded subgroup of the symplectic group is similar to a unitary group; but an analogue and extension to the cited classical result can be obtained by introducing the concept of a "quasicompact" group of transformations on a linear space as one leaving invariant a positive definite symmetric form. As a corollary to the proof of the generalized theorem of Cook, it is shown that any quasicompact subgroup of the symplectic group in an infinite number of dimensions is conjugate to a unitary group.

II. MAIN THEOREM

Before stating the main theorem, we define formally the concept of a group with the "fixed-point" property:

Definition 1. A topological group G has the *fixed-point* property, provided its continuous convex action on a compact convex subset of a locally convex, topological, linear space admits a fixed point (for every such action).

By virtue of classical results (von Neumann, Haar, Markov), solvable groups and compact groups have the fixed-point property; but not all groups do (e.g., the free group on two generators does not, a fact related to the Banach-Tarski paradox).

A. Theorem

Let B be a nondegenerate, skew-symmetric, bilinear form on the real linear vector space L . Let G be a topological group having the fixed-point property. Let the mapping $a \rightarrow V(a)$ be a representation of G by symplectic transformations on (L, B) . Then there exist a complex Hilbert space H , a continuous representation $a \rightarrow U(a)$ of G by unitary transformations on H , and a unitary mapping Σ of L onto a dense subset of H which is invariant under the $U(a)$ and is such that

$$V(a) = \Sigma^{-1}U(a)\Sigma, \quad B(x, y) = \text{Im}[\langle \Sigma x, \Sigma y \rangle]$$

$$(a \in G; \quad x, y \in L; \quad \langle \cdot, \cdot \rangle \text{ denotes inner product on } H),$$

if and only if:

(1) $B[V(a)x, y]$ is a bounded and continuous function of a , for all fixed vectors x and y ;

(2) There exists a real positive definite symmetric form S on L such that:

$$(a) \quad B \text{ is continuous relative to } S: \quad B(x, y)^2 \leq \text{const. } S(x, x) \times S(y, y);$$

(b) S is closed relative to B : if $S(x_n - x_m, x_n - x_m) \rightarrow 0$, and if $\sup_{S(y, y) < 1} |B(x_n, y)| \rightarrow 0$, then $S(x_n, x_n) \rightarrow 0$;

(c) V is continuous relative to S : for all vectors x , $S(V(a)x - x, V(a)x - x) \rightarrow 0$ as $a \rightarrow e$.

Proof of Theorem. The "only if" part of the Theorem is immediate on taking $S(x, y)$ to be the real part of the inner product $\langle \Sigma x, \Sigma y \rangle$ in a system (H, U, Σ) of the indicated nature. To prove the "if" part, let $\bar{L}$ denote the completion of L relative to $S(x, y)$ as a real inner product on L ; by continuity relative to S , B extends uniquely to a skew-symmetric bilinear form $\bar{B}$ on $\bar{L}$; S itself also extends in the same way to a positive definite symmetric form $\bar{S}$ on $\bar{L}$. By a well-known representation theorem, there exists a unique

continuous linear operator T on $(\bar{L}, \bar{S})$ such that

$$\bar{B}(x, y) = \bar{S}(Tx, y);$$

by virtue of the skew-symmetry of $\bar{B}$, $T = -T^*$, where the $*$ indicates the adjoint relative to $\bar{S}$. Now note that T^{-1} exists; i.e., T annihilates no non-zero vector in $\bar{L}$. For suppose that $Tx = 0$, $x \in \bar{L}$. Let x_n be a sequence in L such that $x_n \rightarrow x$ relative to S ; then

$$\sup_{S(y, y) < 1} |B(x_n, y)| = \sup_{S(y, y) < 1} |S(Tx_n, y)| \leq S(Tx_n, Tx_n)^{1/2},$$

employing Schwarz's inequality; from hypothesis (2c) of the Theorem, it results that $S(x_n, x_n) \rightarrow 0$; i.e., $x_n \rightarrow 0$ in $\bar{L}$. It follows also that T has a dense range; for if $S(Tx, y) = 0$ for some fixed vector y and all vectors x , then $S(x, Ty) = 0$ for all x , which implies that $Ty = 0$, so that y must vanish.

We now apply the polar decomposition for a bounded linear operator on a real Hilbert space having a dense range. According to this result, T has the form $T = RA$, where R is an orthogonal and A is a positive, semidefinite, self-adjoint operator; moreover, R and A may be chosen so as to commute with all orthogonal transformations commuting with T , and are then unique. It follows directly that $TT^* = RA^2R^{-1}$ and $T^*T = A^2$; employing the property that $T^* = -T$, we obtain the result that R commutes with A^2 ; by the orthogonal invariance of the operational calculus (here applied to the square root), R must also commute with A . On the other hand, $T^* = AR^{-1}$, so that $RA = -AR^{-1}$; combining this equation with the equation $RA = AR$, we obtain $-AR^{-1} = AR$, which in turn implies, on taking adjoints, that $RA = -R^{-1}A$. From the fact that A has a dense range, it now follows that $R = -R^{-1}$; i.e., $R^2 = -I$.

Since R commutes with A , it also commutes with T . It follows that it leaves $\bar{B}$ invariant:

$$\bar{B}(Rx, Ry) = \bar{S}(TRx, Ry) = \bar{S}(RTx, Ry) = \bar{S}(Tx, y) = \bar{B}(x, y),$$

where the orthogonality of R is again used. Now defining S_1 as the bilinear form given by the equation

$$S_1(x, y) = B(Rx, y),$$

we obtain

$$S_1(x, y) = B(Rx, y) = S(TRx, y) = S(Ax, y) = S(x, Ay) = S(Ay, x),$$

where the self-adjointness of A is used. These equations imply in particular that $S_1(x, y)$ is a positive, definite, symmetric, bilinear form on $\bar{L}$.

We now introduce a complex linear structure in L by making the definition

$$(\alpha + i\beta)x = \alpha x + \beta Rx$$

for an arbitrary complex number $\alpha + i\beta$ and arbitrary vector $x \in \bar{L}$; since $R^2 = -I$, $\bar{L}$ is endowed with the structure of a complex linear vector space by this

definition, as is well known. We next define a complex-valued function $\langle x, y \rangle$ on $L \times L$ by the equation

$$\langle x, y \rangle = S_1(x, y) + iB(x, y).$$

We assert that $\langle x, y \rangle$ is a positive definite hermitian form on $\bar{L}$ as a complex linear vector space, which is in fact endowed with the structure of a complex Hilbert space relative to this form and the action of complex numbers already given.[†] For it is obvious that $\langle x, y \rangle$ is a real-bilinear function of x and y ; that $\langle y, x \rangle = \overline{\langle x, y \rangle}$; that $\langle x, x \rangle \geq 0$; and that $\langle x, x \rangle = 0$ if and only if $x = 0$. Topological completeness is also immediate. It remains only to verify that $\langle ix, y \rangle = i\langle x, y \rangle$. Now $ix = Rx$, so that

$$\begin{aligned} \langle ix, y \rangle &= \langle Rx, y \rangle = S_1(Rx, y) + iB(Rx, y) = B(R^2x, y) + iS_1(x, y) \\ &= -B(x, y) + iS_1(x, y) = i[S_1(x, y) + iB(x, y)] = i\langle x, y \rangle. \end{aligned}$$

We now observe that all the hypotheses of the Theorem remain valid if S is replaced by S_1 . Properties (2a) and (2b) are general ones concerning the relation between the imaginary part and the real part of the complex inner product form in a complex Hilbert space, and follow directly. Property (2c) follows directly from the fact that S_1 is bounded relative to S :

$$S_1(x, x) = B(Rx, x) \leq \text{const. } [S(Rx, Rx)S(x, x)]^{1/2} = \text{const. } S(x, x),$$

by virtue of the orthogonality of R . We may therefore now assume that L is given as a dense linear submanifold of a complex Hilbert space $\bar{L}$, and that B and S are, respectively, the forms defined by taking the imaginary and the real parts of the inner product form. The representation V of G is correspondingly a continuous representation by transformations leaving invariant the imaginary part of the inner product.

At this point, we take advantage of hypothesis (1), employing Cook's argument. Noting that $B(x, y) = -S(ix, y)$ for arbitrary vectors x and y , it follows that $B(V(a)x, y) = -S(iV(a)x, y)$; fixing x , a class of linear functionals of y is obtained, whose value for any particular y is bounded; the uniform-boundedness principle then implies that $\|iV(a)x\|$ is uniformly bounded as a function of a . Another application of the same principle implies that $\|V(a)\|$ is itself bounded.

Now applying the fixed-point property to the compact convex set consisting of the set of all normalized, positive, linear functionals on the Banach space B consisting of all bounded functions on G , relative to the supremum of the absolute value as norm, and employing the w^* -topology in the dual of B , it follows that there exists a linear functional M on B with the following properties:

[†]The mutual relations between a skew form, a symmetric form, and a complex linear structure in a real-linear vector space have been developed at length in the finite-dimensional case in the text of Lichnerowicz.² For a real Hilbert space, the case in which the operator T above has a bounded inverse is treated in the notes of Mackey.³

$$(i) \quad M(1) = 1,$$

(ii) $(M f(a)) \geq 0$ if the function $f(a)$ has the property $f(a) \geq 0$, for all a in G , and

$$(iii) \quad M_a(f(ba)) = M_a(f(a)) \text{ for every element } b \text{ of } G.$$

Now defining $S_0(x, y) = M[S(V(a)x, V(a)y)]$, it is clear that $S_0(x, y)$ is a symmetric bilinear form on $\bar{L}$. Further, S_0 is continuous relative to S :

$$\begin{aligned} S_0(x, x) &= M[S(V(a)x, V(a)x)] \leq M[||V(a)||^2 ||x||^2] \\ &\leq ||x||^2 \sup_a ||V(a)||^2, \end{aligned}$$

where we write $||x||$ for $S(x, x)^{1/2}$. On the other hand, S is continuous relative to S_0 :

$$\begin{aligned} S(x, x) &= \langle x, x \rangle = \langle V(a)^{-1}V(a)x, x \rangle = \langle V(a)x, V(a)^{* -1}x \rangle \\ &\leq ||V(a)x|| ||V(a)^{* -1}x||; \end{aligned}$$

applying M , noting that $V(a)^{* -1} = V(a^{-1})^*$, and using Schwarz's inequality, we obtain

$$\begin{aligned} S(x, x) &\leq M[||V(a)x|| ||V(a^{-1})^*x||] \\ &\leq M[||V(a)x||^2] M[||V(a^{-1})^*x||^2]^{1/2}. \end{aligned}$$

Employing the boundedness of $||V(a)||$, or equivalently, that of $||V(a^{-1})^*||$, it follows that

$$S(x, x) \leq \text{const.} [S_0(x, x)S(x, x)]^{1/2},$$

which implies, in turn, that

$$S(x, x)^{1/2} \leq \text{const.} S_0(x, x)^{1/2}.$$

Finally, by virtue of the invariance property (iii) of M , $S_0(x, x)$ is left invariant by the $V(b)$, $b \in G$.

We may now replace S by S_0 , at the cost of losing the complex structure and hermitian form, but these features are readily recovered by once more going through the procedure indicated earlier (in the simpler case in which the operator T is invertible), resulting in a new complex Hilbert space in which the $V(a)$ act in unitary fashion. Specifically, there exists a real-linear operator T_0 on $\bar{L}$, continuous and with continuous inverse relative to S_0 , such that $B(x, y) = S_0(T_0x, y)$, for the same reasons as earlier; that T_0 is invertible follows from the equivalence of the forms S and S_0 . Taking the polar decomposition $T_0 = R_0A_0$, then, as earlier, $R_0^2 = -I$, and a new complex structure may be introduced by the definition $(\alpha + i\beta)x = \alpha x + \beta R_0x$. Defining $S'(x, y) = B(R_0x, y)$ and a new form

$$\langle x, y \rangle' = S'(x, y) + iB(x, y),$$

the argument to show that $\bar{L}$ is thereby endowed with the structure of a complex Hilbert space is identical with that given earlier. In addition, it now follows that T_0 commutes with the $V(a)$, for on expressing the invariance of B , $B(x, y) = B(Vx, Vy)$, where V is an arbitrary one of the $V(a)$, in terms of S' , one obtains the equation $S'(T_0x, y) = S'(T_0Vx, Vy)$; using the invariance of S' under the $V(a)$, the last expression is equal, in turn, to $S'(V^{-1}T_0Vx, y)$; therefore $T_0 = V^{-1}T_0V$. It follows that V commutes with the constituents of T_0 in its polar decomposition, in particular with R_0 ; returning to the definition of S' , it follows that V leaves S' invariant. This means that the $V(a)$ are unitary relative to the inner product $\langle x, y \rangle'$.

It remains only to show that the mapping $a \rightarrow V(a)$ is continuous. This means that $\langle V(a)x, y \rangle'$ should be a continuous function of a for all x and y . Now $\langle V(a)x, y \rangle' = S'(V(a)x, y) + iB(V(a)x, y)$; $B(V(a)x, y)$ is continuous by hypothesis; the remaining term may be put in the form $B(R_0V(a)x, y) = -B(V(a)x, R_0y)$, on noting that R_0 is symplectic, by virtue of the relations $B(R_0x, R_0y) = S_0(R_0A_0R_0x, R_0y) = S_0(A_0R_0x, y) = B(x, y)$; it is evident that $B(V(a)x, R_0y)$ is a continuous function of a .

B. Example

Let L be the space of all real C^{00} solutions of the Klein-Gordon equation $\square\phi = m^2\phi$ which have compact support in space at each time. Let

$$B(\phi, \psi) = \int [\phi(\vec{x}, t)\dot{\psi}(\vec{x}, t) - \psi(\vec{x}, t)\dot{\phi}(\vec{x}, t)] d\vec{x}.$$

Let G denote the additive group of the real numbers, and let $V(t)$ denote for any t the temporal displacement determined by the indicated partial-differential equation. It is not difficult to verify that $B(\phi, \psi)$ as defined above is independent of t ; from which it follows that the temporal displacements acting on L , $V(t)$, leave B fixed. We may apply the Theorem to show that a complex, pre-Hilbert, space structure may be introduced on L relative to which the $V(t)$ are unitary. Specifically, the conservation of energy shows that $B(V(t)\phi, \psi)$ remains bounded; that it is continuous, follows directly from the general theory of such equations. The energy inner product $S(\phi, \psi)$ is given by the equation

$$S(\phi, \psi) = m^2 \int \phi(\vec{x}, t)\psi(\vec{x}, t) d\vec{x} + \int \text{grad}\phi(\vec{x}, t) \cdot \text{grad}\psi(\vec{x}, t) d\vec{x} \\ + \int \dot{\phi}(\vec{x}, t)\dot{\psi}(\vec{x}, t) d\vec{x}.$$

Hypothesis (2) of the Theorem may be verified directly. The hermitian structure, which results now from the application of the Theorem, is in fact identical with the unique Lorentz-invariant hermitian form on L , which is employed in the theory of relativistic fields; (cf. Ref. 4).

The foregoing example could also have been treated through the consideration of four-dimensional Fourier transforms. This is not the case for the space L defined in the same way, except that an external potential term is introduced: $\square\phi = m^2\phi + F(\vec{x})\phi$ (this may be considered to be an algebraically simplified but fundamentally typical equation for such a system as a charged meson field in interaction with an external electromagnetic potential). The form B and the representation $V(t)$ may be defined in the same way.

For simplicity, $F(\vec{x})$ will be assumed to be bounded, continuous, and non-negative. The energy form is altered by the addition of the term $\int F(\vec{x})\phi(\vec{x}, t)\psi(\vec{x}, t)$, but otherwise the analysis is identical with that for the Klein-Gordon equation. The resulting hermitian form on L which is invariant under the $V(t)$ is identical with the one which derives naturally from the two-point function for the associated quantized field.

III. COROLLARIES TO MAIN THEOREM

We consider now the case of a continuous representation V of a general topological group G by symplectic transformations on a space (L, B) . Even when G is a simple Lie group, the conclusion of the Theorem is in all probability invalid; in fact, an open simple Lie group is in a sense the furthest from admitting the fixed-point property. The proof of the Theorem suggests, however, the following characterization of symplectic representations, which are unitary relative to a suitable, complex, hermitian inner product.

A. Corollary 1

With the notation and assumptions of the Theorem, except that G is not assumed to possess the fixed-point property, then the conclusion is valid if and only if hypotheses (1) and (2) hold with a form S which is invariant under V .

The proof is already contained in that of the Theorem, since the fixed-point property is used only to obtain an invariant symmetric form, which is here hypothesized.

As an application, we may add one more item to the complex of results, indicating strongly that the concept of classical simple Lie group and its basic theory extend most naturally to the case of infinite-dimensional groups. One significant property of the finite-dimensional, semisimple groups is their possession of a maximal compact subgroup, which is unique within conjugacy. The following result extends this theorem to the case of an infinite-dimensional, symplectic group; it is highly probable that the same is true for all the classical infinite-dimensional groups, and indeed this may well be provable along generally similar lines, but such general considerations are outside the scope of this report.

B. Corollary 2

Any quasicompact subgroup of the symplectic group on a complex Hilbert space is conjugate to a subgroup of the unitary group.

We recall that a quasicompact group of operators on a Hilbert space is one leaving invariant a positive definite symmetric form, topologically equivalent to the real part of the inner product. In particular, any compact group is quasicompact, as follows by integration over the group. We recall also that the symplectic group over the complex Hilbert space H is the group of all continuous invertible linear transformations on H which leave invariant the form $B(x, y) = \text{Im}[\langle x, y \rangle]$.

Proof of Corollary 2. Let the given inner product on H be written in the form $\langle x, y \rangle = S(x, y) + iB(x, y)$, where $S(x, y)$ and $B(x, y)$ are real forms, and let the action of i be denoted as R , as an orthogonal operator on H as a real Hilbert space relative to $S(x, y)$ as inner product. Let $S_0(x, y)$ be another positive definite inner product on H , which is topologically equivalent to $S(x, y)$, and which is invariant under the group G of symplectic transformations on H . By Corollary 1, there exists a hermitian form $\langle x, y \rangle'$ on H together with a complex structure given by a real-linear operator R' , which is orthogonal relative to the form $S'(x, y) = \text{Re}[\langle x, y \rangle']$, which are invariant under G . From the topological equivalence of S and S_0 , it follows readily that S and S' are likewise equivalent, which implies that H is complete relative to the inner product $\langle x, y \rangle'$. Thus the action of G is unitary on H relative to a modified hermitian form and complex structure, and to complete the proof it is only necessary to show that this form and structure may be transformed into the originally given one on H by a symplectic transformation.

We now set up explicitly a symplectic transformation, which will carry any element V of the group G into a unitary transformation on the original space H . To this end, we set K for the range of $I - R$ and note that RK is (real-)orthogonal to K , and that both K and RK are closed real-linear manifolds; moreover, H is the direct sum of K and RK . Indeed, $[(I - R)/2]^{1/2}$ is orthogonal, and of adjoint (or inverse) $[(I + R)/2]^{1/2}$, as is clear from the fact that $R^2 = -I$; and $RK = R(I - R)H = (I + R)H$; the relation $x = (1/2)[(I - R)x + (I + R)x]$ then shows that $H = K \oplus RK$. Now denoting by $[x, y]$ the element of H whose first component is x and second component is y —i.e., the element z such that $x = (1/2)(I - R)z$ and $y = (1/2)(I + R)z$ —the action of the complex unit has the form

$$i[x, y] = [-y, x],$$

as follows by multiplication of the foregoing equations by R , and the original hermitian form may be given by the equation

$$\langle [x, y], [x', y'] \rangle = S(x, x') + S(y, y') + i[S(x, y') - S(x', y)].$$

Making the identical analysis for derived complex structure and hermitian form $\langle x, y \rangle'$, we have also $H = K' \oplus R'K'$; and denoting by $[x, y]'$ the element of H whose component in K' (relative to S') is x and in $R'K'$ is y , we have the analogous formulas

$$i'[x, y]' = [-y', x]';$$

$$\langle [x, y]', [x', y'] \rangle' = S'(x, x') + S'(y, y') + i[S'(x, y') - S(x', y)].$$

Now the dimension of K as a real Hilbert space is the same as that of H as a complex Hilbert space, and the same is true of K' ; consequently, there exists an orthogonal transformation Y , which carries (K, S) onto (K', S') . Let Z denote the transformation $[x, y] \rightarrow [Yx, Yy]'$; then Z carries (H, S) onto (H, S') ; at the same time, it preserves the form B , since

$$\begin{aligned} B([Yx, Yy]', [Yx', Yy']') &= S'(Yx, Yy') - S'(Yy, Yx') = S(x, y') \\ &\quad - S(x', y). \end{aligned}$$

It follows that if V is any real-linear transformation on H , which is orthogonal relative to S' , then ZVZ^{-1} is a real-linear transformation on H , which is orthogonal relative to S ; if, in addition, V is symplectic, then so also is ZVZ^{-1} ; but a unitary transformation on H is precisely one that is both orthogonal and symplectic; thus, for any element $V \in G$, ZVZ^{-1} is a unitary transformation on H relative to the original inner product, and we have therefore achieved the transformation of the given symplectic group G into a unitary one, via a symplectic transformation.

IV. CONCLUSION

The *uniqueness* of the complex structure, which will be left invariant by a given symplectic group and be instrumental in its transformation to a unitary group, is a natural mathematical question. It is easily seen that if the given symplectic group is irreducible, in the sense of admitting no commuting operators other than scalars, then there is uniqueness of the complex structure within sign. This is the case, for example, with the complex structure in the solution manifold for the Klein-Gordon equation described earlier, relative to the Poincare group. On the other hand, when the invariance group is more restricted, as in the case of the equation with an external potential indicated earlier, there is no irreducibility, and indeed, the complex structure in question is also not unique. Physically, one expects a unique structure, insofar as it is physically relevant, from the dynamics alone, irrespective of the magnitude of the fundamental symmetry group. This is very probably indeed the case, if the physically appropriate restriction (in many cases) of positivity of the energy is imposed; this is connected with the uniqueness of the vacuum for the associated Bose-Einstein quantum field. The treatment of these matters is, however, beyond the scope of this report; for certain aspects, see Ref. 5.

REFERENCES

1. J. M. Cook, *Complex Hilbertian Structures on Stable Linear Dynamical Systems*, typed manuscript (Sept 1965).
2. A. Lichnerowicz, text on *Differential Geometry* (in French).
3. G. W. Mackey, notes (mimeographed) on *Mathematical Foundations of Classical and Quantum Mechanics*, distributed by Math. Dept., Harvard University.
4. I. E. Segal, *Direct Formulation of Causality Requirements on the S-operator*, Phys. Rev. 109 (1958), 2191-2198.
5. *Mathematical Problems of Relativistic Physics*, Amer. Math. Soc., Providence (1963).