324 14/1/65 UCRL-7729

University of California

Ernest O. Lawrence Radiation Laboratory

PATENT CLEARANCE OBTAINED. RELEASE TO
THE PUBLIC IS APPROVED. PROCEDURES
ARE ON FILE IN THE RECEIVING SECTION.

SOME THEORETICAL AND EXPERIMENTAL ASPECTS OF DESIGN WITH BRITTLE MATERIAL

Livermore, California

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

UCRL-7729
Metals, Ceramics,
and Materials, UC-25
TID-4500, (37th Ed.)

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

SOME THEORETICAL AND EXPERIMENTAL ASPECTS
OF DESIGN WITH BRITTLE MATERIAL

Ernest Y. Robinson

August 3, 1964

Printed in USA. Price \$4.00. Available from the Clearinghouse for Federal
Scientific and Technical Information, National Bureau of Standards,
U. S. Department of Commerce, Springfield, Virginia.

TABLE OF CONTENTS

	<u>Page No.</u>
I. Introduction	1
II. Theory	2
A. The Weibull Distribution Functions	3
B. The Effect of Size and Stress Distribution	5
1. The Size Effect	6
2. The Effect of Stress Distribution	7
C. Finding the Weibull Modulus, m	9
D. The Normalized Stress Distribution	12
E. Application of the Normalized Stress Distribution	19
F. Design Considerations	20
1. Extreme Value Statistics	21
2. The Proof Stress and Prestress (Bias)	24
3. Safety Factor	27
4. System Survival	31
5. The Extreme Safety Factor	32
III. Experimental	33
IV. Specimens	37
V. Apparatus and Procedure	43
VI. Results	53
VII. Discussion	67
Conclusions	73
Bibliography	77
Appendix A: The Risk of Rupture for the Bending and Bursting of Tubes	81

TABLE OF CONTENTS(Continued)

	<u>Page No.</u>
Appendix B: Errors Induced by Geometric Irregularities	85
Appendix C: Extreme Value and Mean Value of the Truncated, Unbiased Weibull Distribution	89
Appendix D: Variance of the Unbiased, Untruncated Weibull Distribution and its Least Values	93
Appendix E: Risk of Rupture for Several Simple Stress Distributions	97

List of Tables

Table A. Essential results summarized from Data Tables 1-5	54
Data Table 1. Burst test of one group of round tubes .	57
Data Table 2. Burst test of a second group of round tubes	59
Data Table 3. Combined burst test data from a third group of round tubes together with the data from Data Tables 1 and 2	61
Data Table 4. Modulus of rupture results for BeO tubes .	63
Data Table 5. Results of burst tests of fragments from modulus-of-rupture tests	65

List of Figures

Figure 1. The normalized Weibull distribution (working graph)	16
Figure 2. Theoretical Weibull lines	18
Figure 3. Extreme values (minima) of the reduced Weibull distribution (normalized)	24
Figure 4. Theoretical strength improvement due to bias .	26
Figure 5. Material safety factor vs Weibull modulus, m .	30

List of Figures (Continued)

	<u>Page No.</u>
Figure 6. Risk-of-rupture coefficient for round tubes under internal pressure	36
Figure 7. Risk-of-rupture coefficient for round tubes under three-point symmetric bending	36
Figure 8. Risk of rupture vs normalized stress	37
Figure 9. Roundness charts of test specimens	39
Figure 10. Grain size variation in test specimen	41
Figure 11. Microstructural damage due to proof stress	42
Figure 12. Sealing techniques tried in the burst test	45
Figure 13. Cross section of pressurizing probe	47
Figure 14. The probes and O-rings	47
Figure 15. Burst apparatus - assembly nearly complete	48
Figure 16. Burst apparatus showing induction coils and susceptor-blast shield	48
Figure 17. Burst apparatus ready for elevated temperature test	51
Figure 18. Overall view of test area and equipment	51
Figure 19. Schematic of fluid pressurizing and controlling circuit	52
Figure 20. Fracture probability for BeO bar under pure bending stress	56
Figure 21. Maximum bending coefficient for elliptical tube under internal pressure	87
Figure 22. Error in maximum hoop stress in elliptical tube under internal pressure	87
Figure 23. Strength improvement due to truncation of the Weibull distribution	92

List of Figures (Continued)

Page No.

- Figure 24. Standard deviation of various sample sizes
as a function of the Weibull modulus . . . 96

SOME THEORETICAL AND EXPERIMENTAL ASPECTS OF DESIGN WITH BRITTLE MATERIAL

I. INTRODUCTION

Brittle materials, materials without the capability of plastic deformation and usually characterized by a compressive strength which is several times the tensile strength, are more and more frequently necessary in designs which require a precise knowledge of their strength behavior. Most refractory materials are brittle, and their application, for example as fuel elements in gas-cooled reactors or in hypersonic leading edges and in rocket nozzles, requires a very high reliability. Even "ductile" materials under certain biaxial stress conditions, or at high strain rates, fail in a brittle fashion. The strength of such materials is, unfortunately a variable quantity and must be studied statistically. The classical design safety factors must be redefined in terms of reliability. Effects of size and stress distribution may be critical and must be understood. In large systems cracking may indeed be unavoidable.

The evaluation of the requisite statistical parameters for a set of strength data is expedited by the methods presented here. Aspects of design reliability are considered in relation to the classical safety factor and methods of successfully predicting the extreme values, i. e., the weakest specimens, are established.

The technique of burst-testing of brittle materials developed for this study and some of the analytical results should prove to be of value for the investigation of, and design with, brittle materials.

II. THEORY

The most distinguishing feature of experimental strength data on brittle materials is the wide distribution of the data. Many specimens are found to be significantly different from the mean, even in very closely controlled testing. Such a situation leads to a dangerous uncertainty in designs which are based upon mean values. A design philosophy based on probabilities of failure and reliability requirements has gradually evolved from the foundations of Griffith (12, p. 163-198) and Weibull (25, no. 151; 26, no. 153). Such an approach is possible for ductile materials as well, but the relatively small deviations of strength about the mean value allow a practical design to be based on the mean value alone.

The premises of the theory of strength of brittle materials, as advanced by Weibull, establish the importance of tensile stress as the failure mechanism and the existence of a random distribution of flaws. The flaws are of random intensity, and fracture, initiated at any flaw, is assumed to precipitate failure of the specimen. It is evident that large specimens favor the presence of extreme flaws and hence should be weaker. The fact that stress concentrations tend to remain high in brittle materials (in ductile materials plastic strain improves the stress distribution thereby reducing the stress concentration) implies a development of the fracture at lower values of nominal stress and supports the "weakest link" mechanism of brittle failure, which is fundamental to this theory.

The theory will be reviewed here in sections II-A, -B, and -C and certain new modifications and their applications will be described

in the remaining sections. The calculations which are specific to the experimental portion of this study are given in section III.

A. The Weibull Distribution Functions

Consider a material which, at a uniform stress condition, has a probability of failure per unit volume P_0 . The probability of survival, S , of a volume V of the material is given by

$$S = (1 - P_0)^V \quad (1)$$

for the simultaneous survival of V volume units. Taking logs of equation (1) gives

$$\ln S = V \ln (1 - P_0). \quad (2)$$

Weibull defines the risk of rupture as $R \equiv -\ln S$ and, in terms of an infinitesimal element, gives

$$dR = -\ln (1 - P_0) dV.$$

The term $-\ln (1 - P_0)$ is assumed to be a positive function, $n(\sigma)$, of the tensile stress σ alone. Substitution gives:

$$dR = n(\sigma) dV$$

and

$$R = \int_V n(\sigma) dV.$$

From the definition of R we get:

$$S = \exp \left[- \int_V n(\sigma) dV \right]. \quad (3)$$

A form for $n(\sigma)$ suggested by Weibull and justified by its agreement with experimental results is

$$n(\sigma) = \left(\frac{\sigma}{\sigma_0} \right)^m$$

where σ_0 and m are constants. This leads to an expression for S in the following form.

$$S = \exp \left[- \int_V (\sigma/\sigma_0)^m dV \right]. \quad (4)$$

Equation (4) is used in analyzing strength data from brittle materials and frequently provides satisfactory approximations. It is, furthermore, sufficiently simple to allow closed form solutions for many real stress distributions.

In equation (4) the risk of rupture is

$$R = \int \left(\frac{\sigma}{\sigma_0} \right)^m dV. \quad (5)$$

The risk of rupture varies from 0 to ∞ as the survival probability, S , varies from 1 to 0. This implies a variation in σ from 0 to ∞ ; that is, specimens of 0 and infinite strength exist. It is, of course, evident that an upper bound must exist, and if such a bound is taken as the maximum theoretical molecular strength, values are obtained which are 100 to 1000 times the usual mean values of strength. It is then reasonable (except for very small volumes) to take the upper limit as infinite. Very small volumes will have mean strengths closer to theoretical but subject to great variations (1). Whether or not the lower strength limit for a material exists may be difficult to determine if it is near zero since a continuous process of selection exists as specimens are machined and handled. Because of such handling most specimens will actually exhibit a finite lower bound. If a significantly high lower bound exists, it will be manifested in the analysis of the data, and in such cases an added degree of freedom to the proposed

function $n(\sigma)$, in the form of lower bound stress, σ_u , is required.

This modified form was also proposed by Weibull, as

$$n(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m,$$

and the distribution function becomes

$$S = \exp \left[- \int \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m dV \right] \quad (6)$$

and is known as the general Weibull distribution function. This form of the function greatly complicates the integration for the risk of rupture. However, a rather extensive bit of curve matching can be achieved with it, since there are now three arbitrary parameters.

Another form, proposed to account for a maximum value of the variable above which the probability of survival vanishes (26, no. 153), is

$$n(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_s - \sigma} \right)^m$$

with the provision, of course, that $n(\sigma) \equiv 0$ when $\sigma < \sigma_u$ or $\sigma > \sigma_s$.

But this form is not at all convenient.

The implications of the theory with regard to the effect of size and stress distribution can now be examined.

B. The Effect of Size and Stress Distribution

The importance of size is well known in tests of ceramic materials. The size effect, i. e., the decrease of strength with increase in size, and the dependence of strength on the stress distributions are aspects which may be predicted by this theory, whereas in the past there was no unified explanation. The prediction of a size effect

followed by a comparison of expected strengths for the case of uniform tension and uniform bending will be shown as examples. In the following, the form of equation (4) will be assumed.

1. The Size Effect

Consider the risk of rupture for the case of uniform stress (tension) for two groups of specimens of identical material in which the volumes subject to such stress are V_1 and V_2 . From equation (5) we get, using strengths σ_1 and σ_2 and noting that stress is independent of position,

$$R_1 = \left(\frac{\sigma_1}{\sigma_0} \right)^m V_1, \quad (7a)$$

$$R_2 = \left(\frac{\sigma_2}{\sigma_0} \right)^m V_2. \quad (7b)$$

If we compare the two types of specimens at the same value of failure probability or what amounts to the same thing, at equal risks of rupture, then

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1} \right)^{1/m}. \quad (8)$$

Equation (8) shows that the strength depends upon specimen volume and will decrease as the volume increases. Furthermore, the size effect decreases as the value of m (the Weibull modulus) increases. Equation (8) is more general than is at first apparent. The risk-of-rupture integral, given by equation (5), is expressible in the form $kV(\sigma_r/\sigma_0)^m$, where k may be a normalized geometry integral, and σ_r is a reference stress. Wherever k does not change between the two specimens the size effect as given by equation (8) will result.

As a numerical example, m will be taken as 10 (a reasonable value for beryllia). If the volume ratio between two specimens is 10, then equation (8) gives

$$\frac{\sigma_1}{\sigma_2} = 1.26,$$

hence the large specimens are expected to be only about 80% as strong as the smaller specimens.

2. The Effect of Stress Distribution

The effect of stress distribution will be analyzed for uniform bending as compared to uniform tension. In uniform bending which exists for example between the central loads of a symmetric four-point loading test, the stress varies linearly through the height of the beam only:

The volume element for the portion of the span in uniform bending is

$$dV = bl \, dy.$$

The risk of rupture for this case is given by substitution in equation (5):

$$R_B = \left(\frac{\sigma_B}{\sigma_0} \right)^m \int_0^c \left(\frac{y}{c} \right)^m bl \, dy.$$

Note that integration is carried out only in the region of tensile stress.

The result is

$$R_B = \left(\frac{\sigma_B}{\sigma_0} \right)^m \frac{V_B}{2(m+1)} \quad (9a)$$

where V_B is the total volume of the beam in the span l .

The risk of rupture for uniform uniaxial tension in any span is, as before:

$$R_T = \left(\frac{\sigma_T}{\sigma_0} \right)^m V_T.$$

At the same values of risk of rupture, the ratio of the expected strengths is given as

$$\frac{\sigma_B}{\sigma_T} = \left[2 \frac{V_T}{V_B} (m+1) \right]^{1/m}. \quad (10)$$

If the volumes are equal,

$$\frac{\sigma_B}{\sigma_T} = \left[2(m+1) \right]^{1/m}. \quad (10a)$$

Using again the example of $m = 10$ gives for the expected strength difference

$$\frac{\sigma_B}{\sigma_T} = 1.36,$$

and the tensile strength should thus be about 73.5% of the bending strength.

These two cases illustrate the marked effect which size and stress distribution can exert upon expected strength values, and serve to emphasize the importance of control over the test, coupled with a proper analysis of the data.

Differences between the results of various stress distributions observed in practice can now be explained, in many cases with

satisfactory quantitative agreement, on the basis of the Weibull model.

In order to define the statistical strength distribution it is necessary to evaluate the various parameters appearing in equation (4), of which the most important is m .

C. Finding the Weibull Modulus, m

To evaluate the Weibull modulus from experimental data, equation (4) is expressed in logarithmic form:

$$\ln S = - \int \left(\frac{\sigma}{\sigma_0} \right)^m dV.$$

If the integral can be expressed as $kV \left(\frac{\sigma_{\max}}{\sigma_0} \right)^m$ where $\sigma_{\max}$, the reference stress, is taken as the maximum fiber stress, then, taking logs again,

$$\ln \ln \frac{1}{S} = m \ln \sigma_{\max} + \ln \frac{kV}{\sigma_0^m}. \quad (11)$$

Experimental values of $\sigma_{\max}$ and the corresponding values of survival probability may be used to deduce m . A plot of equation (11) with coordinates $\ln \ln \frac{1}{S}$ and $\ln \sigma_{\max}$ will produce a straight line of slope m . If, however, the data does not produce a satisfactory straight line then the accepted practice is to use a function, $n(\sigma)$, modified to include a threshold value as in equation (6), giving

$$\ln \ln \frac{1}{S} = m \ln (\sigma_{\max} - \sigma_u) + \ln \frac{kV}{\sigma_0^m}. \quad (12)$$

Using the graphical coordinates $\ln \ln \frac{1}{S}$ and $\ln (\sigma_{\max} - \sigma_u)$, one tries various values of σ_u until a straight line is obtained. This corresponds to the general Weibull distribution, given in equation (6),

but only for cases where the risk of rupture is expressible as

$$R = kV \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m.$$

To this point the exposition is, in general, similar to the various presentations in the literature. Weibull himself suggests that data which produce curved lines, when treated as in equation (11), be re-plotted with various trial values of σ_u , as in equation (12), until a straight line is obtained. In general, however, the risk of rupture does not integrate to the form $kV \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m$. It should be noted that equation (12) evolves from the assumption of equation (6) only if σ_u is assumed to follow the same spatial distribution as σ , although Weibull treats σ_u as a constant. The assumption of σ_u being a constant in space has the consequence that equation (12) applies only to the case of uniform tension.

The risk of rupture for the generalized Weibull distribution will now be studied with the purpose of finding m , and to show the incorrectness in categorically using the method of equation (12).

The function to be integrated is

$$R = \int \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m dV$$

where

σ_u = threshold,

σ_0 = scaling factor.

Clearly the case of uniform tension in a volume V results in

$$R = \left(\frac{\sigma_T - \sigma_u}{\sigma_0} \right)^m V \quad (13)$$

where σ_T = tensile stress. Equation (13) leads to the form of equation (12). The case of uniform bending, e. g., symmetric four-point loading, gives

$$R = \int_V \left(\frac{\sigma_B \frac{y}{c} - \sigma_u}{\sigma_0} \right)^m dV.$$

In this case the risk of rupture is assumed to be zero at the stress σ_u ; hence the integration must be taken over the region where the stress is equal to or greater than σ_u , and σ_u is a constant independent of position.

The integration gives

$$k = \frac{V}{2\sigma_B \sigma_0^m} \left| \frac{\left(\sigma_B \frac{y}{c} - \sigma_u \right)^{m+1}}{m+1} \right|_c^{\frac{\sigma_u}{\sigma_B}}.$$

Putting in the limits:

$$R = \frac{V}{2(m+1)} \left(\frac{\sigma_B - \sigma_u}{\sigma_0} \right)^m \left(1 - \frac{\sigma_u}{\sigma_B} \right). \quad (14)$$

Writing this risk of rupture into a logarithmic form similar to equation (12), and rearranging a bit:

$$\ln \ln \frac{1}{S} = (m+1) \ln (\sigma_B - \sigma_u) - \ln \sigma_B + \ln \frac{V}{2(m+1)\sigma_0^m}. \quad (15)$$

The extra term involving the variable σ_B shows that there is no reason to expect a straight line on the coordinates of equation (12). The graphical determination of m , from uniform bending data, requires coordinates of $\ln \ln \frac{1}{S} + \ln \sigma_B$ and $\ln (\sigma_B - \sigma_u)$ to produce a straight line of slope $m+1$ (4, 5).

The risk of rupture for the case of three-point center loading, in which the bending is varying linearly in two dimensions, leads to a nonintegrable form, since

$$R = 2b \iint \left[\frac{\sigma_B \frac{2xy}{lc} - \sigma_u}{\sigma_0} \right]^m dx dy.$$

Integrating x first between the limits $\frac{lc}{2y} \frac{\sigma_u}{\sigma_B}$ and $\frac{l}{2}$ results in

$$R = \frac{V}{\sigma_B \sigma_0^m 2(m+1)} \int_{\frac{c}{\sigma_B} \sigma_u}^c \left(\sigma_B \frac{y}{c} - \sigma_u \right)^{m+1} \frac{dy}{y},$$

which is not evidently integrable in closed form for general m .

Again it is incorrect to deduce m and σ_0 from a requirement such as equation (12) imposes and data of three-point bending. The use of Weibull's more general distribution function is, therefore, frequently inconvenient; thus one is led to rationalize, if possible, the use of the distribution with $\sigma_u = 0$ to approximate more complex distributions.

D. The Normalized Stress Distribution

If attention is confined to the reduced distribution, $\sigma_u = 0$, a certain degree of simplification and insight into the nature of the parameters is afforded by the use of mean strength, σ_a , as the normalizing factor, suitably substituted for σ_0 in the distribution function. In the following, the equations of Weibull's theory will be re-expressed in terms of the mean strength, and it will be shown that the failure probability distribution is explicitly dependent only on σ_a and m , regardless of the stress distribution or the specimen volume. This

relation allows a straightforward and simple determination of m in cases where the integration of the applied stress distribution is not possible, and data of various specimens of a material are at hand.

Equation (4) is assumed to apply and is repeated here for convenience.

The expression for survival probability is equation (4),

$$S = \exp \left[- \int (\sigma/\sigma_0)^m dV \right]. \quad (16)$$

We replace σ with $\sigma_f f(\xi)$ where the reference stress, σ_f , is taken as the maximum tensile stress (though it could be any experimental reference stress), and $f(\xi)$ is an appropriate normalized geometry function describing the distribution of stress. This gives

$$S = \exp \left\{ - \left(\sigma_f / \sigma_0 \right)^m V \int [f(\xi)]^m d\xi \right\}$$

or

$$\frac{1}{\sigma_f^m} \ln \frac{1}{S} = V \int \frac{[f(\xi)]^m d\xi}{\sigma_0^m}. \quad (17)$$

The material and specimen geometry fix the right side of this equation, and therefore

$$\frac{1}{\sigma_f^m} \ln \frac{1}{S} = \text{constant}.$$

If the value of mean strength, σ_a , is chosen as the reference stress, to which corresponds the survival probability, S_a , then

$$\frac{1}{\sigma_a^m} \ln \frac{1}{S_a} = V \int \frac{[f(\xi)]^m d\xi}{\sigma_0^m}$$

which equated with equation (17), gives

$$S = \exp \left[\ln S_a \left(\frac{\sigma_f}{\sigma_a} \right)^m \right]. \quad (18)$$

Equation (18) is a simpler and more convenient expression of equation (4), which may be applied directly to any test result regardless of the stress distribution. It replaces the parameter σ_0 by the more familiar σ_a (mean strength). The integration is replaced by $\ln S_a$, a value obtainable from the data.

$\ln S_a$ can, however, be given as an explicit function of m only.

The mean value of strength σ_a , is

$$\sigma_a = \int_0^{\infty} S \, d\sigma$$

and, using equation (18),

$$\sigma_a = \int_0^{\infty} \exp \left[\ln S_a \left(\frac{\sigma}{\sigma_a} \right)^m \right] d\sigma. \quad (19)$$

If we set

$$x = \ln \frac{1}{S_a} \left(\frac{\sigma}{\sigma_a} \right)^m$$

then

$$\sigma_a = \frac{\sigma_a}{m \left(\ln \frac{1}{S_a} \right)^{1/m}} \int_0^{\infty} e^{-x} x^{1/m-1} dx$$

and, since the integral is $\Gamma(1/m)$,

$$\ln \frac{1}{S_a} = \left[\frac{1}{m} \Gamma \left(\frac{1}{m} \right) \right]^m.$$

From the recurrence relation

$$\Gamma(1+x) = x\Gamma(x)$$

we get

$$\ln \frac{1}{S_a} = \left[\Gamma \left(1 + \frac{1}{m} \right) \right]^m. \quad (20)$$

Putting back in equation (18):

$$S = \exp \left\{ - \left[\frac{\sigma_f}{\sigma_a} \Gamma \left(1 + \frac{1}{m} \right) \right]^m \right\}. \quad (21)$$

Equation (21) is general for the reduced Weibull distribution and will prove much more useful than the form of equation (16). For comparison, the logarithmic form is

$$\ln \ln \frac{1}{S} = m \ln \frac{\sigma_f}{\sigma_a} + m \ln \Gamma \left(1 + \frac{1}{m} \right). \quad (22)$$

If we use $\beta = \sigma_f / \sigma_a$ and the corresponding survival probability is denoted by S_β , then a combination of equations (20) and (21) yields

$$S_\beta = S_a^{\beta^m}. \quad (23)$$

With equation (21) the Weibull modulus m can be plotted against survival fraction for various values of β . This equation is valid for any stress distribution, provided the initial premises are met. The graphs of equation (21) are shown in Fig. 1, where $\beta = \sigma_f / \sigma_a$. Figure 2 is a plot of equation (22). Note again that the foregoing considerations have replaced σ_0 by the mean value, σ_a , and eliminated the integration of the stress distribution.

An analogous approach on the general distribution function having a bias value, σ_u , can be employed. It is necessary to assume that the threshold stress, σ_u , varies through space in the same way as the applied stress, so the two bear a constant relation to one another. This allows the quantity $(\sigma - \sigma_u)^m$ to be factored out of the geometry

Fig. 1. The normalized Weibull distribution (working graph).

Fig. 1. (Continued)

GLL-6411-3246

Fig. 2. Theoretical Weibull lines.

integral, as with equation (16), and eventually leads to

$$S = \exp \left\{ - \left[\left(\frac{\sigma - \sigma_u}{\sigma_a - \sigma_u} \right) \Gamma \left(1 + \frac{1}{m} \right) \right]^m \right\}. \quad (24)$$

E. Application of the Normalized Stress Distribution

The correspondence of the experimentally obtained distribution to the form of the theoretical distributions of equation (21), Fig. 1, or of equation (22), Fig. 2, or the more complex form of equation (24), will confirm the theory. The recommended method of evaluating a set of data is to normalize the experimentally obtained strength values, dividing by the mean strength, and rank them in a monotonic order of size. The cumulative probabilities are computed by $r_i/(N_{\text{total}} + 1)$ where r_i is the rank of the i th entry. These are tabulated and plotted on the graph of Fig. 1. An uncertainty interval is plotted parallel to the nearest β parameter line (see data plots in Results).

The value in the region approaching $\beta = 1$ must be treated with caution since the β lines become nearly horizontal and a small uncertainty in probability has associated with it a very wide band of m values.

If the plot resulting on Fig. 1 be approximated by a vertical line, then the reduced Weibull distribution with a single m value describes the data.

If the line curves either to the right or left, a complex distribution is indicated, involving at least two values of m , of the form (for a duplex distribution)

$$S = A \exp \left\{ \left[\beta \Gamma \left(1 + \frac{1}{m_1} \right) \right]^{m_1} \right\} + (1 - A) \exp \left\{ \left[\beta \Gamma \left(1 + \frac{1}{m_2} \right) \right]^{m_2} \right\}$$

or a threshold value is necessary, such as equation (24).

There are several interesting features to the graph of Fig. 1. It is used, in effect, to make several determinations of m by testing various portions of the data. If the data is not precisely fitted by a single value of m , the quantitative errors in estimating failure probability may be directly evaluated when an approximate m value is assigned. Such an immediate perspective of the errors of approximation is not afforded by the plot of Fig. 2, nor is this plot as sensitive to differences in m .

F. Design Considerations

The fundamental precept of design with brittle materials would seem to be a definition of an acceptable performance probability. Thus, the requirement may be, for example, for less than 0.1% probability of fracture. Such a statement should also be accompanied by further levels of assurance such as confidence limits and other statistical paraphernalia. On the other hand, due regard must be given to the concept of nondestructive fracture. This concept refers to conditions of material fracture which do not produce system failure. Such events occur, for example, with thermal stress fracture of brittle components which retain their overall geometric stability although they may be permeated with thermal stress cracks. It is evident that for nondestructive fractures to exist, such cracking must provide relief from load. Therefore the loading must be redundant. This element of nondestructive cracking seems analogous to the plastic deformation of ductile materials which promotes improved stress distribution and

hence provides local relief. The philosophy of design with brittle materials must recognize the possibility of "crack-tolerant" design while attempting, as a rule, to prevent fracture. It seems reasonable to conjecture, therefore, that design with brittle materials will evolve to more redundant loading configurations so that tensile cracking will not precipitate failure. Design with ductile material employs the safety factor or safety margin which is usually applied without regard to material properties other than mean strength. Weibull's modulus, m , for metals is usually greater than 40, and the size effect is therefore very small. Thus, if size is increased to reduce working stress, no penalty results. It is an entirely different situation with brittle materials where safety factor must be considered in light of the size effect. Another new concern of design with brittle material is the statistical behavior of the extremes, i. e., the weakest specimens in a given group.

The implications of these considerations will be examined below.

1. Extreme-Value Statistics

When data evaluation must be based on statistical methods because of a significant variance in some measured parameter, a number of unique problems present themselves. These result from the necessity of interpolating or extrapolating test results to provide a forecast of operational behavior. The first problem is to express quantitatively the degree of certainty associated with the data and to establish a satisfactory design philosophy based on this knowledge. The second problem is the use of a parameter which appropriately describes the

central tendency of the data. Third is the problem of extreme values, their magnitude, distribution, and expectancy. The third problem is the most crucial aspect of a design which must be based on significantly variable data, and in which the survival of a system depends entirely on the statistics of the extreme values, i. e., those lowest values which are sufficient to cause failure. In these terms the central tendency (mean, median, most probable value) is almost irrelevant. A few aspects of extreme value statistics will be given, to emphasize this kind of analysis for development of design philosophies with variable phenomena, and particularly where the occurrence of an extreme value is likely to be a costly or dangerous proposition.

It is required to estimate the magnitude of the extreme (in this case the lowest extreme) value which may be expected in a sample of size n . For this we follow the development of Epstein (8, p. 140).

If $S(x)$ is the survival probability at x (cumulative distribution), the probability that a series of n specimens will contain a value less than x is

$$G(x) = 1 - S^n. \quad (25)$$

The differential or frequency distribution, $g(x)$, is found by differentiating,

$$g(x) = -nsS^{n-1}$$

where s is $\frac{dS}{dx}$. The mode, or most probable value, of the extreme-value frequency distribution is found by determining its maximum:

$$g'(x) = -ns' S^{n-1} - n(n-1)s^2 S^{n-2} = 0.$$

or

$$s'S = -(n - 1)s^2 \quad (26)$$

The value of the variate satisfying this equation will be the mode of the extreme-value distribution. The normalized Weibull distribution with $\sigma_u = 0$ (equation 21) can be used in this formula to find the mode of its extreme values.

$$\begin{aligned} S &= e^{-[\beta\Gamma(1 + 1/m)]^m} = e^{-c\beta m}, \\ s &= S = cm\beta^{m-1} S, \\ s' &= cm\beta^{m-2} S[cm\beta^m - (m - 1)]. \end{aligned}$$

Putting in equation (26), gives for the mode of extreme values of β ,

$$\beta^* = \left(\frac{m - 1}{mn}\right)^{1/m} \left[\frac{1}{\Gamma(1 + 1/m)} \right]. \quad (27a)$$

A similar form for the case $\sigma_u > 0$, with $\sigma_u/\sigma_a = a$, can be obtained, using the assumption at the end of section D and equation (24):

$$\beta_u^* = (1 - a) \left(\frac{m - 1}{mn}\right)^{1/m} \left[\frac{1}{\Gamma(1 + 1/m)} \right] + a. \quad (27b)$$

Using equation (27a) where $\sigma_u = 0$, the graph of the mode of extreme values for varying sample size and modulus m is given in Fig.

3. Here is seen the very core of the difficulty with brittle materials. As the number of specimens is increased the extremely low value drops fairly rapidly, particularly for the lower values of m . Note, for example, that for $m = 6$ a sample of 100 specimens will probably contain one specimen whose strength is only half of the mean. Furthermore this least value is itself a statistical parameter and is distributed according to equation (25) which may be written as

$$S^* = e^{-n\beta^m [\Gamma(1 + 1/m)]^m}$$

Fig. 3. Extreme values (minima) of the reduced Weibull distribution (normalized).

2. The Proof Stress and Prestress (Bias)

For purposes of design the existence of a threshold strength may be of great importance since there is theoretically no failure at stresses below this level. The possibility of artificially establishing a threshold, by a bias or a proof test for example, is of great interest. Furthermore, the set of specimens containing a threshold should exhibit an enhanced average strength, provided no internal damage has been sustained, since the weaker specimens in the initial population will be biased by σ_u or eliminated by the proof stress.

The increase in strength due to a bias σ_u on a group of specimens in uniform bending may be estimated by equating the risks of rupture with and without a bias, equations (9a) and (14).

Using $\bar{\sigma}_B$ as the strength with a zero bias and σ_B as the strength with a bias of σ_u , where σ_u is a constant in space:

$$\left(\sigma_B - \sigma_u\right)^m \left(1 - \frac{\sigma_u}{\sigma_B}\right) = \bar{\sigma}_B^m.$$

If we set $s_B = \sigma_u / \bar{\sigma}_B$, the "bias ratio," then

$$s_B = \frac{\sigma_B}{\bar{\sigma}_B} \left[1 - \left(\frac{\sigma_B}{\bar{\sigma}_B} \right)^{-m/m+1} \right]. \quad (28)$$

The term $\sigma_B / \bar{\sigma}_B$ may be called the "strength-improvement" ratio.

Thus, if a uniform bending strength σ_B is desired of a population exhibiting a strength $\bar{\sigma}_B$ (with a "zero" threshold) and modulus m , the bias needed is $s_B \bar{\sigma}_B$ where s_B is given in equation (28). If the assumptions at the end of section D are imposed on σ_u the bias ratio will be given in all cases by

$$s = \frac{\sigma_B}{\bar{\sigma}_B} - 1.$$

The analysis of bias threshold in tension follows analogously and gives

$$\sigma_T - \sigma_u = \bar{\sigma}_T$$

where $\bar{\sigma}_T$ is uniaxial strength with zero threshold and σ_T is the strength with threshold σ_u .

To achieve a desired strength σ_T , the bias required is

$$\sigma_u = \sigma_T - \bar{\sigma}_T.$$

Dividing by $\bar{\sigma}_T$ as in the previous case and using $s_T = \sigma_u / \bar{\sigma}_T$ gives

$$s_T = \frac{\sigma_T}{\bar{\sigma}_T} - 1. \quad (29)$$

Equations (28) and (29) are plotted on Figure 4.

The second means of creating a threshold, the screening or proof test, results in a truncation of the initial distribution. Appendix C describes some analytical aspects of such a distribution, when it is initially unbiased. The most probable extreme value is given by a formula identical with equation (27a) provided the truncation ψ is less than the predicted value β^* . A critical number of specimens can be deduced from this formula by inserting ψ for β^* . For sample sizes above this number the most likely least value is ψ itself. The mean value of the truncated distribution relative to the initial (unbiased) mean value is given by

$$\hat{\beta} = \psi + e^{\psi^m \Gamma^m} \int_{\psi}^{\infty} e^{-(\beta \Gamma)^m} d\beta.$$

No effort was made in the present investigation to study experimentally the effect of truncation. A graph of $\hat{\beta}$ vs ψ is shown in Appendix C.

Fig. 4. Theoretical strength improvement due to proof testing.

It is not expected, however, that all of the improvement suggested by these equations can be realized. It has been observed that brittle materials subjected to a stress below the ultimate will exhibit micro-cracking and slip-band formation and upon subsequent testing will exhibit a reduced strength (19, p. 36-45). An apparent example of this is observed in the present investigation where the burst strength of fragments from the bending tests is considerably below that of previously unstressed specimens. This is described in more detail in section VI.

3. Safety Factor

The relationship between reliability and safety factor is given by equation (21) in which the factor $\beta = \sigma/\sigma_a$ is the inverse of the safety factor. Now, whereas σ_a is essentially invariant with size for metals, which are characterized by high values of m , as the size of a ceramic structure is changed to reduce the working stress, the value of σ_a must be re-evaluated for the new volume and stress distribution, because a relatively low value of m implies a significant size and stress distribution effect, as discussed in section B-1.

Let

σ_a = mean strength,

V_a = volume at which σ_a is evaluated (from test specimen configuration).

As volume is varied the nominal stress will change in accordance with

$$\sigma = \sigma_a f\left(\frac{V_a}{V}\right)$$

where f is an appropriate function of volume. The safety factor, however, must be based on the mean strength of components which have a volume of V , and the stress distribution.

Referring to equation (8) for the size effect,

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1} \right)^{1/m}.$$

The new reference mean strength $\bar{\sigma}$ for components of volume V is given by

$$\bar{\sigma} = \sigma_a \left(\frac{V_a}{V} \right)^{1/m}. \quad (30)$$

Now, whereas the classical safety factor is

$$k_c = \frac{\sigma_a}{\sigma} = \frac{1}{f(V_a/V)},$$

the "true" safety factor will be given by

$$k_t = \frac{\bar{\sigma}}{\sigma} = \frac{(V_a/V)^{1/m}}{f(V_a/V)} \quad (31)$$

or, in terms of the classical factor,

$$k_t/k_c = \left(\frac{V_a}{V} \right)^{1/m},$$

and substituting equation (31) into equation (21) we get

$$S = \exp \left\{ - \left[\frac{1}{k_t} \Gamma(1 + 1/m) \right]^m \right\}$$

or

$$S = \exp \left\{ - \frac{V}{V_a} \left[f \left(\frac{V_a}{V} \right) \Gamma(1 + 1/m) \right]^m \right\}. \quad (32)$$

The graph of Fig. 5 (reproduced from ref. 17) gives the material safety factors for two levels of reliability for various m values. These values of safety factor must be applied to a mean value which is corrected for size.

Consider, for example, the case of a rectangular beam in which the stress is varied by varying the height only. The stress will be related to volume by

$$\sigma = \frac{\text{const}}{V^2}.$$

If the mean strength is measured in specimens of volume V_a , the stress in the working component of volume V is related to the mean strength by

$$\sigma = \sigma_a \left(\frac{V_a}{V} \right)^2.$$

If size effect is neglected, the classical safety factor will be given by

$$k_c = \frac{\sigma_a}{\sigma} \left(\frac{V}{V_a} \right)^2. \quad (33)$$

The size effect, however, specifies that the mean strength of the component be less than the reference specimen by virtue of its greater volume, in accordance with equation (30); therefore, the true safety factor, equation (31), becomes, in terms of the classical safety factor,

$$k_t = k_c^{(1-1/2m)}. \quad (34)$$

Fig. 5. Material safety factor vs Weibull modulus, m (after McDonough, ref. 17).

4. System Survival

Let us assume a mean strength σ_a , for a sufficient number of test specimens of unit volume with a Weibull modulus m . The typical operational component is B times the size of a test specimen, and n such components make up a system. An estimate of the survival probability for the system is required.

First we assume for this example that the size effect is given by equation (8) (the risk of rupture is $kV(\sigma/\sigma_0)^m$). The mean strength of the operational component is, then,

$$\bar{\sigma} = \sigma_a \left(\frac{1}{B} \right)^{1/m}.$$

In a group of n such components, the weakest specimen is expected, by equation (27a), to have a strength of

$$\bar{\sigma}^* = \sigma_a \left(\frac{m-1}{mnB} \right)^{1/m} \frac{1}{\Gamma(1+1/m)}. \quad (35)$$

If B is taken as unity, this equation becomes identical with equation (27) and is described by Fig. 3.

The survival probability of the system, as a function of applied stress, in terms of the component mean strength is

$$S = \exp \left\{ -n \left[\frac{\sigma}{\bar{\sigma}} \Gamma(1+1/m) \right]^m \right\}$$

or, using the test specimen mean strength and the relative component volume (see equation (32)),

$$S = \exp \left\{ -nB \left[\frac{\sigma}{\sigma_a} \Gamma(1+1/m) \right]^m \right\}. \quad (36)$$

or, approximately for low failure probability F ,

$$F \approx n\beta^m [\Gamma(1+1/m)]^m.$$

5. The Extreme Safety Factor

It has been assumed that failure of the weakest specimen, at σ^* , will precipitate system failure. An "extreme safety factor" - as contrasted to the classical safety factor of part 3 - can be defined on this basis as the ratio of the most probable extreme, σ^* , to the operational stress, σ .

$$k_x = \frac{\sigma^*}{\sigma}.$$

Putting in for σ^* its value from equation (35),

$$k_x = \frac{\sigma_a}{\sigma} \left(\frac{m-1}{mnB} \right)^{1/m} \frac{1}{\Gamma(1+1/m)}.$$

In σ_a/σ we recognize the classical safety factor, k_c , so

$$\frac{k_x}{k_c} = \left(\frac{m-1}{mnB} \right)^{1/m} \frac{1}{\Gamma(1+1/m)}. \quad (37)$$

This equation is identical in form with equation (27a) if the product nB is taken together, and is described by the graph of Fig. 3.

To establish the "extreme safety factor" for a given value of system reliability we use equation (36)

$$\ln \frac{1}{S} = nB \left[\frac{\sigma}{\sigma_a} \Gamma(1+1/m) \right]^m$$

where S is the desired survival probability. Again, $k_c = \sigma_a/\sigma$, and

$$k_c = (nB)^{1/m} \Gamma(1+1/m) (\ln 1/S)^{-1/m}.$$

Combining with equation (37),

$$k_x = \left(\frac{m-1}{m} \right)^{1/m} \left(\ln \frac{1}{S} \right)^{-1/m}. \quad (38)$$

For the very low probability of failure, F , which must be used in design the following approximation can be made,

$$\ln \frac{1}{S} \approx F.$$

This gives, approximately, for the "extreme safety factor,"

$$k_x \approx \left(\frac{m-1}{mF} \right)^{1/m}. \quad (39)$$

This equation shows how the true margin of safety decreases as F increases. As a numerical example we take $F = 10^{-7}$ for a man-rated system and $m = 10$. The extreme safety factor is, then, about 6. If the total volume, nB , relative to test specimen volume is, say, 100, the classical safety factor, from Fig. 3, will be about 9!

This concludes these limited analytical studies. Some of the theoretical implications were investigated in an experimental study of the strength behavior of BeO specimens, and are described below.

III. EXPERIMENTAL

In order to study the applicability of the foregoing considerations to a particular material, a series of tests were carried out using beryllia (BeO) specimens in the form of thick-walled tubes, 0.308 in. o.d., 0.230 in. i.d., and approximately 5 in. long. These specimens are described in section IV.

The effect of stress distribution and volume could be studied by using a standard three-point (center loading) bending test for modulus of rupture and comparing with strength values obtained by pressurizing the interior to failure (burst test). The burst-test apparatus which was developed for this study is unique and appears to be a useful and practical technique. The apparatus used is described in detail in section V.

There were three goals to the experimental study:

1. The application of the Weibull statistics to a group of strength tests, and demonstration of the use of the normalized expressions.
2. The result of pooling several groups of such tests and the reliability of extreme-value prediction (i. e., forecasting the weakest observed values).
3. Prediction of the relative burst and bending strengths.

As the experiment progressed, some of the fragments of the bending tests were burst-tested. The unusually low values of strength observed seemed inconsistent with the burst strength of full-length tubes. At this point a brief study was made to see if microstructural damage could be induced by moderate stresses, without gross fracture, in the BeO. The results seemed to be affirmative, and therefore significant to the aspect of proof testing as a means of improving strength behavior, by creating a prescribed threshold.

The computations for the comparison of burst and bending strength require the evaluation of the integrals in the non-normalized Weibull distribution. The details are given in Appendix A, and the results, using a subscript P to denote pressure and B for bending, are given in the form

$$R_P = \left(\frac{\sigma_P}{\sigma_{0P}} \right)^{m_P} r_0^2 \ell_P C_P, \quad (40)$$

$$R_B = \left(\frac{\sigma_B}{\sigma_{0B}} \right)^{m_B} r_0^2 \ell_B C_B, \quad (41)$$

where r_0 is the outside radius and ℓ the stressed span.

The graphs of the constants C_P and C_B are given in Figs. 6 and 7. If we invoke the relation (see Fig. 8)

$$R = [\beta \Gamma(1 + 1/m)]^m \quad (42)$$

where $\beta = 1$ at the mean value, and solve equation (41) and (42) for the desired strength relation, the following is obtained:

$$\frac{\sigma_P}{\sigma_B} = \frac{\sigma_{0P}}{\sigma_{0B}} \frac{\Gamma\left(1 + \frac{1}{m_P}\right)}{\Gamma\left(1 + \frac{1}{m_B}\right)} \frac{\left(r_0^2 \ell_B C_B\right)^{1/m_B}}{\left(r_0^2 \ell_P C_P\right)^{1/m_P}}. \quad (43)$$

As an example, assume $m_B = m_P = 10$; $\ell_P = \ell_B$, ξ_i (i. e., $\frac{r_i}{r_0}$) = 0.74; $\sigma_{0P} = \sigma_{0B}$ and take C_P and C_B from the graphs of Figs. 7 and 8. The expected strength ratio would be (see Fig. 9):

$$\frac{\sigma_P}{\sigma_B} = \left(\frac{C_B}{C_P}\right)^{1/m} = 0.665.$$

We find, therefore, quite a significant difference to be expected between the burst and bending strengths.

To summarize, then, the m -value is found for each series of tests by normalizing the data and plotting on Fig. 1. The extreme value graph is used to check the prediction of the extreme value; and to compare absolute magnitudes of the bending and burst strength, equation (44) is used with, if possible, the assumption that $\sigma_{0P} = \sigma_{0B}$.

Fig. 6. Risk-of-rupture coefficient for round tubes under internal pressure.

$r_0 = \text{OUTER RADIUS}$

$R = \text{RISK OF RUPTURE}$

$l = \text{LENGTH}$

$$\xi_i = \frac{r_i}{r_0}$$

$C = \text{COEFFICIENT}$

Fig. 7. Risk-of-rupture coefficient for round tubes under three-point symmetric bending.

Fig. 8. Risk of rupture vs normalized stress ($\sigma_u = 0$).

IV. SPECIMENS

The specimens used in this investigation were fabricated by extrusion. This process requires that a suitable binder, which also serves as a lubricant to reduce extrusion pressures, be added to the BeO powder and be thoroughly mixed with it. The resulting doughy mass is then extruded and segments of selected lengths are cut off as they emerge from the extrusion die (for convenience, the axis of extrusion is usually vertical). These segments are then fired to a temperature sufficiently high to remove the organic binder. An alternate process employs various kinds of furnaces which surround the material emerging from the extrusion die and perform this prefiring prior to any cutoff operation. This prefired (or "starch-burned") material is introduced into a

high temperature kiln where it is fired for several hours at temperatures approaching 1700°C . This is the sintering process and the parts emerge in a hard, densified condition. Let it be noted that this kind of manufacturing technique is still very much an art and apparently minor manipulations at various steps may produce profound changes. This is particularly true at the mixing stage.

In order to produce the tubular shape a mandrel is used which is supported "upstream" of the die by a "spider." The spiders used had either three or six legs. The flow gradients around these legs and the specific shape of the die inlet will influence the final fired geometry. The tubular specimens had an inside diameter of about 0.230 in. and outside diameter of 0.308 in. A number of test specimens were measured very precisely for concentricity and roundness on an Indi-Ron machine to examine wall thickness variations and ellipticity. The results exhibit a striking regularity which is shown in Fig. 9. These photographs show the charts obtained from two specimens, at two locations, near an end and at the center. Each radial division represents 0.0005 in. A round specimen would give a circular trace, though not necessarily centered. Non-uniformities in wall thickness show up as differences in radial distance between inner and outer traces. The thickness variations are on the order of 6-7% of the wall thickness and the pattern is thought to be caused by the supports of the spider or the die inlet. Clearly a precise treatment of this geometry must take account of its polygonal character, but, for this study, the nonuniformity was disregarded.

Fig. 9. Roundness charts of test specimens.

Another important factor in contributing variability is the grain size distribution which varied in a regular way from the interior to exterior surfaces as shown in Fig. 10. It would be unreasonable to assign a single value to grain size and yet grain size is a well-known influence in the strength of materials (15). This introduces still another element of uncertainty into the present study.

The mechanical aspect of proof testing which is described in section II (Theory) was examined briefly because of the disparity observed between the strengths of fragments from the bending tests (data table 5, section VI) and specimens which were previously untested (data tables 1, 2, 3). The method used was to grind a flat about 1/8 in. wide, axially on a specimen and then to polish and etch it (using hydrofluosilicic acid at 90°C). After taking photographs of this "unperturbed" surface (Fig. 10 a, b) the specimen was loaded in bending to various increasing stress levels and re-examined after each stress cycle until evidence of damage was observed, as shown in Fig. 11. These photomicrographs show one of the specimens after loading to a nominal stress of about 35 ksi, at the polished surface. Cracked and broken grains are visible. This early breakup may indicate the advent of structural damage at stresses below the ultimate and that a proof test as a means of raising or creating a lower threshold must be approached with wariness. Figure 10c, at 2000X, shows what seems to be a microcrack about 0.0005 in. long. At the present it is not certain that these features were definitely caused by the stress cycle.

The observed strength anisotropy between bending and bursting (which exceeded the predicted difference) may be the manifestation of

(a)

500X
NEAR INSIDE BORE

(b)

500X
NEAR OUTER SURFACE

(c)

GLB-6411-7020

2000X

Fig. 10. Grain size variation in test specimen.

Fig. 11. Microstructural damage due to proof stress. Direction of stress is along axis.

an observed microstructural anisotropy. The BeO powder used in this fabrication was of the UOX grade supplied commercially by the Brush Beryllium Corporation. This particular material is derived from beryllium sulfate and characteristically contains many needle-shaped particles (laths) which, during the extrusion process, are oriented parallel to the axis of extrusion. Subsequently, during the sintering cycle grain growth occurs about these laths producing a preferred orientation in the final grain structure.

Prior to testing all specimens were inspected by fluorescent penetrant in order to eliminate any which contained visible cracks.

V. APPARATUS AND PROCEDURE

Two kinds of tests were run on the BeO tubes: a standard three-point bending (center loading) test, and a unique burst test for which a special apparatus was devised. The major requirement of the test burst apparatus was to provide means of pressurizing the interior of a tubular specimen and measuring the pressure at failure. This must be accomplished in a reproducible fashion so that the only significant source of variability will be the strength of the specimen.

The overall apparatus consists of a source of pressurized fluid, gages, valves and controls, furnace or heater for elevated temperature work, and the specimen holder. All but the specimen holder are fairly straightforward components. The specimen holder required some development.

There are, of course, a number of ways to pressurize the interior of a tube, and the judgment for selecting a particular method required the following criteria to be met:

1. The performance of each run must not be fraught with complexity nor demand a meticulous and precise test procedure to give reproducible results.
2. The time to perform an individual run must be relatively short: on the order of a few minutes for room temperature testing and perhaps four or five times this for high temperature tests.
3. The apparatus must be adaptable to testing at various temperature levels.
4. The apparatus should be of inherently flexible design, allowing the use of various size specimens, and it should be suitable for standardized parts.

The specimen holder and apparatus which finally evolved for this investigation passed through a number of distinct phases. There were four basic designs tried, which are illustrated in Fig. 12. The first (Fig. 12a) employed a surgical rubber tube within the test specimen and sealed at the ends with special barbs. This method required a long setup time and was frequently hampered by leaks at the collar or the actual extrusion of the rubber tube around the collar. The second design (Fig. 12b) used a Stat-O-Seal, a commercial sealing device, at the end faces of the tubular specimen. Sealing was accomplished by imposing a sufficiently large axial force and, on occasion, these forces became undesirably large. The third design (Fig. 12c)

NOTE: It was difficult to prevent extrusions around the collar, and in addition, the collar caused leaks. Long set-up time.

CONSTRUCTION OF
STAT-O-SEAL

NOTE: High end-loads were required to achieve a seal.

NOTE: The tolerance in the specimen bore caused a variation in the sealing load.

NOTE: Most reproducible sealing load.

GLL-6312-3276A

Fig. 12. Sealing techniques tried in the burst test.

consisted of an internal probe with O-ring seals. If the tolerances were correct, the specimen slipped over the O-rings with just enough force to produce sealing. However, the existing variations in dimension caused a wide difference in the sealing forces. In the final and present design (Fig. 12d), this internal probe was improved to allow adjustment of the seal. This apparatus is shown in greater detail in Fig. 13. Here the adjusting screw causes the O-ring to be compressed and produce a seal against the specimen; the sealing force, in addition, is controlled by the degree to which the adjusting screw is tightened.

This sealing design was incorporated into a test apparatus which is shown in Figs. 13 to 16. The essential element of this design is the probe assembly, shown in cross section Fig. 13 and visible in the other photographs. This part was also intended for use at elevated temperatures and hence was finned near its massive portion. The fin roots were tapered to produce moderate variable cooling and to restrict the major axial temperature gradients to the region within the fins. These parts were threaded to allow adjustment in assembly, although, as Fig. 13 shows, the threaded holders are hinged to allow rapid assembly and disassembly. These hinged parts were fastened by "wing-bolts," Allen-head bolts to which were brazed wing nuts to expedite the setup and disassembly (Figs. 15 and 16).

The sliding probe which compressed the O-rings to achieve a seal was fabricated out of bar stock for one side and from high-pressure tubing for the pressurizing side. The detail of the tubular sliding probe can also be seen in Fig. 13. Since this tube required connection to a high pressure supply, and, in addition, the disassembly

Fig. 13. Cross section of pressurizing probe.

Fig. 14. The probes and O-rings.

Fig. 15. Burst apparatus — assembly nearly complete.

Fig. 16. Burst apparatus showing induction coils and susceptor-blast shield.

requirements imposed a size restriction, a miniature high-pressure coupling was devised and successfully used for this connection. This coupling, a simple sliding O-ring assembly, was pinned by a key which, when the coupling was pressurized, became locked and prevented accidental disassembly (see Fig. 13).

In order to perform high temperature tests, a suitable sealing material was needed. The requisite properties were corrosion resistance and softness. Although such tests are not within the scope of the present study it is worthwhile to mention some encouraging results. It was found that gold O-rings worked very well in the temperature region 1300 to 1900°F and nickel O-rings from 2200 to 2450°F. The procedure was to raise the temperatures above 1200°F and gradually tighten the adjustment, to cause the metal to flow and seal the annular gap between the specimen and the probe. Approximately 30 minutes was required from start to complete disassembly at the highest temperature. This time interval is satisfactory. The high temperatures were achieved by the induction heating of a susceptor which surrounded the test specimen. Various steels and superalloys were tried but the best results were obtained with a very thick-walled silicon carbide susceptor, with which temperatures of 3000°F could be reached in less than 30 minutes. The susceptor also served as a blast shield during specimen failure. Again, in order to expedite the test procedures, the induction coil was wound in two transverse sections and a hinge was devised which allowed one side to be raised (see Fig. 16). This coil, though adequate to about 1850°F, was too

inefficient to achieve the higher temperatures in reasonable times, so another full spiral coil with varying pitch was used for temperature in excess of this limit. The apparatus as it appears ready for an elevated temperature run is shown in Fig. 17.

This final design is shown partially disassembled and assembled in Figs. 14, 15, and 16. Figure 15 shows the position of the specimen, with the O-ring probes inserted, as it would be set up for a room temperature test. The blast shield, which is seen surrounding the specimen in Fig. 16, restricts the violent scattering of fragments and, for elevated temperature testing, serves as a susceptor for the induction coils. The blast shield is centered by ceramic end caps. The assembled view (Fig. 16) shows the blast shield and caps in place but the upper coil not lowered for an elevated temperature run. Close-ups of the specimen holding the seal probes in Fig. 14 show the distension of the O-ring (B) when the sliding member (A) is tightened against the fixed tube (C).

Because of the toxicity of BeO the test apparatus was enclosed in a ventilated glove box. Figure 18 shows this box in an overall view of the test apparatus. A_1 and A_2 are the high pressure supplies of argon gas and hydraulic fluid respectively. The gas bottle A_1 , is a Lawrence Radiation Laboratory (LRL) unit, made and filled at LRL. A_2 is a commercial pneumatically operated high pressure pump capable of delivering 20,000 psi. The control valves are at B and the gages at C. The apparatus is inside the ventilated box, D. The induction power supply, E, is controlled by a Leeds and Northrup Controller, F, and the three specimen thermocouples are recorded by two pens at G and one at F.

Fig. 17. Burst apparatus ready for elevated temperature test.

Fig. 18. Overall view of test area and equipment.

The instrumentation used in room temperature tests consisted only of pressure gages. Two Heise pressure gages were used to measure pressure in two ranges (15,000 psi full scale and 50,000 psi full scale). The gages were calibrated every two weeks to 1/4% and exhibited no significant changes during the period of the test work. In order to measure the pressure difference between the gages and the specimens, a transducer was connected in place of a specimen and the maximum pressure difference found to be about 50 psi at 5000 psi.

For the elevated temperature runs to 2400 °F three Chromel-Alumel thermocouples were placed in contact with the specimen, at either end and the center as seen in Fig. 17. Any one of the thermocouples could be used as the reference by the L & N controller which controlled the induction power supply. The outputs of all three were recorded continuously, one on the L & N controller and the other two on a Honeywell Electronik-17 two-pen recorder. Temperature gradients rarely exceeded 2%. Figure 20 gives a schematic of the fluid pressurizing and controlling circuit.

Fig. 19. Schematic of fluid pressurizing and controlling circuit.

VI. RESULTS

The experimental results are summarized in Table A and given in full in the Data Tables 1 to 5. The data is presented in the order observed and is also ranked in size order, to facilitate normalization and computation of probabilities. Data Tables 1 and 2 represent two groups of specimens, which were tested in burst, several months apart. A third group was tested still later and all three were pooled to give Data Table 3. Data Table 4 gives the results of modulus-of-rupture (M-R) tests in symmetric three-point bending. Some of the fragments from these tests were subsequently burst tested and the results are shown in Data Table 5. With each of the tables is a working graph similar to Fig. 1 and a plot of the data and approximate theoretical distribution.

It is possible that an unusually high failure probability at the mean value is a manifestation of a relatively near upper bound. The portion of the distribution which is of prime importance in design, however, is the region below the mean, where for the burst-test-results the approximations are quite satisfactory. The M-R data contain structure and imply a complex probability density, for which the single Weibull distribution is only a rough approximation. The formula of equation (44) was still used to compare the theoretically expected difference in mean strength with the observed value, as shown in the last entry of the results table (Table A).

TABLE A. Essential results of strength tests on BeO at room temperature.

Data table no.	No. of specimens	Mean strength, ksi	Observed extreme	m-value	Predicted extreme		Type of test	Remarks
					β^*	Stress, ksi		
1	30	21.08	15.64	9.5	0.73	15.4	Burst, 4.5-in. length	If the mean and m-value of Table 1 are used to predict for Table 3, the extreme value is 13.3ksi
2	37	21.81	14.28	9.5	0.7	15.3	Burst, 4.5-in. length	
3	106	20.1	12.2	7	0.55	11.1	Burst, 4.5-in. length	Pooled data
4	38	42.1	32.4	9.5	0.7	29.5	Bend, 3-in. span	Rough fit
5	29	15.8	1.7	3	0.32	5.1	Burst of M-R frag- ments	Rough fit, no mode.

The ratio of burst to bend strength is predicted to be 0.64, observed to be 0.48.

The difference is sufficiently large that causes for real strength anisotropy should be looked for, and indeed they appear to lie in the nonisotropic grain orientation.

It should, in principle, be possible to pool the normalized results of the bend and burst tests, but, because of the rough fit to the bending data, this was not done. The three separate groups of burst data were pooled in Data Table 3 and seem to be well approximated by the Weibull distribution, although a slightly lower m results.

Insofar as proof testing is concerned, the microstructural damage which was apparently induced by stresses below the ultimate (see Fig. 10c and 11), implies that the benefits of proof testing are likely to be mixed or even nonexistent.

The extreme value predicted from the Weibull modulus is, in every case (except the M-R fragments), in satisfactory agreement with the observed extreme value. This prediction would be used when projecting to large populations and it is therefore a notable success that predictions for the 106 specimens of Data Table 3, based on the results of Data Table 1, are in reasonable agreement.

This method of extreme value prediction was also applied to other published data, Fig. 20, taken from Daniel and Weil (5). They give $m = 7.25$ for 140 BeO specimens tested in four-point bending. The least value observed was 8870 psi. The value predicted by the graph of Fig. 3 leads to a value of 8400 psi. Their low values of strength reflect the garbage they used.

EXPERIMENTAL RESULTS OF BERYLLIUM OXIDE
AT ROOM TEMPERATURE

Specimen Type	A	B	C
Dimensions (in.)	$\frac{3}{16} \times \frac{1}{3} \times 4$	$\frac{1}{4} \times \frac{1}{4} \times 4$	$\frac{1}{3} \times \frac{3}{16} \times 4$
Cross section (in.)	$\frac{1}{3}$ $\frac{3}{16}$	$\frac{1}{4}$ $\frac{1}{4}$	$\frac{1}{3}$ $\frac{3}{16}$
Gage volume (cu in.)	0.125	0.125	0.125
Number of specimens tested, N	48	42	50
Mean failure stress, σ_m , (psi)	15,740	15,540	15,650
Standard deviation, σ , (psi)	2,320	2,540	2,530
Coefficient of variation, v , (%)	14.74	16.34	16.17
Mean stress gradient at failure, (psi/in.)	94,440	124,320	166,930
Lowest Failure Stress, σ_{low} , (psi)	11,830	8,870	10,210
Highest Failure Stress, σ_{high} , (psi)	19,900	19,320	21,000

Fig. 20. Fracture probability for BeO bar under pure bending stress (after Daniel and Weil, ref. 5); $m = 7.25$

DATA TABLE 1. Burst test of one group of round tubes, i. d. = 0.230 in., o. d. = 0.310 in., $\xi_1 = 0.74$, $\sigma/P = 3.4$, stressed length = 4.5 in.

Observed data			Ordered data			
Run no.	Burst pressure, psi	Burst pressure, psi	Max. stress, ksi	Norm. stress	Cumulative failure probability	Cumulative survival probability
1	6200	4600	15.64	0.743	0.032	0.968
2	5600	5000	17.0	0.808	0.064	0.936
3	6000	5200	17.68	0.840	0.097	0.903
4	6500	5300	18.04	0.857	0.129	0.871
5	5200	5500	18.7	0.888	0.161	0.839
6	5600	5600	19.04	0.904		
7	6900	5600	19.04	0.904		
8	6600	5600	19.04	0.904	0.250	0.742
9	6500	5700	19.38	0.921	0.290	0.710
10	5500	5900	20.06	0.953	0.322	0.678
11	6400	6000	20.4	0.969	0.355	0.645
12	5300	6100	20.74	0.985	0.387	0.613
13	6500	6200	21.08	1.00	0.419	0.581
14	7000	6300	21.42	1.02	0.452	0.548
15	6900	6400	21.76	1.03		
16	6800	6400	21.76	1.03	0.516	0.484
17	6900	6500	22.10	1.05		
18	6700	6500	22.10	1.05		
19	5000	6500	22.10	1.05		
20	4600	6500	22.10	1.05	0.645	0.355
21	5600	6600	22.44	1.07		
22	5900	6600	22.44	1.07	0.709	0.291
23	6300	6700	22.78	1.08	0.742	0.258
24	5100	6800	23.12	1.10	0.774	0.226
25	6100	6900	23.46	1.11		
26	6900	6900	23.46	1.11		
27	6500	6900	23.46	1.11		
28	7400	6900	23.46	1.11	0.903	0.097
29	6400	7000	23.80	1.13	0.935	0.065
30	6600	7400	25.16	1.19	0.968	0.032
		Mean burst pressure	Mean strength			
		6200	21.08

Working graph and failure probability plot corresponding to Data Table 1.

DATA TABLE 2. Burst test of a second group of round tubes, i. d. = 0.230 in., o. d. = 0.310 in., $\xi_i = 0.74$, $\sigma/P = 3.4$, stressed length = 4.5 in.

Observed data			Ordered data			
Run no.	Burst pressure, psi	Burst pressure, psi	Max. stress, ksi	Norm. stress	Cumulative failure probability	Cumulative survival probability
1	6000	4200	14.28	0.655	0.026	0.974
2	6000	5100	17.34	0.795	0.053	0.947
3	6500	5200	17.68	0.811	0.079	0.921
4	6600	5400	18.36	0.842	0.105	0.895
5	6600	5600	19.04	0.873	0.132	0.868
6	6700	5700	19.38	0.889	0.158	0.842
7	5400	6000	20.40	0.935		
8	6500	6000	20.40	0.935		
9	6400	6000	20.40	0.935	0.236	0.764
10	7200	6200	21.08	0.967		
11	6400	6200	21.08	0.967	0.289	0.711
12	6300	6300	21.42	0.982		
13	6300	6300	21.42	0.982		
14	6600	6300	21.42	0.982		
15	6200	6300	21.42	0.982	0.395	0.605
16	7100	6400	21.76	0.998		
17	6300	6400	21.76	0.998	0.447	0.553
18	5100	6500	22.10	1.013		
19	6600	6500	22.10	1.013	0.500	0.5
20	6000	6600	22.44	1.029		
21	7000	6600	22.44	1.029		
22	5200	6600	22.44	1.029		
23	7000	6600	22.44	1.029		
24	4200	6600	22.44	1.029	0.631	0.469
25	6300	6700	22.78	1.044		
26	6700	6700	22.78	1.044		
27	6200	6700	22.78	1.044	0.710	0.290
28	7300	6800	23.12	1.060	0.737	0.263
29	7400	6900	23.46	1.076	0.763	0.233
30	6700	7000	23.80	1.091		
31	6800	7000	23.80	1.091	0.816	0.184
32	5700	7100	24.14	1.107	0.842	0.158
33	6600	7200	24.48	1.122	0.868	0.132
34	7400	7300	24.82	1.138	0.895	0.105
35	7600	7400	25.16	1.154		
36	5600	7400	25.16	1.154	0.947	0.053
37	6900	7600	25.84	1.185	0.973	0.027

Av pressure = 6416 psi
Av strength = 21.81 ksi

Working graph and failure probability plot corresponding to Data Table 2.

DATA TABLE 3. Combined burst test data from a third group of round tubes together with the data from Data Tables 1 and 2, i. d. = 0.230 in., o. d. = 0.310 in., $\xi_1 = 0.74$, $\sigma/P = 3.4$. Combined data of 106 specimens.

Burst press.	Norm. value	Surv. prob.	Burst press.	Norm. value	Surv. prob.	Burst press.	Norm. value	Surv. prob.
3600			5700			6500		
3600	0.608	0.981	5700			6500		
4000	0.675	0.971	5700	0.962	0.616	6500		
4200			5800	0.979	0.607	6500		
4200			5900			6500		
4200			5900			6500	1.098	0.271
4200			5900			6600		
4200	0.709	0.925	5900	0.996	0.570	6600		
4400	0.743	0.915	6000			6600		
4600			6000			6600		
4600			6000			6600		
4600	0.776	0.887	6000			6600		
4700	0.793	0.878	6000	1.013	0.523	6600	1.112	0.206
4800			6100			6700		
4800	0.810	0.859	6100	1.030	0.505	6700		
4900			6200			6700		
4900	0.827	0.841	6200			6700	1.130	0.168
5000			6200			6800		
5000			6200			6800	1.145	0.150
5000	0.844	0.813	6200			6900		
5100	0.861	0.803	6200			6900		
5200			6200			6900		
5200			6200	1.047	0.430	6900		
5200	0.878	0.775	6300			6900	1.163	0.103
5300			6300			7000		
5300	0.895	0.757	6300			7000		
5400			6300			7000	1.180	0.075
5400	0.912	0.734	6300			7100		
5500			6300	1.063	0.374	7200		
5500	0.928	0.719	6400			7300		
5600			6400			7400		
5600			6400			7400		
5600			6400			7400	1.249	0.02
5600			6400	1.080	0.328	7600	1.283	0.009
5600								
5600								
5600								
5600	0.945	0.644						

Av burst pressure = 5924 psi

Av maximum stress = 20.1 ksi

Working graph and failure probability plot corresponding to Data Table 3.

DATA TABLE 4. Modulus of rupture results for BeO tubes, three-point loading, 3-in. span, $\xi_1 = 0.74$, $\sigma/F = 500$. Room temperature.

Run no.	Max load	Max stress	Ordered stress data	Normalized stress	Survival probability
1	98.4	49.2	32.4	0.769	0.973
2	64.9	32.4	33.2	0.788	0.947
3	90.4	45.2	33.5	0.796	0.922
4	79.0	39.5	33.7	0.800	0.896
5	70.2	35.1	34.2	0.812	0.870
6	89.7	44.9	35.1	0.834	0.845
7	91.9	48.0	35.5	0.843	0.819
8	84.0	47.0	36.5	0.867	0.794
9	95.7	47.8	37.8	0.898	0.768
10	85.1	42.6	39.5	0.938	0.742
11	67.0	33.5	40.7	0.967	0.717
12	82.1	41.1	41.1	0.976	0.691
13	91.4	45.7	41.2	0.978	0.665
14	101.3	50.7	41.5	0.986	0.614
15	87.6	43.8	41.6	0.988	0.589
16	87.0	43.5	42.0	0.997	0.563
17	87.5	43.7	42.6	1.012	0.538
18	96.7	48.3	42.9	1.019	0.512
19	92.2	46.1	43.0	1.021	0.486
20	92.5	46.2	43.0	1.021	0.486
21	91.4	45.7	43.4	1.031	0.461
22	86.8	43.4	43.5	1.033	0.435
23	82.4	41.2	43.7	1.038	0.410
24	66.4	33.2	43.8	1.040	0.384
25	86.0	43.0	44.7	1.062	0.358
26	95.2	47.6	44.9	1.066	
27	73.0	36.5	45.2		
28	75.6	37.0	45.7		
29	83.3	41.6	45.7		
30	89.4	44.7	46.0	1.09	0.205
31	81.5	40.7	46.1		
32	83.0	41.5	46.2		
33	86.0	43.0	47.2		
34	98.5	47.2	47.6		
35	85.8	42.9	47.8	1.135	0.102
36	67.5	33.7	48.3		
37	68.4	34.2	49.2	1.169	
38	71.0	35.5	50.7	1.204	0.098
Av load		Av max stress			
84.2		42.1

Working graph and failure probability plot corresponding to Data Table 4.

DATA TABLE 5. Results of burst tests of fragments from modulus-of-rupture tests, $\xi_i = 0.74$, $\sigma/P = 3.4$. Room temperature.

Observed data		Ordered data		
No.	Burst pressure, psi	Burst pressure	Normalized value	Survival probability
1	5500	500		
2	3900	500	0.108	0.932
3	3600	2200	0.473	0.900
4	3600	2400	0.516	0.867
5	3300	2750	0.591	0.833
6	2750	3300	0.709	0.800
7	6000	3600		
8	500	3600	0.774	0.733
9	6200	3900	0.838	0.700
10	6000	4200	0.903	0.667
11	4200	4400	0.946	0.632
12	5800	4600	0.989	0.600
13	6200	5000		
14	6200	5000	1.075	0.533
15	5000	5200	1.118	0.500
16	5200	5400	1.16	0.466
17	500	5500		
18	6200	5500	1.182	0.400
19	6000	5800		
20	5800	5800	1.247	0.333
21	6200	6000		
22	5000	6000		
23	5500	6000	1.291	0.233
24	2200	6200		
25	2400	6200		
26	5400	6200		
27	4400	6200		
28	4600	6200		
29	6200	6200	1.333	0.034
Mean pressure = 4630				
Mean strength = 15.7 ksi

Working graph and failure probability plot corresponding to Data Table 5.

VII. DISCUSSION

The present study was concerned with certain analytical modifications and experimental checks on the statistical strength distribution of brittle materials. Unfortunately, the literature dealing with strength studies of such materials frequently fails to give a table of all results, and the published graphs are difficult to interpret with any precision. Very often, also, an adequate characterization of the specimens, the material, and fabrication is lacking, and in some cases conclusions are drawn on the basis of extremely meager data (22, p. 298-300). Since there are many parameters that may contribute to the variability of strength - grain size, density, fabrication, and test technique, to mention a few - it seems appropriate here to specify the type of information that should accompany strength tests on brittle material. Identification of the variability due to each of these factors will contribute to more efficient engineering applications of ceramics and other brittle materials. It is therefore recommended that strength studies on brittle materials (or ductile material below the transition temperature) contain the following information:

A. Starting material description

1. Ceramics

- a. Source of powder
- b. Extraction and preparation of powder
- c. Relevant properties of the powder
- d. Quantitative analysis of constituents

2. Non-ceramics

- a. Source

- b. Initial preparation and fabrication history
- c. Quantitative analysis of constituents

B. Fabrication

1. Ceramics

- a. Powder preparation
- b. Binder addition
- c. Pressing or extrusion conditions
- d. Firing sequence
- e. Final test specimen fabrication and geometry
- f. Grain size as well as size and spatial distribution
and the method of measurement
- g. Density

2. Non-ceramics

- a. Final fabrication procedures
- b. Heat treatment (anneals, etc.)
- c. Final test specimen machining and position in
stock billet
- d. Grain size as well as size and spatial distribution
and measurement method

C. Testing

- 1. Loading method in detail
- 2. Load measurement in detail
- 3. Specimen selection
- 4. Detailed procedure

D. Results

1. All the raw data in the order observed, or at least the following: the mean, the number of specimens, and the extremes.
2. The data ordered monotonically in magnitude of the variable
3. The appropriate statistical parameters describing the data, and the method of analysis

Although many of these requirements are obvious, there is probably not a single paper (including this one) which contains all of the above. The preoccupation with the starting material and fabrication may serve to elucidate the variability which is frequently observed between specimens of different batches or lots of the "same" material. The importance of grain size and density can be appreciated from their effect on elastic properties and strength (14, p. 452-453; 15, p. 376-387). The problem is therefore further complicated by the general lack of meaningful grain size measurement techniques particularly as regards grain size distribution. The problem of porosity, that is density, measurement must ultimately contend with pore size and shape. The investigation of these "fine structure" effects, however, must await the elimination of gross causes of variability such as the geometric nonuniformity of the BeO tubes in this study, and the proper treatment of the starting material and experimental procedure.

The strength distribution suggested by Weibull seems to fit the data observed by many investigators (25, 26, 27) and the normalized form described here provides a quick and useful test for the reduced Weibul distribution ($\sigma_u = 0$). This simplified form is directly useful in

consideration of extreme value statistics and the formula for the mode of extreme values. The results of burst testing of BeO specimens seem to be satisfactorily approximated by this theory.

Theoretically some improvement in the mean and extreme value can be achieved by a threshold, e. g., as shown in Fig. 4 and 23. The necessary truncation is, however, relatively high and experience seems to show that proof testing may cause a decrease in strength, which is evidently due to microscopic damage induced in the proof test (section IV, p. 41). The use of proof testing to create a threshold must be cautiously done.

The section on design reliability gives a link between the "classical" safety factor and a new and physically more significant quantity, the "extreme" safety factor (equations (37) and (39)). This, with the formulas for the mode of extreme values, equations (27), should prove very usable for design estimates with brittle materials when a suitable value of the Weibull modulus m is selected. Application of the extreme-value formula to the experimental data (Table of Results) as well as to published data in the literature confirmed its usefulness. This extreme (least) value has uncertainty attached to it. Its distribution is, however, known from equation (25). Since the distributions are known, it is a straightforward matter to compute standard deviations; however, it is felt that a better measure of dispersion is given by the least value. Nevertheless, the standard deviation of the unbiased, untruncated Weibull distribution and its least values are given in Appendix D. Computations of risk of rupture at various stress distribution may be facilitated by the formulas in Appendix E.

Before the experimental data are discussed the difficulties associated with testing brittle material (Appendix B) need to be emphasized. The problems of the standard tensile test are of course well known and extensive attempts to eliminate eccentricity in the specimen and load train have not met with much success. Various other schemes in the literature (2, 7) have suffered from the complexity of specimens, and finally, the work of Sedlacek and Halden (22, p. 298-300) on the burst testing of thin-wall rings seemed to promise a new era of precision in brittle material testing (coefficients of variation $\sim 2\%$). Unfortunately, this precision rests on very few specimens (four in one group and eight in a second, of alumina), and attempts to repeat the tests at IITRI and at LRL have produced the usual range of about $\pm 10\%$ of the mean. This is easy to understand when the implications of slight amounts of ellipticity and nonuniformity (Appendix B) are considered. The same criticisms, that is of inherent variability because of geometric irregularities, apply as well to the burst test performed here on the relatively thick-walled long tubes.

The experimental results of this study generally confirm the application of Weibull statistics. The extreme value forecast is very satisfactory, giving agreement which is frequently within a few percent of the observed value. The data of burst testing for two groups of specimens (drawn from the same lot but tested several months apart) yielded similar m -values but differed from that observed for a combination of these two test series with a third group of similar size; however the extreme value was predictable with satisfactory accuracy.

The distribution of bend strength data is complex and the m-value chosen is only a fair approximation. It may be that the bend data have a lower bound, but in any case the bend strength is greater than the theoretical prediction by an apparently significant margin. This theoretical prediction is obtained with equation (44), and the assumption which it contains may not apply. This "true" anisotropy may be due in part to grain orientation caused by the lathlike particles in the powder which oriented during extrusion, but there may be other mechanisms as well causing the irregular bend-strength distribution. The fragments of the M-R tests which were burst-tested gave very erratic results and a separate, short investigation indicated that prestressing may cause microscopic damage. This prestressing study was based on a similar one done on alumina at IITRI (19). Such a result is of major significance if proof testing, as a method of improving strength behavior, is contemplated. A careful study of this point should precede the adoption of proof stress.

The burst-test apparatus and method developed embody a number of notable features. The specimen is subjected to a tensile stress over a large majority of the volume (although the condition is one of biaxial stress and nonuniform tension), and the specimen is of a simple geometry. In addition the apparatus is capable of burst test at elevated temperatures, and incorporates several features to minimize the test time. It provides a new method for "tensile" testing of brittle materials, in which deviation from the nominal stress distribution is due almost entirely to specimen geometry and can therefore be controlled to some extent by careful fabrication and selection. This test is easy

to perform and requires little skill. It should be useful, in conjunction with standard bend testing, for the study of and design with brittle materials.

CONCLUSIONS

The experimental investigation of the strength of BeO tubes confirmed several aspects of the Weibull distribution.

The observed strength distribution was well approximated by a theoretical distribution containing no bias or truncation. The Weibull modulus for BeO was found to range between about 7 and 10, the lower value corresponding to other published results (5). The strength distribution of BeO using the lower value is given by

$$S \simeq e^{-0.625\beta^7}.$$

Essentially the same value of m was obtained for several samples tested over a time interval of several months, and it is concluded that there are no significant time-dependent effects on m occurring at room temperature. The effects of stress distribution and volume under tensile stress were qualitatively confirmed by the bending data. The bending strength, however, was higher than predicted, and its distribution was not as well approximated by the theoretical curve. This was attributed to grain orientation which resulted from extrusion and material effects. The normalized plot (Fig. 1) for data analysis and approximation was found to be very convenient. The normalized form of the distribution function is very useful since the mean value and m explicitly define the function.

The observed extreme values were predicted with satisfactory precision by the theoretical expression

$$\beta^* = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma(1 + 1/m)}$$

and its graph in Fig. 3. This gives further assurance to the application of the Weibull theory and is of most interest in problems of design where information on least-value behavior is mandatory.

It is felt that the burst test technique should be of interest in the study of brittle strength behavior. The test at room temperature was found to be simple and quick, and although elevated temperature tests are more difficult they are possible using suitable seals (metal O-rings). But whatever strength measurement is used, care must be taken to consider the quantitative effects of eccentricity as discussed for example in Appendix B, and, in cases where proof testing is contemplated, the problems of low-stress damage.

Respecting design with brittle materials, a number of reasonable design rules may be proposed:

1. Experiment and material must be carefully characterized.
2. Data should be approximated by the unbiased, untruncated function if at all possible. The approximation to least values is more important than to values near the mean since, as shown in Appendix D, the least values are statistically more reliable.
3. The extreme value and its range should be defined by a formula of the form of equation (27a).

4. The degree of conservatism used in design can be expressed by the "extreme safety factor" defined by equations (37) and (39).
5. Attempts to influence the distribution by screening or bias can be estimated by the analysis of truncation (Appendix C) or the idealized bias as in equation (24).
6. Several rough estimates can be made using approximations to various equations as "rules of thumb"; for example:
 - a. The value of m can be guessed at using the least observed value and Fig. 3.
 - b. The failure probability at a stress β in a sample of n is guessed at by

$$F = n\beta^m.$$
 - c. The extreme safety factor is given approximately by

$$k_x \approx F^{-1/m}.$$
 - d. If a sample contains bias, the m -value can be estimated from the survival probability at the mean value.
7. As a design principle, it is sometimes desirable to give up information on stress distribution and magnitude in order to obtain the benefits of redundant supports as discussed in Section II-F.
8. Although data is only approximated by the Weibull distribution, the theory can lead to statistically rigorous design formulation and is therefore vastly superior to design based on mean strength and a large factor of ignorance.

A number of extensions of this study suggest themselves. The problems of proof testing, particularly when proof stress and operational stress have different distributions, should be studied. The effects of time and temperature on m are of interest, and finally, the application of this distribution to other phenomena such as stress rupture and fatigue (as Weibull has done) may prove of practical value.

BIBLIOGRAPHY

1. Anthony, F. M. and A. L. Minstretta. Leading edge design with brittle materials. Aerospace Engineering 20:20-1, 44-52. Nov. 1961.
2. Barnett, R. L. Review of structural design techniques for brittle components under static loads. Chicago, 1963. 123 p. (Illinois Institute of Technology. Armour Research Foundation vol. 2, AD-410262, ARF report no. 8259, Final Report Feb. - May 1963)
3. Bodine, E. G. Design with rate-sensitive ceramics. Aerospace Engineering 22:100-107. Jan. 1963.
4. Bortz, S. A. and N. A. Weil. Studies on the predictive value of statistical theories for the fracture of brittle materials. Chicago, 1962. 33 p. (Illinois Institute of Technology. Armour Research Foundation ASD-TR-61-628, Air Force Contract AF33(616)-7465)
5. Daniel, I. M. and N. A. Weil. Influence of stress gradient upon fracture of brittle materials. ASME Paper 63-WA-228; presented at Winter Annual Meeting, American Society of Mechanical Engineers, Philadelphia, Nov. 17-22, 1963.
6. Duckworth, W. H. and Alfred Rudnick. The strength of ceramic materials. Battelle Technical Review 10:3-8. April 1961.
7. Durelli, A. J., S. Morse and V. J. Parks. The theta specimen for determining tensile strength of brittle materials. Materials Research and Standards 2:114-117. 1962.

8. Epstein, B. Statistical aspects of fracture problems. *Journal of Applied Physics* 19:140. 1948.
9. Fisher, J. C. and J. H. Hollomon. Statistical theory of fracture. *Metals Technology* 14:1-16. Aug. 1947. (American Institute of Mining and Metallurgical Engineers. Technical Publication no. 2218)
10. Frankel, J. P. Relative strengths of Portland cement mortar in bending under various loading conditions. *Journal of the American Concrete Institute* 20:21-32. 1948.
11. Goode, H. P. and J. H. K. Kao. Sampling procedures and tables for life and reliability testing based on the Weibull distribution (mean life criteria). Ithaca, Sept. 30, 1961. 58 p. (Cornell University. Sibley School of Mechanical Engineering. Quality Control and Reliability Technical Report TR-3, Office of Assistant Secretary of Defense, Installations and Logistics)
12. Griffith, A. A. The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London, ser. A*, 221:163-198. 1920.
13. Gumbel, E. J. Statistical theory of extreme values and some practical applications. Washington, 1943. 51 p. (U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 33)
14. Hasselman, D. P. H. On the porosity dependence of the elastic moduli of polycrystalline refractory materials. *Journal of the American Ceramic Society* 45:452-453. 1962.

15. Knudsen, F. P. Dependence of mechanical strength of brittle polycrystalline specimens on porosity and grain size. *Journal of the American Ceramic Society* 42:376-387. 1959.
16. Manson, S. S. and R. W. Smith. Theory of thermal shock resistance of brittle materials based on Weibull's statistical theory of strength. *Journal of the American Ceramic Society* 38:18-27. 1955.
17. McDonough, N. T. Structural reliability of components using brittle materials. Dallas, Oct. 1961. 41 p. (Chance Vought Aircraft, Dallas, Texas. Strength analysis report AST/EIR-13431)
18. Midgley, E. and F. T. Peirce. Tensile tests for cotton yarns, theorems on strength of long and composite specimens. *Journal of the Textile Institute* 17:T305-T368. 1926.
19. Parikh, N. M. Studies of the brittle behavior of ceramic materials. June 1964. (Wright-Patterson Air Force Base, Air Force Systems Command, report no. ASD-TR-61-628 Part III. Office of Technical Services, U. S. Department of Commerce)
20. Pears, C. D. and S. Oglesby. Evaluation of tensile data for brittle materials obtained with gas bearing concentricity. May 1963. (Wright-Patterson Air Force Base, Air Force Systems Command, report no. ASD-TDR-63-245. Office of Technical Services, U. S. Department of Commerce)
21. Roark, R. J. Formulas for stress and strain. 2d ed. New York, McGraw-Hill, 1954. 381 p.

22. Sedlacek, R. and F. A. Halden. Method for tensile testing of brittle materials. Review of Scientific Instruments 33:298-300. 1962.
23. Stokes, R. J. and C. H. Li. Dislocations and the strength of polycrystalline ceramics. 1962. 42 p. (U. S. Government Research Reports 37:54. HR-62-262 or AD 275-273)
24. Tippett, L. H. C. On the extreme individuals and the range of samples taken from a normal population. Biometrika 17:364-387. 1925.
25. Weibull, Waloddi. A statistical theory of the strength of materials (in English). Ingeniors Vetenskaps Akadamien-Handlingar, Stockholm, no. 151. p. 1-45. 1939.
26. _____. The phenomenon of rupture in solids. Ingeniors Vetenskaps Akadamien-Handlingar, Stockholm, no. 153. p. 1-55. 1939.
27. _____. A statistical distribution function of wide applicability. Journal of Applied Mechanics 18:293-297. 1951.

APPENDIX A
THE RISK OF RUPTURE FOR THE BENDING AND
BURSTING OF TUBES

I. The Three-point Bending of Round Tubes

The risk of rupture will be found from the superposition of two solutions for the round rod.

Consider the cross section:

The stress in terms of the maximum outer fiber stress is

$$\sigma = \sigma_B \frac{y}{r_0} \frac{z}{\ell}$$

where ℓ is the half-span. The risk of rupture for this rod is

$$R_0 = \left(\frac{\sigma_B}{\sigma_0} \right)^m 4 \int_0^{r_0} \int_0^\ell \left(\frac{y}{r} \frac{z}{\ell} \right)^m \sqrt{r_0^2 - y^2} dy dz.$$

Integrating the z term,

$$R_0 = \left(\frac{\sigma_B}{\sigma_0} \right)^m \frac{4\ell}{m+1} \int_0^{r_0} \left(\frac{y}{r} \right)^m \sqrt{r_0^2 - y^2} dy.$$

Using $\frac{y}{r_0} = \sin \theta$

$$R_0 = \left(\frac{\sigma_B}{\sigma_0} \right)^m \frac{4\ell}{m+1} r_0^2 \int_0^{\pi/2} \sin^m \theta (1 - \sin^2 \theta) d\theta.$$

We have from integral tables

$$\begin{aligned} \int_0^{\pi/2} \sin^{m+2} \theta \, d\theta &= \left| -\frac{\sin^{m+1} \theta \cos \theta}{m+2} + \frac{m+1}{m+2} \int \sin^m \theta \, d\theta \right|_0^{\pi/2} \\ &= \frac{m+1}{m+2} \int_0^{\pi/2} \sin^m \theta \, d\theta \end{aligned}$$

and

$$\int_0^{\pi/2} \sin^m \theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)}.$$

Putting in, we get, finally:

$$R_0 = \left(\frac{\sigma_B}{\sigma_0}\right)^m \frac{4\ell}{m+1} r_0^2 \left(1 - \frac{m+1}{m+2}\right) \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)}. \quad (A-1)$$

To find the risk of rupture for the tube the calculation is repeated for the case of a rod having a radius r_i , the inner radius of the tube, and for which the maximum fiber stress is $\sigma_{B_i} = \frac{r_i}{r_0} \sigma_B$, where σ_B is the maximum outer fiber stress given in equation (A-1):

$$R_i = \left(\frac{\sigma_B}{\sigma_0}\right)^m \left(\frac{r_i}{r_0}\right)^m \frac{4\ell}{m+1} r_i^2 \left(1 - \frac{m+1}{m+2}\right) \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)}. \quad (A-2)$$

The difference of the two gives

$$R_{B_{\text{tube}}} = \left(\frac{\sigma_B}{\sigma_0}\right)^m \left(\frac{1}{m+1} - \frac{1}{m+2}\right) \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2} + 1\right)} \frac{1}{\sqrt{\pi}} V_{R_0} \left[1 - \left(\frac{r_i}{r_0}\right)^{m+2}\right]. \quad (A-3)$$

V_{R_0} is the volume of a solid rod of radius r_0 . Equation (A-3) may be re-expressed in the form

$$R_{B_{\text{tube}}} = \left(\frac{\sigma_B}{\sigma_0}\right)^m r_0^2 \ell C_B. \quad (A-4)$$

Figure 7 gives C_B as a function of ξ_i , the radius ratio $\frac{r_i}{r_0}$.

II. The Round Tube with Internal Pressure

The tangential stress distribution in thick-wall tubes with internal pressure is

$$\sigma = p \frac{r_i^2}{r^2} \frac{(r_0^2 + r^2)}{(r_0^2 - r_i^2)}.$$

Using $\frac{r_i}{r_0} = \xi_i$ and $\frac{r}{r_0} = \xi$,

$$\sigma = p \frac{\xi_i^2}{\xi^2} \frac{(1 + \xi^2)}{(1 - \xi_i^2)}.$$

The maximum stress, at ξ_i , is

$$\sigma_{\max} = p \frac{1 + \xi_i^2}{1 - \xi_i^2}$$

which gives for the variable stress, in terms of $\sigma_{\max}$,

$$\sigma = \sigma_{\max} \left(\frac{\xi_i^2}{1 + \xi_i^2} \right) \left(\frac{1 + \xi^2}{\xi^2} \right).$$

The differential volume in the annulus is

$$dV = 2\pi l r dr$$

where l is the total span under pressure.

The risk of rupture is

$$R_p = \int_{\xi_i}^1 \left[\frac{\sigma_{\max}}{\sigma_0} \left(\frac{\xi_i}{1 + \xi_i^2} \right) \left(\frac{1 + \xi^2}{\xi^2} \right) \right]^m 2\pi r_0^2 l \xi d\xi.$$

Let $\sigma_{\max} = \sigma_p$.

$$R_p = \left(\frac{\sigma_p}{\sigma_0} \right)^m \left(\frac{\xi_i^2}{1 + \xi_i^2} \right)^m 2\pi r_0^2 l \int_{\xi_i}^1 \left(1 + \frac{1}{\xi^2} \right)^m \xi d\xi. \quad (A-5)$$

The integral in equation (A-5) must be integrated by numerical methods. An IBM 650 computer code was written to perform this computation. The results were used to express the risk of rupture in the form

$$R_p = \left(\frac{\sigma_p}{\sigma_0} \right)^m r_0^2 l C_p. \quad (A-6)$$

The constant C_p in equation (A-6) is plotted against thickness ratio, ξ_1 , in Fig. 6.

APPENDIX B

ERRORS INDUCED BY GEOMETRIC IRREGULARITIES

I. Errors Due to Eccentric Load in Uniaxial Tensile Specimens

The eccentricity cannot be eliminated by precision fabrication of the specimen alone and requires precise loading and gripping apparatus. The true stress is composed of a bending component superimposed on the nominal tensile stress:

$$\begin{aligned}\sigma_t &= \frac{P}{A} + \frac{Mc}{I} \quad (\text{where } M = P\epsilon) \\ &= \frac{P}{A} + 6 \frac{P\epsilon}{bh^2} .\end{aligned}\tag{B-1}$$

If $\frac{P}{A} = \sigma_{\text{nom}}$, the nominal stress, then, for the rectangular specimen with simple eccentricity,

$$\frac{\sigma_t}{\sigma_{\text{nom}}} = 1 + 6 \frac{\epsilon}{h} .\tag{B-2}$$

For a circular specimen $\frac{I}{c} = \frac{\pi r^3}{4}$, and

$$\frac{\sigma_t}{\sigma_{\text{nom}}} = 1 + 8 \frac{\epsilon}{d} .\tag{B-3}$$

II. Eccentric Annulus Under Internal Pressure

For the eccentric thin wall annulus under internal pressure, we consider as a limiting case the difference in stress between annuli of minimum and maximum wall thickness:

$$t_1 = t_2 + 2\epsilon_a$$

$$\epsilon_a = \frac{t_1 - t_2}{2}$$

Using the thin-wall approximation gives

$$\frac{\sigma_2}{\sigma_1} = \frac{t_1}{t_2},$$

and

$$\frac{\sigma_2}{\sigma_1} = 1 + 2 \frac{\epsilon_a}{t_2}. \quad (\text{B-4})$$

This equation is a limiting case, and direct control can be exercised over ϵ_a by specification of tolerance and/or specimen selection.

III. Thin-Wall Elliptical Annulus Under Internal Pressure

For the thin-wall elliptical annulus of major radius r under internal pressure P , a bending stress is induced which may be estimated from the formulas and tables given by Roark (21, p. 166). We have for a pressurized elliptical tube a bending moment $M = KPr^2$. The maximum value of K vs the ellipticity b/a is given in Fig. 22. The

Fig. 21. Maximum bending coefficient for elliptical tube under internal pressure (after Roark, ref. 21).

Fig. 22. Error in maximum hoop stress in elliptical tube under internal pressure.

perturbing stress (for unit width) will be given by

$$\begin{aligned}\sigma &= \frac{6M}{t^2} & M &= KPr^2 \\ \frac{\sigma}{p} &= 6K\left(\frac{r}{t}\right)^2\end{aligned}\tag{B-5}$$

If the nominal stress is given by the thin-wall approximation and equation (B-5) is considered as a perturbation then the ratio of the two will be an estimate of error:

$$\begin{aligned}\frac{\left(\frac{\sigma}{p}\right)_{\text{pert}}}{\left(\frac{\sigma}{p}\right)_{\text{nom}}} &= \frac{6KPr\left(\frac{r}{t}\right)^2}{Pr\frac{r}{t}} \\ \text{Error} &= 6K\frac{r}{t}.\end{aligned}\tag{B-6}$$

This equation gives the fractional error in stress due to the presence of bending. This error is plotted in Fig. 23. These equations are applicable to the thin-wall tubes only, and represent a maximum error due to ellipticity.

APPENDIX C

EXTREME VALUE AND MEAN VALUE OF THE TRUNCATED, UNBIASED WEIBULL DISTRIBUTION

The survival probability of a truncated Weibull distribution with zero bias ($\sigma_u = 0$) is given as the ratio of the initial survival probability to the survival probability at the truncation point,

$$\frac{\hat{S}}{S} = \frac{S(\beta)}{S(\psi)}$$

where ψ is the normalized proof stress (truncation point) and $\beta > \psi$.

If reasoning similar to that of section F-1 is followed, the extreme value is found to be identical to that of the nontruncated distribution since the quantity $1/S(\psi)$ is equivalent to multiplication by a constant.

The result, then, for the most probable least value of the truncated distribution is

$$\beta^* = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma(1 + 1/m)}$$

provided that

$$\psi \leq \beta^*.$$

If ψ does not meet this condition then ψ is the most likely extreme value.

The reason for this form of the extreme value is that the frequency distribution retains the same mode regardless of truncation value, until truncation value, ψ , exceeds the mode, in which case ψ defines the mode of the truncated distribution. Several ways of detecting the truncation are suggested. The presence and location of an

inflection point indicate existence and approximate magnitude of a mode. If the cumulative distribution exhibits no inflection, the slope at the abscissa leads to an estimate of ψ .

Unlike the extreme value, the mean value of the truncated distribution is always a function of ψ . Referring to a sketch of the failure frequency,

$\hat{\beta}$ is the mean after screening (truncating) at ψ . The mean value theorem gives

$$\hat{\beta} = \psi + e^{\psi^m [\Gamma(1+1/m)]^m} \int_{\psi}^{\infty} e^{-\beta^m [\Gamma(1+1/m)]^m} d\beta.$$

Note that $\hat{\beta}$ is relative to the mean value of the initial distribution.

The experimental confirmation of this equation was not within the scope of the present study. It is clearly easier to predict the consequences of a screening test than to deduce the parameter ψ or the mean value of the initial distribution. A possibly fruitful approach is to test several samples of increasing size and plot the predicted and observed extreme values as a function of the sample size, as in the sketch below:

The asymptotic extreme value observed will estimate ψ .

Values of $\hat{\beta}$ were computed using Tables of the Exponential Integral* by transforming to

$$\hat{\beta} = \psi + \frac{\psi}{m} e^{\psi^m [\Gamma(1+1/m)]^m} \int_1^{\infty} \left\{ e^{-\psi^m [\Gamma(1+1/m)]^m u} \right\} u^{-(1-1/m)} du$$

which in standard notation is

$$\hat{\beta} = \psi + \frac{\psi}{m} e^x E_{(1-1/m)}(x).$$

A graph of the strength improvement due to truncation is shown in Fig. 23.

* Pagurova, V. I. Tables of the exponential integral. New York, Pergamon Press, 1961.

Fig. 23. Strength improvement due to truncation of the Weibull distribution.

APPENDIX D

VARIANCE OF THE UNBIASED, UNTRUNCATED WEIBULL DISTRIBUTION AND ITS LEAST VALUES

Before computing the variances, it will be found useful to obtain the integral of the following function:

$$H(p, n) = \int_0^{\infty} \beta^p e^{-\beta^m n \Gamma^m} d\beta. \quad (D-1)$$

Here $\Gamma = \Gamma(1 + 1/m)$ and the result will be in terms of the parameters p and n . Parameter p will depend on the order of the moment of the distribution which is being computed, and n is the number of specimens in the sample for which the statistics of the least values are desired. Note that the behavior of the mean value is obtained when $n = 1$, that of the extreme value when $n =$ the sample size. In this sense the number of specimens n corresponds to relative volume, since the least value in n specimens should represent the strength of the n -sized specimen. If we let

$$\begin{aligned} \omega &= \beta^m \Gamma^m n, \quad \beta = \frac{1}{\Gamma} \left(\frac{\omega}{n} \right)^{1/m}, \\ d\beta &= \frac{1}{mn^{1/m} \Gamma} \omega^{1/m-1} d\omega, \\ \beta^p &= \frac{1}{\Gamma^p n^{p/m}} \omega^{p/m}. \end{aligned}$$

Putting back,

$$H(p, n) = \int_0^{\infty} \frac{1}{\Gamma^p n^{p/m}} \omega^{p/m} e^{-\omega} \frac{1}{m \Gamma n^{1/m}} \omega^{1/m-1} d\omega.$$

Combining,

$$H(p, n) = \frac{1}{mn^{(p+1)/m} \Gamma^{p+1}} \int_0^\infty e^{-\omega^{(p+1)/m}} d\omega.$$

The integral is $\Gamma\left(\frac{p+1}{m}\right)$, giving

$$H(p, n) = \frac{1}{mn^{(p+1)/m}} \frac{\Gamma\left(\frac{p+1}{m}\right)}{[\Gamma(1 + 1/m)]^{p+1}}. \quad (D-2)$$

Now the variance of the unbiased, untruncated Weibull Distribution is given by

$$a^2 = \int_0^\infty (\beta - \beta_{\text{mean}})^2 f d\beta \quad (D-3)$$

where the probability density f is given by $-\frac{dS}{d\beta}$.

Using

$$S = e^{-n\beta^m \Gamma^m},$$

which is the distribution of least values in a sample of size n , gives for f ,

$$f = -\frac{dS}{d\beta} = n\Gamma^m m\beta^{m-1} e^{-\beta^m \Gamma^m n}.$$

The mean of this density function is at $n^{-1/m}$ and, putting in (D-3),

$$a^2 = mn\Gamma^m \int_0^\infty \left(\beta - n^{-1/m}\right)^2 \beta^{m-1} e^{-n\beta^m \Gamma^m} d\beta.$$

We can expand this into three integrals each of which may be immediately solved by the formula (D-2):

$$a^2 = mn\Gamma^m \left\{ \underbrace{\int_0^\infty \beta^{m+1} e^{-\beta^m \Gamma^m n} d\beta}_I - \underbrace{\frac{2}{n^{1/m}} \int_0^\infty \beta^m e^{-\beta^m \Gamma^m n} d\beta}_{II} + \underbrace{\frac{1}{n^{2/m}} \int_0^\infty \beta^{m-1} e^{-\beta^m \Gamma^m n} d\beta}_{III} \right\},$$

$$H_I(m+1, n) = \frac{1}{mn^{1+2/m}} \frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^{m+2}}$$

$$\frac{2}{n^{1/m}} [H_{II}(m, n)] = \frac{2}{mn^{1+2/m}} \frac{1}{[\Gamma(1+1/m)]^m},$$

$$\frac{1}{n^{2/m}} [H_{II}(m-1, n)] = \frac{1}{mn^{1+2/m}} \frac{1}{[\Gamma(1+1/m)]^m}.$$

Multiplying by $mn\Gamma[(1+\frac{1}{m})]^m$ gives,

$$a^2 = \frac{1}{n^{2/m}} \left\{ \frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^2} - 1 \right\}. \quad (D-4)$$

This formula gives the variance of the least value in a sample of size n and demonstrates that the least values are more reliable than the mean values. Its square root, the standard deviation, is plotted in Fig. 24 for various n ; that is:

$$\text{Std. dev.} = \frac{1}{n^{1/m}} \sqrt{\frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^2} - 1}. \quad (D-5)$$

Fig. 24. Standard deviation of various sample sizes as a function of the Weibull modulus.

APPENDIX E

RISK OF RUPTURE FOR SEVERAL SIMPLE STRESS DISTRIBUTIONS

The risk of rupture is given for several stress distributions. In the case of the bending stress, an additional condition, static equilibrium, has been imposed to provide a comparison between various distributions which must satisfy this requirement.

A dimensionless length parameter is used by dividing the x-dimensions by h, the width of the beam. In all cases V is the total beam volume, i. e., ℓhb where ℓ is length and b is thickness. The stress is assumed to be identical over the span ℓ .

$$\bar{\alpha} = \frac{\bar{x}}{h},$$

$$\bar{\beta} = \frac{\bar{s}}{h},$$

$$\bar{\alpha}_n = \frac{\bar{x}_n}{h} \text{ or } \frac{\bar{s}_n}{h}.$$

All the other symbols should be self-explanatory. m is Weibull's modulus, σ is stress.

The results may be useful in approximating certain real stress distributions and providing estimates.

The risk of rupture as computed here is based on the simplified definition:

$$R = \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dV.$$

STRESS DISTRIBUTION (BENDING)	RISK OF RUPTURE REGION A, R_A	RISK OF RUPTURE REGION B, R_B	R_A/R_B FOR STATIC EQUIL.
$\frac{V}{m+1} \left(\frac{\sigma_A}{\sigma_0} \right)^m \bar{a}$	$V \left(\frac{\sigma_B}{\sigma_0} \right)^m (1 - \bar{a})$	$\frac{2^m}{m+1} \left(\frac{1 - \bar{a}}{\bar{a}} \right)^{m-1}$
$\frac{V}{m+1} \left(\frac{\sigma_A}{\sigma_0} \right)^m \bar{a}$	$V \left(\frac{\sigma_B}{\sigma_0} \right)^m \left[1 - \bar{a} - \bar{\beta} \left(\frac{m}{m+1} \right) \right]$	$\frac{[2(1 - \bar{a}) - \bar{\beta}]^m}{\bar{a}^{m-1} [(1 - \bar{a})(m+1) - \bar{\beta}^m]}$
$\frac{2V}{m+1} \left(\frac{\sigma_A}{\sigma_0} \right)^m \bar{a}$	$\frac{2V}{m+1} \left(\frac{\sigma_B}{\sigma_0} \right)^m \bar{\beta}$	$\left(\frac{\bar{\beta}}{\bar{a}} \right)^{m-1}$
$V \left(\frac{\sigma_A}{\sigma_0} \right)^m \bar{a}$	$V \left(\frac{\sigma_B}{\sigma_0} \right)^m (1 - \bar{a})$	$\left(\frac{1 - \bar{a}}{\bar{a}} \right)^{m-1}$

GLL-6411-3268

STRESS DISTRIBUTION, MISC.	RISK OF RUPTURE
$\frac{V}{m+1} \left(\frac{\sigma_{A0}}{\sigma_0} \right)^m \left[\frac{\bar{a}_0 \left(\frac{\sigma_{A1}}{\sigma_{A0}} \right)^{m+1} - 1}{\frac{\sigma_{A1}}{\sigma_{A0}} - 1} + \bar{a}_1 \left(\frac{\sigma_{A1}}{\sigma_{A0}} \right)^m \right]$
$\frac{V}{m+1} \sum_{n=0}^N \left(\frac{\sigma_{A_n}}{\sigma_0} \right)^m \left\{ \frac{\bar{a}_n \left(\frac{\sigma_{A_{n+1}}}{\sigma_{A_n}} \right)^{m+1} - 1}{\frac{\sigma_{A_{n+1}}}{\sigma_{A_n}} - 1} \right\}$

GLL-6411-3269

STRESS DISTRIBUTION	RISK OF RUPTURE
<p>$\sigma = KX^2$</p>	$\frac{V}{2m+1} \left(\frac{Kh^2}{\sigma_0} \right)^m \left(\bar{a}_2^{2m+1} - \bar{a}_1^{2m+1} \right)$
<p>$\sigma = KX^n$</p>	$\frac{V}{nm+1} \left(\frac{Kh^n}{\sigma_0} \right)^m \left(\bar{a}_2^{nm+1} - \bar{a}_1^{nm+1} \right)$

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.