AN APPLICATION OF THE J-INTEGRAL TO OBTAIN
SOME SIMILARITY RELATIONS*

Shih-Jung Chang and F. Joel Witt
Oak Ridge National Laboratory
Oak Ridge, Tennessee

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

ABSTRACT

For two linear elastic bodies which are geometrically similar to each other and of the same thickness, it is easy to show that the ratio of their total energies up to the time of fracture is equal to the ratio of their linear dimensions. Experimental evidence seems to suggest that this similarity condition in fracture energies may still exist for the specimens tested in their elastic-plastic range, independent of the amount of work hardening to be associated with the material. We are able to show this result, as well as a set of other similarity conditions, from the method of the J-integral. On the application of the J-integral we made one assumption, that along the respective contours of integration within the two specimens the ratio of the stress components is assumed to be a constant value for each particular pair of contours. This assumption is then justified for the region close to the crack tip from the singular solution of Rice and Rosengren and of Hutchinson. It is also justified for both close to and far from the crack tip from a solution previously obtained by Neuber from his mapping method. If the total energy of fracture is not significantly influenced by the contribution from the transition region, then the similarity relation for the total fracture energy in the elastic-plastic range will be a close approximation.

MASTER

*Research sponsored by the U. S. Atomic Energy Commission under contract with Union Carbide Corporation.

INTRODUCTION

In performing fracture tests, it is often desirable to predict the fracture conditions for a larger specimen from the results of a smaller but geometrically similar specimen. For linear elastic bodies, this does not pose any difficulty. In fact, for specimens which are geometrically similar to each other and of the same thickness, it is easy to show that the ratio of their total energies up to the time of fracture is proportional to the ratio of their linear dimensions. It is not obvious, however, whether any similarity conditions can still be found if the specimens have been deformed to their elastic-plastic range before fracture. Experimental evidence^{1,2} seems to suggest that the similarity in fracture energies described above for the elastic case may still exist for the specimens tested in their inelastic range. It is interesting in the sense that the observation is independent of the amount of work hardening to be associated with the material.

We are able to show this result, as well as other results, from the method of the J-integral,³ which has recently been demonstrated experimentally by Begley and Landes to be an appropriate parameter for elastic-plastic fracture.^{4,5} On the application of the J-integral we made one assumption, that along the respective contours of integration within the two specimens the ratio of the stress components is assumed to be a constant value for each particular pair of contours. This assumption is then justified for the region close to the crack tip from the singular solution of Rice and Rosengren⁶ and of Hutchinson.^{7,8} It is also justified for regions both far from and close to the crack tip from a solution previously obtained by Neuber from his mapping method.⁹ Therefore, if the total energy of fracture is not significantly influenced by the contribution from the transition region, then our similarity relation for the

total fracture energy in the elastic-plastic range will be a close approximation. An interesting investigation along this line was also made recently by Merkle.¹⁰

PRELIMINARY RELATIONS

In the analysis we assume that the material is nonlinear elastic so that the strain energy function exists and is a well-defined quantity. We are interested in deriving the fracture conditions between two similar specimens under similar loading conditions by the method of the J-integral.³ At present we limit our analysis to the two-dimensional plane strain condition and to the case of flat fracture. The J-integral in its two-dimensional form is

$$J = \int_{\Gamma} W \, dx_2 - T_i \frac{\partial u_i}{\partial x_1} \, ds, \quad (1)$$

where W denotes the strain energy density, T_i is the traction vector, and u_i is the displacement vector. In equation (1) the crack is assumed to be parallel to the x_1 -direction and Γ is any contour of integration enclosing the crack tip and terminating on the straight edges of the crack.

We shall restrict ourselves by assuming the material to be the Ramberg-Osgood type whose stress-strain relation is the well-known form

$$E \, \epsilon_{ij} = (1 + \nu) \, s_{ij} + \frac{1 - 2\nu}{3} \, \sigma_{kk} \, \delta_{ij} + \frac{3}{2} \, \alpha \, \frac{1}{\sigma_y^{n-1}} \, \sigma_e^{n-1} \, s_{ij}. \quad (2)$$

By choosing this particular type of material, we shall treat the elastic-plastic solid as nonlinear elastic under the assumption of the deformation theory of plasticity as was done by Rice and Rosengren⁶ and Hutchinson.⁷ We have to, of course, assume that at any point within the contour of integration the material does not exhibit unloading and that the stress components remain to be proportional with fixed values as required by the deformational theory of plasticity.

In equation (2) s_{ij} is the stress deviator

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{jj} \delta_{ij} , \quad (3)$$

σ_y is the yielding stress, and σ_e is the effective stress defined by

$$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij} . \quad (4)$$

For $\sigma_e = \sigma_y$ the Mises yield condition is recovered. E , ν , n , and α in equation (2) are Young's modulus, Poisson's ratio, the hardening coefficient, and a material constant, respectively. It is convenient to separate equation (2) into two parts: the first two terms of equation (2) being denoted as the elastic strain $E \epsilon_{ij}^e$ and the last term as the plastic strain $E \epsilon_{ij}^p$.

The strain energy function based on equation (2) is defined to be

$$w = \int \sigma_{ij} d \epsilon_{ij} = \sigma_{ij} \epsilon_{ij} - \int \epsilon_{ij} d \sigma_{ij} , \quad (5)$$

which can be calculated as

$$w = \frac{1}{2E} [(1 + \nu) \sigma_{ij} \sigma_{ij} + \nu \sigma_{kk}^2] + \frac{n}{n+1} \frac{\alpha}{E \sigma_y^{n-1}} \sigma_e^{n+1} . \quad (6)$$

The first term denotes the elastic work w_e and the second denotes the plastic work w_p .

SIMILARITY RELATIONS

Suppose that we have two geometrically similar specimens of identical material and with the same thickness. The ratio of their linear dimensions is denoted by a constant referred to as k_g . We shall derive the fracture conditions owing to the effect of the size of the specimen by the method of

the J-integral. To evaluate the integral, we shall choose the contours of integration to be located at the same relative position within the specimens. The ratio of the total lengths of the contours, as well as any pair of segments of them, is obviously the same constant, k_g . We shall make an assumption, however, that at any corresponding points along any pair of contours the ratio of their stress components is a constant value

$$k_s = \frac{\sigma_{ij}}{\sigma'_{ij}}, \quad (7)$$

where a prime is used to denote the quantities for the second specimen. For $n=1$, that is, the linear elastic case, the ratio k_s is always a constant value, but it is not necessarily so for the nonlinear materials.

We shall illustrate in the next two sections that the assumption shown in equation (7) is a close approximation for regions both close to and far from the crack tip. For this reason we shall apply the J-integral to the two extreme cases that the pair of the contours are both located in the plastic zones or both located in the elastic zones. For either case, we derive a set of similarity relations.

Consider the case in which the J-integral is evaluated within the plastic region, so that the effect due to the elastic strain is negligible as compared with that due to the plastic strain. The first term of the strain energy function in equation (6) is negligible and, consequently, the J-integral in equation (1) should assume a form with the effect of plastic strain only. The ratio k_s as expressed in terms of the strain components is

$$k_s = \frac{\sigma_{ij}}{\sigma'_{ij}} = \left(\frac{\epsilon_{ij}}{\epsilon'_{ij}} \right)^{\frac{1}{n}}. \quad (8)$$

Substituting equations (6) and (8) into equation (1), we obtain

$$J = \int_{\Gamma} (w \, dx_2 - T_i \frac{\partial u_i}{\partial x_1} \, ds) = k_s^{n+1} k_g J' . \quad (9)$$

Since the two specimens are made of identical material, we have

$$J = J' = k_s^{n+1} k_g J' , \quad (10)$$

which leads to a simple relation

$$k_s = \left(\frac{1}{k_g} \right)^{\frac{1}{n+1}} . \quad (11)$$

Consequently, the ratio of the critical energies per unit area is

$$k_w = \frac{w}{w'} = k_g^{-1} , \quad (12)$$

and the ratio of the critical strains is

$$k_e = \frac{\epsilon_{ij}}{\epsilon'_{ij}} = \left(\frac{1}{k_g} \right)^{1+\frac{1}{n}} . \quad (13)$$

A set of similarity relations is therefore obtained.

Recall that n is the work hardening coefficient of the material. For increasing n , the material will tend to become perfectly plastic. If we assume that the second specimen is smaller, $k_g > 1$, then the fracture stress for this specimen will be larger, $k_g < 1$, as it should be. However, with fixed k_g but increasing n , k_g will approach the limit, one. Obviously, it indicates that for perfectly plastic material the fracture stresses for both specimens remain the same. For critical fracture strain, equation (13) shows that, for increasing n , k_e tends to k_g^{-1} ; this provides a measure on the critical

condition which is more sensitive than their fracture stresses, which are almost equal.

The more interesting part, however, seems to be the result of the energy ratio shown in equation (12). It is invariably proportional to $1/k_g$, irrespective of the work hardening parameter n . For some time this result has appeared repeatedly from experimental results conducted by Witt^{1,2} which initiated the equivalent energy method.

Consider the case that the contour of integration is located completely within the elastic region. The result should be simply a set of relations which correspond to that for the linear elastic case. In equation (6), the second term due to the plastic strain is negligible and

$$k_s = \frac{\sigma_{ij}}{\sigma'_{ij}} = \frac{\epsilon_{ij}}{\epsilon'_{ij}} \quad (14)$$

After substituting equations (6) and (14) into the J-integral, we obtain a set of linear elastic conditions

$$k_s = \frac{1}{\sqrt{k_g}} \quad (15)$$

$$k_w = \frac{1}{k_g} \quad (16)$$

and

$$k_e = \frac{1}{\sqrt{k_g}} \quad (17)$$

which correspond to the ratios for the fracture stresses, the fracture energies, and the fracture strains.

SINGULAR SOLUTION NEAR THE CRACK TIP

The similarity results obtained in the preceding section depend mainly on our assumption in equation (7) that k_g is a constant value along the corresponding contours, since, otherwise, we shall not be able to factor out the ratio k_g from the integral sign. This assumption, unfortunately, is not easily shown for nonlinear materials. At present, we are able to show that it is approximately correct for the region close to the crack tip where the singular solution of Rice-Rosengren⁶ and Hutchinson⁷ applies.

In fact, this singular solution near the crack tip is a general result for the specimen of Romberg-Osgood material. The solution is not restricted to the case of small-scale yielding and is equally valid for the case when the yielding zone is of considerable size. For small-scale yielding, the nonlinear stress concentration factor can be determined easily from the linear elastic solution. Instead, for the large-scale yielding the nonlinear stress concentration factor can be determined as follows. Similar to Hutchinson's result for the two-dimensional plane stress case, the stress concentration factor for this case is

$$K = \left(\frac{1}{\alpha} \right)^{\frac{1}{n+1}} \left(\frac{J}{I} \right)^{\frac{1}{n+1}}, \quad (18)$$

where J is the J value of the material in nondimensionalized form and I is a constant which depends only on the work hardening coefficient n . With this value of K , the stress components near the crack tip can be expressed in terms of the stress function

$$\varphi = K r^S \tilde{\varphi}(\theta), \quad (19)$$

where both r and $\tilde{\varphi}(\theta)$ are nondimensional quantities, not affected by the size of the specimen. In equation (19)

$$s = \frac{2n + 1}{n + 1} . \quad (20)$$

The stress components are

$$\sigma_r = K r^{s-2} \left(s \tilde{\varphi} + \frac{d^2 \varphi^2}{d\theta^2} \right) \sigma_y \quad (21)$$

$$\sigma_\theta = K r^{s-2} s(s-1) \tilde{\varphi} \sigma_y \quad (22)$$

$$\sigma_{r\theta} = K r^{s-2} (1-s) \frac{d\tilde{\varphi}}{d\theta} \sigma_y \quad (23)$$

where σ_y is the yielding stress. Obviously, the assumption made in equation (7) is satisfied by the set of stress components listed above.

NEUBER'S SOLUTION FOR THE NONLINEAR SOLID

It was shown in the preceding section that our similarity results are implied by the singular solution of Rice and Hutchinson when the region is close to the crack tip. We are interested in whether these similarity relations can be extended beyond this region. From a solution which was obtained by Neuber from his mapping method,⁹ we are able to demonstrate in this section that our assumption, equation (7), is valid both far from and close to the crack tip and consequently our similarity relations are valid in both areas, at least in this particular problem. It should be noted that in the preceding section our assumption is actually proved in general, but in the present section we can only demonstrate that the assumption is consistent with this particular example.

Neuber's method, and therefore this example, is restricted by the particular stress-strain relation

$$G\epsilon = F(r) = \frac{\tau}{\sqrt{1 - (\tau/\tau^*)^2}} \quad (24)$$

between the effective stress and the effective strain.

In equation (24) G is the linear shear modulus and τ^* is the yielding stress. The strain ϵ will tend to become large as the stress increases to its yielding stress. Furthermore, his method is limited to the antiplane shear problems only.

His method for the linear elastic case is simply a method of conformal mapping. From a given complex function, whose mapping pattern is supposed to be known, we can therefore determine the boundary, or the shape, of the specimen. For this special shape, we can derive a set of stress and strain components in terms of the complex function. For nonlinear materials, the complex function needs some modification which depends on a parameter, k , that is a function of the yielding stress τ^* and the external force applied along the boundary. The real part of the function is the warping function of the specimen, and the imaginary part is the stress function. These two families of curves are perpendicular to each other for both linear and nonlinear materials.

We shall consider a specimen which assumes the shape of two-dimensional half space with a crack of finite length extended into the solid from the flat surface. First we use a complex function for a linear problem of the following form:

$$z_0 = e^{-\beta} \sinh w_1, \quad (25)$$

$$w = \cosh (w_1 - \beta) . \quad (26)$$

In equations (25) and (26), $z_0 = x_0 + iy_0$ and $w = u + iv$, where i is the unit imaginary number, u is the warping function, and v is the stress function defined by

$$\tau_{xz} = G \frac{\partial v}{\partial y} \quad (27)$$

$$\tau_{yz} = -G \frac{\partial v}{\partial x} \quad (28)$$

In equations (25) and (26) w_1 is an intermediate parameter which is introduced so that z_0 is understood to be a function of the complex variable w . Both w and z_0 in equations (25) and (26) are nondimensionalized by some reference length. The crack is located parallel to the y_0 axis and extended from $x_0 = 0$ on the free surface, $y_0 = 0$, toward the positive y_0 axis. The specimen is located in the region $y_0 > 0$.

For a nonlinear problem with the stress-strain relation shown in equation (24), the complex function should be modified by the parameter k as

$$\begin{aligned} z(w, \bar{w}) &= z_0(w) - k \int \frac{\frac{d\bar{w}}{d\tau_0}}{\frac{d\bar{w}}{d\bar{w}}} \\ &= z_0(w_1) - k \int \frac{\left(\frac{d\bar{w}}{d\bar{w}_1}\right)^2 d\bar{w}_1}{\frac{d\bar{\tau}_0}{d\bar{w}_1}} \\ &= e^{-\beta} \sinh w_1 - k \int \frac{\sinh^2(\bar{w}_1 - \beta) d\bar{w}_1}{e^{-\beta} \cosh \bar{w}_1} \quad (29) \end{aligned}$$

In equation (29), a bar is used to denote the complex conjugate of the quantity and k is defined by

$$k = \frac{1}{4} \left(\frac{C}{\tau^*} \right)^2, \quad (30)$$

where C is related to the shear stress τ_n at infinity. The new function, equation (29), still gives the boundary of the specimen approximately the same shape. For $k = 0$ it reduces to the linear case. As obtained by Neuber, the condition for an infinitely sharp crack tip is

$$k = e^{-2\beta} \sinh^2 \beta . \quad (31)$$

The corresponding crack length is

$$\frac{1}{2i}(z)_{w=0} = t = e^{-\beta} \cosh \beta \{1 - \sinh^2 \beta [1 - \cosh \beta \tanh^{-1} \beta]\} \quad (32)$$

in which k has been replaced by the condition equation (31). The crack length t is only a function of the constant β .

The stress distribution is given by

$$\frac{\tau}{\tau^*} = \frac{2}{\frac{h}{\sqrt{k}} + \frac{\sqrt{k}}{h}} , \quad (33)$$

where h is calculated from equations (25) and (26) as

$$h^2 = \frac{dz_0}{dw} \frac{d\bar{z}_0}{dw} = \frac{e^{-2\beta} (\sinh^2 u_1 + \cos^2 v_1)}{\sinh^2(u_1 - \beta) + \sin^2 v_1} . \quad (34)$$

At the crack tip $w = 0$, which implies $w_1 = \beta + i \frac{\pi}{2}$. At this point, from equation (34),

$$(h)_{w_1 = \beta + i \frac{\pi}{2}} = h_0 = e^{-\beta} \sinh \beta , \quad (35)$$

and at infinity

$$h_{\infty} = 1 . \quad (36)$$

From equation (33), since the stress at infinity is τ_n , a relation between k and τ_n is

$$\frac{\tau_n}{\tau^*} = \frac{2\sqrt{k}}{1+k} \quad (37)$$

which is equivalent to equation (30) and determines β in our complex function if we assume that the tip of the crack is infinitely sharp. The variables z and w are understood to be nondimensionalized by some reference length.

Therefore we have two free choices τ_n and the linear dimension of the specimen. If we use subscripts 1 and 2 to denote the two geometrically similar specimens, then from the expression of the stress distribution, equation (33),

$$\frac{\tau_1}{\tau_2} = \frac{h_2^2 + k_2}{h_1^2 + k_1} \cdot \frac{\sqrt{h_1}}{\sqrt{h_2}} \quad (38)$$

For the locations far from the crack tip $h_\infty = 1$, and close to the crack tip $h = h_0$. The ratios of the stresses are, respectively,

$$\frac{\tau_1}{\tau_2} = \frac{1 + k_2}{1 + k_1} \quad (39)$$

and, from equations (31) and (35),

$$\frac{\tau_1}{\tau_2} = \frac{h_{01}^2 + k_2}{h_{01}^2 + k_1} = 1 \quad (40)$$

which, of course, corresponds to the perfectly plastic condition. Therefore our assumption equation (7) is satisfied. It is worthwhile to mention that on applying Neuber's method the geometrical similarity is only described approximately owing to the fact that for the two specimens the different τ_n 's, and consequently β 's, will make the two mapping functions different. Presumably their shapes are not much different in the present case, since both crack tips are infinitely sharp and narrow.

DISCUSSION

From the singular solution of Rice and Rosengren and of Hutchinson, we have shown the validity of the similarity relation in fracture energy for regions close to the crack tip. In fact the independence of this similarity relation on the work hardening coefficient is not surprising, as can be seen from their result that the structure of the energy density is invariably proportional to $1/r$ where r is the distance from the point to the crack tip, irrespective of the nonlinearity of the material. This similarity relation can also be regarded as an immediate consequence from Neuber's result¹¹ that for any nonlinear material the product of the stress concentration factor and the strain concentration factor is invariably proportional to the square of the product of their Hookian counterparts. It seems to us that these three results, although derived in different forms and for different purposes, are actually not different from each other.

Our similarity relation on the total fracture energy depends heavily on whether the contribution due to the transition region will significantly influence the assumption made in equation (7). It will be interesting to investigate the question quantitatively. We shall make some calculation for this purpose later.

On the demonstration of our result by Neuber's solution, it is understood that Neuber's method of mapping is limited to the antiplane problems. Consequently, our result is also dictated by the same limitation. We shall investigate later on any possible generalization to the inplane case similar to the case discussed by Gowda and Topper.¹² It may also be interesting to investigate the effect of large deformation.¹³

ACKNOWLEDGMENT

We are indebted to Dr. J. G. Merkle of Oak Ridge National Laboratory and Professor H. T. Corten of the University of Illinois for many helpful discussions.

REFERENCES

1. F. J. Witt and T. R. Mager, "Fracture Toughness Values at Temperatures Up to 550°F for ASTM A533 Grade B, Class I Steel," Nucl. Engr. Design 17, 91-102, 1971.
2. F. J. Witt, "Fracture Behavior of Reactor Pressure Vessel Steel in the Frangible, Transitional, and Tough Regimes," Nucl. Engr. Design 20, 237-249, 1972.
3. J. R. Rice, "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks," J. Appl. Mech. 35, 379-386, 1968.
4. J. A. Begley and J. D. Landes, "The J-Integral as a Failure Criterion," ASTM STP 514, 1972.
5. J. D. Landes and J. A. Begley, "The Effect of Specimen Geometry on J_{1C} ," ASTM STP 514, 1972.
6. J. R. Rice and G. F. Rosengren, "Plain Strain Deformation Near a Crack Tip in a Power-Law Hardening Material," J. Mech. Phys. Solids 16, 1-12, 1968.
7. J. W. Hutchinson, "Singular Behavior at the End of a Tensile Crack in a Hardening Material," J. Mech. Phys. Solids 16, 13-31, 1968.
8. J. W. Hutchinson, "Plastic Stress and Strain Fields at a Crack Tip," J. Mech. Phys. Solids 16, 337-347, 1968.
9. H. Neuber, Kerbspannungslehre, Springer, Berlin, 1958.
10. J. G. Merkle, "Analytical Applications of the J-Integral," Progress in Flaw Growth and Fracture Toughness Testing, ASTM STP 536 (to appear).
11. H. Neuber, "Theory of Stress Concentration for Shear-Strained Prismatical Bodies with Arbitrary Nonlinear Stress-Strain Law," J. Appl. Mech. 28, 544-550, 1961.
12. C. V. Byre Gowda and T. H. Topper, "On the Relation Between Stress and Strain-Concentration Factors in Notched Members in Plane Stress," J. Appl. Mech. 37, 77-84, 1970.
13. S. J. Chang, "Path Independent Integral for Rupture of Perfectly Elastic Materials," J. Appl. Math. Phys. (ZAMP) 23, 149, 1972.