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Calculation of High Beta Toroidal Plasma Equilibrium

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CALCULATION OF HIGH BETA TOROIDAL PLASMA EQUILIBRIUM

by

Thomas A. Oliphant

ABSTRACT

This report describes a numerical procedure for calculating a high-beta, thin-skinned, toroidal plasma equilibrium. The mathematical problem involved is the solution of Maxwell's equations with a boundary between a perfectly conducting plasma and the magnetic field. This boundary must take such a shape that the magnetic field-plasma pressure balance relation holds. The mathematical problem is formulated in detail and a numerical example is presented which possesses very reasonable geometric properties, and which has external fields similar to those contemplated for the Scyllac toroidal θ pinch.

I. INTRODUCTION

Recently, a proposal¹ has been made for a toroidal theta-pinch (Scyllac). Among the theoretical problems associated with the design of this device is the $\beta = 1$ equilibrium configuration of the plasma confined in a toroidal magnetic field. This report presents and discusses a solution to such a problem.

In Section II we formulate the magnetostatic free-boundary problem and discuss the methods of solution; in Section III we develop from the basic theory the equations relevant to toroidal geometry; in Section IV we discuss the computational procedures; and in Section V we present the results of an actual numerical calculation.

II. THE FREE-BOUNDARY PROBLEM OF PLASMA EQUILIBRIUM

The problem we consider is the determination of the static equilibrium configuration of the sharp boundary between a confined, perfectly conducting, high- β plasma and a magnetic field produced by external currents. The mathematical problem involved is the solution of Maxwell's equations for fixed ex-

ternal sources and a free boundary (current-carrying plasma surface). This free boundary is required to take a shape which satisfies not only the boundary conditions of a perfect conductor, but also the plasma pressure balance relation. This is a magnetostatic problem in that there will be a static magnetic field $\vec{B}$, but no electric field $\vec{E}$.

The usual boundary conditions for $\vec{B}$ just outside a conducting surface are

$$\lim_{\epsilon \rightarrow 0^+} \hat{n} \cdot \vec{B}(\vec{r}_0 + \epsilon \hat{n}) = 0 \quad (1)$$

and

$$\vec{J} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{4\pi} \hat{n} \times \vec{B}(\vec{r}_0 + \epsilon \hat{n}). \quad (2)$$

Here $\vec{r}_0$ is a vector from the origin of our coordinate system to an arbitrary point P_0 on the plasma surface, $\hat{n} = \hat{n}(\vec{r}_0)$ is the outwardly directed unit normal to the plasma boundary at $\vec{r}_0$, and ϵ is a small positive constant which is allowed to approach

zero as indicated in the above expressions. The quantity $\vec{J}$ is the current density per unit length induced on the plasma surface which shields the interior of the plasma from the magnetic field. The plasma surface is required to assume such a shape that the solution $\vec{B}$ satisfying Eqs. (1) and (2) also satisfies the pressure balance relation

$$\lim_{\epsilon \rightarrow 0_+} |\hat{n} \times \vec{B}(\vec{r}_0 + \epsilon \hat{n})|^2 = 8\pi p \quad (3)$$

The complete Maxwell boundary conditions for the magnetostatic problem are then given by Eqs. (1), (2), and (3).

Beard² proposed that, for computational reasons, we replace the boundary condition Eq. (1) by the requirement that the component of $\vec{B}$ tangential to the plasma surface vanish just inside the surface; i.e.,

$$\lim_{\epsilon \rightarrow 0_+} \hat{n} \times \vec{B}(\vec{r}_0 - \epsilon \hat{n}) = 0. \quad (4)$$

The Beard problem is then the same as the Maxwell problem except for the boundary conditions which are Eqs. (2), (3), and (4) instead of Eqs. (1), (2), and (3).

It is relevant to ask whether we can demonstrate the equivalence of the Beard and Maxwell problems, but this question can be better approached after we have expressed our boundary conditions in a form more convenient for computation.

The magnetic field $\vec{B}(\vec{r})$ at any point P in space ($\vec{r}$ is the vector drawn from the origin of the coordinate system to the point P) can be written as the sum of the field $\vec{B}_e(\vec{r})$ arising from external sources, and the field $\vec{B}_c(\vec{r})$ arising from the current density induced on the plasma surface:

$$\vec{B}(\vec{r}) = \vec{B}_e(\vec{r}) + \vec{B}_c(\vec{r}). \quad (5)$$

The contribution $\vec{B}_c(\vec{r})$ is obtained by using the Biot-Savart law, which is written,

$$\vec{B}_c(\vec{r}) = \oint_S \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dS, \quad (6)$$

where S refers to the plasma surface. Here $\vec{r}$ is the field point and $\vec{r}'$ is the source point. Substitut-

ing Eq. (5) into Eqs. (2) and (4) we obtain

$$\lim_{\epsilon \rightarrow 0_+} \hat{n} \times \vec{B}_e(\vec{r}_0 + \epsilon \hat{n}) + \hat{n} \times \vec{B}_c(\vec{r}_0 + \epsilon \hat{n}) = 4\pi \vec{J}(\vec{r}_0) \quad (7)$$

and

$$\lim_{\epsilon \rightarrow 0_+} \hat{n} \times \vec{B}_e(\vec{r}_0 - \epsilon \hat{n}) + \hat{n} \times \vec{B}_c(\vec{r}_0 - \epsilon \hat{n}) = 0. \quad (8)$$

We note that

$$\lim_{\epsilon \rightarrow 0} \vec{B}_e(\vec{r}_0 - \epsilon \hat{n}) = \lim_{\epsilon \rightarrow 0} \vec{B}_e(\vec{r}_0 + \epsilon \hat{n}) = \vec{B}_e(\vec{r}_0) \quad (9)$$

and define

$$\vec{B}_c(\vec{r}_0) \equiv \lim_{\epsilon \rightarrow 0_+} \frac{\vec{B}_c(\vec{r}_0 - \epsilon \hat{n}) + \vec{B}_c(\vec{r}_0 + \epsilon \hat{n})}{2}. \quad (10)$$

The definition Eq. (10) takes account of the fact that we encounter a discontinuity in going from just inside the plasma surface to just outside it. Adding Eqs. (7) and (8) and making use of Eqs. (9) and (10) we obtain

$$\begin{aligned} \vec{J}(\vec{r}_0) &= \frac{1}{2\pi} \left[\hat{n} \times \vec{B}_e(\vec{r}_0) + \hat{n} \times \vec{B}_c(\vec{r}_0) \right] \\ &= \frac{1}{2\pi} \hat{n} \times \vec{B}(\vec{r}_0). \end{aligned} \quad (11)$$

Substituting Eq. (5) into Eq. (3) we obtain

$$\lim_{\epsilon \rightarrow 0_+} |\hat{n} \times \vec{B}_e(\vec{r}_0 + \epsilon \hat{n}) + \hat{n} \times \vec{B}_c(\vec{r}_0 + \epsilon \hat{n})|^2 = 8\pi p; \quad (12)$$

combining Eq. (8) with Eq. (12) we obtain,

$$\begin{aligned} |\hat{n} \times \vec{B}_e(\vec{r}_0) + \hat{n} \times \vec{B}_c(\vec{r}_0)|^2 &= |\hat{n} \times \vec{B}(\vec{r}_0)|^2 \\ &= 2\pi p. \end{aligned} \quad (13)$$

Our computational equations are now (11), (13), and (6) for the Beard problem. For the Maxwell problem we replace Eq. (13) by

$$\hat{n} \cdot \vec{B}(\vec{r}_0) = 0 \quad (14)$$

and add the constraint that

$$|\vec{B}(\vec{r}_0)| = \sqrt{2\pi p} \equiv b. \quad (15)$$

Thus, the equations for the Maxwell problem are Eqs. (11), (14), (15), and (6).

We now approach the question of the equivalence of the Beard and Maxwell problems. We write $\vec{B}$ (leaving the argument $\vec{r}_0$ understood) in the form

$$\vec{B} = \hat{n}(\hat{n} \cdot \vec{B}) + \hat{n} \times (\vec{B} \times \hat{n}) . \quad (16)$$

Thus

$$\vec{B} \cdot \vec{B} = (\hat{n} \cdot \vec{B})^2 + (\hat{n} \times \vec{B})^2 . \quad (17)$$

Now, if we insert the solution to the Maxwell problem, the first term on the right vanishes, leaving

$$|\vec{B}|^2 = |\hat{n} \times \vec{B}|^2 .$$

But using Eq. (15) on the left then yields Eq. (13) so that the Beard boundary conditions are satisfied.

On the other hand, if we insert the solution to the Beard problem, the second term on the right of Eq. (17) becomes b^2 , leaving

$$|\vec{B}|^2 = (\hat{n} \cdot \vec{B})^2 + b^2 . \quad (18)$$

But we cannot say from the Beard boundary conditions that $|\vec{B}| = b$ as we did in Eq. (15). Therefore, we cannot rigorously prove that $\hat{n} \cdot \vec{B}$ has to vanish. Hence, a numerical solution to the Beard problem does not necessarily satisfy the Maxwell boundary conditions. This seems to be a manifestation of the fact that Eq. (13) is a weaker statement than the combination of Eqs. (14) and (15). Therefore, if the Beard problem is solved it would seem advisable to ascertain that Eq. (14) is satisfied.

III. TOROIDAL GEOMETRY

In order to apply the equations of the previous section to the problem of interest, we must develop them fully in terms of toroidal coordinates, which are illustrated in Fig. 1. The major toroidal radius is R and the coordinates of the field point can be taken to be either (r, θ, z) or (r, θ, ϕ) where

$$z = R\phi . \quad (19)$$

The set of orthogonal vectors is also shown in Fig. 1; $\hat{e}_z$ and $\hat{e}_\phi$ are the same. We note that in the limit $R \rightarrow \infty$, the toroidal coordinates reduce to cylindrical coordinates. Our calculations are confined to the region within the torus of minor radius ρ of Fig. 1.

Let us assume that the plasma surface S surrounds the minor axis of the torus as shown in Fig. 2, and that it resides well within the minor radius ρ . Thus we can express the position of S in terms

Fig. 1. Toroidal coordinate system for use in studying the toroidal θ pinch.

of its distance a from the minor axis. This distance is assumed to be a single-valued function $a(\theta, z)$ of the two independent variables (θ, z) . At each point on the surface there will be a set of orthogonal unit vectors of the type illustrated in Fig. 1. The directions of these unit vectors vary with the variables (θ, z) .

Appearing prominently in most of our expressions is the unit outward normal vector $\hat{n} = \hat{n}(\theta, z)$ written as

$$\hat{n} = n_r \hat{e}_r + n_\theta \hat{e}_\theta + n_z \hat{e}_z , \quad (20)$$

where

$$\begin{aligned} n_r &= \frac{1}{D} \\ n_\theta &= \frac{1}{D} \left(-\frac{1}{a} \frac{\partial a}{\partial \theta} \right) \\ n_z &= \frac{1}{D} \left(-\frac{1}{1 - \frac{a}{R} \cos \theta} \frac{\partial a}{\partial z} \right) , \end{aligned} \quad (21)$$

and

$$D(\theta, z) = \sqrt{1 + \left(\frac{1}{a} \frac{\partial a}{\partial \theta} \right)^2 + \left(\frac{1}{1 - \frac{a}{R} \cos \theta} \frac{\partial a}{\partial z} \right)^2} . \quad (22)$$

The current $\vec{J}$ induced at any point $a(\theta, z)$ on S is given in component form as

$$J_i(\theta, z) = \frac{1}{2\pi} (n_j B_k - n_k B_j) . \quad (23)$$

The subscripts (i, j, k) refer to components along the

MINOR AXIS OF TORUS

Fig. 2. Schematic drawing of the plasma surface S.

directions of $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$, respectively, or cyclic permutations thereof.

To solve the integral Eq. (6), we must have expressions for the field point vector $\vec{r}$ and the source point vector $\vec{r} - \vec{r}''$. We choose the origin of our coordinate system to be at the intersection of the major axis of the torus with the plane which

contains its minor axis. The position vectors $\vec{r}$, $\vec{r}''$, and $\vec{r} - \vec{r}''$ are then as indicated in Fig. 3. The position vector $\vec{r}$ is written

$$\vec{r} = \hat{e}_r (a - R \cos \theta) - \hat{e}_\theta R \sin \theta. \quad (24)$$

A similar expression holds for $\vec{r}''$ in terms of the unit vectors $(\hat{e}_r'', \hat{e}_\theta'')$. Thus, $\vec{r} - \vec{r}''$ is given in terms of mixed systems of unit vectors. Equation (23) gives the components of $\vec{J}$ in terms of the unit vectors at the point of the argument of $\vec{J}$. From the point of view of the integrand in Eq. (6) the argument is (θ'', z'') and hence the unit vectors are $(\hat{e}_r'', \hat{e}_\theta'', \hat{e}_z'')$. However, from the point of view of the left-hand side of Eq. (6), we must have our answer in terms of $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$. Therefore, it is convenient to express the set $(\hat{e}_r'', \hat{e}_\theta'', \hat{e}_z'')$ in terms of the set $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$. For this purpose we introduce the transformation

$$\hat{e}_i'' = \sum_j \hat{e}_j \alpha_{ji}''; \quad (25)$$

thus

$$\alpha_{ji}'' = \hat{e}_i \cdot \hat{e}_j''. \quad (26)$$

Writing out the transformation we obtain,

$$\begin{aligned} \hat{e}_r'' &= \hat{e}_r \alpha_{rr}'' + \hat{e}_\theta \alpha_{r\theta}'' + \hat{e}_z \alpha_{rz}'' \\ \hat{e}_\theta'' &= \hat{e}_r \alpha_{r\theta}'' + \hat{e}_\theta \alpha_{\theta\theta}'' + \hat{e}_z \alpha_{z\theta}'' \\ \hat{e}_z'' &= \hat{e}_r \alpha_{rz}'' + \hat{e}_\theta \alpha_{\theta z}'' + \hat{e}_z \alpha_{zz}'' \end{aligned} \quad (27)$$

where

$$\begin{aligned} \alpha_{rr}'' &= \cos \theta \cos \left(\frac{z'' - z}{R} \right) \cos \theta'' + \sin \theta \sin \theta'' \\ \alpha_{r\theta}'' &= -\sin \theta \cos \left(\frac{z'' - z}{R} \right) \cos \theta'' + \cos \theta \sin \theta'' \\ \alpha_{rz}'' &= \sin \left(\frac{z'' - z}{R} \right) \cos \theta'' \\ \alpha_{r\theta}'' &= -\cos \theta \cos \left(\frac{z'' - z}{R} \right) \sin \theta'' + \sin \theta \cos \theta'' \\ \alpha_{\theta\theta}'' &= \sin \theta \cos \left(\frac{z'' - z}{R} \right) \sin \theta'' + \cos \theta \cos \theta'' \\ \alpha_{z\theta}'' &= -\sin \left(\frac{z'' - z}{R} \right) \sin \theta'' \\ \alpha_{rz}'' &= -\cos \theta \sin \left(\frac{z'' - z}{R} \right) \\ \alpha_{\theta z}'' &= \sin \theta \sin \left(\frac{z'' - z}{R} \right) \\ \alpha_{zz}'' &= \cos \left(\frac{z'' - z}{R} \right) \end{aligned} \quad (28)$$

Fig. 3. Illustration of the field point (r, θ, z) and source point (r'', θ'', z'') for use in the Biot-Savart integral.

Using the above transformation we write $\vec{r} - \vec{r}''$ in the form

$$\begin{aligned} \vec{r} - \vec{r}'' = & \hat{e}_r \left\{ a + \left[R \left(1 - \cos \frac{z'' - z}{R} \right) \right. \right. \\ & \left. \left. - a \cos \left(\frac{z'' - z}{R} \right) \cos \theta \right] \cos \theta \right. \\ & \left. + a \sin \theta'' \sin \theta \right\} \\ & + \hat{e}_\theta \left\{ \left[-R \left(1 - \cos \frac{z'' - z}{R} \right) + a \cos \left(\frac{z'' - z}{R} \right) \cos \theta \right] \sin \theta \right. \\ & \left. - a \sin \theta'' \cos \theta \right\} \\ & + \hat{e}_z \left\{ -R \sin \left(\frac{z'' - z}{R} \right) \left(1 + \frac{a}{R} \cos \theta'' \right) \right\}. \end{aligned} \quad (29)$$

In the integrand we also wish to use the field point projections of the source point current given by

$$J_i(\theta'', z'') = \sum_{i''} a_{ii''} J_{i''}(\theta'', z''), \quad (30)$$

thus the cross product in the integrand is given in terms of components by

$$[\vec{J} \times (\vec{r} - \vec{r}'')]_i = J_j(\theta'', z'') \times (\vec{r} - \vec{r}'')_k. \quad (31)$$

The expression for the surface element $dS(\theta'', z'')$ is

$$\begin{aligned} dS(\theta'', z'') = & a(\theta'', z'') \left[1 - \frac{a(\theta'', z'')}{R} \cos \theta'' \right] \\ & \times D(\theta'', z'') d\theta'' dz''. \end{aligned} \quad (32)$$

Thus, in component form, Eq. (6) is written

$$\vec{B}_{c,i}(\theta, z) = \int \int \frac{[\vec{J} \times (\vec{r} - \vec{r}'')]_i}{|\vec{r} - \vec{r}''|^3} dS(\theta'', z''), \quad (33)$$

and from the expressions derived above we can evaluate this integral.

As pointed out earlier, the Beard boundary condition Eq. (13) can be regarded as a differential equation which determines the surface shape. To see this we substitute Eqs. (20-22) into Eq. (13), ob-

$$\begin{aligned} & \left(\frac{1}{a} \frac{\partial a}{\partial \theta} \right)^2 (B_z^2 + B_r^2 - b^2) + \left(\frac{\partial a}{\partial z} \right)^2 \frac{(B_\theta^2 + B_r^2 - b^2)}{(1 - \frac{a}{R} \cos \theta)^2} \\ & - \frac{2B_\theta B_z}{(1 - \frac{a}{R} \cos \theta)} \left(\frac{1}{a} \frac{\partial a}{\partial \theta} \right) \left(\frac{\partial a}{\partial z} \right) + 2B_r B_\theta \left(\frac{1}{a} \frac{\partial a}{\partial \theta} \right) \\ & + \frac{2B_r B_z}{1 - \frac{a}{R} \cos \theta} \left(\frac{\partial a}{\partial z} \right) + B_\theta^2 + B_z^2 - b^2 = 0. \end{aligned} \quad (34)$$

IV. COMPUTATIONAL PROCEDURES

The most time-consuming step of the calculation is the calculation of the Biot-Savart double integral Eq. (33) at an adequate number of field points. Since the field function $B_{c,i}(\theta, z)$ is rather smooth, we will not need to use very many points in (θ, z) to obtain a good representation of it. However, the integrand is singular and the evaluation of the integral is a bit tricky. The mesh points at which $\vec{B}_c(\theta, z)$ is to be evaluated are at intervals $\Delta\theta$ and Δz as indicated in Fig. 4. Consider the field point (θ, z) located at the point a . We integrate first the variable z'' along all the lines ℓ at constant θ'' . The singularity in the integrand will be troublesome only for the line ℓ which passes through a . To avoid the difficulty we move the field point, a , slightly along the θ -dependent curve c . For a approaching its original position, the integral is still finite and changes little so that our evaluation is a good approximation. The integral over θ is then straightforward. In the numerical integrals, the presence of the singularity dictates that we need the integrand at many more points than we need the function $\vec{B}_c$. Thus, we use spline fits³ on $\vec{B}_c$ for the points between the mesh points and use a variable-interval Simpson's rule routine to integrate as accurately as required.

Solution of the differential Eq. (34) is difficult and for the present calculation we have resorted to a variation at the surface by a cut-and-try method. As a temporary expedient this represents a

Fig. 4. Schematic representation of the numerical meshing.

substitute for actually solving Eq. (34).

We have written theoretical expressions Eqs. (23), (33), and (34) and have given some idea of how each is to be solved or evaluated. However, there is no explicit way to find the simultaneous solution to these equations. As indicated by Beard,⁴ some sort of iterative or self-consistent method must be used. We may begin by solving the Beard Eq. (34) with the guess $\vec{B}_c = 0$ everywhere. However, in our torus problem $\vec{B}_c$ will be rather large. For the circular cylinder in a constant B_z external field, $\vec{B}_c$ is exactly one-half the value of $\vec{B}_e$ and oppositely directed. Thus, it is better to start with a $\vec{B}_c$ of about half the magnitude of $\vec{B}_e$ and oppositely directed. Then we can either guess a beginning surface or solve Eq. (34). Having our first surface guess, we can do as many iterations as we wish with Eqs. (23) and (33) (called Biot-Savart iterations). Then we get a second surface by solving Eq. (34), and so on until $\vec{B}_c$ and the surface shape remain constant with further iterations. Then, if the normal component at $\vec{B}$ vanishes as it should, we have our solution.

V. SPECIFIC EXAMPLE

We now present the principal results of an illustrative numerical calculation of a toroidal equilibrium. The plasma boundary is shown schematically in Fig. 5. For an externally imposed field, we take a low order harmonic expansion solution of Maxwell's equations in toroidal coordinates, including the effects of modulations, of periodicity length λ , along the minor toroidal axis. The actual expression for this external field is

$$\begin{aligned} \vec{B}_e = \hat{e}_z \left(1 - \frac{r}{R} \cos\theta \right) B_0 + G_1 \sin \frac{z}{\lambda} \left[\hat{e}_r \left(\cos\theta - \frac{1}{2} \frac{r}{R} \right) - \hat{e}_\theta \sin\theta \right] \\ + \left\{ \hat{e}_r \left(-\frac{1}{2} r \frac{B_m}{\lambda} \sin \frac{z}{\lambda} + \frac{3}{8} G_1 \left(\frac{r}{\lambda} \right)^2 \sin \frac{z}{\lambda} \cos\theta \right) - \hat{e}_\theta \frac{1}{8} G_1 \left(\frac{r}{\lambda} \right)^2 \sin \left(\frac{z}{\lambda} \right) \sin\theta \right. \\ \left. + \hat{e}_z \left[-B_m \cos \frac{z}{\lambda} - \frac{1}{4} B_m \left(\frac{r}{\lambda} \right)^2 \cos \frac{z}{\lambda} + G_1 \frac{r}{\lambda} \cos \frac{z}{\lambda} \cos\theta \right] \right\} \\ - \frac{1}{R} \left\{ \hat{e}_r \left[\cos\theta \left(\frac{3}{2} B_m \frac{r^2}{\lambda} \sin \frac{z}{\lambda} + \frac{13}{2} \lambda B_m \sin \frac{z}{\lambda} \right) - \frac{21}{4} r G_1 \sin \frac{z}{\lambda} \sin 2\theta \right] \right. \\ \left. + \hat{e}_z \left[-\sin\theta \left(\frac{1}{2} B_m \frac{r^2}{\lambda} \sin \frac{z}{\lambda} + \frac{13}{2} B_m \lambda \sin \frac{z}{\lambda} \right) + \frac{21}{4} r G_1 \sin \frac{z}{\lambda} \sin 2\theta \right] \right. \\ \left. + \hat{e}_z \left[\frac{3}{4} G_1 \frac{r^2}{\lambda} \cos \frac{z}{\lambda} + \frac{11}{2} B_m r \cos \frac{z}{\lambda} \cos\theta - \frac{5}{8} G_1 \frac{r^2}{\lambda} \cos \frac{z}{\lambda} \cos 2\theta \right] \right\}. \end{aligned} \quad (35)$$

The most prominent contribution from Eq. (35) consists of the following three terms:

$$B_e \approx \hat{e}_z B_0 - B_m \cos \frac{z}{\lambda} + G_1 \frac{r}{\lambda} \cos \frac{z}{\lambda} \cos\theta. \quad (36)$$

Fig. 5. The confined toroidal plasma is illustrated schematically. The sharp plasma surface is as seen looking down along the major axis of a torus of major radius R . The period of variation of the external field along the minor toroidal axis is λ .

A first guess for the surface shape may be parameterized as follows:

$$a(\theta, z) = \bar{a} \left[1 + \epsilon_1 \cos \frac{z}{\lambda} (1 + \epsilon_2 \cos\theta) \right]. \quad (37)$$

The parameters $\bar{a}$, ϵ_1 , and ϵ_2 are adjusted to minimize $\vec{B}$ where $\vec{B}_e$ has the form of Eq. (36). The field Eq. (35) simulates that which would actually be produced by the external currents in the toroidal Scyllac apparatus. This expression can be obtained

by using off-axis expansions of the solutions I_n and K_n of Bessel's equation as well as perturbation treatment⁵ in $1/R$. A similar expression involving the actual Bessel functions instead of

Fig. 6. The boundaries discussed in connection with Fig. 5 are plotted in terms of their deviation Δr from the corresponding boundaries of a smooth torus of minor radius 1 cm as a function of the distance z along the minor axis of the torus for one period λ . The outward (inward) deviation of the outside (inside) boundary are plotted upward (downward).

off-axis expansions can be obtained and is described in Reference 6.

The outside boundary of the plane view section of the plasma shown in Fig. 5 is approximately circular, while the inside has a bumpy (corrugated) shape. This is a reasonable configuration in view of the fact that the current conservation and pressure equilibrium condition lead to the requirement that field lines lying on the equilibrium surface have constant length independent of their location.

For our sample toroidal calculation, we chose a major toroidal axis of $R = 200$ cm and allowed 313 periodicity lengths in the external field so that $\lambda = 2\pi R/313 = 4.01$ cm. We used an external field strength such that the plasma formed a bumpy torus having an average minor radius of about 1 cm. The actual parameters fed into the expression for $\vec{B}_e$ were: $B_o = 200$ kG, $B_m = 13$ kG, and $G_1 = -13$ kG. In each period λ we placed 12 computational mesh points on the surface. A calculation of this sort takes from five to seven hours of CDC-6600 computer time.

Our solution bears a striking qualitative resemblance to the Meyer and Schmidt "bumpy torus" configuration.⁷ The correspondence is probably not quantitative because of restrictions imposed on the planes of current flow in the latter solution. As indicated above, a convenient test of the accuracy of our solution is the constancy of the lengths of magnetic lines of force on the plasma surface. The inside and outside lengths corresponding to Fig. 5 agree with each other to within 0.10% whereas the

corresponding deviations from the smooth torus peripheries average 0.44%. The plasma pressure corresponds to a surface magnetic field strength of 200 kG, and the maximum deviation from this value on the surface is 5 kG. (The variations in the externally imposed field strengths are as great as 40 kG).

In Fig. 6 we have plotted the deviations of the outer and inner peripheries from the corresponding peripheries of a smooth torus having a minor radius of 1 cm. An additional item of qualitative interest is the cross-sectional shape of the plasma column at various points z along the mirror toroidal axis. These cross sections are shown in Fig. 7.

The above result serves primarily to indicate the existence of an equilibrium configuration with an external field $\vec{B}_e$ qualitatively like that to be used experimentally. Since the calculation is somewhat unwieldy, no attempt has been made to use the numerical program for design parameter studies. It is hoped that improved methods can be incorporated for this purpose.

Fig. 7. Cross sections of the plasma surface perpendicular to the minor toroidal axis at indicated distances along the minor toroidal axis. The dotted line is the cross section of the smooth torus of minor radius 1 cm.

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