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FOUNDATIONS OF S-MATRIX THEORY I
THEORY AND MEASUREMENT

Henry P. Stapp

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ABSTRACT

This report is the first of a series devoted to a systematic development of S-matrix theory. This first report deals with the general problem of the connection between theory and experiment, and hence with the specific problems of measurement and interpretation in relativistic quantum theory. Also included in this report is an introduction to the first three reports of the series.

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INTRODUCTION

S-matrix theory is a relativistic quantum theory. Its foundations are thus rooted in those of both relativity theory and quantum theory. The foundations of relativity theory demand no special attention here, but those of quantum theory do. For they are the focus of a growing controversy that arises in part precisely from the difficulty of reconciling quantum theory with relativity theory. S-matrix theory is based upon--and in a broad sense constitutes--a resolution of the problems underlying this controversy.

The central problem in the interpretation of quantum theory is to reconcile the claim that quantum theory provides a complete description of atomic phenomena with the fact that the wave functions of quantum theory have the mathematical properties appropriate to functions that determine only probabilities. According to quantum theory, these wave functions give a complete description of atomic and subatomic particles. But macroscopic objects are evidently composed of atomic and subatomic particles. Thus the claim of completeness is sometimes construed to imply that macroscopic objects are also completely described by wave functions. However, this conclusion entails either that the basic principles of quantum theory fail to hold universally, or that the world is radically different from what it appears to be.

This difficulty has been widely discussed.¹⁻⁴ It is sufficient to note here that the basic principles of quantum theory ensure that the square of the absolute value of the wave function behaves like a probability function. But a probability function evolves in the course of time into a function that is a sum of parts corresponding to the

various alternative possibilities. Thus, for example, the wave function that represents a scientist observing the time of decay of a radioactive atom will evolve into a superposition of parts corresponding to the various possible times of decay: it cannot evolve into a form that would correspond to the scientist's having observed the decay to have occurred during one particular time interval, without violating the superposition principle. Yet experience shows that the decay does occur during some one particular time interval.

Several ways of coping with this situation have been proposed:

(1) Assert that the superposition principle fails in the macroscopic domain (Ludwig²); (2) Assert that the superposition principle fails when consciousness becomes a factor (Wigner³); (3) Assert that the superposition principle holds universally and, accordingly, that the scientist himself actually evolves into a superposition of different editions of himself, with the different editions corresponding to different times of decay (Everett-Wheeler⁴).

The first two proposals demand a breakdown of the superposition principle, and hence go beyond quantum theory. The third proposal is not immediately refuted by experience because the parts of the wave function that correspond to the various editions of the scientist would move into different regions of the multiparticle phase-space and become effectively noninterfering. Consequently, each edition of the scientist would be unaware of the existence of the others. This lack of effective interference makes it apparently impossible to test whether the "other editions" of the scientists really exist or not. Thus according to this third proposal quantum theory would have two parts: it would have a metaphysical part, which would affirm the

existence of these "other editions," and also a practical or pragmatic part, which would be a procedure for calculating correlations between observations.

These three proposals are all based on the same basic idea, namely the idea that the wave function of quantum theory describes the world itself. All three proposals are attempts to reconcile this idea with the fact that the mathematical properties of wave functions are those appropriate to functions that describe probabilities rather than the world itself. The most natural conclusion to draw from this mathematical fact would be simply that wave functions describe probabilities, rather than the world itself. But this raises the question: How is the world itself described?

The Bohr-Heisenberg solution of this problem is based on a pragmatic view of science. According to this view the basic aim of science is not to construct a mental or mathematical image of the world itself. Rather, it is to order our experience, and to devise methods for forming expectations that are fulfilled by our experience. From this point of view the claim that quantum theory provides a complete description of atomic phenomena does not entail either that macroscopic objects are described by wave functions, or that quantum theory describes the world itself. It entails only that no other theory can bring additional order to our experience, in the realm of atomic phenomena, or allow us to form additional expectations that are fulfilled by our experience in that realm.

The pragmatic orientation of Bohr's view is fixed in the opening words of his first book:¹ "The task of science is both to extend the range of our experience and reduce it to order,...." The

focus here is not on the description of nature itself; it is rather on man's experience and his efforts to order it. Bohr says later, in a summary of his mature views: "Strictly speaking, the mathematical formalism of quantum theory merely offers rules of calculation for the deduction of expectations about observations obtained under well-defined experimental conditions specified by classical physical concepts."

This pragmatic view is formalized in the opening section of Chapter I. That formalization of the pragmatic view forms the logical basis of the present work.

A basic aim of S-matrix theory is to reconcile quantum theory with relativity theory. In the domain of nonquantum phenomena the theory of relativity introduced a view of the world profoundly different from that of pre-relativity physics. One naturally expects relativistic quantum theory to be based on the relativistic world-view, rather than that of pre-relativity physics.

In classical nonrelativistic physics the world is considered to be evolving or unfolding in the course of time. At any instant of time the world as a whole is considered to have evolved up to a certain stage. Certain events are considered to have already occurred, and others are considered to have not yet occurred. These two classes of events are separated by the existing world, which is represented by the instantaneous values of a set of time dependent variables.

This evolutionary view of the world conflicts with the ideas of the theory of relativity. The relativistic world-view is an overall space-time view. Events are labelled simply by their space-time position. The notion that the world as a whole has evolved only

up to a certain stage is not introduced, because this would entail introducing a family of special space-like surfaces to represent the evolutionary advance of the world as a whole. The existence of such surfaces would confer special status upon the corresponding special definition of simultaneity, contrary to the ideas of relativity theory.

Rejection of the evolutionary world-view creates conceptual problems,⁶ insofar as scientific theories are considered to describe the world itself. However, these problems dissolve if the pragmatic viewpoint is adopted. The pragmatic viewpoint is thus well-suited to both relativity theory and quantum theory.

Nonrelativistic quantum theory, like nonrelativistic classical theory, is based on the evolutionary world-view. In the transition from nonrelativistic classical theory to nonrelativistic quantum theory the variables whose instantaneous values constitute the classical representation of the world are replaced by corresponding operators, which are related to measurements performed at instants of time. And the state of the world is represented by an evolving state vector $\psi(t)$. This state vector, or the corresponding density matrix $\rho(t)$, is essentially a compendium of statistical predictions about results of possible measurements performed at the instant of time t .

Real measurements, however, are not instantaneous: they take finite amounts of time. The assumption that measurements are instantaneous has long been recognized as an objectionable feature of quantum theory.⁷ Within the framework of a nonrelativistic approximation this assumption is tolerable, because the passage to the nonrelativistic limit $c \rightarrow \infty$ effectively compresses finite time intervals into instants of time. However, in a relativistic theory

the assumption is objectionable. Yet if the concept of instantaneous measurements is rejected then the concept of the evolving state vector $\psi(t)$ --or $\psi[\sigma]$, where σ is a space-like surface--becomes obscure, since this state is then a compendium of statistical predictions about results of measurements that are recognized not to exist. In fact, the whole concept of an evolving state of the world seems to be an alien concept from pre-relativity physics.

S-matrix theory is based on the overall space-time viewpoint. It harbors no concept of an evolving state of the world. Space and time enter only as coordinates assigned to physical measurements. These coordinates are macroscopic, in the sense that they are determined by readings on clocks and other real measuring devices. Exact equations of motion are obtained for the observables associated with these measurements. But these equations entail the existence neither of an evolving state of the world as a whole nor of a microscopic description of the flow of probability--or any other conserved quantity. The basic intent of S-matrix theory is, in fact, precisely to construct a relativistic theory of atomic and subatomic phenomena that is not based on the concept of an evolving state of the world, or on a microscopic space-time description of the flow of conserved quantities. This intent is motivated by the severe difficulties that have beset all efforts to construct a mathematically consistent relativistic quantum theory based upon these ideas.

A satisfactory quantum theory must resolve the long-standing problem of measurement. Basically this problem is to reconcile the following two facts: (1) The nonzero value of the quantum of action

entails that the interaction between the measured system and the measuring devices cannot be ignored: this interaction transfers a full quantum of action to the measuring device, and hence its effects cannot be arbitrarily reduced or diminished. Thus the measuring instruments should be taken into consideration. Ideally the processes of measurement should be treated as integral parts of the whole phenomena. In fact, the demand that the entire phenomena, including the processes of measurement, be treated as a whole is a principal tenet of the Bohr-Heisenberg interpretation of quantum theory.

(') Real measuring devices are too complicated to be treated with qualitative accuracy by present-day quantum theoretical methods. Thus in practice the wave function is always the wave function of a system that does not include the measuring devices: it is the wave function of an idealized isolated system that is not being disturbed by measuring devices. In fact, a second basic tenet of the Bohr-Heisenberg interpretation is the idea that the wave function is in principle the wave function of an isolated system that is not being disturbed by measurements: real measuring devices cannot be considered isolated from their surroundings, and must be described classically.

The problem is therefore to reconcile the demand that the entire phenomena, including the measurements, be treated as a whole, with the demand that wave functions describe isolated systems that are not being disturbed by measurements.

The S-matrix reconciliation of these two demands is based on the interpretation of wave functions as images of the classical descriptions of preparations and examinations in the space of the degrees of freedom of long-range components of the interaction between

the preparing and examining systems. This matter is discussed in Chapter I, in the context of a general discussion of the problems of measurement and interpretation.

S-matrix theory forsakes microscopic space-time description. It introduces instead certain analytic requirements. Two questions then arise: (1) How is the apparent causal space-time structure of nature incorporated into the theory? (2) How is arbitrariness in the choice of analytic properties avoided?

The general principle governing the introduction of analytic properties is the principle of maximal analyticity. This principle asserts that scattering functions have only those singularities that are required by the general logical structure of the theory itself. This general logical structure is based on the quantum principles of superposition and unitarity, and on the concept of particles. The concept of particles, and in fact the whole interpretation of the theory, is based on the assumption that the interactions of longest range are carried by stable systems. That is, interactions not ascribable to transfers of stable systems are assumed to fall off faster, under space-time dilation, than those ascribable to transfers of stable systems. This property is formalized in Chapter II as the property of "macrocausality." Macrocausality expresses, in particular, the condition that any interaction not ascribable to a transfer of stable particles has a probability to occur that falls off exponentially under space-time dilation. This exponential fall-off condition is akin to the requirement that the potentials of nonrelativistic Schroedinger theory have Yukawa-type tails. Macrocausality unites this finite-range condition with certain space-time ordering conditions associated with the motions of stable systems.

The macrocausality property is mathematically equivalent to a set of analytic properties called the normal analytic structure. This analytic structure is described in Chapter III. It is apparently compatible with all the general logical and mathematical principles of relativistic quantum theory. The normal analytic structure is therefore regarded as the first specific consequence of the principle of maximal analyticity. Its equivalence to macrocausality ensures that the theory yields a causal macroscopic space-time structure. Alternatively, if one chooses to adhere to the traditional view that analytic properties should be derived from space-time properties, then this equivalence can be used in the other direction, and the normal analytic structure can be regarded as a consequence of the short-range and macroscopic space-time ordering conditions expressed by macrocausality. In any case, the normal analytic structure is assumed to hold: it constitutes the mathematical basis upon which the whole theory rests.

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* This work was supported by the U. S. Atomic Energy Commission.

I. THEORY AND EXPERIMENT

"The task of science is both to extend the range of our experience and reduce it to order,..."

Niels Bohr¹

"Strictly speaking, the mathematical formalism of quantum mechanics and electrodynamics merely offers rules of calculation for the deduction of expectations about observations obtained under well-defined experimental conditions specified by classical physical concepts."

Niels Bohr¹

A. LOGICAL FRAMEWORK

Quantum theory is a procedure by which scientists calculate predictions about observations obtained under certain types of conditions. These conditions are described by saying that a system is first prepared according to certain specifications, and is then examined according to certain specifications. And this examination is such that it can either yield or not yield various possible specified results.

The calculational procedure is as follows:

Let A represent the specifications on the manner of preparation. And let B represent the specifications on both the manner of examination, and one of its possible results. These specifications A and B are described in the language of the experimental physicist. They are described in a language that allows trained technicians to prepare systems in ways that meet the specifications A , to examine systems in ways that meet the specifications B , and to determine whether the results of such examinations meets the specifications B . The descriptions A and B are thus operational, in the sense that they describe how technicians set up experiments and recognize results. Moreover they are objective, in that they refer not to any individual technician or scientists, but rather to the actions and observations of any properly trained technician. To ensure objectivity these specifications can be formulated in the language of classical physics. In this case the specifications are formulated in terms of the idealizations of classical physics.

These specifications A and B are mapped, by a procedure to be described presently, into corresponding elements of the mathematical formalism. The specifications A on the preparation are

mapped into a density function $\rho_A(\xi'_A; \xi''_A)$ that has the form

$$\rho_A(\xi'_A; \xi''_A) = \psi_A^*(\xi'_A) \psi_A(\xi''_A)$$

in the special "pure" cases. In the general "mixed" cases $\rho_A(\xi'_A; \xi''_A)$ is a sum of terms of the pure-case form. The functions $\psi_A(\xi_A)$ are called wave functions of the prepared system, and ξ_A is a set of variables called the degrees of freedom of the prepared system.

Similarly, the specifications B on the examination are mapped into a density function $\rho_B(\xi'_B; \xi''_B)$, where ξ_B is a set of variables called the degrees of freedom of the examined system. These degrees of freedom ξ_A and ξ_B are discussed in the next section.

The transition function $S(\xi_A; \xi_B)$ is constructed from theoretical principles to be described later. Then the transition probability $\mathcal{P}(A; B)$ is defined by

$$\begin{aligned} \mathcal{P}(A; B) &= \int d\xi'_A d\xi''_A d\xi'_B d\xi''_B \\ &\times \rho_A(\xi''_A; \xi'_A) S(\xi'_A; \xi'_B) \rho_B(\xi'_B; \xi''_B) S^\dagger(\xi''_B; \xi''_A) \\ &= \text{Tr } \rho_A S \rho_B S^\dagger \end{aligned} \quad (\text{A.1})$$

where

$$S^\dagger(\xi_B; \xi_A) = S^*(\xi_A; \xi_B). \quad (\text{A.2})$$

The transition probability $\mathcal{P}(A; B)$ is the predicted limit of the relative frequency of results of the kind specified by B as the number of systems prepared in a manner meeting specifications A and examined in a manner meeting specifications B tends to infinity.

If ρ_A and ρ_B are both of the pure-case form then the transition probability formula (A.1) reduces to

$$P(A; B) = |S[\psi_A, \psi_B^*]|^2, \quad (A.3)$$

where $S[\psi_A, \psi_B^*]$ is defined by

$$S[\psi_A, \psi_B^*] \equiv \int d\xi_A d\xi_B \psi_A(\xi_A) S(\xi_A; \xi_B) \psi_B^*(\xi_B). \quad (A.4)$$

The quantity $S[\psi_A, \psi_B^*]$ is called the transition amplitude.

B. THE DEGREES OF FREEDOM

Quantum theory predicts statistical correlations between complex preparing and examining systems. These systems consist of the preparing and examining devices together with their environments, which include in the final analysis the technicians themselves. Yet the predictions are calculated from a formula based on the degrees of freedom of the relatively simple prepared and examined systems. To apply quantum theory one must identify these degrees of freedom and must specify the conditions under which the empirical correlations between the complex preparing and examining systems are expected to have the simple predicted form.

The empirical correlations are expected to have the simple predicted form only if certain conditions are met. First, the space-time region associated with the process of preparation should be separated from the space-time region associated with the process of examination. These two processes are represented in the transition probability formulas by independent factors, ρ_A and ρ_B . But the corresponding physical processes cannot be expected to be physically independent unless the space-time regions in which they occur are separated. That is, the theoretical structure will not correspond to the physical structure unless these two space-time regions are separated. Second, the locations of these two space-time regions should be compatible with the idea that the prepared system travels from the region of the preparation to the region of the examination, changing on its journey to a system of the type being examined. The empirical correlations cannot be expected to have the predicted form unless the information about the preparation can be conveyed by the quantum system from the preparing devices to the examining devices.

Third, the effect of the interaction between the preparing and examining devices associated with the exchanged quantum system should not be masked by the effects of other interactions between these devices. The effect of the exchanged quantum system must dominate over other effects, if it is to observably manifest itself.

These demands can be met only if the interaction between preparing and examining devices associated with the exchanged quantum system is long range. The interaction must be long range both in order to bridge the separation between the space-time regions associated with the processes of preparation and examination, and in order to dominate over the short-range interactions that effectively constitute the measuring devices themselves. This dominance arises from the relative damping of these short-range interactions brought about by the space-time separation of the regions of preparation and examination. The quantum system is therefore, in effect, the representation of a long-range component of the interaction between the preparing and examining systems. This situation is depicted in Fig. 1.

Fig. 1. The space-time regions associated with the processes of preparation and examination must be separated in order for these processes to be independent, and in order for the effect of the exchanged quantum system to dominate over the effects of the short-range interactions that effectively constitute the devices themselves. The quantum system represents, in effect, a long-range component of the interaction between the preparing and examining systems.

The interactions of longest range are those associated with stable systems. Correspondingly, the degrees of freedom of the quantum systems are taken to be degrees of freedom associated with stable systems. Stable systems are considered to evolve eventually into sets of elementary stable systems, called "particles." The degrees of freedom of particle i consists of a triad (p_i, μ_i, t_i) . Here t_i is the type index, which specifies the "type" of particle i --whether it is a proton, or electron, or positron, etc. The index μ_i is the spin index, which specifies the component of spin with respect to a prescribed spin axis. The variable p_i is an energy-momentum four vector, which satisfies the mass-shell constraint

$$p_1^0 = (m_1^2 + (\vec{p}_1)^2)^{1/2} \quad (B.1)$$

Here $m_1 \equiv m(t_1)$ is a real constant called the mass of particles of type t_1 .

The variables ξ_A and ξ_B are taken to be sets of single-particle variables; they are taken to be variables of the form $((p_1, \mu_1, t_1; \dots; p_n, \mu_n, t_n))$. The integration over ξ in (A.1) denotes an integration over the various $\vec{p}_i$, and a sum over the various discrete variables n , μ_i , and t_i .

The spin and type variables will generally be suppressed when they are not relevant to the matters being discussed: it will then be understood that each p_j is associated with a μ_j and a t_j .

It is convenient to express the integration over $\vec{p}_i$ in the covariant form

$$\begin{aligned} \int_{p_i} &\equiv \int \frac{d^4 p_i}{(2\pi)^4} \, 2\pi \, \delta(p_i^2 - m_i^2) \, \theta(p_i^0) \, 2m_i \\ &= \int \frac{m_i}{p_i^0} \frac{d^3 p_i}{(2\pi)^3} \, , \end{aligned} \quad (B.2)$$

where p_i^0 is defined by (B.1). The choice of weight factor in (B.2) has no physical significance; it merely determines the normalization of the functions $\rho_A(\xi'_A; \xi''_A)$, $\rho_B(\xi'_B; \xi''_B)$ and $S(\xi'_A; \xi'_B)$.

Massless particles require special consideration. This is because infinite numbers of them must be considered together. They will be discussed later. Until then they are ignored, along with the closely related electromagnetic, weak, and gravitational interactions.

Quantum systems have been associated here with stable systems. Variables corresponding to unstable or transient systems are not included among the quantum theoretical degrees of freedom because transients do not satisfy the unitarity requirement ordinarily imposed by quantum dynamics. On the other hand, if, as in nonrelativistic quantum theory, there is a unitary operator that generates the temporal evolution of the quantum system, then this evolution can be followed into the asymptotic domain without loss of information. Thus in this situation the prepared and examined systems can, without loss of information, be represented by the asymptotic systems into which they would eventually evolve, if left undisturbed, or from which they would evolve.

This procedure of representing the preparations and examinations by asymptotic systems can apply also in more general circumstances: it can apply even if there is no temporal evolution operator that transforms the "state at one instant of time" into the "state at some other instant of time". The concept of the state of a system at some instant of time--or on some space-like surface--ascribes special status to time (as opposed to space) and may thus be inappropriate in the relativistic domain. The concept of the asymptotic system, on the other hand, ascribes no special status to time. In fact, neither time nor space occur among the degrees of freedom ξ_A and ξ_B .

C. SPACE-TIME DISPLACEMENTS AND TEMPORAL EVOLUTION

The degrees of freedom ξ_A and ξ_B contain no time variables. Nevertheless, the theory yields precise predictions about temporal evolution. For time variables occur in the specifications A and B, and the transition probabilities $P(A; B)$ depend in a simple way on these times.

The temporal evolution of $\mathcal{O}(A; B)$ is determined in the following way. Suppose A represents a set of specifications on the manner of preparation. Then let A^u represent the same set of specifications, except that each space-time four-vector x occurring in A is replaced by $x + u$. Similarly, let B^u be the same as B, except that each space-time four-vector x occurring in B is replaced by $x + u$. The following assumptions are then introduced:

$$\psi_{A^u}(p_1, \dots, p_n) = \psi_A(p_1, \dots, p_n) e^{iu \sum p_i}$$

and

$$\psi_{B^u}(p_1, \dots, p_n) \equiv \psi_B(p_1, \dots, p_n) e^{iu \sum p_i}.$$

$$\text{Here } u p_i = \sum u_\mu p_i^\mu = u^0 p_i^0 - \vec{u} \cdot \vec{p}_i.$$

In terms of the ρ_A and ρ_B these rules become

$$\begin{aligned} \rho_{A^u}(p_1'', \dots, p_n''; p_1', \dots, p_m') \\ = \rho_A(p_1'', \dots, p_n''; p_1', \dots, p_m') e^{iu[\sum p_i' - \sum p_i'']} \end{aligned} \quad (C.1a)$$

and

$$\begin{aligned} \rho_{B^u}(p_1', \dots, p_n'; p_1'', \dots, p_m'') \\ = \rho_B(p_1', \dots, p_n'; p_1'', \dots, p_m'') e^{iu[\sum p_i'' - \sum p_i']} \end{aligned} \quad (C.1b)$$

The formulas (C.1a) and (C.1b), taken together, are regarded as a fundamental postulate. The units of u are fixed so that Planck's constant is unity. The temporal evolution of the transition probability $P(A; B)$ is obtained by inserting (C.1) into (A.1).

The degrees of freedom ϵ_A and ϵ_B are those appropriate to an asymptotic system of freely moving particles. However, there is no stipulation that the preparations and examinations specified by A and B be performed at asymptotic times. For example, the temporal evolution of $P(A; B)$ calculated from (C.1) and (A.1) holds even if the examination is performed long before the prepared system evolves into a system of essentially freely moving stable particles. The temporal evolution of $P(A; B)$ calculated from (A.1) and (C.1) holds for all times. And it is expected to represent the evolution of the empirical correlations to the extent that the space-time regions associated with the processes of preparation and examination are located in a way such that the effects of the exchanged quantum system are not masked by other effects.

The temporal evolution of the transition probability $P(A; B)$ is calculated in this way, for all times, by a procedure that does not use--or depend on the existence of--the temporal evolution operator of Schroedinger theory. To see how this has come about it is useful to compare the Schroedinger procedure for calculating the temporal evolution of $P(A; B)$ with the S-matrix procedure. The Schroedinger procedure is based on the idea that there is a unitary operator $U(t_1, t_2)$ that converts the state vector $\psi(t_1)$ that represents the system at time t_1 into the state vector $\psi(t_2)$ that represents the system at time t_2 :

$$\psi(t_1) U(t_1, t_2) = \psi(t_2) . \quad (C.2)$$

[Please read from left to right.]

In the Schroedinger formulation the specifications A on the preparation are mapped into a state vector $\psi_A(t_A)$ that represents the prepared system at the time of the preparation:

$$A \rightarrow \psi_A(t_A) . \quad (C.3a)$$

And the specifications B on the examination are mapped into a state vector $\psi_B(t_B)$ that represents the detected system at the time of the examination:

$$B \rightarrow \psi_B(t_B) . \quad (C.3b)$$

The state vector $\psi_A(t_A)$ is then converted to $\psi_A(t_B)$ by $U(t_A, t_B)$, and the transition amplitude is given by

$$\begin{aligned} S_{AB} &= \psi_A(t_A) U(t_A, t_B) \psi_B^\dagger(t_B) \\ &= \psi_A(t_B) \psi_B^\dagger(t_B) . \end{aligned} \quad (C.4)$$

In the S-matrix formulation the preparation A is mapped onto a time independent wave function (or state vector):

$$A \rightarrow \psi_A . \quad (C.5a)$$

And the examination B is mapped onto a time independent wave function (or state vector):

$$B \rightarrow \psi_B . \quad (C.5b)$$

If the Schroedinger operator $U(t_1, t_2)$ is assumed to exist then these states ψ_A and ψ_B can be formally defined in terms of it, and the corresponding free-particle operator $U_0(t_1, t_2)$, in the following way:

$$\psi_A = \lim_{t \rightarrow -\infty} \psi_A(t) U_0(t, 0) \quad (C.6a)$$

and

$$\psi_B = \lim_{t \rightarrow \infty} \psi_B(t) U_0(t, 0) , \quad (C.6b)$$

where for any t

$$\psi_A(t) = \psi_A(t_A) U(t_A, t) \quad (C.7a)$$

and

$$\psi_B(t) = \psi_B(t_B) U(t_B, t) . \quad (C.7b)$$

And the operator S is defined by

$$S = \lim_{t \rightarrow \infty} U_0(0, -t) U(-t, t) U_0(t, 0) . \quad (C.8)$$

These equations combine to give

$$\begin{aligned} S_{AB} &= \psi_A(t_B) \psi_B^\dagger(t_B) \\ &= \psi_A S \psi_B^\dagger \\ &\equiv S[\psi_A, \psi_B^*] . \end{aligned} \quad (C.9)$$

Thus the S-matrix expression for the transition probability is equivalent to the Schroedinger theory expression, provided the time development operator $U(t_1, t_2)$ of Schroedinger theory exists, and provided the various limits defined in terms of it exist (in a weak sense). And this equivalence between the Schroedinger and S-matrix

formulas holds without restrictions on the times t_A and t_B of the preparation and examination.

Inspection of these equations shows that the need for the temporal evolution operator $U(t_1, t_2)$ of Schroedinger theory is eliminated in S-matrix theory by mapping the specifications A and B onto the "asymptotic" states ψ_A and ψ_B , instead of onto the states $\psi_A(t_A)$ and $\psi_B(t_B)$, that represent the preparation and result of examination at the times t_A and t_B of the preparation and examination. That is, the need for the temporal evolution operator of Schroedinger theory is eliminated by reorganizing the problem of measurement.

The question immediately arises whether this reorganization of the problem of measurement provides a bona fide solution of the problem of temporal development, or merely shifts this problem onto the already complex problem of measurement.

This question is examined in the following sections, where it is shown how this reorganization of the problem of measurement in fact resolves various logical difficulties, and renders it amenable to practical solution.

D. THEORY OF MEASUREMENT

Bohr said: "On the one hand, the definition of the state of a physical system, as ordinarily understood, claims the elimination of all external disturbances. But in that case, according to the quantum postulate, any observation will be impossible,... On the other hand, if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible, and there can be no question of causality in the ordinary sense of the word."

These words point to the crux of the problem of measurement: The quantum theoretical formulas are built on the degrees of freedom of the prepared and examined systems, and they do not describe the microscopic degrees of freedom of the preparing and examining systems. Consequently, the processes that produce the prepared system and that detect the examined system are not described by these formulas.

The functions ρ_A and ρ_B occurring in the quantum formulas represent the completed preparations and examinations. These formulas acquire physical meaning only to the extent that the process of preparation can be considered to have produced a system that can evolve as a dynamically independent system, and only to the extent that the process of examination can be considered to detect a system that could have evolved as a dynamically independent system. And both of these systems are in fact treated in the quantum formalism as dynamically independent entities: the interactions that produce the prepared system and that detect the examined system are not incorporated into the dynamics of either system. Thus within the framework of the quantum formalism the connection of the preparation and examination to

their representations σ_A and σ_B is not dynamical. Rather it is logical: σ_A and σ_B are the images in the mathematical formalism of the operational specifications A and B that are met by the preparation and examination.

In certain cases the processes of preparation and examination can be partially analyzed within the framework of the quantum formalism by shifting to a new level of description in which the original preparing and examining devices are included as parts of the new quantum system. Such a procedure allows the self-consistency of the theory to be studied. But it does not resolve the basic problem of measurement. It simply shifts to another level the basic problem, which is to determine at some level the mappings $A \rightarrow \sigma_A$ and $B \rightarrow \sigma_B$.

The mappings $A \rightarrow \sigma_A$ and $B \rightarrow \sigma_B$ translate one language into another. They translate operational specifications, formulated in a language that allows trained technicians to set up experiments and recognize results, into mathematical expressions, formulated in the language of the quantum formalism.

These mappings are, in practice, determined empirically: experimentalists calibrate their devices. They must calibrate their devices because there exists no quantitatively accurate theory of measuring devices. Thus the bridge between the operational language of the experimentalist and the mathematical language of the theorist is built in practice upon accumulated empirical evidence. The problem of measurement is thus shifted to the problem of empirical calibration.²

The simplest calibrations are those of devices that prepare and examine single particles. Accordingly, let A specify a manner of

preparation of a single particle of specified type. And let B specify both a manner of examination of a particle of the same type, and also a possible result of this examination. Then the transition probability formula (A.1) reduces to

$$\begin{aligned} \mathcal{P}(A; B) &= \int \frac{m}{p \cdot 0} \frac{d^3 p'}{(2\pi)^3} \frac{m}{p'' \cdot 0} \frac{d^3 p''}{(2\pi)^3} \cdot A(p''; p') \cdot B(p'; p'') \\ &\equiv \int_{p' p''} \rho_A(p''; p') \rho_B(p'; p'') \\ &\equiv \text{Tr } \rho_A \rho_B . \end{aligned} \quad (D.1)$$

Here it has been assumed that S reduces to unity when acting on a single-particle wave function. This assumption is essentially a definition, since the product of the one-particle wave function ψ_A with S can be defined to be the new ψ_A .

The introduction of (C.1b) into (D.1) gives

$$\mathcal{P}(A; B) = \int_{p' p''} \rho_A(p''; p') \rho_B(p'; p'') e^{iu(p''-p')} . \quad (D.2)$$

The following theorem can then be invoked:

Theorem 1D Suppose $\psi_A(p') \psi_B^*(p')$ is continuous, along with its first two derivatives, and vanishes for $|\vec{p}'| > R$ for some R . Let $p \equiv mv$ be any fixed mass-shell vector. Then

$$\lim_{\tau \rightarrow \infty} \left(\frac{2\pi i \tau}{m} \right)^{3/2} e^{i m \tau} \int_{p'} \psi_A(p') e^{-i \tau v \cdot p'} \psi_B(p') = \psi_A(p) \psi_B^*(p) . \quad (D.3)$$

(Proof in Appendix IA.)

Corollary Under the same conditions

$$\lim_{\tau \rightarrow \infty} \left(\frac{\tau^3}{m} \right)^{-1} \mathcal{P}(A; B^{v\tau}) = \rho_A(p; p) \cdot \rho_B(p; p). \quad (D.4)$$

This corollary says that the product $\rho_A(p; p) \cdot \rho_B(p; p)$ is determined by the transition probability $\mathcal{P}(A; B^u)$, for large displacements $u = v\tau$ in the direction $v = p/m$.

Formula (D.4) suggests the following classical asymptotic space-time interpretation: for sufficiently large τ the space-time region around $X = v\tau$ contains mainly those prepared particles that have momentum-energy near $p = mv$; $\rho_A(p; p)$ is the probability density for producing particles of momentum-energy $p = mv$ in a preparation that meets specifications A ; $\rho_B(p; p)$ is the efficiency for detecting such particles in an examination that meets specifications B ; the cubic fall-off with τ of the transition probability is due to the cubic growth with τ of the volume of space occupied by particles in a fixed momentum interval d^3p --as compared to the fixed volume of space in which detection effectively occurs.

Formula (D.4) gives information about $\rho_A(p''; p')$ and $\rho_B(p'; p'')$ only for $p' = p''$. A full calibration of the devices requires these functions to be determined also for $p' \neq p''$. To achieve this the information available from nonasymptotic measurements must be used. This information is conveniently organized by transcribing the quantum mechanical formulas into classical form. This transcription converts the asymptotic classical space-time interpretation suggested by (D.4) into a precise form valid for all times.

E. SPACE-TIME INTERPRETATION

A space-time interpretation of the quantum theoretical quantities can be obtained by transcribing the quantum theoretical formulas into classical form. This transcription converts the density functions ρ_A and ρ_B into functions that represent statistical ensembles of freely moving classical particles.

A statistical ensemble of freely moving classical particles can be represented by a density function $w(p, x)$ defined as follows: The probability $\mathcal{P}(\Delta^3 p, \Delta^3 x, t)$ that a particle of the ensemble with momentum in the range $\Delta^3 p$ lies in the spatial interval $\Delta^3 x$ at time $t = x^0$ is given by

$$\mathcal{P}(\Delta^3 p, \Delta^3 x, t) = \int_{\Delta^3 p} \frac{d^3 p}{(2\pi)^3} \int_{\Delta^3 x} d^3 x w(p, x). \quad (E.1)$$

Here x is a space-time four-vector with zeroth component $x^0 = t$, and p is a momentum-energy four-vector with zeroth component $p^0 = (\vec{p} \cdot \vec{p} + m^2)^{1/2}$. As before, $v \equiv p/m$.

This classical density function $w(p, x)$ enjoys the following properties:

$$(a) \quad w(p, x) \text{ is real.} \quad (E.2a)$$

$$(b) \quad \text{For any real } \tau \quad w(p, x) = w(p, x + v\tau). \quad (E.2b)$$

$$(c) \quad \text{For any real } \tau \text{ and real } t$$

$$\begin{aligned} \mathcal{P}(A; B) &= \int \frac{d^3 p}{(2\pi)^3} \int d^3 x w_A(p, x) w_B(p, x) \\ &= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \int d^4 x \delta(x \cdot v - \tau) w_A(p, x) w_B(p, x). \end{aligned} \quad (E.2c)$$

Property (a) follows from the physical meaning of $w(p, x)$. Property (b) follows from the fact that classical freely moving particles of momentum p move on space-time trajectories having direction $v \equiv p/m$. The first form of property (c) follows from the definitions of $w_A(p, x)$ and $w_B(p, x)$; $w_A(p, x)$ represents the statistical ensemble corresponding to the specifications A on the manner of preparation, and $w_B(p, x)$ represents the detection efficiency for particles of momentum p at point x in an examination that conforms to specifications B . The second (covariant) form of property (c) follows from the first form, together with property (b) and the fact that the Jacobian of the transformation from constant t to constant τ is (m/p^0) .

The quantum mechanical analogue of this classical $w(p, x)$ is³

$$w(p, x) = \int \alpha(Mv - \frac{1}{2}q; Mv + \frac{1}{2}q) e^{-iqx} \left(\frac{M}{m}\right)^{1/2} 2\pi\delta(q \cdot v) \frac{d^4q}{(2\pi)^4} \quad (E.5)$$

where $M = (m^2 - \frac{1}{4}q^2)^{1/2}$. This function enjoys the classical properties (a), (b), and (c) listed above. The reality of $w(p, x)$ follows from the representation of $\alpha(p'; p'')$ as a sum of terms of the pure case form $\psi^*(p')\psi(p'')$: this form entails that $\alpha(Mv - \frac{1}{2}q; Mv - \frac{1}{2}q) = \alpha^*(Mv - \frac{1}{2}q; Mv + \frac{1}{2}q)$. The invariance of $w(p, x)$ under $x \rightarrow x + v\tau$ follows from the fact that x occurs only in $q \cdot x$, together with the condition $q \cdot v = 0$ imposed by the factor $\delta(q \cdot v)$. Property (c) is derived in Appendix IB.

The quantum theoretical $w(p, x)$ has also the following desired properties:

$$(d) \quad w_{Au}(p, x + u) = w_A(p, x) \quad (E.2d)$$

and

$$(e) \quad \int d^4x \delta(x \cdot v - \tau) w_A(p, x) = n_A(p; p), \quad (E.2e)$$

and the similar ones with B in place of A. Property (d) follows from the insertion of (C.1a) into (E.3). Property (e) follows from the insertion of (E.3) into the left-hand side of (E.2e).

The quantum mechanical $w(p, x)$, unlike its classical counterpart, is not in all cases everywhere positive. The occurrence of negative probability densities in (p, x) space is characteristic of quantum theory. It is demanded by interference phenomena. However, if the classical statistical framework is expanded to accommodate negative probability densities in (p, x) space then there is an exact correspondence between the classical and quantum formulas. Thus in the calibration of the single-particle preparing and examining devices the preparations and examinations can be represented by ensembles of freely moving classical particles: All the quantum theoretical formulas used in the calibration are equivalent to ones occurring in the classical theory. Negative probability densities must in principle be allowed, but they can be shown⁴ to occur effectively only in cases where there is a rapid variation over phase space or spatial regions that are not large in comparison to $\hbar$ or $\hbar/mc$ respectively.

The calibration procedures can be considered procedures for determining the functions $w(p, x)$ (see Appendix IB).

The single-particle density functions $w_i(p_i, x_i)$ determined by these calibration procedures are used in the study of many-particle processes. In the case of scattering experiments the initial and final density functions are assumed to be products of the density

functions of the individual particles. To perform the calculations it is not necessary to convert the empirically determined forms $w_i(p_i, x_i)$ of these density functions into the corresponding quantum theoretical forms $\rho_i(p_i; p_i')$. Instead, the transition probability formula (A.1) can be cast into the classical form

$$P(A; B) = \int \prod_{j=1}^n \left[\frac{d^3 p_j}{(2\pi)^3} d^3 x_j w_j(p_j, x_j) \right] S(p, x) \quad (E.4a)$$

where $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$, and

$$S(p, x) = \int \prod_{j=1}^n \left[\frac{d^4 q_j}{(2\pi)^4} 2\pi \delta(q_j \cdot v_j) e^{-iq_j \cdot x_j} \left(\frac{M_j}{m_j} \right)^{\frac{1}{2}} \right] \\ \times S\left(\{M_j v_j \mp \frac{1}{2} q_j\}\right) S^*\left(\{M_j v_j \pm \frac{1}{2} q_j\}\right). \quad (E.4b)$$

The upper sign is for initial particles, and the lower sign if for final particles. Equation (E.4) is derived in Appendix IB.

Equation (E.4a) has precisely the form that arises in the classical treatment of scattering processes. There $S(p, x)$ represents the probability density for a set of freely moving final particles on trajectories specified by the final p_i and x_i if the initial particles move on trajectories specified by the initial p_i and x_i .

Because (E.4a) has precisely the form that occurs in the classical statistical treatment of scattering processes, the entire formalism is now cast into a classical form, based on the classical concept of statistical ensembles of freely moving classical particles. If the operational specifications A and B are also formulated in terms of the idealizations of classical physics, then all the elements of the theory of measurement are expressed in terms of classical

concepts--slightly generalized to allow for negative probability densities in (p, x) space.

A simple application of this classical formulation is the derivation of the cross section formula. Classically, the cross section can be defined by considering a target particle 1 at rest at the origin, and a projectile of momentum p_2 directed along the $\mu = 1$ axis. The time t at which the projectile reaches $x_1^1 = 0$ can be defined to be $t = 0$. Then the classically defined cross section for the production of a set of final particles of momenta $p_f (f \in F)$ in Δp_F is

$$\sigma(p_1, p_2; \Delta p_F) = \int_{\Delta p_F} \prod_{f \in F} \left[\frac{d^3 p_f}{(2\pi)^3} d^3 x_f \right] dx_2^2 dx_2^3 S(p, x) \quad (E.4)$$

where x_1 is fixed at zero, x_2^0 and x_2^1 are fixed at zero, p_1 is $(m_1, 0, 0, 0)$, and p_2 is $m(v_2^0, v_2^1, 0, 0)$.

Introduction of (E.4b) into (E.5) gives, after some simple algebra,

$$\sigma(p_1, p_2; \Delta p_F) = \frac{1}{v_2^1} \int_{\Delta p_F} \prod_{f \in F} \left(\frac{m_f}{p_f^0} \frac{d^3 p_f}{(2\pi)^3} \right)$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - \sum p_f) |S_r(p_1, p_2; p_F)|^2, \quad (E.6)$$

where v_2^1/v_2^0 is the velocity of the projectile in the rest frame of the target, p_F is the set of $p_f (f \in F)$, and $S_r(p_1, p_2; p_F)$ is defined by

$$S(p_1, p_2; p_F) = S_r(p_1, p_2; p_F) (2\pi)^4 \delta^4(p_1 + p_2 - \sum p_f). \quad (E.7)$$

The conservation-law delta function appearing in (E.7) is required by the law of conservation of energy-momentum. It will be discussed later from a technical point of view. The origin of (E.5) and the calculation of (E.6) are described in Appendix IB.

F. NONSCATTERING PROBLEMS AND THE NONRELATIVISTIC APPROXIMATION

The primary focus of S-matrix theory is on relativistic scattering problems. However, nonscattering problems and the non-relativistic approximation are also encompassed.

In scattering problems the prepared system at the time of preparation is a set of essentially independent particles. The general prepared system does not have this form. However, it will evolve eventually into a system of freely-moving particles. Examination of these particles by means of the already calibrated single-particle detecting devices will yield information about the asymptotic final state of the prepared system. Thus the representation of the prepared system by the density function $\rho'_A = S^\dagger \rho_A S$, will have direct physical significance: the transition probabilities $P(A; B) = \text{Tr } \rho'_A \rho_B$ can be compared directly to experiment. However, the information provided by this kind of measurement is probably not sufficient to determine ρ'_A itself: the microscopic many-particle structure of ρ'_A probably cannot be extracted from the macroscopic information provided by such measurements.

The principal practical measurements on complex systems are measurements of their properties as sources and sinks of massless particles. Thus the further discussion of the measurement problem for complex systems must be deferred until after the introduction of massless particles. Here it is only remarked that the logical structure is not altered to accommodate massless particle effects. The necessary changes are merely the technical modifications needed to accommodate the infinite numbers of massless particles that must be considered. A main consequence of the introduction of massless particles into the formalism will be the incorporation of the

classically described environmental fields into the quantum theoretical framework.

Many important nonscattering applications of quantum theory are to nonrelativistic problems. In the special case of two-particle processes the Schroedinger equation emerges automatically from the S-matrix formalism in the nonrelativistic approximation.' It is probable that the same thing happens in general. In any case, the emergence of Schroedinger theory in the nonrelativistic approximation is compatible with S-matrix ideas. For in the nonrelativistic approximation no particles are created or destroyed, and hence the asymptotic degrees of freedom can become relevant for all times. Moreover, the passage to the nonrelativistic limit infinitely expands the time scale. This contracts to infinitesimals the effectively finite time intervals in which the preparations and examinations take place, and tends also to make the asymptotic representations relevant for all times.

In cases where the Schroedinger equation does emerge from the S-matrix framework the temporal evolution derived from it will agree with that derived from the basic S-matrix formulas, provided the appropriate asymptotic limits exist. This was shown earlier. Thus nonrelativistic Schroedinger theory has a natural place within S-matrix theory as the nonrelativistic limit of the general relativistic theory. The preferred status of time characteristic of Schroedinger theory would naturally be expected to emerge from general relativistic quantum theory only in the nonrelativistic limit.

APPENDIX IA

PROOF OF THEOREM ID

Theorem ID is equivalent to the following theorem:

Theorem Suppose $g(p')$ is continuous, together with its first and second derivatives, and vanishes for $|\vec{p}| > R$ for some R . Let $p \equiv mv$ be any fixed mass-shell four-vector. Then

$$\lim_{\tau \rightarrow \infty} \left(\frac{2\pi i \tau}{m} \right)^{3/2} e^{im\tau} \int g(p') e^{-ip' \cdot v\tau} \frac{m}{p'^0} \frac{d^3 p'}{(2\pi)^3} = g(p) \quad (\text{IA.1})$$

Proof Transform to the variables corresponding to a frame in which $v = (1, 0, 0, 0)$. In terms of these variables one has

$$v \cdot p' = p'^0 = [m^2 + (\vec{p})^2]^{1/2} = m + f[(\vec{p})^2] \quad (\text{IA.2})$$

where

$$f[(\vec{p})^2] = \frac{(\vec{p})^2}{2m} + \dots > 0. \quad (\text{IA.3})$$

The introduction of the variable f in place of $(\vec{p})^2$, followed by an integration over angles, converts (IA.1) to

$$\frac{2}{-i\pi} (i\tau)^{3/2} \int_0^\infty \bar{g}(f) \sqrt{f} e^{-if\tau} df \rightarrow \bar{g}(0) \quad (\text{IA.4})$$

where $\bar{g}(0) = g(0)$, and $\bar{g}(f)$ and its first and second derivatives are continuous at $f > 0$. Since

$$\int_0^\infty e^{-if(\tau - i\epsilon)} \sqrt{f} df = \frac{\sqrt{\pi}}{2} \frac{1}{[i(\tau - i\epsilon)]^{3/2}} \quad (\text{IA.5})$$

and $\bar{g}(f)$ is continuous with compact support, the required result (IA.4) equivalent to

$$\lim_{\tau \rightarrow 0} \tau^{3/2} \lim_{\epsilon \rightarrow 0+} \int_0^{\infty} [\bar{g}(f) - \bar{g}(0)] e^{-if(\tau-i\epsilon)} \sqrt{f} df = 0. \quad (\text{IA.6})$$

Bounds on $(\bar{g}(f) - \bar{g}(0))$ and its first two derivatives can be obtained by writing

$$g(p) = g(r, \Omega) = g(0) + \vec{\nabla} g(0) \cdot \vec{r} + \int_0^r dr' \int_0^{r'} dr'' \frac{\partial^2 g}{\partial r'^2}(r'', \Omega) \quad (\text{IA.7})$$

where $\vec{r} = \vec{p}$, and $r = |\vec{p}|$. The integration over angles eliminates the linear term and gives

$$\frac{\bar{g}(f)}{\sqrt{1 + \frac{f}{2m}}} - \bar{g}(0) = \int \frac{d\Omega}{4\pi} \int_0^{r(f)} dr' \int_0^{r'} dr'' \frac{\partial^2 g}{\partial r'^2}(r'', \Omega). \quad (\text{IA.8})$$

Since the second derivative of $g(p)$ is bounded,

$$\left| \frac{\partial^2 g}{\partial r'^2} \right| \leq c, \quad (\text{IA.9})$$

one has

$$\left| \frac{\bar{g}(f)}{\sqrt{1 + \frac{f}{2m}}} - \bar{g}(0) \right| \leq \frac{1}{2} c r^2. \quad (\text{IA.10})$$

Letting F be such that

$$\bar{g}(f) = 0 \quad \text{for } f \geq F,$$

and defining $\bar{m} = F + m$, so that $\partial r^2 / \partial f = 2f + 2m \leq 2\bar{m}$ for $f \leq F$, one obtains, for $f \geq 0$,

$$|\bar{g}(f) - \bar{g}(0)| \leq f \bar{c} m^2 / m \quad (\text{IA.11})$$

Equation (IA.8) also yields, for $f \geq 0$,

$$|\bar{g}'(f)| \equiv \left| \frac{d}{df} \bar{g}(f) \right| \leq \bar{c} m^2 / m \quad (\text{IA.12})$$

and, for $f > 0$,

$$|\bar{g}''(f)| \leq \left(\frac{1}{f} + \frac{2}{m} \right) (\bar{c} m^2 / m) \quad (\text{IA.13})$$

An integration by parts on the integral in (IA.6) gives

$$\begin{aligned} & \int_0^\infty [\bar{g}(f) - \bar{g}(0)] \sqrt{f} e^{-if(\tau - i\epsilon)} df \\ &= \frac{-1}{-i(\tau - i\epsilon)} \int_0^\infty e^{-if(\tau - i\epsilon)} \frac{d}{df} [\bar{g}(f) - \bar{g}(0)] \sqrt{f} df \\ &= \frac{1}{i(\tau - i\epsilon)} \int_0^\infty e^{-if\tau} h_\epsilon(f) df \end{aligned} \quad (\text{IA.14})$$

where

$$h_\epsilon(f) = e^{-\epsilon f} \frac{d}{df} ([\bar{g}(f) - \bar{g}(0)] \sqrt{f}) . \quad (\text{IA.15})$$

However,

$$\begin{aligned} & \int_0^\infty e^{-if\tau} h_\epsilon(f) df \\ &= \int_0^{\pi/\tau} e^{-if\tau} h_\epsilon(f) df - \int_0^\infty e^{-if\tau} h_\epsilon(f + \pi/\tau) df \end{aligned}$$

[Equation (IA.16) continued]

[Equation (IA.16) continued]

$$= \int_0^{\infty} e^{-if\tau} \frac{1}{2} [h_{\epsilon}(f) - h_{\epsilon}(f + \pi/\tau)] df + \frac{1}{2} \int_0^{\pi/\tau} e^{-if\tau} h_{\epsilon}(f) df. \quad (\text{IA.16})$$

The last term in (IA.16) has, by virtue of (IA.11) and (IA.12), the bound $\frac{1}{2} \bar{c} \bar{m}(\pi/\tau)^{3/2}$. Thus this contribution, inserted into (IA.14), satisfies (IA.6).

The first term in (IA.16) can be written as a sum of two terms. The first is

$$\frac{1}{2} \int_0^F e^{-if\tau} [h_{\epsilon}(f) - h_{\epsilon}(f + \frac{\pi}{\tau})] df$$

$$< \frac{1}{2} \left(\frac{\pi}{\tau}\right) \int_0^F |\max h'_{\epsilon}(f)| df \quad (\text{IA.17})$$

where $|\max h'_{\epsilon}(f)|$ is the maximum of the absolute value of $dh_{\epsilon}(f)/df$ for $f \geq f$. The bounds (IA.11), (IA.12), and (IA.13) ensure that the integral on the right-hand side of (IA.17) has a finite bound that is independent of ϵ . Thus this contribution, inserted into (IA.14), also satisfies (IA.6).

The remaining part of (IA.16) is

$$\begin{aligned}
& \frac{1}{2} \int_F^{\infty} e^{-if\tau} [h_{\epsilon}(f) - h_{\epsilon}(f + \frac{\pi}{\tau})] df \\
&= \frac{-\bar{g}(0)}{4} \int_F^{\infty} e^{-if\tau} \left[\frac{e^{-\epsilon f}}{\sqrt{f}} - \frac{e^{-\epsilon(f+\pi/\tau)}}{\sqrt{f + \pi/\tau}} \right] df \\
&= \frac{-\bar{g}(0)}{4} \int_F^{\infty} e^{-if(\tau-i\epsilon)} \left[\frac{1}{\sqrt{f}} - \frac{1}{\sqrt{f + \pi/\tau}} \right] \\
&\quad + \frac{(-\bar{g}(0))}{4} \int_F^{\infty} e^{-if(\tau-i\epsilon)} [1 - e^{-\epsilon\pi/\tau}] \frac{1}{\sqrt{f + \pi/\tau}} . \quad (IA.18)
\end{aligned}$$

The first term on the right-hand side of (IA.18) is bounded in magnitude by

$$\frac{|\bar{g}(0)|}{4} \left(\frac{\pi}{\tau}\right) \int_F^{\infty} \left(\frac{d}{df} \frac{1}{\sqrt{f}}\right) df = \frac{|\bar{g}(0)|}{4} \left(\frac{\pi}{\tau}\right) F^{-1/2} . \quad (IA.19)$$

Thus this contribution, inserted into (IA.14) also satisfies (IA.1).

The second term on the right-hand side of (IA.18) can be written

$$\begin{aligned}
 & \frac{-g(0)}{h} [1 - e^{-\epsilon\pi/\tau}] e^{-i(\pi/\tau)(\tau-i\epsilon)} \int_{F+\pi/\tau}^{\infty} e^{-if(\tau-i\epsilon)} \frac{1}{\sqrt{f}} df \\
 &= \frac{g(0)}{h} [e^{\epsilon\pi/\tau} - 1] \\
 & \times \left[\int_0^{\infty} e^{-if(\tau-i\epsilon)} \frac{1}{\sqrt{f}} df - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)} \frac{df}{\sqrt{f}} \right] \\
 &= \frac{g(0)}{h} [e^{\epsilon\pi/\tau} - 1] \left[\sqrt{\frac{-}{-i(\tau-i\epsilon)}} - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)} \frac{df}{\sqrt{f}} \right].
 \end{aligned}$$

This term vanishes when we take the limit $\epsilon \rightarrow 0$ in (IA.6). Thus all the contributions satisfy (IA.6).

APPENDIX IB

CORRESPONDENCE TO CLASSICAL FORMULAS

1. Derivation of (E.2c).

A useful identity is

$$\delta(a - \frac{1}{2} b) \delta(a + \frac{1}{2} b) = \delta(a) \delta(b) . \quad (IB.1)$$

Thus the introduction of the variables

$$Q = \frac{1}{2}(p' + p'') \quad (IB.2a)$$

and

$$q = p' - p'' \quad (IB.2b)$$

into (D.1) gives [see (B.2)]

$$\mathcal{P}(A; B) = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 Q}{(2\pi)^4} 2\pi \delta[(Q + \frac{1}{2} q)^2 - m^2] 2\pi \theta(Q^0 + \frac{1}{2} q^0)$$

$$\times 2\pi \delta[(Q - \frac{1}{2} q)^2 - m^2] 2\pi \theta(Q^0 - \frac{1}{2} q^0)$$

$$\times \rho_A(Q - \frac{1}{2} q; Q + \frac{1}{2} q) \rho_B(Q + \frac{1}{2} q; Q - \frac{1}{2} q)$$

$$= \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 Q}{(2\pi)^4} 2\pi \delta(Q^2 + \frac{1}{4} q^2 - m^2)$$

$$\times 2\pi \delta(2Q \cdot q) (2m)^2 \theta(Q^0 + \frac{1}{2} q^0) \theta(Q^0 - \frac{1}{2} q^0)$$

$$\times \rho_A(Q - \frac{1}{2} q; Q + \frac{1}{2} q) \rho_B(Q + \frac{1}{2} q; Q - \frac{1}{2} q)$$

Equation (IB.5) Continued.

$$\begin{aligned}
 &= \int \frac{d^4(Mv)}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \delta(M^2 - (m^2 - \frac{1}{4} q^2)) \\
 X &= \delta(Mv \cdot q) (m^2 - \frac{1}{4} q^2) \delta(q^0 + \frac{1}{2} q^3) \delta(q^0 - \frac{1}{2} q^3) \\
 X &= \delta_A(Mv - \frac{1}{2} q; Mv + \frac{1}{2} q) \delta_B(Mv + \frac{1}{2} q; Mv - \frac{1}{2} q) \\
 &= \int \frac{M}{m} \frac{m}{p^0} \frac{d^4(mv)}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \delta(v \cdot q) \\
 X &= \delta_A(Mv - \frac{1}{2} q; Mv + \frac{1}{2} q) \delta_B(Mv + \frac{1}{2} q; Mv - \frac{1}{2} q) \\
 &= \int \frac{m}{p^0} \frac{d^4 p}{(2\pi)^4} \frac{d^4 q'}{(2\pi)^4} \delta(v \cdot q') \frac{d^4 q''}{(2\pi)^4} \delta(v \cdot q'') \\
 X &= \int e^{-i(q' \cdot q'')x} d^4 x \delta(v \cdot x) \frac{M}{m} \\
 X &= \delta_A(Mv - \frac{1}{2} q'; Mv + \frac{1}{2} q') \delta_B(Mv + \frac{1}{2} q''; Mv - \frac{1}{2} q'') \\
 &= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \int d^4 x \delta(v \cdot x) v_A(p, x) w_B(p, x) \\
 &= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 x}{v^0} w_A(p, x) w_B(p, x) \Big|_{x^0 = \vec{v} \cdot \vec{x} / v^0} \\
 &= \int \frac{d^3 p}{(2\pi)^3} \int d^3 x w_A(p, x) w_B(p, x) \Big|_{x^0 = t} \quad (IB.5)
 \end{aligned}$$

Equation (E.2b) is used to obtain the last line.

2. Derivation of (E.4)

Equations (A.1) and (B.1') give

$$\begin{aligned}
 \mathcal{P}(A; B) &= \int \prod_{j=1}^n \left[\frac{d^4 p_j}{(2\pi)^4} \delta^4(p_j^{\text{in}} - p_j^{\text{out}}) \theta(p_j^0) m_j \right. \\
 &\quad \left. \times \frac{d^4 p_j^{\text{in}}}{(2\pi)^4} \delta^4(p_j^{\text{in}} - p_j^{\text{out}}) \theta(p_j^0) m_j \right] \\
 &\times \prod_{i \in I} c_i(p_i^{\text{in}}; p_i^{\text{out}}) \prod_{f \in F} c_f(p_f^{\text{in}}; p_f^{\text{out}}) S(p^+) S^*(p^-) \quad (\text{IB.4})
 \end{aligned}$$

where I and F are the index sets for the initial and final particles.

Introduction of (IB.2) gives, as in (IB.3),

$$\mathcal{P}(A; B) = \int \prod_{j=1}^n \left[\frac{d^3 p_j}{(2\pi)^3} d^4 x_j w_j(p_j, x_j) \right] S(p, x),$$

where $w_j(p_j, x_j)$ is defined by (E.3) and $S(p, x)$ is defined by the analogous (E.4b).

3. Derivation of (E.6)

Equation (E.4) can be written

$$\begin{aligned}
 \sigma(p_1, p_2; \Delta p_f) &= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} d^4 x_f \delta(x_f \cdot v_f) \right] \\
 &\times dx_2^2 dx_2^3 S(p, x). \quad (\text{IB.5})
 \end{aligned}$$

The introduction of (E.4b) gives

$$\begin{aligned}
& \sigma(p_1, p_2; \Delta p_F) = \\
& = \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} d^4 x_f \delta(x_f \cdot v_f) \frac{d^4 q_f}{(2\pi)^3} \delta(q_f \cdot v_f) \right. \\
& \times \left. e^{-i q_f \cdot x_f \left(\frac{M_f}{m_f} \right)^{\frac{1}{2}}} \right] \prod_{i \in I} \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] \\
& \times S(M_j v_j \pm \frac{1}{2} q_j) S^*(M_j v_j \pm \frac{1}{2} q_j) dx_2^2 dx_2^3 \\
& = \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S(M_1 v_1 - \frac{1}{2} q_1; p_F) S^*(M_1 v_1 + \frac{1}{2} q_1; p_F) \\
& \times \prod_{i \in I} \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] dx_2^2 dx_2^3 \\
& = \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S_r(M_1 v_1 - \frac{1}{2} q_1; p_F) S_r^*(M_1 v_1 + \frac{1}{2} q_1; p_F) \\
& \times (2\pi)^4 \delta^4(M_1 v_1 + M_2 v_2 - \sum p_f) (2\pi)^4 \delta^4(q_1 + q_2) \\
& \times \prod_{i=1}^2 \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] dx_2^2 dx_2^3
\end{aligned}$$

Equation (B.6) Continued

$$\begin{aligned}
&= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S_r(M_1 v_i - \frac{1}{2} q_i; p_F) S_r^*(M_1 v_i + \frac{1}{2} q_i; p_F) \\
&\quad \times (2\pi)^4 \delta^4(M_1 v_1 + M_2 v_2 - \sum p_f) 2\pi \delta(q_2^0) \\
&\quad \times \frac{d^4 q_2}{(2\pi)^3} \delta(q_2^1 v_2^1) e^{-i q_2 \cdot x_2} \left(\frac{M_1}{m_1} \frac{M_2}{m_2} \right)^{\frac{1}{2}} dx_2^2 dx_2^3 \\
&= \frac{1}{v_2} \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{dp_f}{(2\pi)^3} \right] S_r(p) S_r^*(p) (2\pi)^4 \delta^4(\sum \pm p_j) .
\end{aligned}
\tag{IB.6}$$

This is the result (E.6).

The derivation of the cross section formula given above is straightforward, in the sense that there is no argumentation regarding shapes of wave packets, or any equating of delta functions $\delta(0)$ to volumes of space or time.

4. Origin of (E.5)

In classical physics each initial system evolves into a uniquely determined final system. Thus for any fixed pair of trajectories of the two initial particles the probability for the final system to lie in Δp_F is either zero or one. Hence if one fixes the target particle at the origin and integrates over all spatial positions of the projectile trajectory (i.e., over x_2^2 and x_2^3), weighting each projectile trajectory by the probability that the final systems will lie in Δp_F , then one obtains the cross sectional area ΔA that corresponds to final systems lying in Δp_F .

In the semi-classical approach the deterministic structure of the classical physics is replaced by a statistical structure: the probability for scattering into Δp_F is allowed to be different from zero or one. In this case the integral described above gives the "effective" cross-sectional area for producing a final system in Δp_F . In other words, it gives the area ΔA such $\Delta A/A$ is the probability for a particle in a uniform beam of cross-sectional area A to produce a final system lying in Δp_F --provided, of course, that trajectories lying outside the beam have a negligible change of producing a final system in Δp_F .

5. Measurements at Finite Times

Equation (D.4) relates the values of $\rho(p'; p'')$ at $p' = p'' = p$ to asymptotic measurements. Information about $\rho(p'; p'')$ for $p' \neq p''$ is contained in measurements performed at finite times. For example, (D.2) gives

$$\begin{aligned} & \int d^3u \, \mathcal{P}(A, B^u) e^{-iu\Delta} \\ &= \int_{p' p''} \rho_A(p''; p') \rho_B(p'; p'') (2\pi)^3 \delta^3(p'' - p' - \Delta) \\ &= \int_p \frac{m}{p^0} \frac{m}{(p^0 + \Delta^0)} \frac{d^3p}{(2\pi)^3} \rho_A(p + \Delta; p) \rho_B(p; p + \Delta) . \quad (1B.7) \end{aligned}$$

This equation shows that measurements at finite times give, in principle at least, information about the nondiagonal parts of $\rho(p', p'')$. The problem of calibration is the problem of reconstructing $\rho_A(p'', p')$ and $\rho_B(p'; p'')$ from experimental information.

6. Calibration of Measuring Devices

Consider N_p preparing devices and N_e examining devices. Suppose continuity properties entail that each device can be adequately represented by N parameters. Then the equations

$$\mathcal{P}(A_i, B_j) = \text{Tr } S \rho_{A_i} S^\dagger \rho_{B_j},$$

with $1 \leq i \leq N_p$ and $1 \leq j \leq N_e$ are $N_p \times N_e$ equations on $(N_p + N_e)N$ unknowns. So if one takes $N_p = N_e > 2N$ then the number of unknowns is less than the number of equations. This fact provides the essential leverage needed for empirical calibration.

This procedure depends, however, on finding a representation in which ρ is smooth or, more generally, can be represented, to a useful degree of approximation, by conjectured forms involving only a finite number of parameters. These conjectured forms are based in practice on classical interpretations of the classical descriptions A and B .

All real quantum measuring devices give information about both space-time and energy-momentum. That is, they have characteristic sensitivities that depend both on the location of the trajectory of the measured particle, and on its momentum-energy. The human eye is a good example: its response depends not only on the location of the photon but also on its momentum-energy. The representation $w(p, x)$ is useful because it displays the simultaneous dependence on both these quantities. The equivalent function $\rho(p'; p'')$ buries the space-time information in its phase.

A real measuring device that provides a good determination of $w(p,x)$, for charged particles in an appropriate energy range, is a pair of thin, well-separated spark chambers with accurate timers. By determining the positions and times of the two sparks one obtains a fairly accurate determination of the trajectory of the particle. By using such a device to examine the distribution of the trajectories of the particles produced in an ensemble of preparations that meet a set of specifications A , one gets a determination of $w_A(p,x)$. In practice one must find a parametrization of $w_A(p,x)$ that allows a good fit to the empirical data, and that also fulfills the theoretical requirement that $w_A(p,x)$ be equivalent to a quantum theoretical $\rho_A(p'; p'')$.

For neutral particles of lower energy the determination of $w_A(p,x)$ is not so easy. However, the human eye is an example of a device that determines with fair accuracy the $w_A(p,x)$ of a preparation A of optical photons. These examining devices are essentially macroscopic in character. And what they determine are statistical weights associated with entire space-time trajectories, rather than with isolated space-time points. This assigning of statistical weights to entire trajectories, rather than isolated space-time points is characteristic of S-matrix theory.

If $w_A(p,x)$ oscillates rapidly over the characteristic intervals of the best available detecting devices, then the simple calibration procedures described above break down. In any case the fine-scale calibration of the measuring devices depends ultimately on testing them in a variety of situations involving other preparing and detecting devices, and exploiting the essential leverage mentioned

earlier, rather than on any direct transcription of the specifications A and B onto their quantum theoretical images.

7. Normalization

It is assumed that the w_A can all be normalized so that if the preparation definitely produces precisely one particle, then

$$\int \frac{d^3p}{(2\pi)^3} d^4x w_A(p,x) = 1 . \tag{IB.8}$$

8. Alternative Form of $w(p,x)$

Let $p = mv$ be a mass-shell vector, and define

$$\psi_p(p') = v(p') [p \cdot p' / m]^{5/4} . \tag{IB.9}$$

The corresponding space-time wave function is

$$\tilde{\psi}_p(x) = \int \frac{m}{p \cdot 0} \frac{d^3p'}{(2\pi)^3} \psi_p(p') e^{-ip' \cdot x} . \tag{IB.10}$$

Then

$$w(p,x) = \int \tilde{\psi}_p^*(x - \frac{1}{2} \Delta) \tilde{\psi}_p(x + \frac{1}{2} \Delta) \delta(\Delta \cdot v) d^4\Delta . \tag{IB.11}$$

This form shows, for example, that $w(p,x)$ will not be large unless both $|\tilde{\psi}_p(x + \frac{1}{2} \Delta)|$ and $|\tilde{\psi}_p(x - \frac{1}{2} \Delta)|$ are large, for some Δ satisfying $\Delta \cdot v = 0$.

To prove (IB.11) introduce, as before, the variables

$$Q = (p' + p'')/2 \tag{IB.12a}$$

and

$$q = p' - p'', \quad (\text{IB.12b})$$

and write

$$\begin{aligned} & \int \tilde{\psi}_p^*(x - \frac{1}{2} \Delta) \tilde{\psi}_p(x + \frac{1}{2} \Delta) \delta(\Delta \cdot v) d^4 \Delta \\ &= \int \psi_p^*(p'') e^{+ip''(x - \frac{1}{2} \Delta)} \psi_p(p') e^{-ip'(x + \frac{1}{2} \Delta)} \end{aligned}$$

$$\times \int d^4 p'' \delta((p'')^2 - m^2) \theta(p''^0)_{2m} d^4 p'' / (2\pi)^4$$

$$\times \int d^4 p' \delta((p')^2 - m^2) \theta(p'^0)_{2m} d^4 p' / (2\pi)^4$$

$$\times \delta(\Delta \cdot v) d^4 \Delta$$

$$= \int \psi_p^*(Q - \frac{1}{2} q) \psi_p(Q + \frac{1}{2} q) e^{-iqx}$$

$$\times \int d^4 Q \delta(Q^2 - M^2) e^{-iQ \cdot \Delta} (2\pi) \delta(2Q \cdot q) \theta(Q^0 + \frac{1}{2} q^0) \theta(Q^0 - \frac{1}{2} q^0)$$

$$\times \int d^4 Q \delta(Q^2 - M^2) \theta(Q^0 + \frac{1}{2} q^0) \theta(Q^0 - \frac{1}{2} q^0)$$

$$\times \int d^4 Q \delta(Q^2 - M^2) \theta(Q^0 + \frac{1}{2} q^0) \theta(Q^0 - \frac{1}{2} q^0)$$

$$= \int \psi_p^*(Mv - \frac{1}{2} q) \psi_p(Mv + \frac{1}{2} q) e^{-iqx}$$

$$\times \int d^4 q \delta(v \cdot q) \frac{d^4 q}{(2\pi)^4}$$

Equation (IB.13) continued

Equation (IB.13) continued

$$= \int \psi^* \left(Mv - \frac{1}{2} q \right) \psi \left(Mv + \frac{1}{2} q \right) e^{-iqx}$$

$$\times \left(\frac{p \cdot \left(Mv - \frac{1}{2} q \right)}{m^2} \right)^{5/4} \left(\frac{p \cdot \left(Mv + \frac{1}{2} q \right)}{m^2} \right)^{5/4} \left(\frac{m}{M} \right)^2$$

$$\times 2\pi \delta(v \cdot q) \frac{d^4 q}{(2\pi)^4}$$

$$= w(p, x) \quad .$$

(IB.13)

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