JAN 21 1962

BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, Long Island
New York

BNL 6420
IN Formal

Accelerator Department
Cosmotron Division

Internal Report

EHH-1
May 1, 1962

Facsimile Price \$ 1.60

Microfilm Price \$.80

Available from the
Office of Technical Services
Department of Commerce
Washington 25, D. C.

SYNCHROTRON OSCILLATIONS

E. H. Hochman and M. Q. Barton

MASTER

Some numerical calculations have been done which permit the deduction of many properties of the synchrotron oscillation problem. For the presentation of the numerical results, it is useful to review the formulation framework to which the results are to be fitted. These are worked out for an arbitrary circular accelerator. The algebra would have to be modified slightly to include straight sections but the numerical results are still useful.

A particle is described by the parameters, radius $R + x$, momentum $p_s + \Delta p$, energy $E_s + \Delta E$, velocity $v_s + \Delta v$ and angular velocity $\omega_s + \frac{\partial \omega}{\partial E} \Delta E$. We define each parameter by its deviation from the conditions of the synchronous particle defined by the kinematic identities

$$p_s = \frac{H_s e R}{c}, \quad p_s = \frac{1}{c} \left[E_s^2 - m^2 c^4 \right]^{\frac{1}{2}}, \quad p_s = \frac{v_s}{c^2} E_s$$

$$R\omega_s = v_s$$

$$eV \sin \psi_0 = \frac{e}{c} 2\pi R^2 \frac{\partial H_s}{\partial t}$$

where V is the r.f. gap voltage and ψ_0 the r.f. phase at which the synchronous particle crosses the accelerating gap.

We define two useful parameters

$$\frac{x}{R} = \alpha \frac{\Delta p}{p}, \quad \alpha \text{ is called momentum compaction factor}$$

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and

$$\eta = \alpha - \frac{E_0^2}{E_s^2}$$

Now it can be shown that

$$\frac{\partial \omega}{\partial E} = \frac{-\eta}{R p_s}$$

Using these parameters, and assuming the r.f. is running on the h-th harmonic of ω_0 the equations of motion for oscillations in phase as measured with respect to the r.f. phase are given by the Lagrangian

$$L = \frac{1}{2\omega_0 \left[\frac{\partial \omega}{\partial E} \right] h} \dot{\psi}^2 + \frac{eV}{2\pi} [\cos \psi + \psi \sin \psi_0]$$

The momentum canonical to ψ is

$$p_\psi = \frac{\dot{\psi}}{\omega_0 \left[\frac{\partial \omega}{\partial E} \right] h}$$

If p_ψ is plotted versus ψ , we obtain

This figure indicates that $\oint p_\psi d\psi$ represents the area of the "bucket" in phase space. p_ψ vs. ψ is symmetrical about the ψ -axis, so that

$$\oint p_\psi d\psi = 2 \int_{\psi_1}^{\psi_2} p_\psi d\psi$$

By the Helmholtz Circulation Theorem

$$\oint p_\psi d\psi = \text{constant in phase space throughout the motion.}$$

Liouville's theorem states, "the density of points in phase space in the neighborhood of any particle does not change with time," if the assumption is made that the density of particles in the bucket is uniform then Liouville's theorem and Helmholtz's theorem are equivalent, i.e. $\oint p_\psi d\psi$ represents the number of particles within a bucket, and remains constant in time.

$$\oint p_{\psi} d\psi = \oint \frac{\dot{\psi}}{\omega_0 \frac{\partial \omega}{\partial E} h} d\psi = \frac{2}{\left[-\omega_0 \frac{\partial \omega}{\partial E}\right]^{\frac{1}{2}}} \left[\frac{eV_0}{\pi}\right]^{\frac{1}{2}} \frac{1}{h^{\frac{1}{2}}} \int_{\psi_1}^{\psi_2} [-\cos \psi + \cos \psi_1 - (\psi - \psi_1) \sin \psi_0]^{\frac{1}{2}} d\psi$$

This integral represents the number of particles in a bucket.

The numerical calculations that are tabulated consist of plots of $\psi_{\max}$ versus ψ for various sized buckets centered at ψ_0 . The values of ψ_0 used range from 105° to 175° . The Cosmotron operates at $\psi_0 \approx 150^\circ$. The $\oint p_{\psi} d\psi$ or the number of particles in a bucket are plotted for the cases described above.

Reference to the first set of plots indicates the maximum phase stable bucket that may be used ($\psi_{\max}$ can be converted to the maximum radial excursion from the central orbit)*. Having selected the proper phase stable range, one can then determine the number of particles within this range by referring to the second set of plots.

The plots also indicate the dependence of $\dot{\psi}_{\max}$ and $\oint p_{\psi} d\psi$ on the harmonic number h . The quantities ω_0 , $\frac{\partial \omega}{\partial E}$ refer to the particle.

We will now illustrate the use of these plots with an example.

For the Cosmotron

$V = 2000$ volts

$n = .7$

$\omega_i = 2\pi f_i = 1.13 \times 10^6$ radians/sec.

$R = 30$ feet

$T_i = 3.60$ Mev

$E_0 = 938$ Mev

$L = 10$ feet

The subscript i refers to injection values.

$$\alpha = \frac{1}{1-n} \cdot \frac{1}{1 + \frac{4L}{2\pi R}} = 2.75$$

* The condition for uncoupled synchronous oscillations is

$$\frac{x}{r_0} - \frac{\alpha \dot{\psi}}{\eta h \omega_0} = 0 \quad \text{or} \quad \dot{\psi} = \frac{\eta \omega_0 h}{r_0 \alpha} x$$

so that

$$\dot{\psi}_{\max} = \frac{\eta \omega_0 h}{r_0 \alpha} x_{\max}$$

From this data, we can evaluate our dependent quantities more explicitly,
i.e.

$$\frac{\dot{\psi}_{\max}}{\left[-h \frac{\partial \omega}{\partial E} \omega_0 \frac{eV}{\pi}\right]^{\frac{1}{2}}} = .089 x_{\max} h^{\frac{1}{2}}$$

$$\frac{\oint p_{\psi} d\psi}{\left[-\omega_0 \frac{\partial \omega}{\partial E}\right]^{\frac{1}{2}} \left[\frac{eV}{\pi h}\right]^{\frac{1}{2}}} = \frac{h^{\frac{1}{2}} \oint p_{\psi} d\psi}{57.3 \times 10^{-3}}$$

If we consider the r.f. to correspond to the frequency of the orbiting proton, with a prescribed $x_{\max} = 8$ inches, then

$$\frac{\dot{\psi}_{\max}}{\left[-h \frac{\partial \omega}{\partial E} \omega_0 \frac{eV}{\pi}\right]^{\frac{1}{2}}} = (.089) (8) = .712$$

Referring to the graph of $\psi_0 = 150^\circ$ we find that .712 corresponds to $(\psi_2 - \psi_0) = 59^\circ$. Entering the second set of graphs (i.e. $\oint p_{\psi} d\psi$ vs. $[\psi - \psi_0]$), at $(\psi_2 - \psi_0) = 59^\circ$ we find

$$\oint p_{\psi} d\psi = 1.34k, \text{ where } k = 57.3 \times 10^{-3}$$

We now consider the r.f. to be exactly twice the frequency of the orbiting proton, or $h = 2$.

The upper limit of

$$\frac{\dot{\psi}_{\max}}{\left[-h \frac{\partial \omega}{\partial E} \omega_0 \frac{eV}{\pi}\right]^{\frac{1}{2}}}$$

corresponds to the value at $\dot{\psi}_{\max}$ for the maximum phase stable bucket, with $(\psi_2 - \psi_0) = 69^\circ$.

Then

$$(.080) x_{\max} (2)^{\frac{1}{2}} = .83$$

$$x_{\max} = 6.6 \text{ inches}$$

For $(\psi_2 - \psi_0) = 69^\circ$

$$\oint p_{\psi} d\psi = \frac{1.89k}{(2)^{\frac{1}{2}}} = 1.34k$$

We can summarize as results by stating that while we were able to capture approximately the same number of particles for the r.f. at $h = 1$ and $h = 2$, we did achieve a decrease in the radial excursion from the central orbit from 8 inches to 6.6 inches. This occurred due to the larger bucket width obtained by having $h = 2$.

Extending this result to higher harmonics, we would be able to contain a larger bucket (i.e. one with maximum phase stable limits) while in comparison to $h = 1$ we would only have a smaller bucket i.e. the radial excursion of the largest bucket would extend beyond the chamber.

In general, the number of particles contained in phase stable buckets varies as $\frac{1}{h^2}$. However, since we now have h buckets orbiting the machine, the beam current would then be proportional to $\frac{h}{h^2}$ or $h^{\frac{1}{2}}$.

E. H. Hochman
E. H. Hochman

M. Q. Barton
M. Q. Barton

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG. 9

FIG. 10

FIG. 11

FIG. 12

FIG. 13

FIG. 14

FIG. 15

FIG. 16

FIG. 17

FIG. 18