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PROPAGATION OF AN ELECTROMAGNETIC
WAVE IN A TIME-VARYING MEDIUM,
PART I, GENERAL THEORY

by

K S Kunz

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PROPAGATION OF AN ELECTROMAGNETIC WAVE IN A
TIME-VARYING MEDIUM, PART I, GENERAL THEORY

by

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October 1964

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PROPAGATION OF AN ELECTROMAGNETIC WAVE IN A TIME-VARYING MEDIUM, PART I, GENERAL THEORY

1. Introduction

The purpose of this report is to establish a firm mathematical basis for the treatment of electromagnetic waves in time-varying media. The treatment will appear quite complicated since we permit the medium to be conductive, non-homogeneous, and time-varying. We do, however, restrict the treatment to an isotropic medium with linear relations between $\vec{D}$ and $\vec{E}$, $\vec{B}$ and $\vec{H}$, and $\vec{J}$ and $\vec{E}$. The ratios between these field quantities define the macroscopic properties $\epsilon(\vec{r},t)$, $\mu(\vec{r},t)$, and $\sigma(\vec{r},t)$ of the medium.

The treatment is phenomenological in that the medium is assumed to be specified fully by giving the above macroscopic properties. The question of how the assumed variations in ϵ , μ , and σ might arise in an actual problem is not considered in this report. In other words, it treats the mathematical problem of propagation in a general time-varying medium without regard for whether the given medium would be realizable in a practical situation.

In the latter sections of this report we restrict our treatment to homogeneous media with time-varying parameters ϵ , μ , and σ . Moreover, we illustrate in principle how the solution may be made to satisfy simple boundary conditions.

2. Wave Equations for the Field Vectors

In a general non-dispersive isotropic medium the dielectric constant ϵ , permeability μ , and conductivity σ are arbitrary functions of the position vector $\vec{r}$ and the time t . The problem here considered is to obtain the differential equations satisfied by the field vectors in a source-free region, that is, one in which the current density $\vec{J}$ and the charge density ρ vanish everywhere.

We begin with Maxwell's equations in the MKS system of units

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} , \quad (1)$$

$$\nabla \cdot \vec{D} = 0 , \quad (2)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \sigma \vec{E} , \quad (3)$$

$$\nabla \cdot \vec{B} = 0 , \quad (4)$$

with the constitutive relations

$$\vec{D} = \epsilon \vec{E} , \quad (5)$$

$$\vec{B} = \mu \vec{H} . \quad (6)$$

From (1), using (6), and (3), using (5), we have

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} - \frac{\partial \mu}{\partial t} \vec{H} , \quad (7)$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \left(\sigma + \frac{\partial \epsilon}{\partial t} \right) \vec{E} . \quad (8)$$

Dividing (7) by μ and taking the curl of both sides, we have

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) &= - \frac{\partial}{\partial t} \nabla \times \vec{H} - \nabla \times \left(\frac{\partial \ln \mu}{\partial t} \vec{H} \right) \\ &= - \left[\frac{\partial \ln \mu}{\partial t} + \frac{\partial}{\partial t} \right] \left[\frac{\partial \vec{D}}{\partial t} + \sigma \vec{E} \right] + \vec{H} \times \nabla \frac{\partial \ln \mu}{\partial t} \end{aligned} \quad (9)$$

or

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) + \frac{\partial^2}{\partial t^2} (\epsilon \vec{E}) + \frac{\partial}{\partial t} (\sigma \vec{E}) + \frac{\partial \ln \mu}{\partial t} \left[\frac{\partial}{\partial t} (\epsilon \vec{E}) + \sigma \vec{E} \right] = \vec{H} \times \nabla \frac{\partial \ln \mu}{\partial t} . \quad (10)$$

This can be rewritten

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \left(\sigma + 2 \frac{\partial \epsilon}{\partial t} + \epsilon f \right) \frac{\partial \vec{E}}{\partial t} + \left(\frac{\partial \sigma}{\partial t} + \frac{\partial^2 \epsilon}{\partial t^2} + f \frac{\partial \epsilon}{\partial t} + f \sigma \right) \vec{E} = \vec{H} \times \nabla f , \quad (11)$$

where

$$f = \frac{\partial}{\partial t} \ln \mu = \frac{1}{\mu} \frac{\partial \mu}{\partial t} . \quad (12)$$

The first term of (11) can be expanded as follows:

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) = \left(\nabla \frac{1}{\mu} \right) \times \nabla \times \vec{E} + \frac{1}{\mu} \nabla \times \nabla \times \vec{E} ,$$

$$\nabla \times \nabla \times \vec{E} = \nabla \nabla \cdot \vec{E} - \nabla^2 \vec{E} ,$$

$$\nabla \cdot \vec{D} = \nabla \cdot \epsilon \vec{E} = \vec{E} \cdot \nabla \epsilon + \epsilon \nabla \cdot \vec{E} = 0 ,$$

$$\nabla \cdot \vec{E} = -(\nabla \ln \epsilon) \cdot \vec{E} ,$$

$$\nabla \nabla \cdot \vec{E} = - \left[\nabla \ln \epsilon \cdot \nabla \vec{E} + \vec{E} \cdot \nabla \nabla \ln \epsilon + (\nabla \ln \epsilon) \times \nabla \times \vec{E} \right] .$$

Therefore

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) = - \frac{1}{\mu} \left\{ \nabla^2 \vec{E} + (\nabla \ln \epsilon \mu) \times \nabla \times \vec{E} + \Phi \cdot \vec{E} + (\nabla \ln \epsilon \cdot \nabla) \vec{E} \right\} , \quad (13)$$

where the dyadic Φ is given by

$$\Phi = \nabla \nabla \ln \epsilon$$

or

$$\Phi_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \ln \epsilon = \Phi_{ji} . \quad (14)$$

To obtain the differential equation for $\vec{H}$ we start with (8); thus

$$\begin{aligned} \nabla \times \left(\frac{1}{\epsilon} \nabla \times \vec{H} \right) &= \frac{\partial}{\partial t} \nabla \times \vec{E} + \nabla \times \left[\left(\frac{\sigma}{\epsilon} + \frac{\partial}{\partial t} \ln \epsilon \right) \vec{E} \right] \\ &= -\vec{E} \times \nabla \left(\frac{\sigma}{\epsilon} + \frac{\partial}{\partial t} \ln \epsilon \right) - \left[\frac{\sigma}{\epsilon} + \left(\frac{\partial}{\partial t} \ln \epsilon \right) + \frac{\partial}{\partial t} \right] \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

or

$$\nabla \times \left(\frac{1}{\epsilon} \nabla \times \vec{H} \right) + \frac{\partial^2}{\partial t^2} (\mu \vec{H}) + g \frac{\partial}{\partial t} (\mu \vec{H}) = -\vec{E} \times \nabla g , \quad (15)$$

where

$$g = \frac{\sigma}{\epsilon} + \frac{\partial}{\partial t} \ln \epsilon = \frac{1}{\epsilon} \left(\sigma + \frac{\partial \epsilon}{\partial t} \right) . \quad (16)$$

Now (15) can be expanded to give

$$\nabla X \left(\frac{1}{\epsilon} \nabla X \vec{H} \right) + \mu \frac{\partial^2 \vec{H}}{\partial t^2} + \left[2 \frac{\partial \mu}{\partial t} + g\mu \right] \frac{\partial \vec{H}}{\partial t} + \left[\frac{\partial^2 \mu}{\partial t^2} + g \frac{\partial \mu}{\partial t} \right] \vec{H} = -\vec{E} \times \nabla g . \quad (17)$$

If one introduces f as defined in (12), one has

$$\nabla X \left(\frac{1}{\epsilon} \nabla X \vec{H} \right) + \mu \left\{ \frac{\partial^2 \vec{H}}{\partial t^2} + (g + 2f) \frac{\partial \vec{H}}{\partial t} + \left[\frac{\partial f}{\partial t} + f(f + g) \right] \vec{H} \right\} = -\vec{E} \times \nabla g . \quad (18)$$

A similar introduction of g into (11) leads to the equation

$$\nabla X \left(\frac{1}{\mu} \nabla X \vec{E} \right) + \epsilon \left\{ \frac{\partial^2 \vec{E}}{\partial t^2} + \left(f + 2g - \frac{\sigma}{\epsilon} \right) \frac{\partial \vec{E}}{\partial t} + \left[\frac{\partial g}{\partial t} + g \left(f + g - \frac{\sigma}{\epsilon} \right) \right] \vec{E} \right\} = \vec{H} \times \nabla f . \quad (19)$$

In direct analogy to (13)

$$\nabla X \left(\frac{1}{\epsilon} \nabla X \vec{H} \right) = - \frac{1}{\epsilon} \left\{ \nabla^2 \vec{H} + (\nabla \ln \epsilon \mu) \times \nabla X \vec{H} + \Psi \cdot \vec{H} + (\nabla \ln \mu \cdot \nabla) \vec{H} \right\} , \quad (20)$$

where

$$\Psi = \nabla \nabla \ln \mu$$

or

$$\Psi_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \ln \mu = \Psi_{ji} . \quad (21)$$

A sufficient condition for (19) to involve only $\vec{E}$ is $\nabla f = 0$. The necessary and sufficient condition for $\nabla f = 0$ is that μ be the product of a function of position with a function of time. The proof is as follows:

$$\nabla f = \nabla \frac{\partial}{\partial t} \ln \mu = 0 \Rightarrow \frac{\partial}{\partial t} \ln \mu = F(t) ;$$

thus

$$\ln \mu = \int F(t) dt + G(\vec{r})$$

or

$$\mu = e^{\int F dt} e^{G(\vec{r})} = \mu_r(\vec{r}) \mu_t(t) . \quad (22)$$

Likewise the necessary and sufficient condition for $\nabla g = 0$ is that

$$g = \frac{\sigma}{\epsilon} + \frac{\partial}{\partial t} \ln \epsilon = F(t) \Rightarrow \ln \epsilon = - \int_0^t \frac{\sigma}{\epsilon} dt + \int_0^t F(t) dt + \ln \epsilon_0(\vec{r}) ,$$

where ϵ_0 is the value of ϵ at $t = 0$. Thus

$$\epsilon = \epsilon_0(\vec{r}) f(t) e^{- \int_0^t \frac{\sigma}{\epsilon} dt} , \quad (23)$$

where $f(0) = 1$. If $\sigma = 0$, or if σ/ϵ is a function only of the time, then

$$\epsilon = \epsilon_0(\vec{r}) \epsilon_t(t) . \quad (24)$$

Case 1. $\mu = \mu_r(\vec{r})\mu_t(t)$

From (1) and (3)

$$\nabla \times \vec{E} = -\mu_r \frac{\partial}{\partial t} (\mu_t \vec{H})$$

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) = - \frac{\partial}{\partial t} \left[\mu_t \nabla \times \vec{H} \right] = - \frac{\partial}{\partial t} \left[\mu_t \left(\sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \right) \right]$$

or

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) + \frac{\partial}{\partial t} \left[\mu_t \frac{\partial}{\partial t} (\epsilon \vec{E}) \right] + \frac{\partial}{\partial t} (\mu_t \sigma \vec{E}) = 0 . \quad (25)$$

Case 2. $\epsilon = \epsilon_r(\vec{r})\epsilon_t(t)$ and $\sigma = \epsilon_r(\vec{r})\sigma_t(t)$, $\epsilon_r(\vec{r})$ dimensionless

Again from (1) and (3)

$$\frac{1}{\epsilon_r} \nabla \times \vec{H} = \frac{\partial}{\partial t} (\epsilon_t \vec{E}) + \sigma_t \vec{E} ,$$

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) = \frac{\partial}{\partial t} (\epsilon_t \nabla \times \vec{E}) + \sigma_t \nabla \times \vec{E} = - \frac{\partial}{\partial t} \left[\epsilon_t \frac{\partial}{\partial t} (\mu \vec{H}) \right] - \sigma_t \frac{\partial}{\partial t} (\mu \vec{H})$$

or

$$\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) + \frac{\partial}{\partial t} \left[\epsilon_t \frac{\partial}{\partial t} (\mu \vec{H}) \right] + \sigma_t \frac{\partial}{\partial t} (\mu \vec{H}) = \vec{0} . \quad (26)$$

By multiplying by ϵ_r we can also write (26) in the form

$$\epsilon_r \nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) + \frac{\partial}{\partial t} \left[\epsilon \frac{\partial}{\partial t} (\mu \vec{H}) \right] + \sigma \frac{\partial}{\partial t} (\mu \vec{H}) = \vec{0} , \quad (27)$$

and likewise by multiplying by μ_r (25) can be written

$$\mu_r \nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) + \frac{\partial}{\partial t} \left[\mu \frac{\partial}{\partial t} (\epsilon \vec{E}) \right] + \frac{\partial}{\partial t} (\mu \sigma \vec{E}) = \vec{0} . \quad (28)$$

Case 3. $\epsilon = \epsilon(\vec{r})$, $\mu = \mu(\vec{r})$, Static Inhomogeneous Medium

Since from (12) and (16), $f = 0$ and $g = \frac{\sigma}{\epsilon}$, (18) and (19) become

$$\nabla \times \left(\frac{1}{\epsilon} \nabla \times \vec{H} \right) + \mu \frac{\partial^2 \vec{H}}{\partial t^2} + \frac{\sigma}{\epsilon} \frac{\partial \vec{H}}{\partial t} = -\vec{E} \times \nabla \frac{\sigma}{\epsilon} \quad (29)$$

and

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \epsilon \frac{\partial}{\partial t} \left(\frac{\sigma \vec{E}}{\epsilon} \right) = \vec{0} . \quad (30)$$

Case 4. $\epsilon = \epsilon(t)$, $\mu = \mu(t)$, Dynamic Homogeneous Medium

Since this is a subclass of both Case 1 and Case 2, we may rewrite (27)

and (28) taking account of the fact that $\mu_r = 1$ and $\epsilon_r = 1$, respectively; thus

$$\nabla \times \nabla \times \vec{H} + \frac{\partial}{\partial t} \left[\epsilon \frac{\partial}{\partial t} (\mu \vec{H}) \right] + \sigma \frac{\partial}{\partial t} (\mu \vec{H}) = \vec{0} , \quad (31)$$

$$\nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \left[\mu \frac{\partial}{\partial t} (\epsilon \vec{E}) \right] + \frac{\partial}{\partial t} (\mu \sigma \vec{E}) = \vec{0} . \quad (32)$$

By multiplying the upper equation by μ and the lower by ϵ , we have

$$\nabla \times \nabla \times \vec{B} + \mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{B}}{\partial t} \right) + \mu \sigma \frac{\partial \vec{B}}{\partial t} = \vec{0} , \quad (33)$$

$$\nabla \times \nabla \times \vec{D} + \epsilon \frac{\partial}{\partial t} \left(\mu \frac{\partial \vec{D}}{\partial t} \right) + \epsilon \frac{\partial}{\partial t} \left(\frac{\mu \sigma}{\epsilon} \vec{D} \right) = \vec{0} . \quad (34)$$

Making use of the vector algebra relation

$$\nabla \times \nabla \times \vec{A} = \nabla \nabla \cdot \vec{A} - \nabla^2 \vec{A} \quad (35)$$

and the equations (2) and (4), we may write

$$\nabla^2 \vec{B} - \mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{B}}{\partial t} \right) - \mu \sigma \frac{\partial \vec{B}}{\partial t} = \vec{0} , \quad (36)$$

$$\nabla^2 \vec{D} - \epsilon \frac{\partial}{\partial t} \left(\mu \frac{\partial \vec{D}}{\partial t} \right) - \epsilon \frac{\partial}{\partial t} \left(\frac{\mu \sigma}{\epsilon} \vec{D} \right) = \vec{0} . \quad (37)$$

These may also be written

$$\nabla^2 \vec{B} - \epsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} - \mu (\epsilon' + \sigma) \frac{\partial \vec{B}}{\partial t} = \vec{0} , \quad (38)$$

$$\nabla^2 \vec{D} - \epsilon \mu \frac{\partial^2 \vec{D}}{\partial t^2} - (\mu' \epsilon + \mu \sigma) \frac{\partial \vec{D}}{\partial t} - \left(\mu' \sigma + \mu \sigma' - \frac{\mu \sigma \epsilon'}{\epsilon} \right) \vec{D} = \vec{0} , \quad (39)$$

where $\epsilon' = d\epsilon/dt$, $\mu' = d\mu/dt$, and $\sigma' = d\sigma/dt$.

A differential equation for an electric field quantity of the same form as (36) can be obtained in the following way--provided $\mu = \text{constant}$.

Let a vector field $\vec{U}$ be defined by

$$\vec{U} = \frac{\partial \vec{D}}{\partial t} + \frac{\sigma}{\epsilon} \vec{D} ; \quad (40)$$

then (37) becomes

$$\nabla^2 \vec{D} = \mu \epsilon \frac{\partial \vec{U}}{\partial t} . \quad (41)$$

From (40) and (41)

$$\nabla^2 \vec{U} = \left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) \nabla^2 \vec{D} = \left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) \left(\mu \epsilon \frac{\partial \vec{U}}{\partial t} \right)$$

or

$$\nabla^2 \vec{U} - \mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{U}}{\partial t} \right) - \mu \sigma \frac{\partial \vec{U}}{\partial t} = \vec{0} , \quad (42)$$

which is seen to be of the same form as (36), as required.

That $\vec{D}$ is determined by $\vec{U}$ and the initial value of $\vec{D}$ can be seen by solving (40), assuming $\vec{U}$ known. A solution of the homogeneous equation corresponding to (40) is $\vec{D} = \exp \left\{ - \int \frac{\sigma}{\epsilon} dt \right\}$; therefore, we shall assume that a solution of (40) is

$$\vec{D} = \vec{A}(\vec{r}, t) \exp \left\{ - \int \frac{\sigma}{\epsilon} dt \right\} . \quad (43)$$

Substituting this in (40), we have

$$\frac{\partial \vec{D}}{\partial t} + \frac{\sigma}{\epsilon} \vec{D} = \frac{\partial \vec{A}}{\partial t} \exp \left\{ - \int \frac{\sigma}{\epsilon} dt \right\} = \vec{U} ;$$

thus

$$\vec{A} = \int \vec{U} \exp \left\{ \int \frac{\sigma}{\epsilon} dt \right\} dt$$

and

$$\vec{D} = \exp \left\{ - \int \frac{\sigma}{\epsilon} dt \right\} \int \vec{U} \exp \left\{ \int \frac{\sigma}{\epsilon} dt \right\} dt . \quad (44)$$

Replacing the indefinite integrals by definite integrals, we have

$$\vec{D} = \exp \left\{ - \int_0^t \frac{\sigma}{\epsilon} dt - a(\vec{r}) \right\} \left[\vec{b}(\vec{r}) + \int_0^t \vec{U} \exp \left\{ \int_0^t \frac{\sigma}{\epsilon} dt + a(\vec{r}) \right\} dt \right]$$

and its value at $t = 0$ is thus

$$\vec{D}_0 = \vec{b}(\vec{r}) \exp \left\{ -a(\vec{r}) \right\} ;$$

hence

$$\vec{D} = \vec{D}_0 \exp \left\{ - \int_0^t \frac{\sigma}{\epsilon} dt \right\} + \exp \left\{ - \int_0^t \frac{\sigma}{\epsilon} dt \right\} \int_0^t \vec{U} \exp \left\{ \int_0^t \frac{\sigma}{\epsilon} dt \right\} dt . \quad (45)$$

Non-Conductive Media ($\sigma = 0$)

In a non-conductive medium the general equations (36) and (37) for a homogeneous time-varying medium become

$$\nabla^2 \vec{B} - \mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{B}}{\partial t} \right) = \vec{0} , \quad (46)$$

$$\nabla^2 \vec{D} - \epsilon \frac{\partial}{\partial t} \left(\mu \frac{\partial \vec{D}}{\partial t} \right) = \vec{0} . \quad (47)$$

For the case $\mu = \text{constant}$ we let $\vec{V} = \epsilon \frac{\partial \vec{B}}{\partial t}$ and obtain in place of these equations

$$\nabla^2 \vec{V} - \epsilon \mu \frac{\partial^2 \vec{V}}{\partial t^2} = \vec{0} , \quad (48)$$

$$\nabla^2 \vec{D} - \epsilon \mu \frac{\partial^2 \vec{D}}{\partial t^2} = \vec{0} . \quad (49)$$

Since $\vec{V} = \epsilon \frac{\partial \vec{B}}{\partial t}$, we may obtain $\vec{B}$ by the following equation:

$$\vec{B} = \vec{B}_0 + \int_0^t \frac{1}{\epsilon} \vec{V} dt . \quad (50)$$

3. Solution of the Wave Equation (36) for a Homogeneous Medium

If in (36) we assume that $B_i = R(\vec{r})T(t)$; then, since ϵ , μ , and σ are functions of t only,

$$\frac{\nabla^2 R}{R} = \frac{1}{T} \left[\mu \frac{d}{dt} \left(\epsilon \frac{dT}{dt} \right) + \mu \sigma \frac{dT}{dt} \right] = -k^2 = \text{constant} . \quad (51)$$

Thus R and T satisfy the equations

$$\nabla^2 R + k^2 R = 0 , \quad (52)$$

$$T'' + \frac{\sigma + \epsilon'}{\epsilon} T' + \frac{k^2}{\epsilon\mu} T = 0 \quad (53)$$

where the prime indicates the derivative.

Given k we may form a vector

$$\vec{k} = k(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (54)$$

whose magnitude is k and whose direction is described by the spherical coordinates θ and ϕ . It is then seen that a general solution of (52), for a given k, is

$$R(\vec{r}) = \int_0^\pi d\theta \int_0^{2\pi} d\phi A(\theta, \phi) e^{i\vec{k} \cdot \vec{r}} . \quad (55)$$

On the other hand, for a given k, the general solution of (53) is a linear combination of any two independent solutions $T_1(t)$ and $T_2(t)$; thus

$$T(k, t) = C_1 T_1(k, t) + C_2 T_2(k, t) , \quad (56)$$

and therefore we may write for any one of the Cartesian components of $\vec{B}$

$$B_i(\vec{r}, t) = R(\vec{r})T(t) = \int_0^\pi d\theta \int_0^{2\pi} d\phi \left[A_1^i(k, \theta, \phi) T_1(k, t) + A_2^i(k, \theta, \phi) T_2(k, t) \right] e^{i\vec{k} \cdot \vec{r}} .$$

Here $A_1^i(k, \theta, \phi) = A(\theta, \phi)C_1(k)$ and $A_2^i(k, \theta, \phi) = A(\theta, \phi)C_2(k)$, if we stick to our development, but clearly we have a solution to (36) even if $A_1(k, \theta, \phi)$ and $A_2(k, \theta, \phi)$ are left arbitrary. To obtain a general solution of (36) we next integrate over $k = |\vec{k}|$; thus

$$B_i(\vec{r}, t) = \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \left[A_1^i(k, \theta, \phi) T_1(k, t) + A_2^i(k, \theta, \phi) T_2(k, t) \right] e^{i\vec{k} \cdot \vec{r}}. \quad (57)$$

The three equations corresponding to $i = 1, 2, 3$ can be written together in the form of a vector equation

$$\vec{B}(\vec{r}, t) = \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \left[\vec{A}_1(k, \theta, \phi) T_1(k, t) + \vec{A}_2(k, \theta, \phi) T_2(k, t) \right] e^{i\vec{k} \cdot \vec{r}}, \quad (58)$$

where $\vec{A}_1$ and $\vec{A}_2$ are vectors.

These solutions to (36) are in the form of a superposition of plane waves. It remains necessary, of course, to determine the forms of $T_1(k, t)$ and $T_2(k, t)$ and to fit the boundary conditions by the proper choice of $\vec{A}_1(k, \theta, \phi)$ and $\vec{A}_2(k, \theta, \phi)$.

Nature of the Solutions $T_1(k, t)$ and $T_2(k, t)$

Let us assume that we have obtained two independent solutions $T_1(k, t)$ and $T_2(k, t)$ satisfying the condition that

$$\Delta(0) = \begin{vmatrix} T_1(k, 0) & T_2(k, 0) \\ T_1'(k, 0) & T_2'(k, 0) \end{vmatrix} = T_1 T_2' - T_1' T_2 \neq 0; \quad (59)$$

then it is possible to satisfy the equations

$$AT_1(k,0) + BT_2(k,0) = T(k,0) , \quad (60)$$

$$AT_1'(k,0) + BT_2'(k,0) = T'(k,0)$$

for arbitrary $T(k,0)$ and $T'(k,0)$ imposed by the initial conditions. Here the prime indicates a partial derivative with respect to t .

Consider now the behavior of $\Delta(t)$ with time. Note first that

$$\Delta(t) = T_1(k,t)T_2'(k,t) - T_1'(k,t)T_2(k,t)$$

$$\Delta'(t) = T_1(k,t)T_2''(k,t) - T_1''(k,t)T_2(k,t) ,$$

and therefore by (53)

$$\begin{aligned} \Delta'(t) &= - \left\{ T_1(k,t) \left[\frac{\sigma + \epsilon'}{\epsilon} T_2'(k,t) + \frac{k^2}{\epsilon\mu} T_2(k,t) \right] \right. \\ &\quad \left. - T_2(k,t) \left[\frac{\sigma + \epsilon'}{\epsilon} T_1'(k,t) + \frac{k^2}{\epsilon\mu} T_1(k,t) \right] \right\} \\ &= - \frac{\sigma + \epsilon'}{\epsilon} \left[T_1(k,t)T_2'(k,t) - T_1'(k,t)T_2(k,t) \right] = - \frac{\sigma + \epsilon'}{\epsilon} \Delta(t) . \quad (61) \end{aligned}$$

From (61)

$$\Delta(t) = \Delta_o(0) \exp \left[- \int_0^t \frac{\sigma + \epsilon'}{\epsilon} dt \right] = \Delta(0) \frac{\epsilon_o}{\epsilon} \exp \left[- \int_0^t \frac{\sigma}{\epsilon} dt \right], \quad (62)$$

and hence, if $\Delta(0) \neq 0$, then $\Delta(t)$ will not ordinarily become zero in a finite time.

Obtaining a Second Solution of (53) from a First

Assume that $T_1(k, t)$ is known and look for a second solution of (53) of the form

$$T_2(k, t) = U(k, t)T_1(k, t). \quad (63)$$

Equation (53) can be written

$$T'' + 2\alpha T' + \omega^2 T = 0, \quad (64)$$

where $2\alpha = (\sigma + \epsilon')/\epsilon$ and $\omega^2 = k^2/\epsilon\mu$. From (63)

$$\begin{array}{l|l} \omega^2 & T_2 = UT_1 \\ 2\alpha & T_2' = UT_1' + U'T_1 \\ 1 & T_2'' = UT_1'' + 2U'T_1' + U''T_1 \end{array}$$

$$T_2'' + 2\alpha T_2' + T_2\omega^2 = 0 + (2\alpha T_1 + 2T_1')U' + T_1U'',$$

where primes indicate derivatives with respect to t . Thus for T_2 to be a solution of (64) U must satisfy

$$T_1 U'' + 2(aT_1 + T_1')U' = 0$$

or

$$\frac{U''}{U'} = -2a - 2 \frac{T_1'}{T_1}.$$

Therefore

$$\ln U' = -\int_0^t 2adt - 2 \ln T_1 + \ln C,$$

where C is a constant; hence

$$U' = \frac{C}{T_1^2} \exp \left\{ -\int_0^t 2adt \right\}.$$

Integrating both sides of this equation between 0 and t , we have

$$U = U_0 + C \int_0^t T_1^{-2} \exp \left\{ -\int_0^t 2adt \right\} dt,$$

and thus finally

$$T_2 = U_0 T_1 + CT_1 \int_0^t T_1^{-2} \exp \left\{ - \int_0^t 2adt \right\} dt . \quad (65)$$

Without loss of generality in the solution

$$T = AT_1 + BT_2$$

U_0 can be set equal to zero and C equal to unity. With this choice of constants

$$T_2 = T_1 \int_0^t T_1^{-2} \exp \left\{ - \int_0^t \frac{\sigma + \epsilon}{\epsilon} dt \right\} dt . \quad (66)$$

Since ϵ'/ϵ is readily integrated, we may write (66) in the equivalent form

$$T_2 = T_1 \int_0^t \frac{\epsilon(0)}{\epsilon(t)} T_1^{-2} \exp \left\{ - \int_0^t \frac{\sigma}{\epsilon} dt \right\} dt . \quad (67)$$

As a check we note from (66) that

$$T_1 T_2' = T_1' T_2 + \exp \left\{ - \int_0^t \frac{\sigma + \epsilon'}{\epsilon} dt \right\} ;$$

and thus

$$\Delta(t) = T_1 T_2' - T_1' T_2 = \exp \left\{ - \int_0^t \frac{\sigma + \epsilon'}{\epsilon} dt \right\} , \quad (68)$$

which is just (62) with $\Delta(0) = 1$.

Simple Boundary Conditions

To illustrate how the boundary conditions of a problem serve to determine the arbitrary functions $\vec{A}_1(k, \theta, \phi)$ and $\vec{A}_2(k, \theta, \phi)$ in (58) we shall take as boundary conditions a specification of the magnetic field $\vec{B}$ and its time derivative at $t = 0$; thus we assume that $\vec{B}(\vec{r}, 0)$ and $\frac{\partial \vec{B}}{\partial t}(\vec{r}, 0)$ are given.

Let us rewrite (58) in the form

$$\vec{B}(\vec{r}, t) = \int_{\vec{k}} \left[\vec{A}_1(\vec{k}) T_1(k, t) + \vec{A}_2(\vec{k}) T_2(k, t) \right] e^{i\vec{k} \cdot \vec{r}} d^3 k, \quad (69)$$

where $d^3 k = dk_x dk_y dk_z$. The inverse Fourier integral theorem is contained in the equation

$$\frac{1}{(2\pi)^3} \int_{\vec{r}} e^{i(\vec{k} - \vec{k}') \cdot \vec{r}} d^3 r = \delta(\vec{k} - \vec{k}'), \quad (70)$$

where the Dirac delta function $\delta(\vec{k} - \vec{k}')$ has the property

$$\int_{\Omega} f(\vec{k}) \delta(\vec{k} - \vec{k}') d^3 k = \begin{cases} f(\vec{k}'), & \text{if } \Omega \text{ contains } \vec{k}', \\ 0, & \text{if } \Omega \text{ does not contain } \vec{k}'. \end{cases} \quad (71)$$

For $f(\vec{k}) \equiv 1$ this reduces to

$$\int_{\Omega} \delta(\vec{k} - \vec{k}') d^3 k = \begin{cases} 1, & \text{if } \Omega \text{ contains } \vec{k}', \\ 0, & \text{if } \Omega \text{ does not contain } \vec{k}'. \end{cases} \quad (72)$$

Assuming that the order of integration can be reversed and applying (70) and (71) to (69), we have

$$\begin{aligned} \frac{1}{(2\pi)^3} \int_{\vec{r}} \vec{B}(\vec{r}, t) e^{-i\vec{k}' \cdot \vec{r}} d^3 r &= \int_{\vec{k}} \left[\vec{A}_1(\vec{k}) T_1(k, t) + \vec{A}_2(\vec{k}) T_2(k, t) \right] \delta(\vec{k} - \vec{k}') d^3 k \\ &= \vec{A}_1(\vec{k}') T_1(k', t) + \vec{A}_2(\vec{k}') T_2(k', t) . \end{aligned} \quad (73)$$

Dropping the primes, taking a derivative with respect to time, and then letting $t = 0$ in both equations, we have

$$\vec{A}_1(\vec{k}) T_1(k, 0) + \vec{A}_2(\vec{k}) T_2(k, 0) = \frac{1}{(2\pi)^3} \int_{\vec{r}} \vec{B}(\vec{r}, 0) e^{-i\vec{k} \cdot \vec{r}} d^3 r \equiv \vec{F}(\vec{k}) , \quad (74)$$

$$\vec{A}_1(\vec{k}) T_1'(k, 0) + \vec{A}_2(\vec{k}) T_2'(k, 0) = \frac{1}{(2\pi)^3} \int_{\vec{r}} \frac{\partial \vec{B}}{\partial t}(\vec{r}, 0) e^{-i\vec{k} \cdot \vec{r}} d^3 r \equiv \vec{G}(\vec{k}) ,$$

where the primes on T_1 and T_2 denote partial derivatives with respect to t .

Solving these equations, we have

$$\vec{A}_1(\vec{k}) = \frac{1}{\Delta(0)} \begin{vmatrix} \vec{F}(\vec{k}) & T_2(k, 0) \\ \vec{G}(\vec{k}) & T_2'(k, 0) \end{vmatrix} , \quad (75a)$$

$$\vec{A}_2(\vec{k}) = \frac{1}{\Delta(0)} \begin{vmatrix} T_1(k, 0) & \vec{F}(\vec{k}) \\ T_1'(k, 0) & \vec{G}(\vec{k}) \end{vmatrix} , \quad (75b)$$

where $\Delta(0)$ is given by (59).

REFERENCE

Ginzburg, V. L., Propagation of Electromagnetic Waves in Plasma, Gordon and Beach, New York, 1961. Chapter I gives a derivation of the wave equations satisfied by $\vec{E}$ and $\vec{H}$ for inhomogeneous static media.