COO-3533-14
SU-4205-14
January 1973

SYMMETRIC GROUP AND MESON BORN AMPLITUDES*

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* Work supported by the U. S. Atomic Energy Commission

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549

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ABSTRACT

In this paper we construct Born amplitudes for a theory of non-strange meson interactions. It is convenient to summarize our results in two parts.

- (i) Four-particle Born amplitude, A_4 . Here we decide to retain two and reject two of four properties possessed by the unit intercept Euler B function model. We retain (A) no odd daughters and (B) summability over permutations; we reject (C) the supplementary condition on the sum of trajectory functions and (D) the Plahte phase identities. The reason for rejecting (C) is simply that the physical masses do not in general fulfill such a condition; one is then forced to reject (D) because without (C) the phase identities are inconsistent.

Having made these decisions, we show that retention of (A) and (B) enforces that there be new singularities in the integrand. Regge behaviour dictates that these singularities be at the fixed points $x = \exp(\pm i\pi/3)$ of a six-element S_3 group with elements $x \rightarrow (1-x) \rightarrow \frac{x}{x-1}$.

The general form of A_4 is given; it contains a function which is (i) invariant under an S_3 group and (ii) analytic everywhere except for possible simple or multiple poles at the fixed points.

Some examples lead to the selection of a particularly simple form for A_4 as a physically interesting model for $\pi\pi$ elastic scattering. Some phenomenological consequences of this model are given including the elastic resonance widths and the status of the soft-pion consistency condition.

(ii) N-particle Born amplitude, A_N . By introducing redundant variables and rewriting A_4 in terms of projective-invariant anharmonic ratios the integrand becomes invariant under a larger S_4 group. This is convenient because it enables us to extend all factors, including those with fixed-point singularities, from A_4 to A_N . The general formula for A_N is given; its integrand is S_N invariant.

An extension of the specific physically-interesting four-pion model is constructed. Evident in the multipionic amplitude is an anticommutative algebra which we interpret as the signature of underlying fermionic quarks.

In our final remarks we urge that the spectrum uniquely implied by these multipionic amplitudes be analyzed and compared in detail with experimental observations of the non-strange mesonic spectrum.

In an Appendix we point out that the symmetric group may be used to unify three different previous proposals. Special cases, extrapolated to unphysical mass values, of the present approach include: (i) the unit-rho-pi-intercept A_4 of Veneziano and its A_N extension of Bardakci-Chan-Goebel-Koba-Nielsen-Ruegg-Sakita-Tsun-Virasoro; (ii) the unit-rho-intercept A_4 of Lovelace-Shapiro and its A_N extension of Neveu-Schwarz; (iii) the equal-rho-pi-intercept A_4 of Mandelstam and its A_N extension of Gervais-Neveu.

I. INTRODUCTION

In this paper our objective is to construct a strong interaction Born amplitude for mesons.

We shall formulate a general set of rules for the construction of consistent Born amplitudes, and then use these rules to develop an explicit form for an N-meson Born amplitude A_N . Of special interest will be the case of N-pions, where we expect that the on-shell A_N , for all N, should specify a full theory for the non-strange meson sector.

The method used will exploit a group property, and in the present case the relevant group is the symmetric (or permutation) group, S_N .

At the start we should mention two basic assumptions,* namely that (i) the appropriate zeroth-order approximation for strong interactions is the zero-width approximation, and a perturbatively unitary expansion in the width is rapidly convergent and (ii) it is possible to discuss meaningfully the sector of non-strange mesons, taken separately.

* Although these two assumptions are made throughout, it is possible that the second may eventually be relaxed. We are referring to the fact that the symmetric group fixed-point singularities of the mesonic A_4 (discussed later) occur also in simplified models of meson-baryon scattering.

The very simplest proposal of a mesonic Born amplitude was to take the Euler B function of the Regge trajectory functions.⁽¹⁾ Our work will therefore be guided at the earliest stage by the model for four spinless particles, $p_1 + p_2 \rightarrow (-p_3) + (-p_4)$, $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, $u = (p_1 + p_4)^2$

$$T = B(-\alpha_s, -\alpha_t) + B(-\alpha_t, -\alpha_u) + B(-\alpha_u, -\alpha_s) \quad (1)$$

with

$$B(-\alpha_s, -\alpha_t) = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} \quad (2)$$

with $\alpha_s = \alpha(0) + \alpha's$

It is well-known⁽²⁾ that this model can be extended to a multi-particle amplitude. Theoretical consistency^(3,4) requires, however, that there be unit intercept $\alpha(0)=1$, in which case the leading trajectory is physically unacceptable.

Because of this difficulty, it is clear that the Euler B function must be modified if we are to have any hope of establishing a connection with experimental observations. To find the correct modification, we should examine in detail the B-function for the unphysical intercept $\alpha(0)=1$.

We first list four inter-related properties of the B function as applicable to the unit intercept $\alpha(0)=1$ case. They are
(A) The absence of odd daughter trajectories, due to the reflection property of the residues contained in

$$B(-\alpha_s, -\alpha_t) = \sum_{N=0}^{\infty} \frac{R_N(\alpha_t)}{N - \alpha_s} \quad (3)$$

under $\alpha_t \leftrightarrow \alpha_u$ interchange namely

$$R_N(\alpha_t) = (-1)^N R_N(\alpha_u) \quad (4)$$

Thus the linear trajectories are spaced by two units in angular momentum.

(B) Summability⁽⁵⁾ of the three terms of Eq. (2) in terms of a single integral

$$T = \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} \quad (5)$$

(C) The supplementary condition that the sum of the three trajectory functions $\alpha_s, \alpha_t, \alpha_u$ be an integer, in this case

$$\alpha_s + \alpha_t + \alpha_u + 1 = \gamma = 0 \quad (6)$$

(D) The Plahte⁽⁶⁾ phase identities which follow by closing the contour in Eq. (5) and exploiting the analyticity properties of the integrand, namely,

$$B(-\alpha_s, -\alpha_t) - e^{\pm i\pi\alpha_t} B(-\alpha_t, \alpha_u) - e^{\pm i\pi\alpha_s} B(-\alpha_u, -\alpha_s) = 0 \quad (7)$$

and cyclic permutations. Here the exponent signs correspond to closure in the upper, lower half Euler x-plane respectively, when proper account is taken of the branch points at $x = 0, 1$.

As mentioned already these properties are all strongly inter-related. For example, the phase identities (D) imply a supplementary condition $\gamma = 0$, modulo 2 (C). They also imply the two-unit trajectory spacing (A). Summability (B) together with analyticity of the integrand implies the phase identities (D).

Our basic ansatz will be that the realistic model should maintain the property, for the four-particle amplitude, of absence of odd daughters (A). It is desirable, as mentioned in the next paragraph, to retain the related property of summability (B). On the other hand, the supplementary condition (C) $\gamma = \text{integer}$ is not fulfilled in general by physical masses and we must relax it. Since (D) implies (C) we must therefore relax (D) also. This, together with the retention of

summability (B) implies that the analytic properties of the integrand must be changed in an essential way. Remarkably the resultant model A_4 will turn out to be an integral representation only slightly more complicated than the Euler B function (which is so simple, that one cannot hope to further simplify it).

Concerning the summability condition (B) we note that the possibility^(3,4) of theoretical consistency in the earlier A_N depended on the presence of a higher symmetry (generalized projective invariance) than a projective invariance⁽⁷⁾ of the A_N integrand. It was not possible to identify directly the generalized projective group as an invariance group of A_N , at fixed N . It seems to us, however, that the necessary condition for its existence was that the integrand of A_N possess invariance under an S_N group, for all N . This S_N invariance is the straightforward generalization of the summability condition (B) at $N = 4$. This is why we insist on retaining (B) at the four-meson level.

An alternative approach^{*} to amplitude construction is to start from representations of the generalized projective group realized on the Fock space spanned by harmonic oscillator operators; this has the advantage that gauge conditions necessary to allow the overcoming of the indefinite metric ghosts may be built in from the start. The operator approach had one notable success: the dual pion model (albeit unphysical) of Neveu-Schwarz,⁽⁸⁾

^{*} A third approach to amplitude construction is the use⁽⁹⁾ of the rubber string model, pioneered by Nambu, Nielsen and Susskind.

Since we shall often be concerned specifically with pions let us recall how isospin and absence of exotics are dealt with there. For $\pi^a(p_1) + \pi^b(p_2) \rightarrow \pi^c(-p_3) + \pi^d(-p_4)$ we write

$$\begin{aligned} & \langle \pi^c(-p_3) \pi^d(-p_4) | T | \pi^a(p_1) \pi^b(p_2) \rangle = \\ & \delta_{ab} \delta_{cd} (A_4(-\alpha_s, -\alpha_t) - A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s)) \\ & \delta_{ac} \delta_{bd} (A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) - A_4(-\alpha_u, -\alpha_s)) \\ & \delta_{ad} \delta_{bc} (-A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s)) \end{aligned} \quad (8)$$

where $A_4(-\alpha_s, -\alpha_t)$ should contain poles in the same channels as $B(-\alpha_s, -\alpha_t)$ of Eq. (1). To incorporate the same principles in $A_N(N \geq 6)$ it is convenient to use the well-known shorthand of Pauli matrices⁽¹⁰⁾ in the form

$$\begin{aligned} T_N^{a_1 a_2 \dots a_N} = & \sum_{\{q_1 q_2 \dots q_N\}} \frac{1}{2} \text{Tr}(\tau(a_{q_1}) \tau(a_{q_2}) \dots \tau(a_{q_N}))^* \\ & * A_N(p_{q_1} p_{q_2} \dots p_{q_N}) \end{aligned} \quad (9)$$

where the sum is over all permutations of $\{q_1 q_2 \dots q_N\} = 1, 2, \dots, N$. The A_N now has poles only in the planar channels. We should recall that the ansatz⁽¹¹⁾ for four pions

$$A_4(-\alpha_s, -\alpha_t) = \alpha_s \alpha_t (\alpha_s + \alpha_t)^{-1} B(-\alpha_s, -\alpha_t) \quad (10)$$

has some attractive properties for physical mass values. Not surprisingly, therefore, there has been much theoretical effort to extend this A_4 to an A_N ; we would like here to attribute the lack

of success to the presence of odd daughters in this amplitude. To support our view, note that if odd daughters are removed by imposing $\gamma=2$, an unphysical supplementary condition, a consistent extension becomes possible.⁽⁸⁾

The outline of the present paper is as follows: in Section II we discuss the no-odd-daughter requirement, in Section III the analytic properties are determined and in Sect. IV specific forms for A_4 are set up. Sections V and VI explore the properties of soft-pion consistency and reality of coupling constants respectively, and enable selection of a simple form of A_4 , designated $A_4^{(1)}$. In Section VII we make the multiparticle extension for a general case, while in Section VIII the extension to $A_N^{(1)}$ is explicitly stated. Section IX contains the concluding remarks. In an Appendix the relationship to, and a unifying treatment of, earlier work is presented.

The reader interested only in our general approach could read only Sections II, III, VII without loss of continuity.

II. NO ODD DAUGHTERS

We now modify the integrand of $B(-\alpha_s, -\alpha_t)$, by multiplying with an arbitrary function. We choose to write this in the form

$$A_4(-\alpha_s, -\alpha_t) = \int_0^1 dx x^{\alpha_t-1} (1-x)^{-\alpha_s-1} [(x-e^{i\pi/3})(x-e^{-i\pi/3})]^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \quad (11)$$

for reasons which will become clear shortly. We have exhibited branch points⁽¹²⁾ at the fixed points* $x = e^{\pm i\pi/3}$ of order related to the deviation from the supplementary condition $\gamma = 0$. This is necessary in order that the analytic properties of ϕ_4 as a function of x be especially simple (see Part III below).

*Throughout, we use "fixed points" to refer to the points $x = \exp(\pm i\pi/3)$.

Let us now examine the Regge asymptotic behavior by making the change of variables

$$x = 1 - e^{-W} \quad (12)$$

$$= \frac{1}{2}(\alpha_s - \alpha_u) \quad (13)$$

to find a Laplace transform

$$A_4(-\alpha_s, -\alpha_t) = \int_0^\infty dW W^{-\alpha_t-1} e^{\nu W} \left[\sum_{n=0}^\infty \frac{1}{(2n+1)!} \left(\frac{W}{2}\right)^{2n} \right]^{-\alpha_t-1} \\ \left[1 + 2 \sum_{n=1}^\infty \frac{W^{2n}}{(2n)!} \right]^{\gamma/2} \phi_4(\alpha_s, \alpha_t, \alpha_u, 1 - e^{-W}) \quad (14)$$

Now we expand the square brackets, and ϕ_4 , as power series in W and use the fact that

$$\int_0^\infty dW W^{-\alpha_t-1+r} e^{\nu W} = (-\alpha\nu)^{\alpha_t-r} \Gamma(-\alpha_t+r) \quad (15)$$

to see that, since the two square brackets are already even in W , odd daughter terms (odd r) are absent if and only if ϕ_4 satisfies the fundamental requirement

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \phi_4(\alpha_u, \alpha_t, \alpha_s; \frac{x}{1-x}) \quad (16)$$

Combining this with the crossing symmetry requirement that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \phi_4(\alpha_t, \alpha_s, \alpha_u; 1-x) \quad (17)$$

we see that ϕ_4 satisfies an attractive invariance under an S_3 permutation group. It turns out to be extremely useful, for reasons that will be explained later, to introduce an auxiliary function $f(x)$ which has simple poles at the two fixed points $x = e^{\pm i\pi/3}$ and which is invariant under the inversion $x \rightarrow 1/x$ namely

$$f(x) = f(1/x) = x(x-e^{i\pi/3})^{-1}(x-e^{-i\pi/3})^{-1} \quad (18)$$

Now make the identifications

$$\beta_s(x) = f(x) = f(1/x) \quad (19)$$

$$\beta_t(x) = f(1-x) = f(1/(1-x)) \quad (20)$$

$$\beta_u(x) = f(1-1/x) = f(x/(x-1)) \quad (21)$$

to see that

$$\phi_4(\alpha_s, \alpha_t, \alpha_u; x) = \phi_4 \begin{pmatrix} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{pmatrix} \quad (22)$$

must be invariant under all $3!$ permutations of the argument pairs $\alpha_i, \beta_i(x)$ with $i=s, t, u$.

III. ANALYTIC PROPERTIES

In order to derive Regge behaviour for $|v| \rightarrow \infty$ at fixed t in the whole v -plane, apart from the positive v -axis, then according to Watson's Lemma⁽¹³⁾ the integrand should be analytic in the domain $\text{Re } W > 0$, or equivalently $|1-x| < 1$. By crossing symmetry, fixed s Regge behaviour requires analyticity in $|x| < 1$. This then already gives a domain of analyticity which is the union of the interiors of two unit circles of centre $x=0$ and $x=1$.

This analytic domain can be extended by considering simultaneously all of the s, t, u Regge limits. It is convenient to consider the summability condition⁽⁵⁾ directly, since we may write it in the form.

$$\begin{aligned}
& A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s) = \\
& \int_{-\infty}^{\infty} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \quad (23) \\
& = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4(\alpha_s, \alpha_t, \alpha_u; x) \\
& + \int_0^1 d\left(\frac{1}{1-x}\right) \left(\frac{1}{1-x}\right)^{-\alpha_u-1} \left(1-\frac{1}{1-x}\right)^{-\alpha_t-1} \left(\frac{1}{1-x}-e^{i\pi/3}\right)^{\frac{\gamma}{2}} \left(\frac{1}{1-x}-e^{-i\pi/3}\right)^{\frac{\gamma}{2}} \\
& \quad \phi_4(\alpha_t, \alpha_u, \alpha_s, \frac{1}{1-x}) \\
& + \int_0^1 d\left(1-\frac{1}{x}\right) \left(1-\frac{1}{x}\right)^{-\alpha_s-1} \left(1-\left(1-\frac{1}{x}\right)\right)^{-\alpha_u-1} \left(1-\frac{1}{x}-e^{i\pi/3}\right)^{\frac{\gamma}{2}} \left(1-\frac{1}{x}-e^{-i\pi/3}\right)^{\frac{\gamma}{2}} \\
& \quad \phi_4(\alpha_u, \alpha_s, \alpha_t; 1-\frac{1}{x}) \quad (24)
\end{aligned}$$

Consideration now of Regge behaviour in the second and third terms leads to new requirements for the analyticity of ϕ_4 . We find respectively for the three terms that analyticity is required in the domains

$$\mathcal{D}(x) = |x| < 1 \cup |1-x| < 1 \quad (25)$$

$$\mathcal{D}\left(\frac{1}{1-x}\right) = \left|\frac{1}{1-x}\right| < 1 \cup \left|\frac{x}{x-1}\right| < 1 \quad (26)$$

$$\mathcal{D}\left(1-\frac{1}{x}\right) = \left|1-\frac{1}{x}\right| < 1 \cup \left|\frac{1}{x}\right| < 1 \quad (27)$$

The union of all three domains gives the whole complex x -plane including the point at infinity, except for the two points fixed under the six-element S_3 group of transformations on x . These two fixed points are $x = e^{\pm i\pi/3}$. That is

$$\mathcal{D}(x) \cup \mathcal{D}\left(\frac{1}{1-x}\right) \cup \mathcal{D}\left(1-\frac{1}{x}\right) = \{x | x \neq e^{\pm i\pi/3}\} \quad (28)$$

We have chosen to exhibit explicitly branch points of order $\gamma/2$ at these two fixed points in order to simplify the analytic properties of ϕ_4 in the integrand. The analytic properties of ϕ_4 are: $\phi_4(\alpha_s \alpha_t \alpha_u, x)$ is analytic everywhere in the complex x -plane, including the point at infinity, except possibly for (simple or multiple) poles at the points $x = e^{\pm i\pi/3}$.

Clearly the singularities at the fixed points $x = e^{\pm i\pi/3}$ are playing an important role, since they must be present in the integrand if we do not have a (generalized) supplementary condition $\gamma = 0$ (modulo 2).

This result can be made more precise by the following argument. Let us, following Plahte,⁽⁶⁾ close the contour in the summed form for $[A_4(-\alpha_s, -\alpha_t) + A_4(-\alpha_t, -\alpha_u) + A_4(-\alpha_u, -\alpha_s)]$ and apply Cauchy's theorem. If there are branch points at the fixed points $x = e^{\pm i\pi/3}$ we deform the contour around two cuts extending to $x = \frac{1}{2} \pm \infty$ respectively. Calling the contours around the cuts $C_{\pm}$ and dropping the contour at $|x| \rightarrow \infty$ we arrive at

$$\begin{aligned} A_4(-\alpha_s, -\alpha_t) &= e^{\pm i\pi\alpha_t} A_4(-\alpha_t, -\alpha_u) = e^{\pm i\pi\alpha_s} A_4(-\alpha_u, -\alpha_s) \\ &= \int_{C_{\pm}} dx |x|^{-\alpha_t-1} |1-x|^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4 \end{aligned} \quad (29)$$

where the two sign possibilities correspond to closure in the upper, lower half x -planes respectively.

If the singularity at the fixed point were absent, the right-hand side would be zero. By taking a linear combination of the left-hand side with one of its cyclic permutations we could arrive at

$$\begin{aligned} & e^{-i\pi\alpha_t}(A_4(-\alpha_s, -\alpha_t) - e^{i\pi\alpha_t}A(-\alpha_t, -\alpha_u) - e^{-i\pi\alpha_s}A_4(-\alpha_u, -\alpha_s)) \\ & + A_4(-\alpha_t, -\alpha_u) - e^{i\pi\alpha_u}A_4(-\alpha_u, -\alpha_s) - e^{-i\pi\alpha_s}A_4(-\alpha_s, -\alpha_t) = 0 \end{aligned} \quad (30)$$

and hence

$$e^{-i\pi(\alpha_s + \alpha_t)} = -e^{i\pi\alpha_u} \quad (31)$$

or

$$\alpha_s + \alpha_t + \alpha_u + 1 = \gamma = 0 \pmod{2} \quad (32)$$

By these arguments we see that the retention of the summability condition and relaxation of the supplementary condition imply that we are forced to introduce the new singularities.

IV. CONSTRUCTION OF ϕ_4

The construction of candidates for the function $\phi_4(\alpha_s, \alpha_t, \alpha_u; x)$ is an exercise in the use of S_3 , the permutation group on 3 objects.

Our criterion of simplicity will be that the pole of ϕ_4 at $x = e^{\pm i\pi/3}$ should be of low order; to be specific we shall study in detail only those solutions for ϕ_4 which have at most a double pole at these points.

As already noted, it is useful to introduce the auxiliary function $f(x)$ which is invariant under inversion, and which has simple poles at the two fixed points, namely

$$f(x) = x(x-e^{i\pi/3})^{-1}(x-e^{-i\pi/3})^{-1} = f\left(\frac{1}{x}\right) \quad (33)$$

$$= (x+x^{-1}-1)^{-1} \quad (34)$$

$$= \beta_s(x) \quad (\text{definition}) \quad (35)$$

For the other elements of the group one has

$$\beta_t(x) = \beta_s(1-x) \quad (36)$$

$$= \beta_s\left(\frac{1}{1-x}\right) \quad (37)$$

$$= (1-x)(x-e^{i\pi/3})^{-1}(x-e^{-i\pi/3})^{-1} \quad (38)$$

$$\beta_u(x) = \beta_s\left(\frac{x}{x-1}\right) \quad (39)$$

$$= \beta_s\left(1-\frac{1}{x}\right) \quad (40)$$

$$= -x(1-x)(x-e^{i\pi/3})^{-1}(x-e^{-i\pi/3})^{-1} \quad (41)$$

These functions satisfy

$$\beta_s(x) + \beta_t(x) + \beta_u(x) = 1 \quad (42)$$

$$\beta_s(x)^2 + \beta_t(x)^2 + \beta_u(x)^2 = 1 \quad (43)$$

By squaring the first and subtracting from the second it follows that

$$\beta_s(x)\beta_t(x) + \beta_t(x)\beta_u(x) + \beta_u(x)\beta_s(x) = 0 \quad (44)$$

Actually this follows term by term since

$$\beta_s(x)^2 - \beta_s(x) = \beta_t(x)\beta_u(x) \quad (45)$$

$$\beta_t(x)^2 - \beta_t(x) = \beta_u(x)\beta_s(x) \quad (46)$$

$$\beta_u(x)^2 - \beta_u(x) = \beta_s(x)\beta_t(x) \quad (47)$$

We are therefore led to construct the following explicit forms for the function ϕ_4 . Firstly there is the trivial solution

$$\phi_4^{(0)}(\alpha_s, \alpha_t, \alpha_u, x) = 1 \quad (48)$$

Next there are the two possibilities, linear in $\alpha_s, \alpha_t, \alpha_u$

$$\phi_4^{(1)}(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s \beta_s(x) + \alpha_t \beta_t(x) + \alpha_u \beta_u(x) \quad (49)$$

$$\phi_4^{(2)}(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s \beta_t(x) \beta_u(x) + \alpha_t \beta_u(x) \beta_s(x) + \alpha_u \beta_s(x) \beta_t(x) \quad (50)$$

We can also write

$$\phi_4^{(3)}(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s \alpha_t \beta_s(x) \beta_t(x) + \alpha_t \alpha_u \beta_t(x) \beta_u(x) + \alpha_u \alpha_s \beta_u(x) \beta_s(x) \quad (51)$$

The three non-trivial examples given here all avoid tachyons on the leading trajectory. They also avoid ancestor trajectories for each of the three terms separately. The latter condition is unnecessary, however, since the S_3 property which avoids odd daughter trajectories simultaneously removes the first (odd) ancestor. Therefore we consider a fourth possibility

$$\phi_4^{(4)}(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s^2 \beta_s(x) + \alpha_t^2 \beta_t(x) + \alpha_u^2 \beta_u(x) \quad (52)$$

Notice that the following form is a linear combination of $\phi_4^{(2)}$ and $\phi_4^{(3)}$, namely

$$\begin{aligned} & \alpha_s^2 \beta_t(x) \beta_u(x) + \alpha_t^2 \beta_u(x) \beta_s(x) + \alpha_u^2 \beta_s(x) \beta_t(x) \\ &= (\gamma-1) \phi_4^{(2)}(\alpha_s, \alpha_t, \alpha_u, x) + \phi_4^{(3)}(\alpha_s, \alpha_t, \alpha_u, x) \end{aligned} \quad (53)$$

and need not be considered separately.

We may therefore write, up to double poles at the fixed points, the solution

$$\phi_4(\alpha_s \alpha_t \alpha_u x) = \lambda_0 + \lambda_1 \phi_4^{(1)} + \alpha_2 \phi_4^{(2)} + \lambda_3 \phi_4^{(3)} + \lambda_4 \phi_4^{(4)} \quad (54)$$

If we require the absence of a tachyon on the leading positive-intercept trajectory we deduce that $\lambda_0 = 0$.

To construct models with higher-order poles at the fixed points is straight forward because powers of the auxiliary function $f(x)$ satisfy many neat identities, for example

$$f(x)^3 + f(1-x)^3 + f(1-\frac{1}{x})^3 - 3f(x)f(1-x)f(1-\frac{1}{x}) = 1 \quad (55)$$

$$f(x)^4 + f(1-x)^4 + f(1-\frac{1}{x})^4 - 4f(x)f(1-x)f(1-\frac{1}{x}) = 1 \quad (56)$$

$$f(x)^5 + f(1-x)^5 + f(1-\frac{1}{x})^5 - 5f(x)f(1-x)f(1-\frac{1}{x}) = 1 \quad (57)$$

$$f(x)^6 + f(1-x)^6 + f(1-\frac{1}{x})^6 - 6f(x)f(1-x)f(1-\frac{1}{x}) - 3f(x)^2 f(1-x)^2 f(1-\frac{1}{x})^2 = 1 \quad (58)$$

Vanishing symmetric forms then follow as linear combinations, for example

$$f(x)^2(f(1-x) + f(1-\frac{1}{x})) + \text{two permutations} + 3f(x)f(1-x)f(1-\frac{1}{x}) = 0 \quad (59)$$

$$f(x)^2 f(1-x)^2 + \text{two permutations} + 2f(x)f(1-x)f(1-\frac{1}{x}) = 0 \quad (60)$$

and so on.

Further once any ϕ_4 is given, any polynomial in that ϕ_4 also gives an acceptable model.

We do not pursue further these more complicated possibilities here because we shall prefer one of the candidates already in Eq. (54) as the amplitude for strangeness-zero mesonic systems. We therefore keep the four terms ($\lambda_0=0$) in Eq. (54) and proceed.

V. ADLER CONSISTENCY CONDITION

The soft-pion consistency condition, of Adler,⁽¹⁴⁾ requires that when one external pion is extrapolated to zero four momentum e.g. $p_1^\mu \rightarrow 0$ the amplitude $T^{abcd}(s,t,u)$ should vanish (i.e., at the point $s=t=u=\mu^2$). This condition, although often elevated to theorem status in the literature, gives no condition on the on-shell amplitude.

It turns out, however, that all the amplitudes we consider do satisfy one of two versions of the Adler condition. Either (a) the amplitude vanishes at $s=t=u=\mu^2$ to leading order (i.e., the satellite contributions do not vanish identically) provided $\alpha_\rho(\mu^2) = \frac{1}{2}$ or (b) the amplitude precisely vanishes at $s=t=u=\mu^2$ for a range of possible intercepts $\alpha_\rho(0)$.

Thus the Adler zero considerations do not strongly constrain the choice of amplitude, out of those already introduced.

We now give the details underlying these remarks.

Let us first observe that $\phi_4^{(i)}$ has the property

$$\phi_4^{(i)}(\alpha_s=\alpha_t=\alpha_u, x) = \hat{\phi}_4^{(i)} = \text{constant, independent of } x. \quad (61)$$

for the cases $i=1,2,3,4$ considered above. Note that this property cannot be derived from the symmetry properties alone, since had we allowed $\phi_4(\alpha_s, \alpha_t, \alpha_u; x)$ to have a third-order pole at $x=e^{\pm i\pi/3}$

we can find examples such as

$$\phi_4(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s \alpha_t \alpha_u \beta_s(x) \beta_t(x) \beta_u(x) \quad (62)$$

which do not lose the x -dependence at $\alpha_s = \alpha_t = \alpha_u$.

For the simple-pole models $\phi_4^{(1)}$ and $\phi_4^{(4)}$ we find that

$$\hat{\phi}_4^{(1)} = \alpha_\rho(\mu) \quad (63)$$

$$\hat{\phi}_4^{(4)} = \alpha_\rho(\mu)^2 \quad (64)$$

and at the symmetry point the amplitudes become

$$A_4^{(i)}(-\alpha_s, -\alpha_t) \rightarrow \hat{\phi}_4^{(i)} \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \quad (65)$$

$$= \hat{\phi}_4^{(i)} \sum_{r=0}^{\infty} \binom{\gamma/2}{r} (-1)^r B(r-\alpha_s, r-\alpha_t) \quad (66)$$

We consider two cases in turn, here, namely $\alpha(\mu^2) = \frac{1}{2}$ and

$\alpha_\rho(\mu^2) \neq \frac{1}{2}$.

If $\alpha_\rho(\mu^2) = \frac{1}{2}$, then the simple-pole model becomes, at the soft-pion point

$$A_4^{(i)}(-\alpha_s, -\alpha_t) \rightarrow \hat{\phi}_4^{(i)} [B(-\frac{1}{2}, -\frac{1}{2}) + \sum_{r=1}^{\infty} \binom{\gamma/2}{r} (-1)^r B(r-\frac{1}{2}, r-\frac{1}{2})] \quad (67)$$

Thus the leading term vanishes. The remainder does not vanish identically because it may be rewritten

$$\sum_{r=1}^{\infty} \binom{5/4}{r} (-1)^r B(r-\frac{1}{2}, r-\frac{1}{2}) = -\frac{5}{16} \pi {}_2F_1(-\frac{1}{4}, \frac{1}{2}; 2; \frac{1}{4}) \quad (68)$$

and by elementary methods one obtains the bounds

$$\frac{63}{64} > {}_2F_1(-\frac{1}{4}, \frac{1}{2}; 2; \frac{1}{4}) > \frac{47}{48} \quad (69)$$

If $\alpha_\rho(\mu^2) \neq \frac{1}{2}$, even the first term is non-vanishing and we must estimate a series of the form (putting $\alpha_\rho(\mu^2)=a$)

$$aB(-a, -a) {}_2F_1\left(-\frac{3a+1}{2}, -a; \frac{1}{2}-a; \frac{1}{4}\right) \quad (70)$$

$$= aB(-a, -a) \left[1 + \sum_{r=1}^{\infty} I_r\right] \quad (71)$$

with $I_1 = \frac{1}{4}a(3a+1)(1-2a)^{-1}$ and $\frac{1}{4} > I_{r+1}/I_r > 0$ for all $r \geq 2$.

Again by elementary methods we can check that such an expression does not vanish identically for any a in the region $\frac{1}{3} \leq a \leq \frac{2}{3}$ ($a \neq \frac{1}{2}$).

Thus there is an advantage in imposing $\alpha_\rho(\mu^2) = \frac{1}{2}$ in those cases. With $\alpha_\rho(\mu^2) = \frac{1}{2}$ we have the leading order Adler zero.

For the models with double poles at the fixed points, namely

$$\phi_4^{(2)}(\alpha_s, \alpha_t, \alpha_u; x) = \alpha_s \beta_t(x) \beta_u(x) + \alpha_t \beta_u(x) \beta_s(x) + \alpha_u \beta_s(x) \beta_t(x) \quad (72)$$

and

$$\phi_4^{(3)}(\alpha_s, \alpha_t, \alpha_u, x) = \alpha_s \alpha_t \beta_s(x) \beta_t(x) + \alpha_t \alpha_u \beta_t(x) \beta_u(x) + \alpha_u \alpha_s \beta_u(x) \beta_s(x) \quad (73)$$

we find the precise Adler zero condition since the result

$$\phi_4^{(i)}(\alpha_s = \alpha_t = \alpha_u, x) = 0 \quad (74)$$

follows from the identity already mentioned for the symmetrized bilinear expression in the $\beta_k(x)$.

This precise vanishing happens for any value of $\alpha_\rho(\mu^2)$, so that we lose the motivation for imposing the attractive constraint $\alpha_\rho(\mu^2) = \frac{1}{2}$ (which is physically well satisfied).

Since $\alpha_\rho(\mu^2) = \frac{1}{2}$ is an acceptable condition we do not make any selection of models on the basis of these considerations alone; instead we will do this on the basis of reality of the coupling constants, which will turn out to be much more restrictive.

VI. REALITY OF COUPLING CONSTANTS

For a model to be acceptable, it must ultimately be obtainable in a fully factorizable N-particle Born amplitude scheme, where all states can be identifiable by real coupling constants.

A necessary condition to be satisfied first is that the on-shell residua in the four-meson amplitude A_4 be such that the partial-wave projections give rise to a positive-definite imaginary part.

We consider ϕ_4 of the form given by Eq. (54). The corresponding amplitude is

$$A_4(-\alpha_s, -\alpha_t) = \sum_{i=1}^4 \lambda_i A^{(i)}(-\alpha_s, -\alpha_t) \quad (75)$$

$$= \sum_{i=1}^4 \lambda_i \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4^{(i)} \quad (76)$$

$$= \sum_{i=1}^4 \lambda_i \sum_{N=1}^{\infty} R_N^{(i)}(\alpha_t) (N-\alpha_s)^{-1} \quad (77)$$

In the absence of any reliable general argument, we have been forced to make a detailed numerical partial wave analysis of the $R_N^{(i)}(t)$. The analysis was made for the following possibilities

$$\mu^2 = 0 \text{ and } \alpha_p(0) = \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3} \quad (78)$$

The results may be summarized as follows:

(i) For a double-pole model, with precise Adler zeros, of the form

$$A_4 = \lambda_2 A_4^{(2)} + \lambda_3 A_4^{(3)} \quad (79)$$

there are negative squared coupling constants for all possible λ_2, λ_3 and such models are thereby eliminated.

(ii) For single-pole models, it is found that $A_4^{(4)}$ contains negative residues and is eliminated but $A_4^{(1)}$ does not. Only $A_4^{(1)}$ survives and for this model we have further calculated the elastic resonance widths for $\alpha_p(s) = 0.48 + 0.9s$ and $\alpha_p(\mu^2) = 1/2$ normalized to $\Gamma_p = 112$ MeV. The results are given in Table I.

The rest of this subsection contains more details of these results and may be omitted without loss of continuity.

To find the results we first expand the $R_N^{(i)}$ as

$$R_N^{(1)} = \sum_{R=0}^N \binom{\gamma/2-1}{R} (-1)^N \left[\frac{N}{\Gamma(N-R+1)} \frac{\Gamma(R+1-\alpha_t)}{\Gamma(2R+1-N-\alpha_t)} - \frac{\alpha_t}{\Gamma(N-R)} \frac{\Gamma(R-\alpha_t)}{\Gamma(2R+1-N-\alpha_t)} + \frac{(\gamma-1-N-\alpha_t)}{\Gamma(N-R)} \frac{\Gamma(R+1-\alpha_t)}{\Gamma(2R+2-N-\alpha_t)} \right] \quad (80)$$

$$R_N^{(2)} = \sum_{R=0}^{N-1} \binom{\gamma/2-1}{R} (-1)^N \left[- \frac{N}{\Gamma(N-R-1)} \frac{\Gamma(1+R-\alpha_t)}{\Gamma(3+2R-N-\alpha_t)} + \frac{\alpha_t \Gamma(2+R-\alpha_t)}{\Gamma(N-R) \Gamma(3+2R-N-\alpha_t)} - \frac{(\gamma-1-N-\alpha_t)}{\Gamma(N-R)} \frac{\Gamma(1+R-\alpha_t)}{\Gamma(2+2R-N-\alpha_t)} \right]$$

$$R_N^{(3)} = \sum_{R=0}^{N-1} \binom{\gamma/2-2}{R} (-1)^N \left[- \frac{N\alpha_t}{\Gamma(N-R)} \frac{\Gamma(R+1-\alpha_t)}{\Gamma(2R+2-N-\alpha_t)} - \right] \quad (81)$$

$$- (\gamma-1-N-\alpha_t) \frac{\alpha_t \Gamma(R+1-\alpha_t)}{\Gamma(N-R-1) \Gamma(2R+3-N-\alpha_t)} + (\gamma-1-N-\alpha_t) \frac{N\Gamma(R+2-\alpha_t)}{\Gamma(N-R) \Gamma(2R+3-N-\alpha_t)} \quad (82)$$

and $R_N^{(4)}$ is as $R_4^{(1)}$ with the only alteration of squaring N , α_t , and $(\gamma-1-N-\alpha_t)$ respectively in the three terms.

First consider $R_N^{(2)}$ and $R_N^{(3)}$. In Tables 2 and 3 are given the first few levels. The results* are given for $\mu^2 = 0$ and $\alpha_\rho(0) = 1/2$. By considering, for example, the vector resonances at $\alpha_s = 1, 3$ and the scalar at $\alpha_s = 4$ it is found that no linear combination of $A^{(2)}$ and $A^{(3)}$ can be free of ghosts. The same conclusion holds for $\mu^2 = 0$ and $\alpha_\rho(0) = \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}$.

This leaves us with the simple fixed-point pole models $A_4^{(1)}$ and $A_4^{(4)}$.

The partial wave projections* of $R_N^{(4)}$ with $\mu^2 = 0$, $\alpha_\rho(0) = \frac{1}{2}$ are given in Table 4. We see that there are ghosts here also, the lowest-lying being a scalar at $\alpha_s = 4$. The model $A_4^{(4)}$ is thus eliminated.

Finally we come to the beautiful model $A_4^{(1)}$. For it, we find an absence of ghosts up to at least $\alpha_s = 16$. It is straightforward to show that the leading trajectory has the same coupling to $\pi\pi$ as in the conventional $\pi\pi$ model. Therefore, as mentioned earlier, we have tabulated the elastic widths up to $\alpha_s = 10$ for $\alpha_\rho(s) = 0.48 + 0.9s$ and $\alpha_\rho(\mu^2) = \frac{1}{2}$ normalized to $\Gamma_\rho = 112$ MeV; this may then be compared directly to Table 1 on page 1348 of Shapiro.⁽¹¹⁾

This absence of ghosts in $A^{(1)}$ persists if we make the small change to $\mu^2 = 0$ and $\alpha_\rho(0) = \frac{1}{2}$.

*The results in Tables 2,3,4 are proportional to $\pm \int_{-1}^{+1} dz P_\ell(z) R_N(\alpha_t)$ the overall sign being chosen for each Table such that the rho is positive. Unlike Table 1, the overall normalizations have no significance here.

VII. MULTIPARTICLE EXTENSION (GENERAL THEORY)

We should recall that a complete theory (of, for example, the strangeness zero meson sector) can be specified by all the on-shell Born amplitudes A_N . It is towards this goal that we now proceed.

We have suggested that the four-meson Born amplitude be written in the form

$$A_4(-\alpha_s, -\alpha_t) = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4 \quad (83)$$

in which $\phi_4(\alpha_s, \alpha_t, \alpha_u, x)$ is invariant under an S_3 group acting on the objects $\{\alpha_i, \beta_i(x)\}$, $i = s, t, u$.

Here we indicate how such a general form may be extended to a multiparticle amplitude. In particular we shall show how the symmetric group properties of the integrand can be preserved in the multiparticle extension.*

We introduce, for A_4 , the variables⁽⁷⁾ z_i ($i=1,2,3,4$) lying on a circle in the complex z -plane. We make the identification

$$x = (z_2, z_1; z_3, z_4) = (23)/(13) \quad (84)$$

$$(1-x) = (z_1, z_4; z_2, z_3) = \frac{(12)}{(13)} \quad (85)$$

* The ambiguity inherent in any multiparticle extension is being resolved by our insistence on this invariance.

where the notation for an anharmonic ratio is

$$(a,b;c,d) = \frac{(z_a - z_c)(z_b - z_d)}{(z_a - z_d)(z_b - z_c)} \quad (86)$$

and we have introduced the convenient shorthand notation (simply to make some later equations more manageable)

$$(12) = (z_1 - z_2)(z_3 - z_4) \quad (87)$$

$$(23) = (z_2 - z_3)(z_1 - z_4) \quad (88)$$

$$(13) = (z_1 - z_3)(z_2 - z_4) \quad (89)$$

We note first that the Euler B function may be written (for $\alpha_\rho(0) \neq \alpha_\pi(0)$)

$$B(-\alpha_s, -\alpha_t) = \int \prod_{i=1}^4 dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \\ \prod_{i < j} (z_i - z_j)^{-2p_i p_j} \prod_{i=1}^4 (z_i - z_{i+1})^{\alpha_\rho(0) - 1} \\ \left[\prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} \right]^{2(\alpha_\rho(0) - \alpha_\pi(0))} \quad (90)$$

where the z_a, z_b, z_c are fixed values of three out of the four z_i (arbitrarily chosen) and the integration domain is the boundary of the circle, with the restriction that the cyclic ordering of the points z_1, z_2, z_3, z_4 is preserved.

It is useful to note the following identity

$$\frac{\gamma}{2} = -\frac{1}{2}(\alpha_\rho(0) - 1) + 2(\alpha_\rho(0) - \alpha_\pi(0)) \quad (91)$$

This motivates us to find two basic identities to reexpress the fixed-point branch points in terms of z_i variables.

Note first that

$$\prod_{i=1}^4 (z_i - z_{i+1})^2 = (12)^2 (23)^2 \quad (92)$$

and consider the symmetric expression

$$\sum_{\{q_1 q_2 q_3 q_4\}} \prod_{k=1}^4 (z_{q_k} - z_{q_{k+1}})^{-2} \quad (93)$$

where the sum is over the $4!$ permutations of $\{q_1, q_2, q_3, q_4\} = \{1, 2, 3, 4\}$.

This expression is equal to

$$\begin{aligned} & 8 \left[\frac{1}{(12)^2 (23)^2} + \frac{1}{(12)^2 (13)^2} + \frac{1}{(23)^2 (13)^2} \right] \\ &= 16 \prod_{i=1}^4 (z_i - z_{i+1})^{-2} (x - e^{i\pi/3})(x - e^{-i\pi/3}) \end{aligned} \quad (94)$$

This second identity is for the expression

$$\prod_{i < j} (z_i - z_j)^{(-1)^{i-j}} = \frac{(13)}{(12)(23)} \quad (95)$$

The corresponding symmetrized form satisfies

$$\begin{aligned} & \sum_{\{q_1 q_2 q_3 q_4\}} (-1)^P \prod_{k < \ell} (z_{q_k} - z_{q_\ell})^{(-1)^{k-\ell}} = \\ & 8 \left[\frac{(13)}{(12)(23)} + \frac{(23)}{(12)(13)} + \frac{(12)}{(23)(13)} \right] \end{aligned} \quad (96)$$

and hence we may derive the identity

$$\begin{aligned} & (x - e^{i\pi/3})(x - e^{-i\pi/3}) = \frac{1}{16} \prod_{i < j} (z_i - z_j)^{-(-1)^{i-j}} * \\ & * \sum_{\{q_1 q_2 q_3 q_4\}} (-1)^P \prod_{k < \ell} (z_{q_k} - z_{q_\ell})^{(-1)^{k-\ell}} \end{aligned} \quad (97)$$

In these expressions P is the parity ($\pm$) of the permutation.

Collecting these two identities together we arrive at the following

$$(x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} =$$

$$4^{-\gamma} \prod_{i=1}^4 (z_i - z_{i+1})^{1-\alpha_s(0)} \left[\prod_{i<j} (z_i - z_j)^{(-1)^{i-j}} \right]^{-2(\alpha_s(0)-\alpha_\pi(0))}$$

$$\left[\sum_{\{q_1 q_2 q_3 q_4\}} \prod_{k=1}^4 (z_{q_k} - z_{q_{k+1}})^{-2} \right]^{\frac{1-\alpha_s(0)}{2}}$$

$$\left[\sum_{\{q_1 q_2 q_3 q_4\}} (-1)^P \prod_{k<\ell} (z_{q_k} - z_{q_\ell})^{(-1)^{k-\ell}} \right]^{2(\alpha_s(0)-\alpha_\pi(0))} \quad (98)$$

We shall insert this equation into $A_4(-\alpha_s, -\alpha_t)$, but first we should discuss $\phi_4(\alpha_s, \alpha_t, \alpha_u, x)$. What does it become in terms of z variables? The answer is very beautiful, namely that the S_3 invariance of

$$\phi_4(\alpha_s, \alpha_t, \alpha_u, x) \equiv \phi_4 \left(\begin{array}{ccc} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{array} \right) \quad (99)$$

can be expressed as the S_4 invariance of

$$\phi_4 \left(\begin{array}{ccc} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{array} \right) = \phi_4 \left(\begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ z_1 & z_2 & z_3 & z_4 \end{array} \right) \quad (100)$$

under permutations of the argument pairs $\{p_i, z_i\}$. This is then immediately extendable to $N > 4$.

Collecting results, and absorbing numerical factors into an overall normalization, we obtain for the N -particle amplitude,

$$A_N = \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i < j} (z_j - z_i)^{-2p_i p_j} \left[S_N^{(1)}(z_i) \right]^{\frac{1-\alpha_\pi(0)}{2}} \left[S_N^{(2)} \right]^{2(\alpha_\pi(0) - \alpha_\pi(0))} \phi_N \left(\begin{matrix} p_1 & p_2 & p_3 & \dots & p_N \\ z_1 & z_2 & z_3 & \dots & z_N \end{matrix} \right) \quad (101)$$

where the whole integrand, including the final factor ϕ_N , is invariant under an S_N symmetric group acting on the argument pairs $\{p_i, z_i\}$. We have introduced the notations

$$S_N^{(1)}(z_i) = \sum_{\{q_1 q_2 \dots q_N\}} \prod_{k=1}^N (z_{q_k} - z_{q_{k+1}})^{-2} \quad (102)$$

$$S_N^{(2)}(z_i) = \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \prod_{k < \ell} (z_{q_\ell} - z_{q_k})^{(-1)^{\ell-k}} \quad (103)$$

Note that we have little freedom in the multiparticle extension discussed so far if we require S_N invariance. This is already a very strong result.

VIII. MULTIPARTICLE EXTENSION (SPECIFIC AMPLITUDE)

The extension of the specific four-meson Born amplitudes considered earlier has now been reduced to the generalization of a specific ϕ_4 to ϕ_N . Here again the symmetric group fully controls the generalization of the fixed-point poles in ϕ_4 .

Of the four-pion models, the following amplitude

$$A^{(1)} = \int_0^1 dx x^{-\alpha_t-1} (1-x)^{-\alpha_s-1} (x-e^{i\pi/3})^{\frac{\gamma}{2}} (x-e^{-i\pi/3})^{\frac{\gamma}{2}} \phi_4(1) \quad (104)$$

was selected as the best description, and therefore its multipion

extension is the one we will consider here. We should mention that the rejected amplitudes can be extended along similar lines.

We shall first write down, in detail, the final result for $A_N^{(1)}$ and then discuss its derivation. Our result for the multi-pionic Born term is

$$A_N^{(1)} = \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_b)(z_a - z_c)} \right]^{-1} \prod_{i < j} (z_j - z_i)^{-2p_i p_j} (S^{(1)}(z))^{\frac{1-\alpha_\rho(0)}{2}} (S_N^{(2)}(z))^{2(\alpha_\rho(0)-\alpha_\pi(0))-1} \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \frac{\alpha_{q_1 q_2} \alpha_{q_3 q_4} \dots \alpha_{q_{N-3} q_{N-2}}}{|z_{q_2} - z_{q_1}| |z_{q_4} - z_{q_3}| \dots |z_{q_{N-3}} - z_{q_{N-2}}| |z_{q_N} - z_{q_{N-1}}|} \quad (105)$$

where the notation for Regge trajectory functions is

$$\alpha_{ij} = \alpha_\rho(0) + \alpha' (p_i + p_j)^2 \quad (106)$$

and the domain of integration is the boundary of a circle in the complex z -plane, subject to the restrictions that z_a, z_b, z_c are held fixed (and chosen arbitrarily) and the other $(N-3)$ variables are maintained in their correct ordering $z_1, z_2, \dots, z_N, z_1$ corresponding to

the cyclic ordering of the external particles. The sum* is over all $N!$ permutations of $\{q_1 q_2 \dots q_N\} = 1, 2, 3, \dots, N$; and $P = \pm 1$ is permutational parity.

Note that the integrand simplifies if we impose $\alpha_{\rho}(\mu^2) = \frac{1}{2}$, and since this condition was already indicated by soft-pion consistency we advocate that it be met. The two free masses involved in m_{ρ}^2 and $\alpha'^{-1} = 2(m_{\rho}^2 - \mu^2)$ should then be taken from experimental observations; finally the overall normalization may be taken from $g_{\rho\pi\pi}^2$ in $A_4^{(1)}$ and thereafter, for higher N , is dictated by factorization.

In order to derive this expression for $A_N^{(1)}$ we rewrite

$$\phi_4^{(1)} \begin{pmatrix} \alpha_s & \alpha_t & \alpha_u \\ \beta_s(x) & \beta_t(x) & \beta_u(x) \end{pmatrix} = \alpha_s \beta_s(x) + \alpha_t \beta_t(x) + \alpha_u \beta_u(x) \quad (107)$$

$$= \phi_4^{(1)} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \quad (108)$$

$$= (S_4^{(2)})^{-1} \chi^{(1)} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \quad (109)$$

where we have used one of the symmetric functions (chosen both to simplify $\chi_4^{(1)}, \chi_N^{(1)}$ and to allow simplification under $\alpha_{\rho}(\mu^2) = \frac{1}{2}$) of the previous section.

* In the permutation sum of Eq. (105) and in certain later equations it is clear that not all $N!$ terms are different. To see a simple example $q_1 \leftrightarrow q_2$ is an invariance in Eq. (105). It is not difficult to show that the precise number of different terms in Eq. (105) is $N! [(N/2)! 2^{N/2}]^{-1}$.

One finds easily (using our earlier shorthand)

$$\chi_4^{(1)} = \frac{(13)}{(12)(23)} \left[\alpha_{12} \frac{(23)}{(13)} - \alpha_{13} \frac{(12)(23)}{(13)^2} + \alpha_{14} \frac{(12)}{(13)} \right] \quad (110)$$

$$= \frac{\alpha_{12}}{(z_2 - z_1)(z_4 - z_3)} - \frac{\alpha_{13}}{(z_3 - z_1)(z_4 - z_2)} + \frac{\alpha_{14}}{(z_4 - z_1)(z_3 - z_2)} \quad (111)$$

$$= \sum_{\{q_1 q_2 q_3 q_4\}} (-1)^P \frac{\alpha_{q_1 q_2}}{(z_{q_2} - z_{q_1})(z_{q_4} - z_{q_3})} \quad (112)$$

This form for $\chi_4^{(1)}$ is already manifestly S_4 invariant and may be extended directly to the final solution, $\chi_N^{(1)}$, manifestly S_N invariant.

It is worth checking that the final Born amplitude is projective invariant. To do this it is sufficient to show invariance separately under translations $z'_i = z_i + a$, dilations $z'_i = az_i$ and inversion $z'_i = z_i^{-1}$. Invariance under translations is obvious since only z -differences occur. Invariance under dilation follows from the fact that we pick up factors (given in the same order as they occur in our Eq. (105)).

$$a^N a^{-N(\alpha_\pi(0))} (a^{-2N})^{\frac{1-\alpha_\rho(0)}{2}} (a^{-N/2})^{2(\alpha_\rho(0)-\alpha_\pi(0))-1} a^{-N/2} = 1 \quad (113)$$

Under inversion we pick up a factor

$$\prod_{i=1}^N (z_i^{-2}) (z_i^{2\alpha_\pi(0)}) (z_i^4)^{\frac{1-\alpha_\rho(0)}{2}} (z_i)^{2(\alpha_\rho(0)-\alpha_\pi(0))-1} z_i = 1 \quad (114)$$

so that the projective invariance is proved.

In fact, we have just proved that every term in the permutation sum is separately projective invariant.

We shall use this last fact to prove that tachyons are indeed absent from $A_N^{(1)}$. The general mechanism by which tachyons are removed is evident from the formula, namely for every $(z_{q_{k+1}} - z_{q_k})^{-1}$ factor occurring in $\chi_N^{(1)}$ there is, in general, a factor $\alpha_{q_k q_{k+1}}$ which will ensure that no imaginary mass particle occurs at $\alpha_{q_k q_{k+1}} = 0$.

It seems at first sight, however, that we need $N/2$ such α -factors to do the job, while we have only $(N/2-1)$ α -factors. What happens in the channel of energy $(p_{q_{N-1}} + p_{q_N})^2$ where the α -factor is missing? We shall spend some effort to explain what is happening in detail since we believe this is very important.

The point is that by introducing redundant variables to facilitate the multiparticle extension we have led ourselves into this (apparent) difficulty. Actually since every term in the sum is separately projective invariant we may rewrite Eq. (105) as, for example

$$\begin{aligned}
 A_N^{(1)} = & \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \alpha_{q_1 q_2} \alpha_{q_3 q_4} \dots \alpha_{q_{N-3} q_{N-2}} \\
 & \int \prod_{i=1}^N dz_i \left[\frac{dz_{q_1} dz_{q_{N-1}} dz_{q_N}}{(z_{q_{N-1}} - z_{q_1})(z_{q_N} - z_{q_{N-1}})(z_{q_N} - z_{q_1})} \right]^{-1} \frac{1}{(z_{q_N} - z_{q_{N-1}})} \\
 & (S_N^{(1)}(z)) \frac{1 - \alpha_\rho(0)}{2} (S_N^{(2)}(z))^{2(\alpha_\rho(0) - \alpha_\pi(0)) - 1} \\
 & [(z_{q_2} - z_{q_1})(z_{q_4} - z_{q_3}) \dots (z_{q_{N-2}} - z_{q_{N-3}})]^{-1}
 \end{aligned} \tag{115}$$

and now the dangerous factor $(z_{q_N} - z_{q_{N-1}})$ is simply a constant.

This observation that only $(\frac{N}{2}-1)$ rather than $(\frac{N}{2})$ α -factors are sufficient to eliminate tachyons may give an intuition into why $A_N^{(1)}$ is the correct amplitude rather than the closely allied $A_N^{(4)}$ which we eliminated earlier (and which is obtained from Eq. (105) simply by adding an alpha factor $\alpha_{q_{N-1} q_N}$). The point is simplicity and economy: Why should the amplitude have superfluous factors?

The structure of $A_N^{(1)}$ gives evidence* for an underlying anti-commutative algebra, and we interpret this as the sign that we are describing mesons as if made (at least, mathematically) of fermionic quarks.

IX. CONCLUDING REMARKS

It is clearly important that further checks of the consistency of $A_N^{(1)}$ should be made, e.g., one should confirm that multiparticle factorization on a positive space is possible. In these theoretical matters, it is unlikely that we have fully exploited the power of the symmetric group, especially the fact of S_N invariance for every N .

Most important of all, of course, the spectrum uniquely determined by our set of multipionic Born amplitudes Eq. (105) should be analyzed and compared in some detail to the experimental observations of the non-strange mesons. This is one extremely important facet of the theory which can be checked rather directly. It should

*We refer to the similarity of its form to that arising naturally from an expectation value of anticommuting field operators.

be emphasized, however, that such a comparison with experiment of our particular Born amplitudes, Eq. (105), will test in the first instance only our specific simplicity criteria. The general theory of Sections II, III, and VII can be much more accommodating and could (if necessary) provide extra flexibility to allow a successful description of observed mesonic properties.

Finally, on the purely mathematical side, the advantages of our approach can be indicated by its ability to accommodate rather simply certain previously-suggested theoretical models; details of this are presented in the Appendix.

APPENDIX: RELATIONSHIP TO EARLIER WORK

The symmetric group enables us to present a unified view of three earlier different schemes. These earlier schemes arise as special cases, extrapolated to unphysical mass values, of the symmetric group approach.

(i) The unit-rho-pi-intercept A_4 of Veneziano⁽¹⁾ and its A_N extension⁽²⁾ of Bardakci-Chan-Goebel-Koba-Nielsen-Ruegg-Sakita-Tsun-Virasoro correspond to the case $\alpha_\rho(0) = \alpha_\pi(0) = 1$ and $\phi_N = 1$ in our general theory. This proposal was important historically as the first closed-form Born amplitude. Indeed the two extra branch-points in the present A_4 do not change the resonance-pole-producing mechanism already inherent in the Eulerian Integral of the First Kind.

(ii) The unit-rho-intercept A_4 of Lovelace-Shapiro⁽¹¹⁾ and its A_N extension⁽⁸⁾ of Neveu-Schwarz (NS) can be located without great difficulty. The NS model is formulated in terms of certain anti-commuting operators defined by

$$\{b_\mu^{(m)}, b_\nu^{(n)+}\}_+ = -\delta_{mn} g_{\mu\nu} \quad (A1)$$

and of a field

$$H_\mu(z) = \sum_{n=1}^{\infty} (b_\mu^{(n)} z^n + b_\mu^{(n)+} z^{-n}) \quad (A2)$$

These can be removed by rewriting

$$A_N^{(NS)} = \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i < j} (z_j - z_i)^{-2p_i p_j} \prod_{i=1}^N z_i^{-1/2} \langle 0 | p_i \cdot H(z_1) p_2 \cdot H(z_2) \dots p_N \cdot H(z_N) | 0 \rangle \quad (A3)$$

$$= \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i < j} (z_j - z_i)^{-2p_i p_j} \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \frac{\alpha_{q_1 q_2} \alpha_{q_2 q_3} \dots \alpha_{q_{N-1} q_N}}{|z_{q_2} - z_{q_1}| |z_{q_4} - z_{q_3}| \dots |z_{q_N} - z_{q_{N-1}}|} \quad (A4)$$

$$= \int \prod_{i=1}^N dz_i \left[\frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \right]^{-1} \prod_{i < j} (z_j - z_i)^{-2p_i p_j} \sum_{\{q_1 q_2 \dots q_N\}} (-1)^P \frac{\alpha_{q_1 q_2} \alpha_{q_3 q_4} \dots \alpha_{q_{N-3} q_{N-2}}}{|z_{q_2} - z_{q_1}| |z_{q_4} - z_{q_3}| \dots |z_{q_N} - z_{q_{N-1}}|} \quad (A5)$$

where Eq. (A4) follows from application of the Caianello-Fubini algorithm⁽¹⁵⁾ and Eq. (A5) follows most easily from a reformulation of Neveu-Schwarz-Thorn.⁽¹⁶⁾ On extrapolation to the unphysical supplementary condition $\gamma = 2$, therefore, our solutions $A_N^{(1)}$ and $A_N^{(4)}$ of the text coincide both with each other and with the Neveu-Schwarz model. This model was important because its non-strange meson spectrum bore a striking qualitative similarity to that which is experimentally observed.

(iii) The equi-rho-pi-intercept A_4 of Mandelstam⁽¹²⁾ and its A_N extension⁽¹⁷⁾ of Gervais-Neveu correspond to putting $\alpha_p(0) = \alpha_{\pi}(0)$ and $\phi_N = 1$ in our general theory of multiparticle extension. This formula contained the important fixed-point singularities.

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Table 1: Partial widths for $\pi\pi$ decay predicted by the model $A_4^{(1)}$ of the text. The widths are normalized to $\Gamma_\rho = 112\text{MeV}$ and the rho trajectory is $0.49 + 0.9t$. Also $\alpha_\rho(0) - \alpha_\pi(0) = 1/2$. The Table is designed to be compared directly to Table 1 on page 1348 of J. A. Shapiro, Phys. Rev. 179, 1345(1969). From the present table an accurate impression of the smooth behaviour of the Born amplitude residua may be gained simply by removing the isotopic spin factor $3/2$ that favors the even spin resonances.

J^P	Mass (MeV)								
	760	1300	1673	1978	2241	2476	2691	2891	3077
9^-									3
8^+								5	
7^-							6		16
6^+						13		34	
5^-					14		32		8
4^+				34		71		12	
3^-			38		69		8		33
2^+		96		149		13		47	
1^-	112		141		16		24		32
0^+		259		68		15		54	

Table 2. Partial wave projections for $A_4^{(2)}$, as described in the text. We take $\mu^2 = 0$ and, $\alpha_p(0) = 1/2$.

J^P	Mass (MeV)					
	745	1291	1667	1972	2236	2472
6^+						12
5^-					19	
4^+				34		127
3^-			49		195	
2^+		63		315		-628
1^-	67		-536		-925	
0^+		1053		-1160		609

Table 3. Partial wave projections for $A_4^{(3)}$, as described in the text. We take $\mu^2 = 0$ and $\alpha_p(0) = 1/2$.

J^P	Mass (MeV)					
	745	1291	1667	1972	2236	2472
6^+						142
5^-					148	
4^+				165		-884
3^-			177		-989	
2^+		152		-936		1745
1^-	67		-595		705	
0^+		152		-620		2156

Table 4. Partial wave projections for $A_4^{(4)}$, as described in the text. We take $\mu^2 = 0$ and $\alpha_\rho(0) = 1/2$.

J^P	Mass (MeV)					
	745	1291	1667	1972	2236	2472
6^+						17
5^-					23	
4^+				34		90
3^-			47		126	
2^+		62		177		-141
1^-	67		253		-207	
0^+		377		-239		170