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LOWER BOUNDS FOR π - π PARTIAL-WAVE AMPLITUDES*

H. W. Rupertsberger[†]

Physics Department, Syracuse University, Syracuse, N. Y. 13210

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[†]Max Kade Fellow. On leave of absence from Institut für Theoretische Physik der Universität Wien

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ABSTRACT

Lower bounds for the $\ell \geq 2$ $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ and $\pi^+\pi^- \rightarrow \pi^0\pi^0$ partial-wave amplitudes are derived. These bounds are dependent on the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ and $\pi^0\pi^+ \rightarrow \pi^0\pi^+$ scattering amplitudes respectively in the unphysical region. To obtain these inequalities the analytic function approach by Okubo is used.

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I. INTRODUCTION

The high symmetry of the π - π system has made it possible to derive a number of inequalities for the scattering and partial-wave amplitudes by use of analyticity, crossing symmetry and unitarity.¹ It is the purpose of this work to apply a special inequality for analytic functions, derived originally by Meiman² and recently in a generalized form by Okubo,³ to obtain lower bounds for certain combinations of the $l = 0$ and 2 , $l \geq 2$ partial-wave amplitudes. This inequality holds under the following conditions:

Let $f(z)$ be a real analytic function in z with a cut for $z_0 \leq z < \infty$, $w(z)$ be a nonnegative function known along the cut and

$$\frac{1}{\pi} \int_{z_0}^{\infty} dz w(z) |f(z)|^2 \leq I^2 \quad (1)$$

where I is a known constant. Then

$$|f(\beta)|^2 \exp\{\epsilon\} \leq I^2 \text{ for all } \beta < z_0 \quad (2)$$

where

$$\epsilon = \frac{1}{2\pi} \int_0^{2\pi} d\theta \log \tilde{w}(\theta) \quad (3)$$

$$\tilde{w}(\theta) = (z-\beta)(z-z_0)^{\frac{1}{2}}(z_0-\beta)^{-\frac{1}{2}}w(z)$$

and z is connected with θ by the conformal mapping

$$(z-z_0)^{\frac{1}{2}} = i(z_0-\beta)^{\frac{1}{2}} \frac{1+e^{i\theta}}{1-e^{i\theta}} \quad (4)$$

II. INEQUALITIES FOR SCATTERING AMPLITUDES

To apply this inequality to the π - π scattering amplitudes one needs special combinations of the isospin I amplitudes $F^I(s,t)$ in the s -channel, which satisfy the above analyticity properties. There exist two such combinations $F_j(s,t)$ ($j=0,1$) with

$$F_0(s,t) = \frac{1}{3} F^0(s,t) + \frac{2}{3} F^2(s,t) \quad (5)$$

$$F_1(s,t) = F^1(s,t) + F^2(s,t)$$

The first one corresponds to the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$, the second one to the $\pi^0\pi^+ \rightarrow \pi^0\pi^+$ scattering amplitude. As is well known, both amplitudes fulfil a twice subtracted dispersion relation in s for fixed t in the region $0 < t < 4$ [we set the pion mass $m=1$]

$$F_j(s,t) - F_j(s_0,t) = \frac{(s-s_0)(s+s_0+t-4)}{\pi} \int_{s_0}^{\infty} ds' \frac{(2s'+t-4) \text{Im} F_j(s',t)}{(s'-s)(s'+s+t-4)(s'-s_0)(s'+s_0+t-4)} \quad (6)$$

Using instead of s the symmetrizing variable z related to s by

$$s = 2(\sqrt{z}+1) - \frac{t}{2} \quad (7)$$

one sees immediately that $F_j(s,t)$ is a real analytic function in z , provided $-t \leq s_0 \leq 4$, with only one cut for $(1+t/4)^2 \leq z < \infty$. By application of the Cauchy-Schwarz inequality and the unitarity condition

$$\text{Im } a_\ell^I(s) \geq |a_\ell^I(s)|^2 \quad \text{for } 4 \leq s < \infty \quad (8)$$

to the partial-wave expansion

$$F^I(s,t) = \sqrt{\frac{s}{s-4}} \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell^I(s) P_\ell\left(1+\frac{2t}{s-4}\right) \quad (9)$$

one obtains the following inequality for $\text{Im} F_j(s,t)$

$$|F_j(s,0)|^2 \leq h_j(s,t) \text{Im} F_j(s,t) \quad \text{for } 0 < t \leq 4, 4 \leq s < \infty$$

$$h_j(s,t) = \sqrt{\frac{s}{s-4}} \sum_j \frac{2\ell+1}{P_\ell\left(1+\frac{2t}{s-4}\right)} \quad \Sigma_0 = \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{\infty} \quad \Sigma_1 = \sum_{\substack{\ell=0 \\ \text{all } \ell}}^{\infty} \quad (10)$$

Inserting this inequality into the dispersion relation (6) one gets for the analytic functions $F(2(\sqrt{z}+1),0)$ from (2) the inequality

$$\frac{F_j(s,t) - F_j(s_0,t)}{(s-s_0)(s+s_0+t-4)} \geq |F_j(2\sqrt{\beta}+1,0)|^2 \exp\{\epsilon\}$$

for

(11)

$$\beta < 1, -t \leq s \leq 4, -t \leq s_0 \leq 4, 0 < t \leq 4$$

where the weight function $w(z)$, defining ϵ , is given along the cut starting at $z_0 = 1$ by

$$w(z) = (4\sqrt{z}+t) \{ \sqrt{z} [2(\sqrt{z}+1)-s] [2(\sqrt{z}-1)+s+t] \cdot [2(\sqrt{z}+1)-s_0] [2\sqrt{z}-1+s_0+t] h_j(2(\sqrt{z}+1),t) \}^{-1} \quad (12)$$

For $j = 0$ ($\pi^0\pi^0$ scattering amplitude) the above inequality has been exploited by Okubo³ for the special set of values $\beta = 0$, $t=2$, $s_0=4$ and $s=2$ to derive a lower bound for the $\ell=0$, $\pi^0\pi^0$ scattering length. It is evident that taking the limit $s \rightarrow s_0$ in (11) gives an inequality for $dF_j(s,t)/ds$.

III. INEQUALITIES FOR PARTIAL-WAVE AMPLITUDES

Application of relations (10) and (2) to the crossed Froissart-Gribov formula

$$\left(\frac{t}{t-4}\right)^{\frac{1}{2}} a_{j,\ell}(t) = \frac{4}{\pi(4-t)} \int_4^\infty ds \operatorname{Im} F_j(s,t) Q_\ell\left(\frac{2s}{t-4}-1\right) \text{ for } \ell \geq 2 \text{ even} \quad (13)$$

where the $a_{j,\ell}(t)$ are defined by

$$a_{0,\ell}(t) = \frac{1}{3} a_\ell^0(t) + \frac{2}{3} a_\ell^2(t) \quad (\pi^0\pi^0 \rightarrow \pi^0\pi^0 \text{ partial-wave amplitude}) \quad (14)$$

$$a_{1,\ell}(t) = \frac{2}{3} a_\ell^0(t) - \frac{2}{3} a_\ell^2(t) \quad (\pi^0\pi^0 \rightarrow \pi^+\pi^- \text{ partial-wave amplitude})$$

leads to the inequality

$$\left(\frac{t}{t-4}\right)^{\frac{1}{2}} a_{j,\ell}(t) \geq \frac{4}{4-t} |F_j(2(\sqrt{\beta}+1),0)|^2 \exp\{\epsilon\} \quad (15)$$

$$\text{for } 0 < t \leq 4, \ell \geq 2, \beta < 1$$

where the weight function $w(z)$, dependent on ℓ , is now given by

$$w(z) = \frac{1}{\sqrt{z}} \frac{Q\left(\frac{4\sqrt{z}+t}{4-t}\right)}{h_j(2(\sqrt{z}+1), t)} \quad (16)$$

and the cut in the z -plane again starts at $z_0 = 1$. By taking the limit $t \rightarrow 4$ and defining the scattering lengths $\alpha_{j,\ell}$ by

$$\alpha_{j,\ell} = \lim_{t \rightarrow 4} \{a_{j,\ell}(t) \left(\frac{4}{t-4}\right)^{\ell+\frac{1}{2}}\} \quad (17)$$

one obtains from (15)

$$\alpha_{j,\ell} > \frac{\ell! \sqrt{\pi}}{\Gamma(\ell+3/2)} |F_j(2(\sqrt{\beta}+1), 0)|^2 \exp\{\epsilon\} \quad (18)$$

$$w(z) = \frac{1}{\sqrt{z}} \frac{1}{[2(\sqrt{z}+1)]^{\ell+1}} \frac{1}{h_j(2(\sqrt{z}+1), 4)}$$

For $\beta = 0$ and using an estimate for h_j one gets finally the following inequalities for the scattering lengths $\alpha_{j,\ell}$

$$\alpha_{j,\ell} \geq b_\ell c_j |F_j(2, 0)|^2 \quad \text{for } \ell \geq 2 \quad (19)$$

where b_ℓ and c_j are defined by

$$b_\ell = \frac{[\ell!]^2 2^\ell e^{-\frac{4C}{\pi}(\ell+1)}}{(2\ell+1)!} \quad (20)$$

$$c_0 = 0.43 \quad c_1 = 0.32$$

and C is Catalan's constant ($C=0.91597$). Unfortunately there is no information about $F_j(2(\sqrt{\beta}+1), 0)$ and therefore the inequalities (15), (18), and (19) give only general conditions which every model for π - π scattering has to fulfil.

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