FOUNDATIONS OF S-MATRIX THEORY I
THEORY AND MEASUREMENT *

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ABSTRACT

S-matrix theory is systematically developed from basic principles in a series of reports of which this is the first. The question of measurement and space-time interpretation is dealt with here.

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ORIENTATION

"The task of science is both to extend the range of our experience and reduce it to order,...Strictly speaking, the mathematical formalism of quantum mechanics and electrodynamics merely offers rules of calculation for the deduction of expectations about observations obtained under well-defined experimental conditions specified by classical physical concepts."

Niels Bohr¹

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A. LOGICAL FRAMEWORK

Quantum theory is a procedure by which scientists calculate predictions about observations obtained under certain types of conditions. These conditions are described by saying that a system is first prepared according to certain specifications, and is then examined according to certain specifications. And this examination is such that it can either yield or not yield various possible specified results.

The calculational procedure is as follows:

Let A represent the specifications on the manner of preparation. And let B represent the specifications on both the manner of examination, and one of its possible results. These specifications A and B are described in the language of the experimental physicist. They are described in a language that allows trained technicians to prepare systems in ways that meet the specifications A , to examine systems in ways that meet the specifications B , and to determine whether the results of such examinations meets the specifications B . The descriptions A and B are thus operational, in the sense that they describe how technicians set up experiments and recognize results. Moreover they are objective, in that they refer not to any individual technician or scientists, but rather to the actions and observations of any properly trained technician. To ensure objectivity these specifications can be formulated in the language of classical physics. In this case the specifications are formulated in terms of the idealizations of classical physics.

These specifications A and B are mapped, by a procedure to be described presently, into corresponding elements of the mathematical formalism. The specifications A on the preparation are

mapped into a density function $\rho_A(\xi'_A; \xi''_A)$ that has the form

$$\rho_A(\xi'_A; \xi''_A) = \psi_A^*(\xi'_A) \psi_A(\xi''_A)$$

in the special "pure" cases. In the general "mixed" cases $\rho_A(\xi'_A; \xi''_A)$ is a sum of terms of the pure case form. The functions $\psi_A(\xi_A)$ are called wave functions of the prepared system, and ξ_A is a set of variables called the degrees of freedom of the prepared system.

Similarly, the specifications B on the examination are mapped in a density function $\rho_B(\xi'_B; \xi''_B)$, where ξ_B is a set of variables called the degrees of freedom of the examined system. These degrees of freedom ξ_A and ξ_B are discussed in the next section.

The transition function $S(\xi_A; \xi_B)$ is constructed from theoretical principles to be described later. Then the transition probability $\mathcal{P}(A; B)$ is defined by

$$\mathcal{P}(A; B) = \int d\xi'_A d\xi''_A d\xi'_B d\xi''_B$$

$$\times \rho_A(\xi''_A; \xi'_A) S(\xi'_A; \xi_B) \rho_B(\xi'_B; \xi''_B) S^\dagger(\xi''_B; \xi_A)$$

$$= \text{Tr } \rho_A S \rho_B S^\dagger \quad (\text{A.1})$$

where

$$S^\dagger(\xi_B; \xi_A) = S^*(\xi_A; \xi_B). \quad (\text{A.2})$$

The transition probability $\mathcal{P}(A; B)$ is the predicted limit of the relative frequency of results of the kind specified by B as the number of systems prepared in a manner meeting specifications A and examined as the manner meeting specifications B tends to infinity.

If ρ_A and ρ_B are both of the pure case form then the transition probability formula (A.1) reduces to

$$P(A; B) = |S[\psi_A, \psi_B^*]|^2, \quad (A.3)$$

where $S[\psi_A, \psi_B^*]$ is defined by

$$S[\psi_A, \psi_B^*] \equiv \int d\xi_A d\xi_B \psi_A(\xi_A) S(\xi_A; \xi_B) \psi_B^*(\xi_B). \quad (A.4)$$

The quantity $S[\psi_A, \psi_B^*]$ is called the transition amplitude.

B. THE DEGREES OF FREEDOM

Quantum theory predicts statistical correlations between complex preparing and examining systems. These systems consist of the preparing and examining devices together with their environments, which include in the final analysis the technicians themselves. Yet the predictions are calculated from a formula based on the degrees of freedom of the relatively simple prepared and examined systems. To apply quantum theory one must identify these degrees of freedom and must specify the conditions under which the empirical correlations between the complex preparing and examining systems are expected to have the simple predicted form.

The empirical correlations are expected to have the simple predicted form only if certain conditions are met. First, the space-time region associated with the process of preparation should be separated from the space-time region associated with the process of examination. These two processes are represented in the transition probability formulas by independent factors, ρ_A and ρ_B . But the corresponding physical processes cannot be expected to be physically independent unless the space-time regions in which they occur are separated. That is, the theoretical structure will not correspond to the physical structure unless these two space-time regions are separated. Second, the locations of these two space-time regions should be compatible with the idea that the prepared system travels from the region of the preparation to the region of the examination, changing on its journey to a system of the type being examined. The empirical correlations cannot be expected to have the predicted form unless the information about the preparation can be conveyed by the quantum system from the preparing devices to the examining devices.

Third, the effect of the interaction between the preparing and examining devices associated with the exchanged quantum system should not be masked by the effects of other interactions between these devices. The effect of the exchanged quantum system must dominate over other effects, if it is to observably manifest itself.

These demands can be met only if the interaction between preparing and examining devices associated with the exchanged quantum system is long range. The interaction must be long range both in order to bridge the separation between the space-time regions associated with the processes of preparation and examination, and in order to dominate over the short-range interactions that effectively constitute the measuring devices themselves. This dominance arises from the relative damping of these short-range interactions brought about by the space-time separation of the regions of preparation and examination. The quantum system is therefore, in effect, the representation of a long-range component of the interaction between the preparing and examining systems. This situation is depicted in Fig. 1.

Fig. 1. The space-time regions associated with the processes of preparation and examination must be separated in order for these processes to be independent, and in order for the effect of the exchanged quantum system to dominate over the effects of the short-range interactions that effectively constitute the devices themselves. The quantum system represents, in effect, a long-range component of the interaction between the preparing and examining systems.

The interactions of longest range are those associated with stable systems. Correspondingly, the degrees of freedom of the quantum systems are taken to be degrees of freedom associated with stable systems. Stable systems are considered to evolve eventually into sets of elementary stable systems, called "particles." The degrees of freedom of particle i consists of a triad (p_i, μ_i, t_i) . Here t_i is the type index, which specifies the "type" of particle i --whether it is a proton, or electron, or positron, etc. The index μ_i is the spin index, which specifies the component of spin with respect to a prescribed spin axis. The variable p_i is an energy-momentum four vector, which satisfies the mass-shell constraint

$$p_1^0 = (m_1^2 + (\vec{p}_1)^2)^{\frac{1}{2}} \quad (\text{B.1})$$

Here $m_1 \equiv m(t_1)$ is a real constant called the mass of particles of type t_1 .

The variables ξ_A and ξ_B are taken to be sets of single-particle variables; they are taken to be variables of the form $((p_1, \mu_1, t_1; \dots; p_n, \mu_n, t_n))$. The integration over ξ in (A.1) denotes an integration over the various $\vec{p}_1$, and a sum over the various discrete variables n, μ_1 , and t_1 .

It is convenient to express the integration over $\vec{p}_1$ in the covariant form

$$\begin{aligned} \int_{p_1} & \equiv \int \frac{d^4 p_1}{(2\pi)^4} 2\pi \delta(p_1^2 - m_1^2) \theta(p_1^0) 2m_1 \\ & = \int \frac{m_1}{p_1^0} \frac{d^3 p_1}{(2\pi)^3}, \end{aligned} \quad (\text{B.2})$$

where p_1^0 is defined by (B.1). The choice of weight factor in (B.2) has no physical significance; it merely determines the normalization of the functions $\rho_A(\xi'_A; \xi''_A)$, $\rho_B(\xi'_B; \xi''_B)$ and $S(\xi'_A; \xi'_B)$.

Massless particles require special consideration. This is because infinite numbers of them must be considered together. They will be discussed later. Until then they are ignored, along with the closely related electromagnetic, weak, and gravitational interactions.

Quantum systems have been associated here with stable systems. Variables corresponding to unstable or transient systems are not included among the quantum theoretical degrees of freedom because transients do not satisfy the unitarity requirement ordinarily imposed by quantum dynamics. On the other hand, if, as in nonrelativistic quantum theory, there is a unitary operator that generates the temporal evolution of the quantum system, then this evolution can be followed into the asymptotic domain without loss of information. Thus in this situation the prepared and examined systems can, without loss of information, be represented by the asymptotic systems into which they would eventually evolve, if left undisturbed, or from which they would evolve.

This procedure of representing the preparations and examinations by asymptotic systems can apply also in more general circumstances: it can apply even if there is no temporal evolution operator that transforms the "state at one instant of time" into the "state at some other instant of time". The concept of the state of a system at some instant of time--or on some space-like surface--ascribes special status to time (as opposed to space) and may thus be inappropriate in the relativistic domain. The concept of the asymptotic system, on the other hand, ascribes no special status to time. In fact, neither time nor space occur among the degrees of freedom ξ_A and ξ_B .

C. SPACE-TIME DISPLACEMENTS AND TEMPORAL EVOLUTION

The degrees of freedom ξ_A and ξ_B contain no time variables. Nevertheless, the theory yields precise predictions about temporal evolution. For time variables occur in the specifications A and B , and the transition probabilities $P(A; B)$ depend in a simple way on these times.

The temporal evolution of $P(A; B)$ is determined in the following way. Suppose A represents a set of specifications on the manner of preparation. Then let A^u represent the same set of specifications, except that each space-time four-vector x occurring in A is replaced by $x + u$. Similarly, let B^u be the same as B , except that each space-time four-vector x occurring in B is replaced by $x + u$. The following assumptions are then introduced:

$$\psi_{A^u}(p_1, \dots, p_n) = \psi_A(p_1, \dots, p_n) e^{iu \sum p_i}$$

and

$$\psi_{B^u}(p_1, \dots, p_n) \equiv \psi_B(p_1, \dots, p_n) e^{iu \sum p_i}.$$

Here $u p_i = \sum u_\mu p_i^\mu = u^0 p_i^0 - \vec{u} \cdot \vec{p}_i$.

In terms of the ρ_A and ρ_B these rules become

$$\begin{aligned} \rho_{A^u}(p_1'', \dots, p_n''; p_1', \dots, p_m') \\ = \rho_A(p_1'', \dots, p_n''; p_1', \dots, p_m') e^{iu[\sum p_i' - \sum p_i'']} \end{aligned} \quad (C.1a)$$

and

$$\begin{aligned} \rho_{B^u}(p_1', \dots, p_n'; p_1'', \dots, p_m'') \\ = \rho_B(p_1', \dots, p_n'; p_1'', \dots, p_m'') e^{iu[\sum p_i'' - \sum p_i']} \end{aligned} \quad (C.1b)$$

The formulas (C.1a) and (C.1b), taken together, are regarded as a fundamental postulate. The units of u are fixed so that Planck's constant is unity. The temporal evolution of the transition probability $\mathcal{P}(A; B)$ is obtained by inserting (C.1) into (A.1).

The degrees of freedom ξ_A and ξ_B are those appropriate to an asymptotic system of freely moving particles. However, there is no stipulation that the preparations and examinations specified by A and B be performed at asymptotic times. For example, the temporal evolution of $\mathcal{P}(A; B)$ calculated from (C.1) and (A.1) holds even if the examination is performed long before the prepared system evolves into a system of essentially freely moving stable particles. The temporal evolution of $\mathcal{P}(A; B)$ calculated from (A.1) and (C.1) holds for all times. And it is expected to represent the evolution of the empirical correlations to the extent that the space-time regions associated with the processes of preparation and examination are located in a way such that the effects of the exchanged quantum system are not masked by other effects.

The temporal evolution of the transition probability $\mathcal{P}(A; B)$ is calculated in this way, for all times, by a procedure that does not use--or depend on the existence of--the temporal evolution operator of Schrodinger theory. To see how this has come about it is useful to compare the Schrodinger procedure for calculating the temporal evolution of $\mathcal{P}(A; B)$ with the S-matrix procedure. The Schrodinger procedure is based on the idea that there is a unitary operator $U(t_1, t_2)$ that converts the state vector $\psi(t_1)$ that represents the system at time t_1 into the state vector $\psi(t_2)$ that represents the system at time t_2 :

$$\psi(t_1) U(t_1, t_2) = \psi(t_2) . \quad (C.2)$$

[Please read from left to right.]

In the Schroedinger formulation the specifications A on the preparation are mapped into a state vector $\psi_A(t_A)$ that represents the prepared system at the time of the preparation:

$$A \rightarrow \psi_A(t_A) . \quad (C.3a)$$

And the specifications B on the examination are mapped into a state vector $\psi_B(t_B)$ that represents the detected system at the time of the examination:

$$B \rightarrow \psi_B(t_B) . \quad (C.3b)$$

The state vector $\psi_A(t_A)$ is then converted to $\psi_A(t_B)$ by $U(t_A, t_B)$, and the transition amplitude is given by

$$\begin{aligned} S_{AB} &= \psi_A(t_A) U(t_A, t_B) \psi_B^\dagger(t_B) \\ &= \psi_A(t_B) \psi_B^\dagger(t_B) . \end{aligned} \quad (C.4)$$

In the S-matrix formulation the preparation A is mapped onto a time independent wave function (or state vector):

$$A \rightarrow \psi_A . \quad (C.5a)$$

And the examination B is mapped onto a time independent wave function (or state vector):

$$B \rightarrow \psi_B . \quad (C.5b)$$

If the Schroedinger operator $U(t_1, t_2)$ is assumed to exist then these states ψ_A and ψ_B can be formally defined in terms of it, and the corresponding free-particle operator $U_0(t_1, t_2)$, in the following way:

$$\psi_A = \lim_{t \rightarrow -\infty} \psi_A(t) U_0(t, 0) \quad (C.6a)$$

and

$$\psi_B = \lim_{t \rightarrow \infty} \psi_B(t) U_0(t, 0) , \quad (C.6b)$$

where for any t

$$\psi_A(t) = \psi_A(t_A) U(t_A, t) \quad (C.7a)$$

and

$$\psi_B(t) = \psi_B(t_B) U(t_B, t) . \quad (C.7b)$$

And the operator S is defined by

$$S = \lim_{t \rightarrow \infty} U_0(0, -t) U(-t, t) U_0(t, 0) . \quad (C.8)$$

These equations combine to give

$$\begin{aligned} S_{AB} &= \psi_A(t_B) \psi_B^\dagger(t_B) \\ &= \psi_A S \psi_B^\dagger \\ &\equiv S[\psi_A, \psi_B^*] . \end{aligned} \quad (C.9)$$

Thus the S-matrix expression for the transition probability is equivalent to the Schroedinger theory expression, provided the time development operator $U(t_1, t_2)$ of Schroedinger theory exists, and provided the wavefunction limits defined in terms of it exist (in a weak sense). And this equivalence between the Schroedinger and S-matrix

formulas holds without restrictions on the times t_A and t_B of the preparation and examination.

Inspection of these equations shows that the need for the temporal evolution operator $U(t_1, t_2)$ of Schroedinger theory is eliminated in S-matrix theory by mapping the specifications A and B onto the "asymptotic" states ψ_A and ψ_B , instead of onto the states $\psi_A(t_A)$ and $\psi_B(t_B)$, that represent the preparation and result of examination at the times t_A and t_B of the preparation and examination. That is, the need for the temporal evolution operator of Schroedinger theory is eliminated by reorganizing the problem of measurement.

The question immediately arises whether this reorganization of the problem of measurement provides a bona fide solution of the problem of temporal development, or merely shifts this problem onto the already complex problem of measurement.

This question is examined in the following sections, where it is shown how this reorganization of the problem of measurement in fact resolves various logical difficulties, and renders it amenable to practical solution.

D. THEORY OF MEASUREMENT

Bohr said: "On the one hand, the definition of the state of a physical system, as ordinarily understood, claims the elimination of all external disturbances. But in that case, according to the quantum postulate, any observation will be impossible,... On the other hand, if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible, and there can be no question of causality in the ordinary sense of the word."

These words point to the crux of the problem of measurement: The quantum theoretical formulas are built on the degrees of freedom of the prepared and examined systems, and they do not describe the microscopic degrees of freedom of the preparing and examining systems. Consequently, the processes that produce the prepared system and that detect the examined system are not described by these formulas.

The functions ρ_A and ρ_B occurring in the quantum formulas represent the completed preparations and examinations. These formulas acquire physical meaning only to the extent that the process of preparation can be considered to have produced a system that can evolve as a dynamically independent system, and only to the extent that the process of examination can be considered to detect a system that could have evolved as a dynamically independent system. And both of these systems are in fact treated in the quantum formalism as dynamically independent entities: the interactions that produce the prepared system and that detect the examined system are not incorporated into the dynamics of either system. Thus within the framework of the quantum formalism the connection of the preparation and examination to

their representations ρ_A and ρ_B is not dynamical. Rather it is logical: ρ_A and ρ_B are the images in the mathematical formalism of the operational specifications A and B that are met by the preparation and examination.

In certain cases the processes of preparation and examination can be partially analyzed within the framework of the quantum formalism by shifting to a new level of description in which the original preparing and examining devices are included as parts of the new quantum system. Such a procedure allows the self-consistency of the theory to be studied. But it does not resolve the basic problem of measurement. It simply shifts to another level the basic problem, which is to determine at some level the mappings $A \rightarrow \rho_A$ and $B \rightarrow \rho_B$.

The mappings $A \rightarrow \rho_A$ and $B \rightarrow \rho_B$ translate one language into another. They translate operational specifications, formulated in a language that allows trained technicians to set up experiments and recognize results, into mathematical expressions, formulated in the language of the quantum formalism.

These mappings are, in practice, determined empirically: experimentalists calibrate their devices. They must calibrate their devices because there exists no quantitatively accurate theory of measuring devices. Thus the bridge between the operational language of the experimentalist and the mathematical language of the theorist is built in practice upon accumulated empirical evidence. The problem of measurement is thus shifted to the problem of empirical calibration.²

The simplest calibrations are those of devices that prepare and examine single particles. Accordingly, let A specify a manner of

preparation of a single particle of specified type. And let B specify both a manner of examination of a particle of the same type, and also a possible result of this examination. Then the transition probability formula (A.1) reduces to

$$\begin{aligned} \mathcal{P}(A; B) &= \int \frac{m}{p^0} \frac{d^3 p'}{(2\pi)^3} \frac{m}{p'^0} \frac{d^3 p''}{(2\pi)^3} \rho_A(p''; p') \rho_B(p'; p'') \\ &\equiv \int_{p' p''} \rho_A(p''; p') \rho_B(p'; p'') \\ &\equiv \text{Tr } \rho_A \rho_B . \end{aligned} \quad (D.1)$$

Here it has been assumed that S reduces to unity when acting on a single-particle wave function. This assumption is essentially a definition, since the product of the one-particle wave function ψ_A with S can be defined to be the new ψ_A .

The introduction of (C.1b) into (D.1) gives

$$\mathcal{P}(A; B^U) = \int_{p' p''} \rho_A(p''; p') \rho_B(p'; p'') e^{iu(p''-p')} . \quad (D.2)$$

The following theorem can then be invoked:

Theorem ID Suppose $\psi_A(p') \psi_B^*(p')$ is continuous, along with its first two derivatives, and vanishes for $|\vec{p}'| > R$ for some R . Let $p \equiv mv$ be any fixed mass-shell vector. Then

$$\lim_{\tau \rightarrow \infty} \left(\frac{2\pi i \tau}{m} \right)^{3/2} e^{im\tau} \int_{p'} \psi_A(p') e^{-i\tau v \cdot p'} \psi_B^*(p') = \psi_A(p) \psi_B^*(p) . \quad (D.3)$$

(Proof in Appendix IA.)

Corollary Under the same conditions

$$\lim_{\tau \rightarrow \infty} \left(\frac{2\pi\tau}{m} \right)^3 \rho(A; B^{v\tau}) = \rho_A(p; p) \rho_B(p; p). \quad (D.4)$$

This corollary says that the product $\rho_A(p; p) \rho_B(p; p)$ is determined by the transition probability $\rho(A; B^u)$, for large displacements $u = v\tau$ in the direction $v \approx p/m$.

Formula (D.4) suggests the following classical asymptotic space-time interpretation: for sufficiently large τ the space-time region around $X = v\tau$ contains mainly those prepared particles that have momentum-energy near $p = mv$; $\rho_A(p; p)$ is the probability density for producing particles of momentum-energy $p = mv$ in a preparation that meets specifications A ; $\rho_B(p; p)$ is the efficiency for detecting such particles in an examination that meets specifications B ; the cubic fall-off with τ of the transition probability is due to the cubic growth with τ of the volume of space occupied by particles in a fixed momentum interval d^3p --as compared to the fixed volume of space in which detection effectively occurs.

Formula (D.4) gives information about $\rho_A(p''; p')$ and $\rho_B(p'; p'')$ only for $p' = p''$. A full calibration of the devices requires these functions to be determined also for $p' \neq p''$. To achieve this the information available from nonasymptotic measurements must be used. This information is conveniently organized by transcribing the quantum mechanical formulas into classical form. This transcription converts the asymptotic classical space-time interpretation suggested by (D.4) into a precise form valid for all times.

E. SPACE-TIME INTERPRETATION

A space-time interpretation of the quantum theoretical quantities can be obtained by transcribing the quantum theoretical formulas into classical form. This transcription converts the density functions ρ_A and ρ_B into functions that represent statistical ensembles of freely moving classical particles.

A statistical ensemble of freely moving classical particles can be represented by a density function $w(p, x)$ defined as follows: The probability $\mathcal{P}(\Delta^3 p, \Delta^3 x, x)$ that a particle of the ensemble with momentum in the range $\Delta^3 p$ lies in the spatial interval $\Delta^3 x$ at time $t = x^0$ is given by

$$\mathcal{P}(\Delta^3 p, \Delta^3 x, t) = \int_{\Delta^3 p} \frac{d^3 p}{(2\pi)^3} \int_{\Delta^3 x} d^3 x w(p, x). \quad (E.1)$$

Here x is a space-time four-vector with zeroth component $x^0 = t$, and p is a momentum-energy four-vector with zeroth component $p^0 = (\vec{p} \cdot \vec{p} + m^2)^{\frac{1}{2}}$. As before, $v \equiv p/m$.

This classical density function $w(p, x)$ enjoys the following properties:

$$(a) \quad w(p, x) \text{ is real.} \quad (E.2a)$$

$$(b) \quad \text{For any real } \tau \quad w(p, x) = w(p, x + v\tau). \quad (E.2b)$$

$$(c) \quad \text{For any real } \tau \text{ and real } t$$

$$\begin{aligned} \mathcal{P}(A; B) &= \int \frac{d^3 p}{(2\pi)^3} \int d^3 x w_A(p, x) w_B(p, x) \\ &= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \int d^4 x \delta(x \cdot v - \tau) w_A(p, x) w_B(p, x). \end{aligned} \quad (E.2c)$$

Property (a) follows from the physical meaning of $w(p,x)$. Property (b) follows from the fact that classical freely moving particles of momentum p move on space-time trajectories having direction $v \equiv p/m$. The first form of property (c) follows from the definitions of $w_A(p,x)$ and $w_B(p,x)$; $w_A(p,x)$ represents the statistical ensemble corresponding to the specifications A on the manner of preparation, and $w_B(p,x)$ represents the detection efficiency for particles of momentum p at point x in an examination that conforms to specifications B . The second (covariant) form of property (c) follows from the first form, together with property (b) and the fact that the Jacobian of the transformation from constant t to constant τ is (m/p^0) .

The quantum mechanical analogue of this classical $w(p,x)$ is³

$$w(p,x) = \int \rho(Mv - \frac{1}{2}q; Mv + \frac{1}{2}q) e^{-iqx} \left(\frac{M}{m}\right)^{1/2} 2\pi\delta(q \cdot v) \frac{d^4q}{(2\pi)^4}, \quad (E.3)$$

where $M \equiv (m^2 - \frac{1}{4}q^2)^{1/2}$. This function enjoys the classical properties (a), (b), and (c) listed above. The reality of $w(p,x)$ follows from the representation of $\rho(p'; p'')$ as a sum of terms of the pure case form $\psi^*(p') \psi(p'')$: this form entails that $\rho(Mv + \frac{1}{2}q; Mv - \frac{1}{2}q) = \rho^*(Mv - \frac{1}{2}q; Mv + \frac{1}{2}q)$. The invariance of $w(p,x)$ under $x \rightarrow x + v\tau$ follows from the fact that x occurs only in $q \cdot x$, together with the condition $q \cdot v = 0$ imposed by the factor $\delta(q \cdot v)$. Property (c) is derived in Appendix IB.

The quantum theoretical $w(p,x)$ has also the following desired properties:

$$(d) \quad w_{A^u}(p, x+u) = w_A(p, x) \quad (E.2d)$$

and

$$(e) \quad \int d^4x \delta(x \cdot v - \tau) w_A(p, x) = \rho_A(p; p), \quad (E.2e)$$

and the similar ones with B in place of A. Property (d) follows from the insertion of (C.1a) into (E.3). Property (e) follows from the insertion of (E.3) into the left-hand side of (E.2e).

The quantum mechanical $w(p, x)$, unlike its classical counterpart, is not in all cases everywhere positive. The occurrence of negative probability densities in (p, x) space is characteristic of quantum theory. It is demanded by interference phenomena. However, if the classical statistical framework is expanded to accommodate negative probability densities in (p, x) space then there is an exact correspondence between the classical and quantum formulas. Thus in the calibration of the single-particle preparing and examining devices the preparations and examinations can be represented by ensembles of freely moving classical particles: All the quantum theoretical formulas used in the calibration are equivalent to ones occurring in the classical theory. Negative probability densities must in principle be allowed, but they can be shown⁴ to occur effectively only in cases where there is a rapid variation over phase space regions that are not large in comparison to $\hbar$.

Experimentalists use the classical space-time interpretation in calibrating their devices. The detailed techniques by which they perform these calibrations will not be described here.

The single-particle density functions $w_1(p_1, x_1)$ determined by these calibration procedures are used in the study of many-particle processes. In the case of scattering experiments the initial and final density functions are assumed to be products of the density

functions of the individual particles. To perform the calculations it is not necessary to convert the empirically determined forms $w_i(p_i, x_i)$ of these density functions into the corresponding quantum theoretical forms $\rho_i(p_i'; p_i'')$. Instead, the transition probability formula (A.1) can be cast into the classical form

$$\mathcal{P}(A; B) = \int \prod_{j=1}^n \left[\frac{d^3 p_j}{(2\pi)^3} d^3 x_j w_j(p_j, x_j) \right] S(p, x) \quad (E.4a)$$

where $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$, and

$$S(p, x) = \int \prod_{j=1}^n \left[\frac{d^4 q_j}{(2\pi)^4} 2\pi \delta(q_j \cdot v_j) e^{-iq_j \cdot x_j} \left(\frac{M_j}{m_j} \right)^{\frac{1}{2}} \right] \\ \times S((M_j v_j + \frac{1}{2} q_j)) S^*((M_j v_j \pm \frac{1}{2} q_j)) . \quad (E.4b)$$

The upper sign is for initial particles, and the lower sign if for final particles. Equation (E.4) is derived in Appendix IB.

Equation (E.4a) has precisely the form that arises in the classical treatment of scattering processes. There $S(p, x)$ represents the probability density for a set of freely moving final particles on trajectories specified by the final p_i and x_i if the initial particles move on trajectories specified by the initial p_i and x_i .

Because (E.4a) has precisely the form that occurs in the classical statistical treatment of scattering processes, the entire formalism is now cast into a classical form, based on the classical concept of statistical ensembles of freely moving classical particles. If the operational specifications A and B are also formulated in terms of the idealizations of classical physics, then all the elements of the theory of measurement are expressed in terms of classical

concepts--slightly generalized to allow for negative probability densities in (p, x) space.

A simple application of this classical formulation is a derivation of the cross section formula. Classically, the cross section can be defined by considering a target particle 1 at rest at the origin, and a projectile of momentum p_2 directed along the $\mu = 1$ axis. The time t at which the projectile reaches $x_2^1 = 0$ can be defined to be $t = 0$. Then the classically defined cross section for the production of a set of final particles of momenta $p_f (f \in F)$ in Δp_F is

$$\sigma(p_1, p_2; \Delta p_F) = \int_{\Delta p_F} \prod_{f \in F} \left[\frac{d^3 p_f}{(2\pi)^3} d^3 x_f \right] dx_2^2 dx_2^3 S(p, x) \quad (E.5)$$

where x_1 is fixed at zero, x_2^0 and x_2^1 are fixed at zero, p_1 is $(m_1, 0, 0, 0)$, and p_2 is $m(v_2^0, v_2^1, 0, 0)$.

Introduction of (E.4b) into (E.5) gives, after some simple algebra,

$$\sigma(p_1, p_2; \Delta p_F) = \frac{1}{v_2^1} \int_{\Delta p_F} \prod_{f \in F} \left(\frac{m_f}{p_f^0} \frac{d^3 p_f}{(2\pi)^3} \right) \times (2\pi)^4 \delta^4(p_1 + p_2 - \sum p_f) |S_r(p_1, p_2; p_F)|^2, \quad (E.6)$$

where v_2^1/v_2^0 is the velocity of the projectile in the rest frame of the target, p_F is the set of $p_f (f \in F)$, and $S_r(p_1, p_2; p_F)$ is defined by

$$S(p_1, p_2; p_F) = S_r(p_1, p_2; p_F) (2\pi)^4 \delta^4(p_1 + p_2 - \sum p_f). \quad (E.7)$$

The conservation-law delta function appearing in (E.7) is required by the law of conservation of energy-momentum. It will be discussed from a technical point of view in Chapter II. The calculation of (E.6) is given in Appendix IB.

F. NONSCATTERING PROBLEMS AND THE NONRELATIVISTIC APPROXIMATION

The primary focus of S-matrix theory is on relativistic scattering problems. However, nonscattering problems and the non-relativistic approximation are also encompassed.

In scattering problems the prepared system at the time of preparation is a set of essentially independent particles. The general prepared system does not have this form. However, it will evolve eventually into a system of freely-moving particles. Examination of these particles by means of the already calibrated single-particle detecting devices will yield information about the asymptotic final state of the prepared system. Thus the representation of the prepared system by the density function $\rho'_A = S^\dagger \rho_A S$, will have direct physical significance: the transition probabilities $P(A; B) = \text{Tr } \rho'_A \rho_B$ can be compared directly to experiment. However, the information provided by this kind of measurement is probably not sufficient to determine ρ'_A itself: the microscopic many-particle structure of ρ'_A probably cannot be extracted from the macroscopic information provided by such measurements.

The principal practical measurements on complex systems are measurements of their properties as sources and sinks of massless particles. Thus the further discussion of the measurement problem for complex systems must be deferred until after the introduction of massless particles. Here it is only remarked that the logical structure is not altered to accommodate massless particle effects. The necessary changes are merely the technical modifications needed to accommodate the infinite numbers of massless particles that must be considered. A main consequence of the introduction of massless particles into the formalism will be the incorporation of the

classically described environmental fields into the quantum theoretical framework.

Many important nonscattering applications of quantum theory are to nonrelativistic problems. In the special case of two-particle processes the Schroedinger equation emerges automatically from the S-matrix formalism in the nonrelativistic approximation.⁵ It is probable that the same thing happens in general. In any case, the emergence of Schroedinger theory in the nonrelativistic approximation is compatible with S-matrix ideas. For in the nonrelativistic approximation no particles are created or destroyed, and hence the asymptotic degrees of freedom can become relevant for all times. Moreover, the passage to the nonrelativistic limit infinitely expands the time scale. This contracts to infinitesimals the effectively finite time intervals in which the preparations and examinations take place, and tends also to make the asymptotic representations relevant for all times.

In cases where the Schroedinger equation does emerge from the S-matrix framework the temporal evolution derived from it will agree with that derived from the basic S-matrix formulas, provided the appropriate asymptotic limits exist. This was shown earlier. Thus nonrelativistic Schroedinger theory has a natural place within S-matrix theory as the nonrelativistic limit of the general relativistic theory. The preferred status of time characteristic of Schroedinger theory would naturally be expected to emerge from general relativistic quantum theory only in the nonrelativistic limit.

APPENDIX 1A

PROOF OF THEOREM ID

Theorem ID is equivalent to the following theorem:

Theorem Suppose $g(p')$ is continuous, together with its first and second derivatives, and vanishes for $|\vec{p}| > R$ for some R . Let $p \equiv mv$ be any fixed mass-shell four-vector. Then

$$\lim_{\tau \rightarrow \infty} \left(\frac{2\pi i \tau}{m} \right)^{3/2} e^{im\tau} \int g(p') e^{-ip' \cdot v\tau} \frac{m}{p'^0} \frac{d^3 p'}{(2\pi)^3} = g(p) . \quad (IA.1)$$

Proof Transform to the variables corresponding to a frame in which $v = (1, 0, 0, 0)$. In terms of these variables one has

$$v \cdot p' = p'^0 = [m^2 + (\vec{p})^2]^{1/2} = m + f[(\vec{p})^2] \quad (IA.2)$$

where

$$f[(\vec{p})^2] = \frac{(\vec{p})^2}{2m} + \dots > 0 . \quad (IA.3)$$

The introduction of the variable f in place of $(\vec{p})^2$, followed by an integration over angles, converts (IA.1) to

$$\frac{2}{\sqrt{\pi}} (i\tau)^{3/2} \int_0^\infty \bar{g}(f) \sqrt{f} e^{-if\tau} df \rightarrow \bar{g}(0) \quad (IA.4)$$

where $\bar{g}(0) = g(0)$, and $\bar{g}(f)$ and its first and second derivatives are continuous at $f > 0$. Since

$$\int_0^\infty e^{-if(\tau - i\epsilon)} \sqrt{f} df = \frac{\sqrt{\pi}}{2} \frac{1}{[i(\tau - i\epsilon)]^{3/2}} \quad (IA.5)$$

and $\bar{g}(f)$ is continuous with compact support, the required result (IA.4) equivalent to

$$\lim_{\tau \rightarrow 0} \tau^{3/2} \lim_{\epsilon \rightarrow 0+} \int_0^{\infty} [\bar{g}(f) - \bar{g}(0)] e^{-i\tau(\tau - i\epsilon)} \sqrt{f} df = 0. \quad (\text{IA.6})$$

Bounds on $(\bar{g}(f) - \bar{g}(0))$ and its first two derivatives can be obtained by writing

$$g(p) = g(r, \Omega) = g(0) + \vec{\nabla} g(0) \cdot \vec{r} + \int_0^r dr' \int_0^{r'} dr'' \frac{\partial^2 g}{\partial r'^2}(r'', \Omega) \quad (\text{IA.7})$$

where $\vec{r} = \vec{p}$, and $r = |\vec{p}|$. The integration over angles eliminates the linear term and gives

$$\frac{\bar{g}(f)}{\sqrt{1 + \frac{f}{2m}}} - \bar{g}(0) = \int \frac{d\Omega}{4\pi} \int_0^{r(f)} dr' \int_0^{r'} dr'' \frac{\partial^2 g}{\partial r'^2}(r'', \Omega). \quad (\text{IA.8})$$

Since the second derivative of $g(p)$ is bounded,

$$\left| \frac{\partial^2 g}{\partial r'^2} \right| \leq c, \quad (\text{IA.9})$$

one has

$$\left| \frac{\bar{g}(f)}{\sqrt{1 + \frac{f}{2m}}} - \bar{g}(0) \right| \leq \frac{1}{2} c r^2. \quad (\text{IA.10})$$

Letting F be such that

$$\bar{g}(f) = 0 \quad \text{for } f \geq F,$$

and defining $\bar{m} = F + m$, so that $\partial r^2 / \partial f = 2f + 2m \leq 2\bar{m}$ for $f \leq F$, one obtains, for $f \geq 0$,

[Equation (IA.16) continued]

$$= \int_0^{\infty} e^{-if\tau} \frac{1}{2} [h_{\epsilon}(f) - h_{\epsilon}(f + \pi/\tau)] df + \frac{1}{2} \int_0^{\pi/\tau} e^{-if\tau} h_{\epsilon}(f) df . \quad (\text{IA.16})$$

The last term in (IA.16) has, by virtue of (IA.11) and (IA.12), the bound $\frac{1}{2} \bar{c} m(\pi/\tau)^{3/2}$. Thus this contribution, inserted into (IA.14), satisfies (IA.6).

The first term in (IA.16) can be written as a sum of two terms. The first is

$$\begin{aligned} & \frac{1}{2} \int_0^F e^{-if\tau} [h_{\epsilon}(f) - h_{\epsilon}(f + \frac{\pi}{\tau})] df \\ & < \frac{1}{2} (\frac{\pi}{\tau}) \int_0^F |\max h'_{\epsilon}(f)| df \end{aligned} \quad (\text{IA.17})$$

where $|\max h'_{\epsilon}(f)|$ is the maximum of the absolute value of $dh_{\epsilon}(f')/df'$ for $f' \geq f$. The bounds (IA.11), (IA.12), and (IA.13) ensure that the integral on the right-hand side of (IA.17) has a finite bound that is independent of ϵ . Thus this contribution, inserted into (IA.14), also satisfies (IA.6).

The remaining part of (IA.16) is

$$\begin{aligned}
 & \frac{1}{2i} \int_F^{\infty} e^{-if\tau} [h_{\epsilon}(f) - h_{\epsilon}(f + \frac{\pi}{\tau})] df \\
 &= \frac{-\bar{g}(0)}{4} \int_F^{\infty} e^{-if\tau} \left[\frac{e^{-\epsilon f}}{\sqrt{f}} - \frac{e^{-\epsilon(f+\pi/\tau)}}{\sqrt{f + \pi/\tau}} \right] df \\
 &= \frac{-\bar{g}(0)}{4} \int_F^{\infty} e^{-if(\tau-i\epsilon)} \left[\frac{1}{\sqrt{f}} - \frac{1}{\sqrt{f + \pi/\tau}} \right] \\
 &\quad + \frac{(\bar{g}(0))}{4} \int_F^{\infty} e^{-if(\tau-i\epsilon)} [1 - e^{-\epsilon\pi/\tau}] \frac{1}{\sqrt{f + \pi/\tau}} . \quad (IA.18)
 \end{aligned}$$

The first term on the right-hand side of (IA.18) is bounded in magnitude by

$$\frac{|\bar{g}(0)|}{4} \left(\frac{\pi}{\tau}\right) \int_F^{\infty} \left(\frac{d}{d\tau} \frac{1}{\sqrt{f}}\right) df = \frac{|\bar{g}(0)|}{4} \left(\frac{\pi}{\tau}\right) F^{-1/2} . \quad (IA.19)$$

Thus this contribution, inserted into (IA.14) also satisfies (IA.6).

The second term on the right-hand side of (IA.18) can be written

$$\begin{aligned}
 & \frac{-g(0)}{4} [1 - e^{-\epsilon\pi/\tau}] e^{+i(\pi/\tau)(\tau-i\epsilon)} \int_{F+\pi/\tau}^{\infty} e^{-if(\tau-i\epsilon)} \frac{1}{\sqrt{f}} df \\
 &= \frac{g(0)}{4} [e^{\epsilon\pi/\tau} - 1] \\
 & \times \left[\int_0^{\infty} e^{-if(\tau-i\epsilon)} \frac{1}{\sqrt{f}} df - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)} \frac{df}{\sqrt{f}} \right] \\
 &= \frac{g(0)}{4} [e^{\epsilon\pi/\tau} - 1] \left[\sqrt{\frac{\pi}{-i(\tau-i\epsilon)}} - \int_0^{F+\pi/\tau} e^{-if(\tau-i\epsilon)} \frac{df}{\sqrt{f}} \right].
 \end{aligned}$$

This term vanishes when we take the limit $\epsilon \rightarrow 0$ in (IA.6). Thus all the contributions satisfy (IA.6).

APPENDIX IB

CORRESPONDENCE TO CLASSICAL FORMULAS

1. Derivation of (E.2c).

A useful identity is

$$\delta(a - \frac{1}{2} b) \delta(a + \frac{1}{2} b) = \delta(a) \delta(b). \quad (\text{IB.1})$$

Thus the introduction of the variables

$$Q = \frac{1}{2}(p' + p'') \quad (\text{IB.2a})$$

and

$$q = p' - p'' \quad (\text{IB.2b})$$

into (D.1) gives [see (B.2)]

$$\rho(A; B) = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} 2\pi \delta[(Q + \frac{1}{2} q)^2 - m^2] 2m \theta(q^0 + \frac{1}{2} q^0)$$

$$\times 2\pi \delta[(Q - \frac{1}{2} q)^2 - m^2] 2m \theta(q^0 - \frac{1}{2} q^0)$$

$$\times \rho_A(Q - \frac{1}{2} q; Q + \frac{1}{2} q) \rho_B(Q + \frac{1}{2} q; Q - \frac{1}{2} q)$$

$$= \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} 2\pi \delta(q^2 + \frac{1}{4} q^2 - m^2)$$

$$\times 2\pi \delta(2Q \cdot q) (2m)^2 \theta(q^0 + \frac{1}{2} q^0) \theta(q^0 - \frac{1}{2} q^0)$$

$$\times \rho_A(Q - \frac{1}{2} q; Q + \frac{1}{2} q) \rho_B(Q + \frac{1}{2} q; Q - \frac{1}{2} q)$$

Equation (IB.5) Continued.

$$= \int \frac{d^4(Mv)}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} 2\pi \delta[M^2 - (m^2 - \frac{1}{4} q^2)]$$

$$\times 2 \delta(2Mv \cdot q) (2m)^2 \theta(Q^0 + \frac{1}{2} q^0) \theta(Q^0 - \frac{1}{2} q^0)$$

$$\times \rho_A(Mv - \frac{1}{2} q; Mv + \frac{1}{2} q) \rho_B(Mv + \frac{1}{2} q; Mv - \frac{1}{2} q)$$

$$= \int \frac{M}{m} \frac{m}{p^0} \frac{d^3(mv)}{(2\pi)^3} \frac{d^4 q}{(2\pi)^3} \delta(v \cdot q)$$

$$\times \rho_A(Mv - \frac{1}{2} q; Mv + \frac{1}{2} q) \rho_B(Mv + \frac{1}{2} q; Mv - \frac{1}{2} q)$$

$$= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \frac{d^4 q'}{(2\pi)^3} \delta(v \cdot q') \frac{d^4 q''}{(2\pi)^3} \delta(v \cdot q'')$$

$$\times \int e^{-i(q' \cdot q'')x} d^4 x \delta(v \cdot x) \frac{M}{m}$$

$$\times \rho_A(Mv - \frac{1}{2} q'; Mv + \frac{1}{2} q') \rho_B(Mv + \frac{1}{2} q''; Mv - \frac{1}{2} q'')$$

$$= \int \frac{m}{p^0} \frac{d^3 p}{(2\pi)^3} \int d^4 x \delta(v \cdot x) w_A(p, x) w_B(p, x)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \int d^3 x w_A(p, x) w_B(p, x) . \quad (IB.3)$$

2. Derivation of (E.4)

Equations (A.1) and (B.2) give

$$\begin{aligned} \mathcal{P}(A; B) &= \int \prod_{j=1}^n \left[\frac{d^4 p_j'}{(2\pi)^4} 2\pi \delta(p_j'^2 - m_j^2) \theta(p_j'^0) \right] \\ &\quad \times \left[\frac{d^4 p_j''}{(2\pi)^4} 2\pi \delta(p_j''^2 - m_j^2) \theta(p_j''^0) \right] \\ &\quad \times \prod_{i \in I} \rho_i(p_i''; p_i') \prod_{f \in F} \rho_f(p_f'; p_f'') S(p') S^*(p'') \end{aligned} \quad (\text{IB.4})$$

where I and F are the index sets for the initial and final particles. Introduction of (IB.2) gives, as in (IB.3),

$$\mathcal{P}(A; B) = \int \prod_{j=1}^n \left[\frac{d^3 p_j}{(2\pi)^3} d^3 x_j w_j(p_j, x_j) \right] S(p, x),$$

where $w_j(p_j, x_j)$ is defined by (E.3) and $S(p, x)$ is defined by the analogous (E.4b).

3. Derivation of (E.6)

Equation (E.5) can be written

$$\begin{aligned} \sigma(p_1, p_2; p_F) &= \int \prod_{f \in F} \left[\frac{m_f}{p_f^0} \frac{d^3 p_f}{(2\pi)^3} d^4 x_f \delta(x_f \cdot v_f) \right] \\ &\quad \times dx_2^2 dx_2^3 S(p, x). \end{aligned} \quad (\text{IB.5})$$

The introduction of (E.4b) gives

$$\sigma(p_1, p_2; p_F) =$$

$$= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} d^4 x_f \delta(x_f \cdot v_f) \frac{d^4 q_f}{(2\pi)^3} \delta(q_f \cdot v_f) \right.$$

$$\times e^{-i q_f \cdot x_f \left(\frac{M_f}{m_f} \right)^{\frac{1}{2}}} \left. \prod_{i \in I} \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] \right]$$

$$\times S(M_j v_j + \frac{1}{2} q_j) S^*(M_j v_j + \frac{1}{2} q_j) dx_2^2 dx_2^3$$

$$= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S(M_1 v_1 - \frac{1}{2} q_1; p_F) S^*(M_1 v_1 + \frac{1}{2} q_1; p_F)$$

$$\times \prod_{i \in I} \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] dx_2^2 dx_2^3$$

$$= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S_r(M_1 v_1 - \frac{1}{2} q_1; p_F) S_r^*(M_1 v_1 + \frac{1}{2} q_1; p_F)$$

$$\times (2\pi)^4 \delta^4(M_1 v_1 + M_2 v_2 - \sum p_f) (2\pi)^4 \delta^4(q_1 + q_2)$$

$$\times \prod_{i=1}^2 \left[\frac{d^4 q_i}{(2\pi)^3} \delta(q_i \cdot v_i) e^{-i q_i \cdot x_i \left(\frac{M_i}{m_i} \right)^{\frac{1}{2}}} \right] dx_2^2 dx_2^3$$

Equation (IB.6) Continued

$$\begin{aligned}
 &= \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{d^3 p_f}{(2\pi)^3} \right] S_r(M_1 v_1 - \frac{1}{2} q_1; p_F) S_r^*(M_1 v_1 + \frac{1}{2} q_1; p_F) \\
 &\times (2\pi)^4 \delta^4(M_1 v_1 + M_2 v_2 - \sum p_f) 2\pi \delta(q_2^0) \\
 &\times \frac{d^4 q_2}{(2\pi)^3} \delta(q_2^1 v_2^1) e^{-i q_2 \cdot x_2} \left(\frac{M_1}{m_1} \frac{M_2}{m_2} \right)^{\frac{1}{2}} dx_2^2 dx_2^3 \\
 &= \frac{1}{v_2} \int \prod_{f \in F} \left[\frac{m_f}{p_f} \frac{dp_f}{(2\pi)^3} \right] S_r(p) S_r^*(p) (2\pi)^4 \delta^4(\sum \pm p_j) .
 \end{aligned}
 \tag{IB.6}$$

This is the result (E.6).

The formal manipulations used in this appendix can be rigorously justified if the ρ are infinitely differentiable and of compact support, and S is a tempered distribution that is the product of a conservation law delta function times a distribution in the mass shell variables. (See Chapter II.)

The derivation of the cross section formula given above is straightforward, in the sense that there is no argumentation regarding shapes of wave packets, or any equating of delta functions $\delta(0)$ to volumes of space or time.

REFERENCES

1. Niels Bohr, Atomic Theory and the Description of Nature (Cambridge University Press, Cambridge, England, 1934), p. 1; Essays 1958/1962 on Atomic Physics and Human Knowledge (John Wiley and Sons, New York, 1963). p. 60. The pragmatic orientation summarized by these statements is the cornerstone of the Copenhagen interpretation.
See this author's article "The Copenhagen Interpretation" in the American Journal of Physics, to be published.
2. For an extended discussion and critical evaluation of other approaches to the problem of measurement see B. D'Espagnat, Conceptual Foundations of Quantum Theory, (W. A. Benjamin, Menlo Park, California, 1971).
3. This formula is a relativistic generalization of a formula due to Wigner: E. P. Wigner, Phys. Rev. 40, 749 (1932).
4. D. Iagolnitzer, in Lectures in Theoretical Physics (Gordon and Breach Science Publishers, New York, 1968), Ed. K. T. Mahanthappa and W. E. Brittin, p. 251.
5. R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Annals of Physics 10, 62 (1960). See also, J. Charap and S. Fubini, Nuovo Cimento 14, 540 (1959).