The Quasilinear Theory

Revised and Tested*

MASTER

by

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ABSTRACT

The quasilinear theory is revised analytically and its consequences are tested by a numerical simulation for a one-dimensional periodic electron plasma with discrete electrostatic modes. The theory is formulated for unstable and damped modes. The numerical model is a Vlasov-Poisson system where the mode-mode coupling terms have been omitted. The result is that the predictions of the analytical theory are confirmed by simulation. For all cases considered there are always several modes which are initially unstable, but become damped at a later time so that an inclusion of damped modes in the theory is absolutely necessary.

INTRODUCTION

A very lively discussion has arisen recently concerning the quasilinear theory. Its very foundations have been questioned, although the theory has been known for about 10 years and it has been applied widely. After the simultaneous publications in 1961 by Vedenov, Velikov and Sagdeev,¹ and Drummond and Pines,² Bernstein and Engelmann³ gave a more refined representation of the theory, claiming that unstable and stable Fourier modes of the electric field can be treated on an equal footing. They also derived an "H-like" theorem. In 1970 Klozenberg and Bernstein⁴ gave a new derivation of the quasilinear theory which again claims to apply to growing and decaying waves and to agree with the old theory in the limit $\text{Im}(\omega_K) \rightarrow 0$.

Almost simultaneously Vahala and Montgomery⁵ seriously criticized the quasilinear theory, claiming that for damped waves the resulting diffusion equation leads to a negative diffusion coefficient in one dimension and certainly so in two and three dimensions. This violates the continuous dependence of the solution on its initial conditions. Such systems are generally considered inappropriate for describing physical systems and are

termed ill-posed. According to Vahala and Montgomery, the situation can only be salvaged if in one dimension one assumes that initially all $\text{Im}(\omega_K)$ are non-negative and for finite times, remain so.

A series of comments follow these two papers. Montgomery and Bodner⁶ claim that Klozenberg and Bernstein⁴ have improperly treated the perturbed electron distribution function so that they still end up with an ill-posed set of equations. Bernstein and Klozenberg⁷ argue that this is a misunderstanding in part perhaps semantic. Abraham-Shrauner⁸ remarks that the Klozenberg-Bernstein theory may as well be obtained by analytic continuation. Finally Bodner⁹ tries to resolve the difficulties by considering particle orbits stochastically and adding a Fokker-Planck term to the equations. The reader will admit that the situation is rather confusing.

In this communication we try to resolve some of the difficulties by a more practical approach, a combined analytical and numerical check of the quasilinear theory in one dimension for a discrete wave spectrum. A careful derivation of a quasilinear theory, using an "Ansatz" which we infer from our numerical experiments gives us an understanding why we obtain singular distribution functions from equations whose solutions remain finite for all times.

We introduce a somewhat different notion of a "distribution function," namely we define it as a set of prescriptions, which enable us to obtain the correct macroscopic quantities when

integrating over velocity. This definition includes the usual definition of a distribution function as an ordinary function. It may however also include improper functions and/or operators.

The resulting equations are used to derive macroscopic equations. These relations are checked against a computer simulation which differs from a Vlasov-Poisson system only in the neglect of the mode-mode coupling terms and thus represents a version of the quasilinear theory where a minimum of approximations has been made. The result is a satisfactory agreement of theory with experiment.

In Sec. 2 we give an analytical derivation of the theory which, because of the existing differences of opinion has to start at the very beginning, and which goes back to the thesis of one of us.¹⁰ Section 3 reviews the numerical method and Sec. 4 describes the comparison between theory and numerical experiment. Several differences to earlier work is elucidated in Sec. 5.

II. ANALYTICAL FORMULATION

We consider for simplicity the one-dimensional electron plasma with a homogeneous ion background, which is described in dimensionless units by the Vlasov equation and the Poisson equation:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = 0 \quad ; \quad \frac{\partial E}{\partial x} = 1 - \int f \, dv \quad . \quad (1)$$

As usual, the distribution function is Fourier analyzed into its spatial modes

$$f(x, v, t) = \sum_{m=-\infty}^{+\infty} f_m(v, t) \exp(ik_m x) \quad , \quad (2)$$

with $k_m = mk_0$, $k_0 = 2\pi/L$ where L is the periodicity length.

For $m \neq 0$ we find

$$\frac{\partial f_m(v, t)}{\partial t} + ik_m v f_m(v, t) - E_m \frac{\partial f_0(v, t)}{\partial v} = \sum_{\substack{q=-\infty \\ q \neq 0}}^{+\infty} E_{m-q}(t) \frac{\partial f_q(v, t)}{\partial v} \quad ,$$

$$ik_m E_m(t) = - \int_{-\infty}^{+\infty} f_m(v, t) \, dv \quad . \quad (3)$$

The boundary conditions are chosen so that $E_0 \equiv 0$. For the homogeneous part of the distribution function we obtain

$$\frac{\partial f_0(v, t)}{\partial t} = \sum_{m>0} E_{-m}(t) \frac{\partial f_m(v, t)}{\partial v} + \text{c.c.} \quad (4)$$

where c.c. means the conjugate complex of the previous expression.

The quasilinear theory, as we define it, originates from the Vlasov equation by omitting the mode coupling terms on the right-hand side of Eq. (3). A numerical study¹¹ shows that under certain conditions the dominant features of the solutions are indeed qualitatively the same for the truncated and untruncated system (compare also Armstrong¹⁹).

With the right side omitted, but f_0 depending on time, a formal solution for f_m can be written down:

$$f_m(v, t) = \int_0^t dt' \exp[-ik_m v(t - t')] E_m(t') \frac{\partial f_0(v, t')}{\partial v} + \exp(ik_m vt) f_m(v, t = 0) \quad (5)$$

From numerical evidence one finds that $E_m(t)$ behaves proportional to $\exp(-i\omega t)$ if the amplitude of the field is small.

For all stable initial conditions used, this is true after half an oscillation. For unstable initial conditions it holds only if the most unstable eigenmode is stronger excited than all other modes. Otherwise it may take a considerable time (several plasma periods) until an exponential growth develops.

For small electric fields we have $\left| \frac{\partial \ln f_o(v,t)}{\partial t} \right| \ll 1$ and the time dependence of f_o may be neglected. This is the usual linear theory. With

$$E_m(t) = E_m^0 \exp(-i\omega_m t), \quad \omega_m = \omega_m^r + i\gamma_m; \quad (6)$$

where ω may have a positive or imaginary part γ , Eq. (5) may be integrated in time to give

$$f_m(v,t) = \frac{E_m(t) \frac{\partial f_o(v)}{\partial v}}{ik_m(v - \omega_m/k_m)} - \frac{E_m^0 \exp(-ik_m vt)}{ik_m(v - \omega_m/k_m)} \frac{\partial f_o}{\partial v} + \exp(ik_m vt) f_m(v, t=0). \quad (7)$$

In linear and quasilinear theory (compare e.g. Ref. 5) f_m is taken to be

$$f_m(v, t) = \frac{E_m(t)}{ik_m} \frac{\frac{\partial f_o(v)}{\partial v}}{(v - \omega_m/k_m)} \quad (8)$$

A distribution function of form (8) is in effect also used for deriving dispersion relations for various initial conditions in linear theory.

A word of caution, however, is in order: Eq. (8) has to be supplemented by the prescription of the Landau contour. For any macroscopic quantity derived from f_m , the integral path has to go underneath the pole of f_m in Eq. (8). The rigorous derivation with the aid of a Laplace transformation^{12,13} is well known. The question arises: What is the relation of the distribution functions of Eq. (8), which is singular at $v = \omega/k$ and of Eq. (7) which is regular for $v = \omega/k$?

We consider the limit of large time, $t \rightarrow \infty$, and multiply Eq. (7) with $h(v)$, where $h(v)$ is a real, regular function of velocity, as for example moments, and integrate over v . Then the last term in Eq. (7) representing the initial conditions goes to zero at least exponentially and faster than the first term for appropriate¹⁴ and analytic initial conditions and can be neglected.

Because the singularity at $v = \omega/k$ is only an apparent one, we may lower the integration path below $v = \omega/k$ until it crosses the point $(-i\delta)$. δ is the distance of the singularity of $h(v) \partial f_0(v, t)/\partial v$ closest to real axis and we assume $\delta + \gamma/k > 0$. We then arrive at the following estimate.

$$\left| \int_{-i\delta-\infty}^{-i\delta+\infty} \frac{\exp(-ikt v)}{v - \omega/k} h(v) \frac{\partial f_0(v)}{\partial v} dv \right|$$

$$\leq \frac{e^{-k\delta t}}{|\delta - \gamma/k|} \int_{-i\delta-\infty}^{-i\delta+\infty} \left| g(v) \frac{\partial f_0}{\partial v} \right| dv \leq A \exp(-k\delta t),$$

where A is bounded and independent of t .

The time behavior of the first term is governed by

$$|E(t)| \sim e^{\gamma t}.$$

Thus for large times the first term will be the dominant term, irrespective of γ being positive or negative. In this asymptotic sense Eq. (8) is a meaningful distribution function which allows us to evaluate the correct macroscopic quantities. It must be

supplemented by the prescription of the Landau contour. If we had raised the integration path above the singularity or if we had the integration path go through the singularity, using a principal value, the integrated second term in Eq. (7) would not have become small compared to the first and our macroscopic quantities would have turned out to be erroneous. Note that, in general, the moduli of the neglected terms are not at all small compared to the first term, but only their integrals over v .

f_m in Eq. (8) is no longer a distribution function which we may interpret pointwise like a function. Rather, it is a distribution or an operator which gives us the correct macroscopic quantities when integrated over velocity according to certain recipes.

In quasilinear theory f_0 is a weak function of time because of Eq. (4) and we have to allow for a variable, complex $\omega(t)$ to account for the electric field dependence as is borne out by numerical results (compare Figs. 1 and 2). Equations (5) and (6) have to be replaced by

$$f_m(v, t) = E_m^0 \int_0^t dt' \exp[-ik(t - t') v] \exp\left[-i \int_0^{t'} \omega(\tau) d\tau\right]$$

$$\times \frac{\partial f_0(v, t')}{\partial v} \quad (9)$$

We ask again, What are the conditions under which we can obtain a reduced expression for f_m and still obtain the correct macroscopic results? Multiplying Eq. (9) again by $h(v)$ and integrating over v , we obtain after a change of variable

$$\int h(v) f_m dv = \int_0^t G(ks, t-s) \exp \left[-i \int_0^{t-s} \omega(\tau) d\tau \right] ds, \quad (10)$$

where $G(s, t)$ is the Fourier transform of $h(v) \frac{\partial f_0(v, t)}{\partial v}$:

$$G(s, t) = \int_{-\infty}^{+\infty} \exp(-isv) h(v) \frac{\partial f_0(v, t)}{\partial v} dv.$$

Let the half-width of $h(v) \frac{\partial f_0(v, t)}{\partial v}$ be v_h . Then the half-width of $G(s, t)$ with respect to the first variable is v_h^{-1} . If we impose the condition that

$$\left| \frac{\partial \ln G(ks, t)}{\partial t} v_h^{-1} k^{-1} \right| \ll 1, \quad (11)$$

then $G(ks, t-s)$ may be replaced in Eq. (10) by $G(ks, t)$ without noticeable error. This corresponds to replacing t' by t in f_0 in

Eq. (9). We also may push down the integration contour of v from the real v axis to a path parallel to it and a distance δ away from the real axis, because the integrand is well behaved in a strip of width δ around the real v axis.

Considering now the integral over t , we see that we cannot do it exactly. However,

$$\int_0^t \exp \left[i \int_0^{t'} \sigma(\tau) d\tau \right] \approx \exp \left[i \int_0^t \sigma(\tau) d\tau \right] / i\sigma(t) ,$$

if $\left| \frac{d\sigma}{dt} / \sigma^2 \right| \ll 1$. In our case

$$\sigma(\tau) = kv - \omega(\tau)$$

and, taking into account the displaced integration path, we find

$$\left| \frac{d\sigma}{dt} / \sigma^2 \right| \leq \sqrt{(\dot{\omega})^2 + \dot{\gamma}^2} / |k\delta + \gamma|^2 .$$

Thus, we can perform the time integration approximately if

$$\sqrt{(\dot{\omega})^2 + \dot{\gamma}^2} / |k\delta + \gamma|^2 \ll 1 , \quad (12)$$

and $f_m(v, t)$ can be written legitimately as

$$f_m(v, t) = \frac{E_m(t)}{ik_m(v - \omega_m(t)/k_m)} \frac{\partial f_o(v, t)}{\partial v}, \quad (13)$$

where $f_o(v, t)$ and $\omega(t)$ depend slowly on time. Again the prescription of the Landau contour has to be added.

Inserting f_m into Eq. (4) gives

$$\frac{\partial f_o(v, t)}{\partial t} = \frac{\partial}{\partial v} \sum_{m>0} \frac{|E_m(t)|^2}{ik_m(v - \omega_m(t)/k_m)} \frac{\partial f_o(v, t)}{\partial v} + \text{c.c.} \quad (14)$$

This holds as an operator equation for stable or unstable modes and it is understood that the Landau contour is used below the singularity at $v = \omega/k$ in the first term of the right-hand side and above the singularity ω^*/k in the conjugate complex term. It can easily be shown and it follows later from a more general context (see Eq. (25)), that Eq. (14) conserves particle number, momentum and energy, irrespective of the sign of γ .

The equations governing the quasilinear theory are Eq. (14).

$$\frac{d}{dt} \ln |E_m|^2 = 2\gamma_m \quad (15)$$

and the dispersion equation

$$k^2 = \int \frac{\partial f_o(v,t)}{\partial v} (v - \omega/k)^{-1} dv \quad (16)$$

The latter may be obtained from inserting Eq. (13) into Poisson's equation, and demanding that the electric field does not vanish identically. By writing out the conjugate complex terms in Eq. (14) we obtain

$$\frac{\partial f_o(v,t)}{\partial t} = \frac{\partial}{\partial v} \sum_{m>0} |E_m(t)| \frac{\gamma_m/k}{(v - \omega_m^r/k_m) + (\gamma_m/k_m)^2} \frac{\partial f_o(v,t)}{\partial v} \quad (17)$$

In this representation the integral path has to be carried below the pole ω_m/k_m and above the pole ω_m^*/k_m as before. This leads to paths encircling the pole at ω in a positive, and at ω^* in a negative direction, as indicated in Fig. 1. These integration paths were already given by Rogister and Oberman¹⁵ in 1968; those which apply to Eq. (14) were given by Knorr¹⁰ in 1963. It would be erroneous to replace the resonant term in Eq. (17) by a δ -function

term (compare Eq. (20)). Because of the integration path of Fig. 1, Eq. (17) can no longer be considered to be a diffusion type equation. Our further aim is to derive macroscopic relations from Eq. (14) which can be subjected to a numerical check.

Multiplying Eq. (14) with $h(v)$ and integrating over v results in

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} h(v) f_o(v, t) dv = - \sum_{m>0} \int \frac{|E_m(t)|^2}{ik_m(v - \omega_m/k_m)} \frac{\partial h(v)}{\partial v} \\ \times \frac{\partial f_o(v, t)}{\partial v} dv + \text{c.c.} \quad (18) \end{aligned}$$

The integration contour can be shifted in the individual terms such that the new variables u_m

$$u_m = v - i\gamma_m/k_m$$

are real. The integral in a term in Eq. (10) can be written:

$$\int \frac{g(u + i\gamma/k)}{u - \omega^r/k} du, \quad (19)$$

where $g(u) = \frac{\partial h(u)}{\partial u} \frac{\partial f_o(u)}{\partial u}$. The index, m , has been omitted for brevity.

Since u is a real variable, we may apply the Plemelj formulae. Also, assuming γ/k is small compared to the characteristic rate of change of $g(u)$ in the variable u_g , we expand the numerator and keep only linear terms of γ/k . We can then write integral Eq. (18) as:

$$P \int_{-\infty}^{+\infty} du \left\{ \frac{1}{u - V} \left[1 + i\gamma/k \frac{d}{du} \right] g(u) + i\pi\delta(u - V) \left[1 + i\gamma/k \frac{d}{du} \right] \right\} g(u) . \quad (20)$$

The integral path is now along the real u -axis and $V = \omega^r/k$. An application of relation (20) to the dispersion (16) is well known and yields, after collecting equal powers of γ/k

$$k^2 = \oint \frac{f_o'(u)}{u - V} du , \quad (21)$$

$$\gamma/k \oint \frac{f_o''(u)}{u - V} du = -\pi f_o'(V) . \quad (22)$$

Applying relation (20) to Eq. (14) results in

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} h(u) f_0(u, t) du = \sum_{m>0} 2 \frac{|E_m(t)|^2}{k_m} \left\{ \gamma_m/k_m \oint (u - \omega_m^r/k_m)^{-1} \frac{d}{du} [h'(u) f_0'(u)] du + \pi \int h'(u) f_0'(u) \delta(u - v_m) du \right\}. \quad (23)$$

As an example we specify $h(v)$ to represent the moments

$$h(v) = v^n.$$

One can show from Eq. (16) that for the moments

$$W_n(t) = \int_{-\infty}^{+\infty} v^n f_0(v, t) dv$$

the following relations exist

$$\frac{dW_n(t)}{dt} = -n(n-1) \sum_{m \geq 0} \frac{d|E_m(t)|^2}{dt} v_m^{n-2} \left[1 - \frac{1}{(n-1)k_m^2} \sum_{\sigma=0}^{n-4} (1+\sigma)(n-3-\sigma) W_\sigma v^{-(\sigma+2)} \right]. \quad (24)$$

The details of the calculation will be found in the Appendix. Note that the n -th moment depends in this approximation on $|E_m|^2$ and the lower moments only. For $N = 0, 1, 2$, we recover the conservation of particle number, momentum and energy

$$W_0 = \text{const} ,$$

$$W_1 = \text{const} , \quad (25)$$

$$W_2(t) + 2 \sum_{m \geq 0} |E_m(t)|^2 = \text{const} .$$

for $n \geq 3$ we obtain the following nontrivial relations

$$\frac{1}{6} W_3 + \sum_{m>0} V_m |E_m|^2 = C_3 ,$$

$$\frac{1}{12} W_4 + \sum_{m>0} |E_m|^2 V_m^2 [1 - 1/3 k_m^2 V_m^2] = C_4 , \quad (26)$$

$$\frac{1}{20} W_5 + \sum_{m>0} |E_m|^2 \left[V_m^3 - \frac{1}{2k_m^2} (V_m + W_1) \right] = C_5 .$$

In the section on numerical results we will check how well these predictions are numerically satisfied.

III. NUMERICAL METHOD

The equations to be integrated numerically are Eq. (3) with the right-side set equal to zero and Eq. (4). This is, in a way, the most basic form of the quasilinear theory, the only deviation from the Vlasov system being the neglect of the mode-mode coupling terms.

Equations (3) and (4) were integrated with the power transform method, which has been described elsewhere.¹⁸ The advantages of this particular method for our purposes are:

- 1) Different k-modes can easily be added or omitted, contrary e.g., to particle following methods.
- 2) The equations reduce to a system of ordinary differential equations which are convenient to handle.
- 3) The size of the matrix which represents the distribution function can be kept reasonably low by using certain extrapolation procedures at the cutoff of the matrix.

Thus 80 to 120 terms suffice to represent the velocity dependence of $f_m(v, t)$. This is in contrast to the Hermite expansion where no such procedures had been known.¹⁷ To advance the system in time to $60 \omega_p^{-1}$ requires about 1 min of computing time per mode on an IBM 360/65 with this procedure.

- 4) The macroscopic quantities like electric field, density, momentum, etc. of the different modes are represented by elements of the matrix representing the distribution function or a linear combination of them. For example, we have the expansion

$$\int_{-\infty}^{+\infty} f_0(v) e^{i v y} dv = F_0(y, t) = \sum \delta_v A_v(t) y^v \exp\left(-\frac{1}{2} y^2\right), \quad (30)$$

with $\delta_{v+2} = \frac{v+2}{v+1} \delta_v$ and $\delta_0 = 1$, $\delta_1 = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}$.

Then the first moments

$$W_n = \int_{-\infty}^{+\infty} v^n f_0(v) dv$$

are given by $W_0 = A_0 = 1$

$$W_1 = -iA_1$$

$$W_2 = 2A_2 - A_0$$

$$W_3 = 3i\sqrt{\frac{1}{2\pi}} (A_1 - A_3) \text{ etc.}$$

For studying the beam plasma interaction we used a beam of amplitude D_B of temperature 1 superimposed with a relative velocity B on a Maxwellian plasma of normalized temperature 1, as shown in Fig. 2:

$$f(x, v, t = 0) = \frac{1}{N} \exp\left(-\frac{1}{2}v^2\right) [1 + H(v) D_B (v^{B \cdot B} + \sum E_m \cos K_m x)] \quad (31)$$

Here $H(v) = \begin{cases} 1 & v > 0 \\ 0 & v < 0 \end{cases}$, the E_m govern the excitation of the different modes. N is a normalization constant such that $\int_{-\infty}^{+\infty} f_0 dv = 1$.

It can easily be shown that $H(v) v^{B \cdot B} \exp\left(-\frac{1}{2}v^2\right)$ is approximately equal to $H(v) \exp\left(-\frac{1}{2}(v - B)^2\right)$. However, it is easier to write down the initial conditions for the chosen representation than for the displaced Maxwellian.

IV. COMPARISON OF THEORY WITH NUMERICAL RESULTS

When comparing analytic and numerical results about the quasilinear theory, we have to keep in mind that all quasilinear theories are asymptotic in time, i.e., they hold only in the limit $t \rightarrow \infty$. In addition, it is assumed that $|\gamma/\omega^r| \ll 1$ and $|\dot{\gamma}/v_{th}^2 k^2| \ll 1$ which is a simplified version of inequality (11).

Numerically, the physical effects which we wish to check must occur in finite real time. In the nonlinear regime we typically have $|\gamma/\omega| \sim 0.1$ and $|\dot{\gamma}/v_{th}^2 k^2| \sim 0.5$ and since both numbers are not $\ll 1$ and we have limited our runs to $t \sim 50$, we should not expect more than qualitative agreement.

We investigated first the relations (26) by using numerical data for $W_n(t)$, $|E_m(t)|^2$ and the phase velocities $v_n(t)$, and checking if the left-hand side remains indeed constant in time. In order to obtain as accurate a graphical representation as possible, we did not plot the first and second terms in Eq. (23) directly, but their differences with respect to its initial values at $t = 0$. For simplicity we call the kinetic terms ΔK_n and the term involving the electric field ΔEL_n . For example, we have for $n = 3$ from Eq. (26)

$$\Delta K_3(t) = \frac{1}{12} [W_3(t) - W_3(0)] ;$$

$$\Delta EL_3(t) = \sum_{m>0} [(E_m(t))^2 - |E_m(0)|^2] V_m ;$$

$$\Delta C_3(t) = C_3(t) - C_3(0) ; \quad (32)$$

Equation (26) reduces to

$$\Delta K_n(t) + \Delta EL_n(t) = \Delta C_n(t) \quad (33)$$

The phase velocities were determined in the following way:

The electric field of the m-th mode is given by

$$E_m(x,t) = E_m(t) \exp(ikx) + \text{c.c.} = A(t) \exp[i\varphi(t) + ikx] + \text{c.c.}$$

where $E_m(x,t)$, $A(t)$, and $\varphi(t)$ are real, $E_m(t)$ complex quantities.

If there is only one wave per wave number, $k^{-1} \frac{d\varphi}{dt}$ is the phase velocity V_m . Numerically, $k^{-1} \frac{d\varphi}{dt}$ is easily obtained. As Fig. 3 shows, the assumption of one wave is not always true, but at later times in an instability one wave dominates all others and then $k^{-1} \frac{d\varphi}{dt}$ remains reasonably constant. This value was used for V_m in the comparison.

As the quasilinear theory holds asymptotically in time, we expect that the curves ΔK_n and $\Delta E L_n$ behave independently initially, but run parallel to each other later. For all cases considered in this section, $f(v; t = 0)$ is the same as shown in Fig. 2.

Case I: One Unstable Mode

We consider first the case of one mode only with $k = 0.3$ which is unstable. This appears to be unphysical because it has been argued by Vedenov, Velikov and Sagedeev¹ that the mode spectrum should be fairly dense in order to justify the neglect of the mode-mode coupling terms. Recently however, it has been claimed that quasilinear results also apply qualitatively to single wave phenomena.¹⁸ In any case, the one mode example is the simplest case in which the mathematical behavior of the quasilinear equations can be studied. $|E|$ is plotted in Fig. 4 and shows the typical behavior of an instability: initially the linear behavior with more than one wave present, then the linear exponential growth of one wave with $\gamma = 0.11$, and finally saturation and a slow oscillation of the amplitude. There is a definite transition from $\gamma > 0$ to $\gamma < 0$ at $t = 34$. Figure 3 shows $k^{-1} \frac{d\phi}{dt}$ as discussed earlier. Figure 5 gives plots for ΔK_3 and $\Delta E L_3$. They coincide so closely that we have enlarged the difference in Fig. 5b. It can clearly be seen that ΔC_3 changes by $4 \cdot 10^{-5}$ when ΔK_3 changes by $5 \cdot 10^{-2}$;

the relation (33) for $n = 3$ is thus very well satisfied, even in the regime where γ changes sign. Not unexpectedly, we observe here the relatively strongest change of ΔC_3 .

For the fourth and fifth moments (Figs. 6, 7) the curves for ΔK_n and ΔEL_n are still very much the same; the maximal differences amount to about 2% ($n = 4$) and 10% ($n = 5$) at the time of maximum change.

Figure 1 contains also plots of f_0 for different times for this case. At the time of maximum field, $t = 34$, the distribution function is fairly flat at the phase velocity. The slight ripples might originate from the finite numerical array by which f_0 is represented.

Case II: One Stable Mode

The large amplitude of the electric field, $|E| > 10^{-1}$, clearly violates the assumptions of the quasilinear theory because this brings about a rapid change of f_0 in certain regions. The second single wave number system we considered is free from this limitation. It is a linearly damped wave with $k = 0.525$ and $\gamma = -0.0191$ and $|E|$ remains always smaller than $6 \cdot 10^{-3}$ (Fig. 8). Figure 9a shows ΔK_3 and ΔEL_3 which again agree so well that the difference must be blown up in Fig. 9b to be visible. Figures 10 and 11 show the fourth and fifth moment equations. As expected,

they differ widely initially but stay parallel to each other for $t > 4$, thus indicating that the asymptotic state which the theory assumes has more or less been attained at that time. For $t > 20$, all the electrostatic energy has been damped away and nothing changes any more.

Case III: The Multiwave Case

In the multiwave case, one wave interacts indirectly with all others via f_0 . As initial conditions, we have used several excited modes which were all linearly unstable and had comparable growth rates. The distribution function is shown in Fig. 12. The amplitudes of the different modes are plotted in Fig. 13 as functions of time. At the time of saturation, the distribution function is fairly flat at the location of the phase velocities. We observe, very similar to the one wave case, good agreement of Eq. (33) for the third moment in Fig. 14. For the fourth and fifth moment, a wavy structure of the electric field term $\Delta E L_m$ is observed which oscillates around the kinetic term ΔK_n (Figs. 15 and 16). Apart from the superimposed oscillation, the agreement of curves is still quite good.

V. CONCLUSIONS

We have done a comparison between the quasilinear theory developed in Sec. II and a numerical scheme which integrates the Vlasov equation without mode-mode coupling terms and the Poisson equation in one dimension for periodic boundary conditions. The comparison was done by inserting numerical values into a number of relations for the moments derived from theory. In all cases we found good qualitative agreement in spite of the fact that assumptions had been made which were only approximately met by the simulation. We observed in all unstable cases that there were always modes which after initially being unstable turned into damped waves. For these cases, Vahala and Montgomery's requirement that all γ 's remain non-negative for finite times is certainly not true and the consequences drawn therefrom do not apply. It may however be, that when going from a discrete to a continuous set of modes the situation might change. It appeared that the decrease of the electric field for a given mode after saturation was less marked when more modes were present. This point deserves to be investigated further.

It has been argued that damped waves make the equation for f_0 ill posed. In view of the fact that we have to encircle the poles in Eq. (17) according to Fig. 1, we can no longer consider Eq. (17) to be a parabolic differential equation on the real axis. All the conclusions derived from it are physically reasonable and agree with the simulation.

Due to the deformation of the integration path into the complex plane, it is no longer possible to derive an "H-like" theorem for f_0 , in the sense of Bernstein and Engelmann.³ Indeed, we generally do not observe a monotonic approach to a plateau of $|E|^2$ for large times, but a decrease after a maximum has been reached so we are not even able to claim an asymptotic state as $t \rightarrow \infty$. It might, however, be that a plateau develops for a continuous spectrum in k , or for a much smaller beam.

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APPENDIX

With $h(v) = v^n$, Eq. (23) becomes

$$W_n = \sum_{m \geq 0} 2 \frac{|E_m|^2}{k_m} \times \left\{ \frac{\gamma}{k} \oint \frac{n(n-1) u^{n-2} f'_0 + n u^{n-1} f''_0}{u - v_m} du + \pi f'_0(v) v_m^{n-1} \right\}$$

In evaluating the principal value integrals we use

$$u^n - v^n = (u - v) \sum_{\nu=0}^{n-1} u^{n-1-\nu} v^\nu$$

and obtain

$$\oint du \frac{u^{n-1} f''_0}{u - v} = v^{n-1} \oint \frac{f''_0 du}{u - v} + \sum_{\nu=0}^{n-3} (n-2-\nu)(n-3-\nu) W_{n-4-\nu} v^\nu$$

$$\oint du \frac{u^{n-2} f'_0}{u - v} = v^{n-2} \oint \frac{f'_0 du}{u - v} - \sum_{\nu=0}^{n-3} (n-3-\nu) W_{n-4-\nu} v^\nu$$

Using the dispersion Eq. (16) and Eqs. (21) and (22), we find

$$W_n = \sum_{m \geq 0} \frac{d}{dt} |E_m|^2$$

$$\times \left\{ n(n-1) v_m^{n-2} + \frac{n}{k_m^2} \sum_{v=0}^{n-3} (-1-v)(n-3-v) W_{n-4-v} v^v \right\}$$

This can easily be converted into the form (24).

REFERENCES

- ¹A. Vedenov, E. Velikhov and R. Sagdeev, Nucl. Fusion Suppl. Pt. 2, 465 (1962).
- ²W. E. Drummond and D. Pines, Nucl. Fusion Suppl. Pt. 3, 1049 (1962).
- ³I. B. Bernstein and F. Engelmann, Phys. Fluids 9, 937 (1966).
- ⁴J. P. Klozenberg and I. B. Bernstein, J. Plasma Phys. 4, 595 (1970).
- ⁵G. Vahala and D. Montgomery, J. Plasma Phys. 4, 677 (1970).
- ⁶D. Montgomery and S. Bodner, J. Plasma Phys. 5, 131 (1971).
- ⁷I. B. Bernstein and J. P. Klozenberg, J. Plasma Phys. 5, 135 (1971).
- ⁸B. Abraham-Shrauner, J. Plasma Phys. 5, 137 (1971).
- ⁹S. Bodner, J. Plasma Phys. 5, 141 (1971).
- ¹⁰G. Knorr, Z. Naturforsch. 18a, 1304 (1963).
- ¹¹G. Joyce, G. Knorr, T. Burns, Phys. Fluids 14, 797 (1971).
- ¹²L. D. Landau, J. Phys. USSR 10, 25 (1946).
- ¹³G. Backus, J. Math. Phys. 1, 178 (1968).
- ¹⁴T. P. Armstrong, R. C. Harding, G. Knorr and D. Montgomery, Methods in Computational Physics, edited by S. Fernbach and M. Rotenberg (Academic Press, 1970), Vol. 9, p. 36.
- ¹⁵A. Rogister and C. Oberman, J. Plasma Phys. 2, 33 (1968).
- ¹⁶G. Joyce, G. Knorr and H. Meier, to be published in J. Comp. Phys. (1971).

- ¹⁷ T. P. Armstrong and D. Montgomery, Phys. Fluids 12, 2094 (1969).
- ¹⁸ W. E. Drummond, F. H. Malmberg, T. M. O'Neil and F. R. Thompson,
Phys. Fluids 13, 2422 (1970).
- ¹⁹ T. P. Armstrong, Ph.D. Thesis, Univ. of Iowa Res. Rept. No. 66-34
(1966).

FIGURE CAPTIONS

- Figure 1. Integration path to be used when deriving macroscopic quantities from Eq. (14) or (17): a) for unstable, b) for stable modes.
- Figure 2. The homogeneous velocity distribution function $f_0(v,t)$ for the one wave cases. The distribution for $t = 20$ and 34 refers to the case with $K = 0.3$, $V = 3.15$. The case $k = 0.525$, $V = 2.69$ is for a stable mode.
- Figure 3. Plot of $k^{-1} d\phi/dt$ versus time for the unstable $k = 0.3$ case. If the curve is constant in time, we are dealing with one wave only and $k^{-1} d\phi/dt$ is its phase velocity.
- Figure 4. Plot of the amplitude of a single unstable wave with $k = 0.3$.
- Figures 5a,b. The curves of ΔK_3 and $-\Delta EL_3$ for the single mode with $k = 0.3$. In Eq. (33) the right side is zero to a good approximation. The difference has to be enlarged by a factor of 100 to become visible (5b).
- Figures 6,7. The terms of the left hand side of Eq. (33) plotted for $n = 4,5$. Differences now become visible.

Figure 8. Plot of the amplitude for the stable single wave case with $K = 0.525$.

Figures 9a,b. Plot of the terms of Eq. (33) for $n = 3$ for the case of Figure 8, $K = 0.525$.

Figures 10,11. Plot of ΔK_n and $-\Delta EL_n$ for $n = 4.5$. According to theory the differences should become constant.

Figure 12. Distribution function for the multiwave case at saturation ($T = 30$). $f(v,t)$ is fairly flat at the location of the phase velocities.

Figure 13. Electric field amplitudes for 4 excited modes:

_____ $K_1 = 0.225$

.... $K_2 = 0.275$

— — $K_3 = 0.325$

— . — . $K_4 = 0.375$

Figures 14,15,16. Plot of ΔK_n and $-\Delta EL_n$ for $n = 3, 4, 5$ of Eq. (33) for the multiwave case. According to theory the curves should exactly match.

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Figure 1

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Figure 2

Figure 3

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Figure 4

Figure 6

Figure 7

Figure 8

Figure 9a,b

Figure 10

Figure 11

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Figure 12

Figure 13

Figure 14

Figure 15

Figure 16