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THESIS

An Exact Analytical Solution to the
Two-Body Problem with a Separable (Nonlocal)
Central and Tensor Interaction

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National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy \$10.60; Microfiche \$0.95

LA-5274-T

Thesis

UC-34

ISSUED: July 1973

An Exact Analytical Solution to the Two-Body Problem with a Separable (Nonlocal) Central and Tensor Interaction

by

Clarence E. Lee

This report is derived from a dissertation submitted to the Department of Physics and Astrophysics of the University of Colorado in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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AN EXACT ANALYTICAL SOLUTION TO THE TWO-BODY PROBLEM WITH
A SEPARABLE (NONLOCAL) CENTRAL AND TENSOR INTERACTION

by

Clarence Edgar Lee

ABSTRACT

A two-nucleon, non-relativistic, spin-dependent and non-local interaction with a repulsive hard core (of diameter a) is investigated, and exact analytical solutions to the Schrodinger wave equation are obtained for all angular momentum states.

Using a complete orthonormal set of functions on the interval (a, ∞) the spatial and momentum representation wave functions are determined using contour integration techniques.

The known identification of Blatt-Biedenharn, and also Stapp "bar", phase shifts from asymptotic spatial solutions permits determination of all phase-shift parameters in terms of known, explicitly displayed functions.

The wave-function solutions, by further contour integration techniques, are also used in the explicit analytical determination of the deuteron bound-state condition, RMS radius and quadrupole moment.

The low-energy expansions of the S-wave phase-shift expressions for $k \cot \delta$ result in a determination of the scattering length, effective range and shape-dependent parameters in terms of the core radius, and the strengths and ranges of the central and tensor interaction.

Rapid computational methods using continued fractions to evaluate exponential integrals and series expressions with complex

arguments are developed.

Preliminary numerical calculations of the parameters of this exact analytical model are included.

INTRODUCTION

Statement of the Problem and Motivation

The description and understanding of nucleons and nucleons in the nucleus has been a major goal of nuclear physics for four decades. If two-body forces are believed to be the dominant interaction involved, as is indeed suggested by experimental evidence,¹ then the two nucleon problem in the center-of-mass system should be addressed in seeking a solution to the problem.

In 1935 H. Yukawa² proposed the concept of field quanta exchange with nucleons as a representation of nuclear forces, predicting a meson mass of about $208 m_e$ ($m_e \equiv$ electron mass).

In 1947 the pion ($m_\pi \approx 280 m_e$) was discovered.³ In principle one could then possibly deduce the nature of nuclear forces from pion field theory. This has not yet been fully achieved.

One very important result from pion field theory is the One-Pion-Exchange (OPE) interaction, with a local potential (OPEP) given by⁴

$$V_{1\pi}(r) = f^2(m_\pi c^2) \vec{\tau}_1 \cdot \vec{\tau}_2 \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 + S_{12} \left(\frac{x^2 + 3x + 3}{2x} \right) \right] \frac{e^{-x}}{x}, \quad (1)$$

¹Blatt, J. M. and Weisskopf, V., THEORETICAL NUCLEAR PHYSICS, Wiley & Sons, New York, 1952; also Bethe, H. and Morrison, P., ELEMENTARY NUCLEAR THEORY, Wiley & Sons, New York, 1956.

²Yukawa, H., Proc. Phys.-Math Soc. Japan, 17, 1935, p. 17.

³Lattes, Occialini, and Powell, Nature, 160, 1947, pp. 453, 486.

⁴Gunn, J. C., "Meson Theory and Nuclear Forces," in QUANTUM THEORY V.II, RADIATION AND HIGH ENERGY PHYSICS, D. R. Bates ed., Academic Press, New York, 1962, pp. 195-285.

where $x = m_\pi c/h$, $m_\pi c^2 \approx 135$ Mev, $f^2 = .08$ (renormalized πN coupling constant), c = velocity of light, and $\hbar = h/2\pi$ (h = Planck's constant). $V_{1\pi}$ is a valid representation for separation distances greater than about a pion Compton wavelength, $\lambda_\pi \approx 1.4f$. Approximations involving more than one pion, e.g., two pions, other mesons, dispersion relations and subtractions, have not yet proved completely successful, and will not be discussed here. Equation (1) has been utilized extensively in the estimation of high ($l > 3$) angular momentum phase shifts, at low energy. One should note that the usage of Eq. (1) in the standard fashion for Born Approximation phase shift calculations leads to an irregular solution (for the tensor part of the interaction ($\sim 1/x^3$)) except for the long range part.

The concept of the hard core potential was introduced primarily as a mechanism to explain the short range repulsion in the 1S_0 , 3P_0 , 1P_1 , and 3S_1 phase shifts. Jastrow⁵ proposed a hard core in 1951 as a simple explanation of the isotropic p-p scattering at 350 Mev. Usually models utilizing a hard core have diameters of about 0.4 to 0.6F with (in some sense) the height going to infinity (large) so that the wavefunction vanishes at (penetrates) the core surface.

Various examples of numerical phenomenological calculations on large scale computers have been performed by (but not limited to)

⁵Jastrow, R., Phys. Rev., 81, 1951, p. 165.

Gammel and Thaler,⁶ Hamada and Johnston,⁷ Signell and Marshak,⁸ Gartenhaus,⁹ Breit and the Yale Group,¹⁰ Yamaguchi¹¹ and Reid,¹² to mention only a few.

The general form of the phenomenological potentials have usually neglected pion production for energies less than 400 Mev even though the threshold is at about 290 Mev.

The non-central nature of the nuclear force, the tensor force, is evidenced¹ from the deuteron (np) non-zero electric quadrupole moment for a ground state admixture of $\ell = 0$ and $\ell = 2$. There is no $\ell = 0$ contribution because no electric dipole is measured. The binding energy, effective range and scattering length are also measureable. The shape dependent parameter (model dependent) is thereby inferrable.

From the known binding energies in light nuclei, Wigner¹³ argued in 1933 that nuclear forces must be strong, short range, attractive probably (through not conclusive) charge independent. This argument

⁶Gammel, J. and Thaler, R., Phys. Rev., 107, 1957, pp. 291, 1337.

⁷Hamada, T. and Johnston, I.D., Nucl. Phys., 34, 1962, p. 382.

⁸Signell, P. and Marshak, R. E., Phys. Rev., 109, 1958, p. 1229.

⁹Gartenhaus, S., Phys. Rev., 100, 1956, p. 900.

¹⁰Breit, G., Rev. Mod. Phys., 39, 1967, p. 560.

¹¹Yamaguchi, Y., Phys. Rev., 95, 1954, p. 1628.

¹²Reid, R., Thesis (unpublished), Cornell University, 1967.

¹³Wigner, E., Z. Physik, 83, 1933, p. 253; also Wigner, E. and Eisenbud, L., Phys. Rev., 72, 1947, p.29.

was based upon the binding energy per nucleon and the number of two-body bonds per nucleon.

From the binding energy per nucleon and the experimental observation that the volume per nuclei is approximately constant for the majority of the stable nuclei, one concludes that saturation is present and, therefore, that not all of the interaction is attractive. The Pauli-Exclusion principle will not account for the saturation with just two body forces.¹ One must then conclude that the interaction is either repulsive or vanishes in odd angular momentum states and is attractive in even angular momentum states. This leads to an ordinary or space exchange operator as was first introduced by Majorana.¹⁴ Saturation now appears to be a consequence of the hard core rather than the exchange properties of the two nucleon interaction.¹⁵

From an examination of the properties of the mirror nuclei, one expects charge symmetry in the (n,n) and (p,p) interaction.

From the low-energy nucleon-nucleon scattering data, the results of the p-p scattering length determination from cross-section measurements and the n-p determination from the deuteron binding energy are unequal. The implication of this result is that the interaction is different in $S = 0$ and $S = 1$ states. From hydrogen scattering and the a_{np} singlet and triplet scattering length determination,

¹⁴Majorana, E., Z. Physik, 82, 1933, p. 137.

¹⁵Gammel, J. and Thaler, R., "Phenomenology of the Nucleon Nucleon Interaction," PROGRESS IN ELEMENTARY PARTICLE AND COSMIC RAY PHYSICS, J. G. Wilson and S. A. Wouthuysen, eds., North Holland Publishing Co., Amsterdam, 1960, pp. 99-199.

plus the deuteron binding energy, the effective range of the singlet and triplet is determinable. This result implies that there is a spin dependent force which is short range and attractive in $S = 0$ spin states.

From the high energy nucleon-nucleon scattering phase shift analysis (up to 400 Mev), the result that the p-p scattering cross-section (except for Coulomb) is predominately isotropic supports the argument that the interaction is only in $\ell = 0$ (S states). However, at energies greater than 10 Mev there are $\ell = 1$ (P states), so that for energies less than 400 Mev there cannot be S states alone. Hence S and D must be present for the phase shifts to have opposite signs for different energies.

Therefore, the interaction must be attractive at low energies and repulsive at high energies in such a way that saturation can occur despite a long range attraction. A non-local interaction can exhibit these properties.¹⁵

Non-local interactions have the advantages that they can give a reasonable fit to the experimental data,^{11,15,16} that if separable, they may admit to an analytical solution, and that experimental comparisons using analytical solutions, even though complicated, permit a careful study of the solution as a known function of the parameters.

¹⁶Lee, C., Gammel, J., and Thaler, R., "Non-locality in the Two-Nucleon Interaction," NUCLEAR FORCES AND THE FEW-NUCLEON PROBLEM, T. C. Griffith and E. A. Power, eds., Pergamon Press, New York, 1960, p. 43.

-

In this thesis we investigate a two-nucleon, non-relativistic, spin dependent, separable (non-local) interaction with a repulsive hard core of diameter a . We obtain exact, analytical solutions to the Schrodinger equation for all angular momentum states. The scattering phase-shifts, the deuteron bound-state condition, the deuteron RMS radius and quadrupole moment, as well as the low energy scattering parameters (the scattering length, effective range, and shape dependent parameters) for the singlet and triplet S states are all derived explicitly for the model potential described.

This thesis actually represents, in perspective to the major goal under consideration, an intermediate summary of results. The actual problem we originally sought to investigate was the analytical calculation of the thermodynamical potential and ground-state energy of nuclear matter¹⁷ using our non-local central and tensor interaction model, with the model parameters (hard core diameter, the strength and ranges of the central and tensor interaction) adjusted to give a reasonable fit to the scattering data and deuteron properties. The ground state energy can, in fact, be expressed in terms of the many-body reaction matrices which in turn depend upon the detailed knowledge

¹⁷Mohling, F., Phys. Rev., 124, 1961, p. 583.

of the two-body wave-functions. In this thesis we address ourselves primarily to the determination of the two-body wave-functions and only outline their use for determining the reaction matrices.

The possibility of the comparison of all these parameters with experimental numbers by using the high speed computer programs written by the author now exists. These programs were written in MADCAP language for the MANIAC (a research computer) at Los Alamos. The functions, phase shifts, etc. are individually evaluable, and each subroutine was optimized. However, the problem of performing a χ^2 fit of the parameters to the data has not proved feasible at this time on the MANIAC because of the availability, speed, and number significance required. The equations and methods of the code need to be transcribed to a faster more-number-significance computer (like the CDC-7600, 40-80 times faster) in order to achieve a meaningfully significant numerical solution in a reasonable time. The funding for these calculations on the CDC-7600 is not presently available.

Outline of Part I.

In Sec. I.1 we define our two-body problem in both the position and momentum representations, relate an assumed potential to the two-nucleon reaction matrix, and write down partial-wave expansions for the free-particle and true two-nucleon wave functions. The Blatt-Biedenharn states are introduced to clarify the exact form of the two-nucleon wave functions. We then specify the general non-local interaction to be separable, from its partial-wave expansion. Upon choosing the spatial representation of the separable interaction to have a Yukawa form (see Eqs. (39)-(41)) and using the simple properties of the total spin, relative angular momentum and tensor operators, we determine explicit forms for the singlet and triplet wave functions. Finally, explicit coupled differential equations to be solved for the singlet and triplet states are written down.

In Sec. I.2, we instigate the solution of the coupled differential equations for the separable two-body problem by transforming to the momentum representation with a set of functions orthonormal on the interval $a \leq r \leq \infty$. In the momentum representation we assume a general form for the solution to these equations, define some new functions, and then show that we have only to solve a set of coupled linear equations when using the assumed solution. The wave-functions are then determined, apart from normalization, in terms of known functions.

In Sec. I.3, we relate our solution to scattering parameters and determine the wave-function normalization. For the singlet states, the general results of Sec. I.1 for the outgoing wave-functions in conjunction with the solution of Sec. I.2, leads readily to the determination of the normalization of the singlet wave-functions and to an identification of the singlet phase shifts. For the triplet states, we first cast the wave-functions of Sec. I.2 into the conventional form, as defined earlier, for Blatt-Biedenharn pure states.¹⁸ We then use the orthogonality of these states and a general normalization condition to complete the specification of these states. The Blatt-Biedenharn phase-shifts and mixing parameters are derived explicitly for our model interaction. Finally, we make the connection between the solution of Sec. I.2 and the so-called Stapp¹⁹ form for the triplet wave-functions. The Stapp "bar" phase shifts and mixing parameters are also determined explicitly in terms of known functions for our model interaction.

The bound-state wave-function for the deuteron is derived, and the corresponding bound-state eigenvalue condition determined in Sec. I.4. This result may be viewed as follows: Given the deuteron binding energy, a condition is defined between the parameters of the central and tensor interaction for the triplet $J = 1$ bound state. The bound-state wave-function is also used to determine the deuteron

¹⁸ Blatt, J. M. and Biedenharn, L. C., Rev. Mod. Phys., 24, 1952, p. 258.

¹⁹ Stapp, H. P., Ypsilantis, T. M., and Metropolis, N., Phys. Rev., 105, 1957, p. 302.

RMS radius. This in turn represents another functional relationship between the parameters of the central and tensor interaction.

In Sec. 1.5, we use the bound-state wave-function of the previous section to determine an expression for the deuteron quadrupole moment, explicitly in terms of known functions. This result again represents a functional relationship between the parameters of our model interaction. The sensitivity of these parameters to the bound-state condition, RMS radius, and quadrupole moment remains yet to be examined numerically.

The results of various integrals introduced in Part I are summarized in Sec. 1.6. These integrals are all evaluated explicitly in the Appendices in terms of known, computable functions.

Outline of Part II

In Sec. II.1, we begin the development necessary for the numerical determination of our model parameters. We derive the beginning series for the scattering length, effective range and shape-dependent parameters, for singlet and triplet states, by performing low-energy expansion of the expressions derived in Sec. I.3. Here the detailed results from Appendix 5 are also used.

We demonstrate explicitly the techniques involved in the construction of continued-fraction representations of the functions and series encountered in the thesis in Sec. II.2. The procedure includes a generalization to the domain of complex variables and the formulation of contracted continued fractions. A method for the computation of analytic (convergent) infinite series by continued fractions is also outlined. From a computational point of view, the series and functions so represented, and described by explicit MADCAP V procedures at Los Alamos, are fully equivalent to the more common functions such as e^Z , $\ln(Z)$, etc. Thus, we feel justified in referring to these functions as computable and known.

In Sec. II.3, we describe briefly a function optimization technique developed in conjunction with Dr. K. F. Famularo of the Los Alamos Scientific Laboratory. This scheme has proved highly successful when applied to realistic problems. Then in Sec. II.4 we report on preliminary model parameter determinations for the lowest partial wave solutions. A few graphic displays of the basic functions of the thesis for representative parametric values are included.

Outline of Part III

In Sec. III.1 we examine and review the theory of the partial-wave reaction matrices when the tensor interaction is present. We develop an expression for A , the partial-wave reaction matrices, which can be generalized to the case of the non-local interaction of Sec. I.1. In order to relate the A matrices to the scattering parameters we then define four linearly independent free-particle wave-functions and four real reaction matrices. The free-particle wave-functions are solved formally in momentum space, transformed to position space, and the asymptotic forms of the solutions are used to relate the diagonal (on the energy shell) elements of the A matrices to the Blatt-Biedenharn phase shifts. The A matrices are then related to the Stapp α functions and phase shifts.

The reality conditions of the A matrices are demonstrated from the reality of the Blatt-Biedenharn reaction matrices. The unitarity condition, Eqs. (35) - (36) is determined by expressing the wave functions in terms of the A matrices and using wave-function orthogonality and conservation of probability. Upon invoking time-reversal invariance and conservation of total probability of incoming and outgoing waves to evaluate the A matrices on the energy shell, it

is exhibited that the arbitrary real quantity, $\rho_j(k)$, undetermined in the unitarity condition, must be chosen zero. Finally, the general unitarity conditions satisfied by the incoming and outgoing reaction matrices are derived. The closure condition on the A matrices, Eq. (65), is determined using the wave-function orthogonality and completeness conditions.

In Sec. III.2 we review the calculation of the general form of the two-body operator, $R(t_2, t_1)$ matrix elements which appear in Mohling's formulation of Quantum Statistical Mechanics²⁰ and identify certain functions in terms of the A matrices of Sec. II.1 which are involved in the definition of two-body reaction matrix elements in Mohling's Theory of Nuclear Matter.¹⁷

Finally in Sec. III.3 we derive the explicit expressions for the $A^{(+)}$ matrix integrals, and determine the hard-core contribution to the $A^{(+)}$ matrix. We demonstrate that all of the matrix elements for the nuclear matter calculations have been determined explicitly in terms of known evaluable functions. Thus an investigation of the reaction matrix $g_1(k, \ell)$ $Q_{JST,LL}$ is possible now.

²⁰Mohling, F., Phys. Rev., 122, 1961, p. 1043.

PART I: TWO-BODY PROBLEM

Introduction

We develop the two-body problem from the Schrödinger representation of the relative wave equation. Introducing the plane-wave matrix elements of the two particle reaction matrix and the momentum representation of the two particle wave-functions, we exhibit the asymptotic boundary conditions in position and momentum space for outgoing, incoming and standing waves, as well as, the familiar integral and operator forms for the reaction matrix.

The partial wave expansions of free particle and true two nucleon wave functions are derived from the angular momentum wave-functions and the plane wave expansion theorem using the standard Clebsch-Gordon coefficients.

The description of the solution form is then clarified by the introduction of the pure Blatt-Biedenharn states. The connection between the asymptotic solutions and the Blatt-Biedenharn states is made.

From the general form of the non-local spin dependent interaction for singlet and triplet states, the specific forms of the chosen Yukawa-like interaction are exhibited for central and tensor forces with a hard core of diameter a . Then, the radial Schrödinger equations for uncoupled singlet and coupled triplet states are derived using the specific form of the interaction and the orthogonality properties of the wave functions.

In Sec. I.2 these differential-integral equations are solved for

all angular momentum states, singlets and triplets, in the following manner. Introducing momentum transformed wave functions and potentials, and using a set of orthonormal functions on the interval $a \leq r \leq \infty$, a momentum representation equation is obtained. We solve these momenta equations by assuming a general solution form, define some new functions, and then show that we have only to solve a coupled set of linear equations when using the assumed solution. The solutions to the coupled linear equations are then uniquely related to the momentum wave-functions except for a normalization factor which is determined later (in Sec. I.3) and the spatial wave-functions are determined in terms of known functions by inversion. The triplet state calculation is somewhat complicated by a 4×4 algebraic matrix inversion, but, for both singlet and triplet states, we explicitly obtain the spatial wave-functions. From the asymptotic forms of the wave-functions the phase shifts are then identified and correlated with the originally assumed solution form.

The normalization factors for the singlet and triplet states are determined in Sec. I.3 after establishing the connection between the asymptotic solution and the Blatt-Biedenharn phase-shifts (using the pure state orthogonality and wave function normalization condition). We demonstrate that the phase-shifts and normalization factors so obtained are real for the Blatt-Biedenharn pure states and exhibit the phase-shift and mixing parameters explicitly as well as the zero interaction limit. Finally, we make the connection between the solution of Sec. I.2 and the so-called Stapp form for the triplet wave-functions. The Stapp "bar" phase shifts and mixing

parameters are also determined explicitly for our model interaction in terms of known functions.

In Sec. I.4 the bound-state deuteron wave-functions are derived by solving the equations of Sec. I.1 with $k_0^2 \rightarrow -\gamma^2$. In this instance we obtain four linear equations, the determinant of whose coefficients must vanish identically in order for a non-trivial solution to exist. This eigenvalue condition on γ defines the bound-state condition on the deuteron wave-functions and may be viewed as follows: Given the deuteron binding energy, a condition is defined between the parameters of the central and tensor interaction for the triplet $J = 1$ bound state. The bound-state wave-function is also used to determine the deuteron RMS radius. The result for the RMS radius involves integrals obtained in the later evaluation of the deuteron quadrupole moment (Sec. I.5) and these in turn represent another functional relationship between the parameters of the central and tensor interaction.

The deuteron quadrupole moment is determined explicitly in terms of known functions in Sec. I.5 using the bound-state wave-function of Sec. I.4. A number of integrals are introduced in formulating the solution and all are explicitly evaluated in the Appendices. This result represents another functional relationship between the parameters of our model interaction. The sensitivity of these model parameters to the bound-state condition, the RMS radius, and quadrupole moment remains yet to be examined numerically.

Finally, in Sec. I.6, the results of various integrals introduced in Part I are summarized. These integrals are all evaluated explicitly in the Appendices in terms of known, computable functions.

I.1 Development of the Two-Body Problem

We begin by considering a two-nucleon, non-relativistic, spin-dependent local (or point) interaction with a repulsive hard core. The relative wave equation in the Schrödinger representation is given by¹

$$[H_0 - E(k_0)] \langle r | k_0 \rangle^{(\mu)} = -V(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2, \vec{\tau}_1, \vec{\tau}_2) \langle r | k_0 \rangle^{(\mu)}, \quad (1)$$

where $\langle r | k_0 \rangle^{(\mu)} \equiv \langle r, \xi_1, \xi_2, \zeta_1, \zeta_2 | \vec{k}_0; S, T, M_S, M_T \rangle^{(\mu)}$ is the two-particle wave function depending upon the relative position r , the nucleon spins ($\vec{\sigma}_1, \vec{\sigma}_2$ or "coordinate" labels ξ_1, ξ_2), the isospins ($\vec{\tau}_1, \vec{\tau}_2$ or labels ζ_1, ζ_2), the quantum wave number $\vec{k}_0$, the total spin (S) and isospin (T), and the state projections M_S and M_T . The two-body Hamiltonian is given by H_0 . We define the basis vectors $r \equiv (\vec{r}, \xi_1, \xi_2, \zeta_1, \zeta_2)$ and $k \equiv (\vec{k}, S, T, M_S, M_T)$, and the quantum number $k_0 \equiv (\vec{k}_0, S, T, M_S, M_T)$. We also write $E(k) = \hbar^2 k^2 / M$ and $E(k_0) = \hbar^2 k_0^2 / M$ (where $M/2$ is the reduced mass), which should cause no confusion, and where for bound states k_0^2 is $-\gamma^2$.

¹Dirac, P.A.M., PRINCIPLES OF QUANTUM MECHANICS, Princeton University Press, New Jersey, 1955; also Schiff, L. I., QUANTUM MECHANICS, McGraw-Hill Book Co., New York, 1949.

If the interaction operator $V(r, \vec{\sigma}_1, \vec{\sigma}_2, \vec{\tau}_1, \vec{\tau}_2)$ vanishes identically, Eq. (1) has free-particle wave function solutions described by¹

$$\langle r | k_0 \rangle_0 = e^{i \vec{k}_0 \cdot \vec{r}} \chi_{S, M_S}(\xi_1, \xi_2) \chi_{T, M_T}(\zeta_1, \zeta_2), \quad (2)$$

where the total spin (χ_{S, M_S}) and isospin (χ_{T, M_T}) wave functions are defined by²

$$\begin{aligned} \chi_{0,0}(1,2) &\equiv \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) - \alpha(2)\beta(1)], \\ \chi_{1,1}(1,2) &\equiv \alpha(1)\alpha(2), \\ \chi_{1,0}(1,2) &\equiv \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) + \alpha(2)\beta(1)], \\ \chi_{1,-1}(1,2) &\equiv \beta(1)\beta(2), \end{aligned} \quad (3)$$

with

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4)$$

Condon and Shortley phase conventions are used throughout this thesis.

²Condon, E. U. and Shortley, G. H., THEORY OF ATOMIC SPECTRA, Cambridge Univ. Press, 1953, p. 76; also Blatt, J. M. and Weisskopf, V. F., THEORETICAL NUCLEAR PHYSICS, Wiley & Sons, New York, 1952, p. 785ff.

The superscript (μ) attached to the wave functions in Eq. (1) is used to denote the asymptotic boundary condition under consideration. The convention $\mu = (+), (-),$ or (0) describes the following situations:

$$\begin{aligned}
 \langle r | k_o \rangle^{(+)} &\xrightarrow{k_o r \gg 1} \langle r | k_o \rangle_o + (\text{outgoing scattered wave}), \\
 \langle r | k_o \rangle^{(-)} &\xrightarrow{k_o r \gg 1} \langle r | k_o \rangle_o + (\text{incoming scattered wave}), \\
 \langle r | k_o \rangle^{(0)} &\xrightarrow{k_o r \gg 1} \langle r | k_o \rangle_o + (\text{standing scattered wave}).
 \end{aligned} \tag{5}$$

The Two-Particle Reaction Matrix

For subsequent convenience, we introduce at this point plane-wave matrix elements of the two-particle reaction matrix, defined by

$$\langle k | G^{(\mu)} | k_o \rangle = \sum_{\xi, \zeta} \int d^3 r \langle k | r \rangle_o V(r) \langle r | k_o \rangle^{(\mu)}. \tag{6}$$

Upon solving Eq. (6) for $V(r) \langle r | k_o \rangle^{(\mu)}$, one obtains

$$V(r) \langle r | k_o \rangle^{(\mu)} = \frac{1}{(2\pi)^3} \sum_{\substack{S, T \\ M_S, M_T}} \int d^3 k \langle r | k \rangle_o \langle k | G^{(\mu)} | k_o \rangle. \tag{7}$$

We also define the momentum representation of the two-particle wave functions by

$$\langle r | k_o \rangle^{(\mu)} \equiv \frac{1}{(2\pi)^3} \sum_{\substack{S, T \\ M_S, M_T}} \int d^3 k \langle r | k \rangle_o \langle k | k_o \rangle^{(\mu)}. \tag{8}$$

Upon using Eqs. (6) and (8), the momentum representation of the wave equation (1) is then given by

$$[E(k) - E(k_0)] \langle k | k_0 \rangle^{(\mu)} = - \langle k | G^{(\mu)} | k_0 \rangle. \quad (9)$$

The solution to Eq. (9), which satisfies the asymptotic boundary conditions (5), is

$$\langle k | k_0 \rangle^{(\mu)} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_0) \delta_{M_S, M_{S'}} + \frac{\langle k | G^{(\mu)} | k_0 \rangle}{E(k_0) - E(k) + i\mu\epsilon}. \quad (10)$$

When substituted into Eq. (8), this solution yields for the two particle wave functions in position space the result

$$\begin{aligned} \langle r | k_0 \rangle^{(\mu)} &= \langle r | k_0 \rangle_0 \\ &+ \frac{1}{(2\pi)^3} \sum_{\substack{S, T \\ M_S, M_T}} \int d^3k \langle r | k \rangle_0 \frac{\langle k | G^{(\mu)} | k_0 \rangle}{E(k_0) - E(k) + i\mu\epsilon}. \end{aligned} \quad (11)$$

Upon substituting this solution into Eq. (6) we find the familiar integral equation for the reaction matrix,

$$\begin{aligned} \langle k | G^{(\mu)} | k_0 \rangle &= \langle k | V | k_0 \rangle_0 \\ &+ \frac{1}{(2\pi)^3} \sum_{\substack{S, T \\ M_S, M_T}} \int d^3\ell \langle k | V | \ell \rangle_0 \frac{\langle \ell | G^{(\mu)} | k_0 \rangle}{E(k_0) - E(\ell) + i\mu\epsilon}, \end{aligned} \quad (12)$$

where the free-particle matrix elements of the interaction potential are defined by

$$\langle k|V|k_0 \rangle_0 \equiv \sum_{\xi, \zeta} \int d^3r \langle k|r \rangle_0 V(r) \langle r|k_0 \rangle_0. \quad (13)$$

Equation (12) in operator form is

$$G^{(\mu)} = V + V \left(\frac{1}{E - H_0 + i\mu\epsilon} \right) G^{(\mu)}, \quad (14)$$

which is commonly used definition of the reaction matrix.³

Partial-Wave Expansions

We define the angular momentum wave functions (also denoted by ψ_{LSJ}^M - see reference (2)) by

$$\phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2) \equiv \sum_{M_S, M_L} (LSM_L M_S | JM) Y_{LM_L}(\hat{n}_r) \chi_{S, M_S}(\xi_1, \xi_2), \quad (15)$$

where the $Y_{LM_L}(\hat{n}_r)$ are the spherical harmonics. Using the orthogonality properties of the Clebsch-Gordon coefficients,

³Goldberger, J. L. and Watson, K. M., COLLISION THEORY, Wiley & Sons, New York, 1965, p. 215.

$(LSM_L M_S | JM)$, Eq. (15) can be inverted to the form

$$Y_{LM_L}(\hat{n}_r) \chi_{S, M_S}(\xi_1, \xi_2) = \sum_{JM} (LSM_L M_S | JM) \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2). \quad (16)$$

Upon substituting Eq. (16) into the free-particle wave function, Eq. (2), and using the plane wave expansion theorem,

$$e^{i\vec{k} \cdot \vec{r}} = \frac{4\pi}{kr} \sum_{L, M_L} e^{iL\pi/2} F_L(kr) Y_{LM_L}^*(\hat{n}_k) Y_{LM_L}(\hat{n}_k), \quad (17)$$

we obtain

$$\langle r | k \rangle_0$$

$$= \frac{4\pi}{kr} \sum_{LM_L} e^{iL\pi/2} Y_{LM_L}^*(\hat{n}_r) F_L(kr) (LSM_L M_S | JM) \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2) \chi_{T, M_T}(\zeta_1, \zeta_2), \quad (18)$$

or

$$\langle k | r \rangle_0$$

$$= \frac{4\pi}{kr} \sum_{LM_L} e^{-iL\pi/2} Y_{LM_L}(\hat{n}_k) F_L(kr) (LSM_L M_S | JM) \phi_{LSJ}^{M+}(\hat{n}_r, \xi_1, \xi_2) \chi_{T, M_T}^+(\zeta_1, \zeta_2). \quad (19)$$

Here, $F_L(0) = \rho j_L(\rho)$, where $j_L(\rho)$ is the spherical Bessel function,

† indicates Hermitian conjugate and * indicates complex conjugate.

Similarly, the true two-nucleon wave function may be written in the form

$$\begin{aligned} \langle r | k_0 \rangle^{(\mu)} &= \frac{4\pi}{k_0 r} \sum_{\substack{L, L', M_L \\ J M_L}} e^{iL\pi/2} Y_{LM_L}^* (\hat{n}_{k_0}) \\ \langle r | k_0 L' L \rangle^{(\mu)} &= (L' S M_L' M_S | J' M) \phi_{L' S J}^M (\hat{n}_r, \xi_1, \xi_2) Y_{TM_T}(\zeta_1, \zeta_2), \end{aligned} \quad (20)$$

where $\langle r | k_0 L' L \rangle^{(\mu)}$ is a function which is determined by solving the Schrödinger radial wave equation.

The description of the wave function $\langle r | k_0 L' L \rangle^{(\mu)}$ is clarified by the introduction of the so-called Blatt-Biedenharn states.⁴ For the cases: $S = 0, L = J$; $S = 1, L = J$; and $S = 1, L = 1, J = 0$, the Blatt-Biedenharn states are defined by the usual convention as

$$\langle r | k_0 L' L \rangle^{(\mu)} = B_L^{(\mu)}(k_0) \langle r | k_0 L \rangle \delta_{L'L}, \quad (21)$$

where

$$\langle r | k_0 L \rangle \xrightarrow[k_0 r \gg 1]{} \sin \left[k_0 r - \frac{L\pi}{2} + \delta_L(k_0) \right], \quad (22)$$

⁴Blatt, J. M. and Biedenharn, L. C., Rev. Mod. Phys., 24, 1952, p. 258.

and

$$B_L^{(\pm)}(k_0) = e^{\pm i\delta_L(k_0)},$$

$$B_L^{(0)}(k_0) = 1/\cos\delta_L(k_0). \quad (23)$$

We note that our notation has been condensed in the above equations, so that actually $\delta_L(k_0) = \delta_{LJ}^{(S)}(k_0)$ and $\langle r|k_0 L \rangle = \langle r|k_0 L S J \rangle$.

For $S = 1$, $L = J \pm 1$ ($J \neq 0$), and upon suppressing the isospin quantum numbers for the present, the (orthogonal) Blatt-Biedenharn states may be written as

$$\begin{aligned} \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \alpha \rangle &= \cos \epsilon_J(k_0) \langle r | k_0 (J-1); \alpha \rangle \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\ &+ \sin \epsilon_J(k_0) \langle r | k_0 (J+1); \alpha \rangle \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2), \\ \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \beta \rangle &= - \sin \epsilon_J(k_0) \langle r | k_0 (J-1); \beta \rangle \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\ &+ \cos \epsilon_J(k_0) \langle r | k_0 (J+1); \beta \rangle \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2), \end{aligned} \quad (24)$$

where

$$\begin{aligned} \langle r | k_0 L; \alpha \rangle &\xrightarrow[k_0 r \gg 1]{} \sin [k_0 r - \frac{L'\pi}{2} + \delta_{\alpha\alpha}(k_0)], \\ \langle r | k_0 L; \beta \rangle &\xrightarrow[k_0 r \gg 1]{} \sin [k_0 r - \frac{L'\pi}{2} + \delta_{J\beta}(k_0)]. \end{aligned} \quad (25)$$

It is a linear combination of the states given in Eq. (24) which enters into $\langle r | k_0 \rangle^{(\mu)}$ of Eq. (20); namely, for fixed J and M , and with $(L, L') = J \pm 1$,

$$\begin{aligned} & \langle r | k_0 \rangle^{(\mu)} \Big|_{\substack{J, M \\ (L, L') = J \pm 1}} \\ &= \frac{4\pi}{kr} [B'_{J\alpha}{}^{(\mu)}(k_0) \langle \vec{r} \vec{\epsilon}_1 \vec{\epsilon}_2 | k_0 JM; \alpha \rangle + B'_{J\beta}{}^{(\mu)}(k_0) \langle \vec{r} \vec{\epsilon}_1 \vec{\epsilon}_2 | k_0 JM; \beta \rangle]. \quad (26) \end{aligned}$$

Upon using Eqs. (24) and (25), the asymptotic form of Eq. (26) is seen to be

$$\begin{aligned} & \langle r | k_0 \rangle^{(\mu)} \Big|_{\substack{J, M \\ (L, L') = J \pm 1}} \xrightarrow{k_0 r \gg 1} \\ & \frac{4\pi}{kr} \left\{ \begin{aligned} & B'_{J\alpha}{}^{(\mu)}(k_0) \cos \epsilon_J(k_0) \sin[k_0 r - (J-1)\frac{\pi}{2} + \delta_{J\alpha}(k_0)] \\ & - B'_{J\beta}{}^{(\mu)}(k_0) \sin \epsilon_J(k_0) \sin[k_0 r - (J-1)\frac{\pi}{2} + \delta_{J\beta}(k_0)] \end{aligned} \right\} \phi_{J-1, J}^M(\hat{n}_r, \epsilon_1, \epsilon_2), \\ & + \frac{4\pi}{kr} \left\{ \begin{aligned} & B'_{J\alpha}{}^{(\mu)}(k_0) \sin \epsilon_J(k_0) \sin[k_0 r - (J+1)\frac{\pi}{2} + \delta_{J\alpha}(k_0)] \\ & + B'_{J\beta}{}^{(\mu)}(k_0) \cos \epsilon_J(k_0) \sin[k_0 r - (J+1)\frac{\pi}{2} + \delta_{J\beta}(k_0)] \end{aligned} \right\} \phi_{J+1, J}^M(\hat{n}_r, \epsilon_1, \epsilon_2). \quad (27) \end{aligned}$$

For the outgoing (incoming) states, if we match the incoming (outgoing) asymptotic part of Eq. (27) to the incoming (outgoing) asymptotic plane wave states, then we obtain

$$B_{J\alpha}^{(\pm)}(k_o) = e^{\pm i\delta_{J\alpha}(k_o)} \left\{ \begin{array}{l} \cos \epsilon_J(k_o) \sum_{M_L} e^{i(J-1)\pi/2} Y_{(J-1)M_L}^*(\hat{n}_{k_o}) (J-1, 1, M_L, M_S | JM) \\ + \sin \epsilon_J(k_o) \sum_{M_L} e^{i(J+1)\pi/2} Y_{(J+1)M_L}^*(\hat{n}_{k_o}) (J+1, 1, M_L, M_S | JM) \end{array} \right\},$$

$$B_{J\beta}^{(\pm)}(k_o) = e^{\pm i\delta_{J\beta}(k_o)} \left\{ \begin{array}{l} -\sin \epsilon_J(k_o) \sum_{M_L} e^{i(J-1)\pi/2} Y_{(J-1)M_L}^*(\hat{n}_{k_o}) (J-1, 1, M_L, M_S | JM) \\ + \cos \epsilon_J(k_o) \sum_{M_L} e^{i(J+1)\pi/2} Y_{(J+1)M_L}^*(\hat{n}_{k_o}) (J+1, 1, M_L, M_S | JM) \end{array} \right\}.$$

(28)

For the standing wave states, $B_{J\alpha}^{(0)}(k_o)$ and $B_{J\beta}^{(0)}(k_o)$ are determined by matching the $\sin(k_o r - (J \pm 1)\pi/2)$ asymptotic parts of Eq. (27) to the corresponding asymptotic parts of the plane wave states. We obtain $1/\cos \delta_{J\alpha}(k_o)$ and $1/\cos \delta_{J\beta}(k_o)$ in place of $e^{\pm i\delta_{J\alpha}(k_o)}$ and $e^{\pm i\delta_{J\beta}(k_o)}$, respectively, in Eq. (28).

Upon substituting Eqs. (24) and (28) into (26), and comparing with Eq. (20), we obtain the immediate results

$$\begin{aligned}
\langle r | k_0(J-1); J-1 \rangle^{(\mu)} &= B_{J\alpha}^{(\mu)}(k_0) \cos^2 \epsilon_J(k_0) \langle r | k_0(J-1); \alpha \rangle + \\
&+ B_{J\beta}^{(\mu)}(k_0) \sin^2 \epsilon_J(k_0) \langle r | k_0(J-1); \beta \rangle, \\
\langle r | k_0(J+1); J-1 \rangle^{(\mu)} &= (B_{J\alpha}^{(\mu)}(k_0) \langle r | k_0(J+1); \alpha \rangle - \\
&- B_{J\beta}^{(\mu)}(k_0) \langle r | k_0(J+1); \beta \rangle) \cos \epsilon_J(k_0) \sin \epsilon_J(k_0), \\
\langle r | k_0(J-1); J+1 \rangle^{(\mu)} &= (B_{J\alpha}^{(\mu)}(k_0) \langle r | k_0(J-1); \alpha \rangle - \\
&- B_{J\beta}^{(\mu)}(k_0) \langle r | k_0(J-1); \beta \rangle) \cos \epsilon_J(k_0) \sin \epsilon_J(k_0), \\
\langle r | k_0(J+1); J+1 \rangle^{(\mu)} &= B_{J\alpha}^{(\mu)}(k_0) \sin^2 \epsilon_J(k_0) \langle r | k_0(J+1); \alpha \rangle + \\
&+ B_{J\beta}^{(\mu)}(k_0) \cos^2 \epsilon_J(k_0) \langle r | k_0(J+1); \beta \rangle,
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
B_{J\alpha}^{(\pm)}(k_0) &\equiv e^{\pm i \delta_{J\alpha}(k_0)}, \quad B_{J\beta}^{(\pm)}(k_0) \equiv e^{\pm i \delta_{J\beta}(k_0)}, \\
B_{J\alpha}^{(0)}(k_0) &\equiv 1/\cos \delta_{J\alpha}(k_0), \quad B_{J\beta}^{(0)}(k_0) \equiv 1/\cos \delta_{J\beta}(k_0).
\end{aligned} \tag{30}$$

When there is no tensor force, $\epsilon_J(k_0) \rightarrow 0$ and Eqs. (29) and (30)

reduce, in form, to the case $L' = J$ given by Eqs. (21) through

(23).

Next we consider the wave functions, for all L and J values,

$$\langle \vec{r} \xi_1 \xi_2 | k_0 S_{JM}; L \rangle^{(\mu)} \equiv \sum_{L'} \langle r | k_0 L'; L \rangle^{(\mu)} \phi_{L' S_J}^M(\hat{n}_r, \xi_1, \xi_2). \tag{31}$$

Using Eqs. (21), (24) and (29), we obtain

$$\begin{aligned}
 \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; J-1 \rangle^{(\mu)} &= \begin{pmatrix} B_{J\alpha}^{(\mu)}(k_0) \cos \epsilon_J(k_0) \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \alpha \rangle \\ -B_{J\beta}^{(\mu)}(k_0) \sin \epsilon_J(k_0) \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \beta \rangle \end{pmatrix}, \\
 \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; J+1 \rangle^{(\mu)} &= \begin{pmatrix} B_{J\alpha}^{(\mu)}(k_0) \sin \epsilon_J(k_0) \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \alpha \rangle \\ -B_{J\beta}^{(\mu)}(k_0) \cos \epsilon_J(k_0) \langle \vec{r} \xi_1 \xi_2 | k_0 J M; \beta \rangle \end{pmatrix},
 \end{aligned} \tag{32}$$

where $S = 1$ and $J \neq 0$, and

$$\langle \vec{r} \xi_1 \xi_2 | k_0 S J M; J \rangle^{(\mu)} = B_{J S J}^{(\mu)}(k_0) \langle r | k_0 J S J \rangle \phi_{J S J}^M(\hat{n}_r, \xi_1, \xi_2)$$

(with $J \neq 0$ when $S = 1$),

$$\langle \vec{r} \xi_1 \xi_2 | k_0 1 0 0; 1 \rangle^{(\mu)} = B_{110}^{(\mu)}(k_0) \langle r | k_0 1 1 0 \rangle \phi_{110}^M(\hat{n}_r, \xi_1, \xi_2). \tag{33}$$

Upon utilizing the results of Eqs. (31) through (33), the two-nucleon wave function of Eq. (20) takes the form, similar to that of Eq. (18) for free particles,

$$\begin{aligned}
 &\langle r | k_0 \rangle^{(\mu)} \\
 &= \frac{4\pi}{k_0 r} \sum_{\substack{L J \\ M_1 M}} e^{iL\pi/2} Y_{LM_L}^*(\hat{n}_{k_0}) (L S M_L M_S | J M) \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; L \rangle^{(\mu)} \chi_{T M_T}(\zeta_1, \zeta_2).
 \end{aligned} \tag{34}$$

The Non-Local Separable Interaction

A general form of the non-local spin-dependent interaction $U (= \frac{M}{\hbar^2} V)$ can now be specified by the equation $U < \vec{r} \xi_1 \xi_2 | k_0 S J M; L >^{(+)}$

$$\equiv \sum_{\xi'_1, \xi'_2} \int d^3 r' U(\vec{r} \xi_1 \xi_2; \vec{r}', \xi'_1 \xi'_2) < \vec{r}' \xi'_1 \xi'_2 | k_0 S J M; L >^{(+)}, \quad (35)$$

where we use the wave functions (31). We utilize the Wigner arguments⁵ on the most general form of the two-nucleon interaction to write

$$\begin{aligned} U(\vec{r}; \xi_1 \xi_2) &= \frac{1}{4} U^{(1)}(\vec{r}, \xi_1 \xi_2; \vec{r}', \xi'_1 \xi'_2) (1 - \vec{\sigma}_1' \cdot \vec{\sigma}_2') \\ &\quad + \frac{1}{4} U^{(3)}(\vec{r}, \xi_1 \xi_2; \vec{r}', \xi'_1 \xi'_2) (3 + \vec{\sigma}_1' \cdot \vec{\sigma}_2'), \end{aligned} \quad (36)$$

where

$$U^{(1)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) \equiv U_C^{(1)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2),$$

$$U^{(3)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) \equiv U_C^{(3)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2)$$

$$+ U_{LS}^{(3)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) (\vec{L} \cdot \vec{S}') + U_T^{(3)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) S_{12}(\hat{n}', \xi'_1, \xi'_2). \quad (37)$$

⁵ Eisenbud, L. and Wigner, E., Proc. Nat. Acad. Sci., 27, 1941, p. 281; also Okubo, S. and Marshak, R., Ann. of Phys., 4, 1958, p. 166; also Breit, G., Phys. Rev., 111, 1958, p. 652; also Case, K. M. and Pais, A., Phys. Rev., 80, 1950, p. 203; also Gammel, J. L. and Thaler, R. M., PROGRESS IN ELEMENTARY PARTICLE AND COSMIC RAY PHYSICS, Vol. V, North Holland Publishing Co., Amsterdam, 1960, p. 99.

The superscripts 1 and 3 label singlet and triplet states, and the subscripts C, T, and LS refer to Central, Tensor, and Spin-Orbit-interactions. For this investigation we shall, for simplicity, not include a spin-orbit interaction. For the interactions of Eq. (37) we then make the expansions

$$\begin{aligned}
 & U_C^{(S)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) \\
 &= \sum_{LJM} U_C^{(J,S)}(r, r') \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2) \phi_{LSJ}^{M\dagger}(\hat{n}_{r'}, \xi'_1, \xi'_2),
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 & U_T^{(3)}(\vec{r}, \xi_1, \xi_2; \vec{r}', \xi'_1, \xi'_2) \\
 &= \sum_{LJM} U_T^{(J)}(r, r') \phi_{LLJ}^M(\hat{n}_r, \xi_1, \xi_2) \phi_{LLT}^{M\dagger}(\hat{n}_{r'}, \xi'_1, \xi'_2),
 \end{aligned}$$

where for convenience, and simplicity, we choose L-independent non-local interactions, $U_C^{(J,S)}(r, r')$ and $U_T^{(J)}(r, r')$. Finally, we choose the spatial parts of the interaction to be of separable form given by

$$U_C^{(J,S)}(r, r') = \begin{cases} -\sigma_C^{(J,S)} \frac{f_C(r)}{r} \frac{f_C(r')}{r'} & \text{if } (r, r') > a \\ U_{HC} \frac{\delta(r-r')}{rr'} & \text{if } (r, r') < a, \end{cases} \tag{39}$$

(and we are interested only in the limit $U_{HC} \rightarrow \infty$), and

$$\begin{aligned}
& U_T^{(J)}(r, r') \\
& \equiv \begin{cases} - \left[\frac{(2J+1)(1-\delta_{J,0}) + \frac{1}{4}\delta_{J,0}}{6\sqrt{J(J+1)}} \right] \sigma_T^{(J)} r_T \frac{f_T(r)}{r} \frac{f_T(r')}{r'} & \text{if } (r, r') > a \\ 0 & \text{if } (r, r') < a. \end{cases}
\end{aligned} \tag{40}$$

For the function $f_\alpha(r)$ we shall use the exponential form [$f_\alpha(r)/r$ then has a Yukawa form],

$$f_\alpha(r) = \lambda_\alpha^2 e^{-\lambda_\alpha(r-a)} \quad (r > a), \tag{41}$$

with $\lambda_\alpha = 1/r_\alpha$ ($\alpha = C$ or T), and where a is the hard-core (HC) diameter. In Eqs. (39) and (40), σ_α is the interaction "strength" (in dimensionless units), and the factor multiplying $\sigma_T^{(J)}$ in Eq. (40) is included for later convenience.

We next observe the following identities:

I. For the total spin,

$$\vec{S} = \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2), \tag{42}$$

S^2 being a good quantum number implies that

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2S^2 - 3, \quad (43)$$

and hence

$$\begin{aligned} \frac{1}{4} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2) \chi_{0,0}(\xi_1, \xi_2) &= \chi_{0,0}(\xi_1, \xi_2), \\ \frac{1}{4} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2) \chi_{1,M_S}(\xi_1, \xi_2) &= 0, \\ \frac{1}{4} (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \chi_{0,0}(\xi_1, \xi_2) &= 0, \\ \frac{1}{4} (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \chi_{1,M_S}(\xi_1, \xi_2) &= \chi_{1,M_S}(\xi_1, \xi_2). \end{aligned} \quad (44)$$

II. For the total angular momentum,

$$\vec{J} = \vec{L} + \vec{S}, \quad (45)$$

we have

$$\vec{L} \cdot \vec{S} = \frac{1}{2} (\vec{J}^2 - \vec{L}^2 - \vec{S}^2), \quad (46)$$

and hence

$$(\vec{L} \cdot \vec{S}) \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2) = \frac{1}{2} [J(J+1) - L(L+1) - S(S+1)] \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2). \quad (47)$$

III. From the definition of the tensor operator,

$$S_{12}(\hat{n}_r, \xi_1, \xi_2) = 3(\vec{\sigma}_1 \cdot \hat{n}_r)(\vec{\sigma}_2 \cdot \hat{n}_r) - \vec{\sigma}_1 \cdot \vec{\sigma}_2, \quad (48)$$

it can be shown (after performing a somewhat tedious computation) that

$$\begin{aligned} S_{12} \phi_{J0J}^M &= 0, \\ S_{12} \phi_{J1J}^M &= 2\phi_{J1J}^M, \\ S_{12} \phi_{J-1,1J}^M &= -\frac{2(J-1)}{2J+1} \phi_{J-1,1J}^M + \frac{6\sqrt{J(J+1)}}{2J+1} \phi_{J+1,1J}^M, \\ S_{12} \phi_{J-1,1J}^M &= \frac{6\sqrt{J(J+1)}}{2J+1} \phi_{J-1,1J}^M - \frac{2(J+2)}{2J+1} \phi_{J+1,1J}^M. \end{aligned} \quad (49)$$

Coupled Differential Equations

We now write the wave function $\langle \vec{r} \xi_1 \xi_2 | k_0 S J M; L' \rangle^{(+)}$

of Eq. (31) as follows:

$$\begin{aligned} \langle \vec{r} \xi_1 \xi_2 | k_0 1 J M; J-1 \rangle_{S=1}^{(+)} &= u_{Jk_0}^{(-)}(r) \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\ &\quad + w_{Jk_0}^{(-)}(r) \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \quad (\text{with } J \neq 0), \\ \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; J \rangle^{(+)} &= v_{Jk_0}^{(S)}(r) \phi_{J S J}^M(\hat{n}_r, \xi_1, \xi_2) \quad (\text{with } J \neq 0 \text{ when } S = 1), \end{aligned}$$

$$\begin{aligned}
\langle \vec{r} \xi_1 \xi_2 | k_0 1JM; J+1 \rangle_{S=1}^{(+)} &= u_{Jk_0}^{(+)}(r) \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\
&+ w_{Jk_0}^{(+)}(r) \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2),
\end{aligned} \tag{50}$$

where, in the last equation, $u_{Jk_0}^{(+)}(r) = 0$ when $J = 0$. The coupled triplet states can equally well be written as the Blatt-Biedenharn states of Eq. (24); namely

$$\begin{aligned}
\langle \vec{r} \xi_1 \xi_2 | k_0 JM; \alpha \rangle_{S=1} &= u_{Jk_0 \alpha}(r) \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\
&+ w_{Jk_0 \alpha}(r) \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\langle \vec{r} \xi_1 \xi_2 | k_0 JM; \beta \rangle_{S=1} &= u_{Jk_0 \beta}(r) \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) \\
&+ w_{Jk_0 \beta}(r) \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2).
\end{aligned}$$

The important point to notice about either Eqs. (50) or (51), is that the coupled triplet states involve four functions which must be determined from the asymptotic boundary conditions which are set for the problem. At this point, we drop the $(+)$ or (α, β) subscripts. We shall return to their identification in Sec. I.3.

Upon using Eqs. (35) through (51) and the orthonormality properties of the $\phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2)$, the (radial) differential equations which need to be solved are obtained as follows: For the states in which there is no coupling ($S = 0$ or $S = 1$ with $L = J$)

$$\begin{aligned}
& \left[\frac{d^2}{dr^2} + k_o^2 - \frac{J(J+1)}{r^2} \right] v_{Jk_o}^{(S)}(r) \\
& = \left\{ \begin{aligned} & -\sigma_C^{(J,S)} r_C f_C(r) \int_a^\infty dr' f_C(r') v_{Jk_o}^{(S)}(r') \\ & - \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1}(1-\delta_{J,0}) \sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') v_{Jk_o}^{(S)}(r') \end{aligned} \right\}, \quad (52)
\end{aligned}$$

and for the coupled triplet states ($S = 1$ with $L = J-1$ and $L = J+1$) we have the coupled equations

$$\begin{aligned}
& \left[\frac{d^2}{dr^2} + k_o^2 - \frac{J(J-1)}{r^2} \right] u_{Jk_o}(r) \\
& = \left\{ \begin{aligned} & -\sigma_C^{(J)} r_C f_C(r) \int_a^\infty dr' f_C(r') u_{Jk_o}(r') \\ & -\sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') \left[w_{Jk_o}(r') - \frac{(J-1)}{3\sqrt{J(J+1)}} u_{Jk_o}(r') \right] \end{aligned} \right\},
\end{aligned}$$

and

$$\begin{aligned}
& \left[\frac{d^2}{dr^2} + k_o^2 - \frac{(J+1)(J+2)}{r^2} \right] w_{Jk_o}(r) \\
& = \left\{ \begin{aligned} & -\sigma_C^{(J)} r_C f_C(r) \int_a^\infty dr' f_C(r') w_{Jk_o}(r') \\ & -\sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') \left[u_{Jk_o}(r') - \left[\frac{(1-\delta_{J,0})(J+2)}{3\sqrt{J(J+1)}} + \delta_{J,0} \right] w_{Jk_o}(r') \right] \end{aligned} \right\}, \quad (53)
\end{aligned}$$

When $J = 0$ only the second of Eqs. (53), with $u_{JK_0}(r) = 0$, applies.

1.2 Theory of Solution of the Separable Two-Body Problem

We have assumed simple non-local interactions of the form

$$U_c^{(J,S)}(r,r') \equiv \begin{cases} -\sigma_c^{(J,S)} r_c \frac{f_c(r)}{r} \frac{f_c(r')}{r'} & \text{if } r, r' > a \\ U_{HC} \delta(r - r') & \text{if } r, r' < a, \end{cases} \quad (1)$$

where we are only interested in the limit $U_{HC} \rightarrow \infty$, and

$$U_T^{(J)}(r,r') \equiv \begin{cases} -\left[\frac{(2J+1)(1-\delta_{J,0}) + \frac{1}{2}\delta_{J,0}}{6\sqrt{J(J+1)}} \right] \sigma_T^{(J)} r_T f_T(r) f_T(r') & \text{if } r, r' > a \\ 0 & \text{if } r, r' < a, \end{cases} \quad (2)$$

where

$$f_\alpha(r) \equiv \lambda_\alpha^2 e^{-\lambda_\alpha(r-a)}, \quad \text{if } r > a, \quad (3)$$

$\lambda_\alpha = 1/r_\alpha$ ($\alpha = C$ or T), and the factor in brackets is included in Eq. (2) for subsequent convenience.

The Schrödinger equation for the uncoupled states ($S = 0$ or $S = 1$ with $J = L$) is

$$\left[\frac{d^2}{dr^2} + k_0^2 - \frac{J(J+1)}{r^2} \right] v_{Jk_0}^{(S)}(r) = \begin{cases} -\sigma_c^{(J,S)} r_c f_c(r) \int_a^\infty dr' f_c(r') v_{Jk_0}^{(S)}(r') \\ -\frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} \left(1 - \delta_{J,0} \right) \sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') v_{Jk_0}^{(S)}(r') \end{cases}, \quad (4)$$

and for the triplet states ($S = 1$, with $L = J-1$ and $L = J+1$), we have the coupled equations

$$\left[\frac{d^2}{dr^2} + k_o^2 - \frac{J(J-1)}{r^2} \right] u_{Jk_o}(r) = \left\{ \begin{array}{l} -\sigma_c^{(J)} r_c f_c(r) \int_a^\infty dr' f_c(r') u_{Jk_o}(r') \\ -\sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') \left[w_{Jk_o}(r') - \frac{(J-1)}{3\sqrt{J(J+1)}} u_{Jk_o}(r') \right] \end{array} \right\}, \quad (5)$$

and

$$\left[\frac{d^2}{dr^2} + k_o^2 - \frac{(J+1)(J+2)}{r^2} \right] w_{Jk_o}(r) = \left\{ \begin{array}{l} -\sigma_c^{(J)} r_c f_c(r) \int_a^\infty dr' f_c(r') w_{Jk_o}(r') \\ -\sigma_T^{(J)} r_T f_T(r) \int_a^\infty dr' f_T(r') \left[u_{Jk_o}(r') - \left[\frac{(J+2)(1-\delta_{J,0})}{3\sqrt{J(J+1)}} + \delta_{J,0} \right] w_{Jk_o}(r') \right] \end{array} \right\}, \quad (6)$$

(when $J=0$ only Eq. (6), with $u_{Jk_o}(r) \equiv 0$ applies).

We first indicate the solution to Eq. (4) for the singlet states and then, in an analogous fashion, the solution to Eqs. (5) and (6) for the coupled states.

Define the momentum-transformed wave functions $v_{Jk_0}^{(S)}(k)$ by

$$v_{Jk_0}^{(S)}(r) \equiv \int_0^\infty dk F_J'(kr) v_{Jk_0}^{(S)}(k), \quad (7)$$

where $F_J'(kr)$ (and correspondingly $G_J'(kr)$) forms a complete orthonormal set of functions on the interval $a \leq r \leq \infty$ and is defined by

$$F_J'(kr) \equiv \frac{1}{\Delta_J(ka)} [F_J(kr)G_J(ka) - F_J(ka)G_J(kr)], \quad (8)$$

$$G_J'(kr) \equiv \frac{1}{\Delta_J(ka)} [F_J(kr)F_J(ka) + G_J(kr)G_J(ka)], \quad (9)$$

with

$$\Delta_J^2(ka) \equiv F_J^2(ka) + G_J^2(ka),$$

$$F_J(\rho) \equiv \rho j_J(\rho), \quad (10)$$

$$G_J(\rho) \equiv -\rho \eta_J(\rho).$$

In Eq. (10), the functions $j_J(\rho)$ and $\eta_J(\rho)$ are the regular and irregular spherical Bessel functions, respectively.

Upon substituting Eq. (7) into Eq. (4), multiplying

by $\frac{2}{\pi} F_J'(\ell r)$, integrating over the range $a \leq r \leq \infty$, and using the orthogonality condition

$$\frac{2}{\pi} \int_a^\infty dr F_J'(kr) F_J'(\ell r) = \delta(k - \ell), \quad (11)$$

(where $\delta(x)$ is the Dirac delta function) along with the definition

of the momentum transformed potential

$$g_J^\alpha(k) = \int_a^\infty dr F_J'(kr) f_\alpha(r), \quad \alpha = C \text{ or } T, \quad (12)$$

one readily obtains the result that

$$\begin{aligned} & (k_o^2 - \ell^2) v_{Jk_o}^{(S)}(\ell) \\ &= \left\{ \begin{aligned} & -\frac{2}{\pi} \sigma_c^{(J,S)} r_c g_J^{(c)}(\ell) \int_0^\infty dk g_J^{(c)}(k) v_{Jk_o}^{(S)}(k) \\ & -\frac{2}{\pi} \cdot \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} (1-\delta_{J,0}) \sigma_T^{(J)} r_T g_J^{(T)}(\ell) \int_0^\infty dk g_J^{(T)}(k) v_{Jk_o}^{(S)}(k) \end{aligned} \right\}. \end{aligned} \quad (13)$$

If we assume a solution to Eq. (13) of the form

$$v_{Jk_o}^{(S)}(\ell) = N_v^{(J,S)}(k_o) \left\{ \delta(k_o - \ell) + P \left(\frac{\langle \ell | B_v^{(J,S)} | k_o \rangle}{\ell^2 - k_o^2} \right) \right\} \quad (14)$$

where P denotes that the principle value is to be taken, then upon substituting Eq. (14) into Eq. (13), we obtain

$$\begin{aligned}
 < \ell | B_v^{(J,S)} | k_o > \\
 = & \left\{ \frac{2}{\pi} \sigma_c^{(J,S)} r_c g_J^{(c)}(\ell) \left\{ g_J^{(c)}(k_o) + \int_0^\infty dk g_J^{(c)}(k) P \left[\frac{< k | B_v^{(J,S)} | k_o >}{k^2 - k_o^2} \right] \right\} \right. \\
 & \left. + \frac{2}{\pi} \cdot \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} (1-\delta_{J,0}) \sigma_T^{(J)} r_T g_J^{(T)}(\ell) \right. \\
 & \left. \left\{ g_J^{(T)}(k_o) + \int_0^\infty dk g_J^{(T)}(k) P \left[\frac{< k | B_v^{(J,S)} | k_o >}{k^2 - k_o^2} \right] \right\} \right\}
 \end{aligned}
 \tag{15}$$

Examination of Eq. (15) suggests that we define functions $B_c^{(J,S)}$ and $B_T^{(J,S)}(k_o)$ by the relationship,

$$\begin{aligned}
 < \ell | B_v^{(J,S)} | k_o > = & \frac{2}{\pi} \sigma_c^{(J,S)} r_c g_J^{(c)}(\ell) B_c^{(J,S)}(k_o) + \\
 & + \frac{2}{\pi} \frac{2J+1}{3\sqrt{J(J+1)}} \sigma_{S,1} (1-\delta_{J,0}) \sigma_J^{(T)} r_T g_J^{(T)}(\ell) B_T^{(J)}(k_o).
 \end{aligned}
 \tag{16}$$

Then, since $g_J^{(c)}(k)$ and $g_J^{(T)}(k)$ are independent functions, substitution of Eq. (16) into Eq. (15) yields two coupled linear equations in the functions $B_c^{(J,S)}(k_o)$ and $B_T^{(J)}(k_o)$; namely,

$$\begin{aligned}
B_c^{(J,S)}(k_o) &= g_J^{(c)}(k_o) + A_{J(S)}^{(c)}(k_o) B_c^{(J,S)}(k_o) \\
&+ \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} (1-\delta_{J,0}) \left(\frac{\sigma_T^{(J)} r_T}{\sigma_c^{J,S} r_c} \right) A_J^{(c,T)}(k_o) B_T^{(J)}(k_o)
\end{aligned} \tag{17}$$

$$\begin{aligned}
B_T^{(J)}(k_o) &= g_J^{(T)}(k_o) + A_J^{(c,T)}(k_o) B_c^{(J,S)}(k_o) \\
&+ \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} (1-\delta_{J,0}) A_J^{(T)}(k_o) B_T^{(J)}(k_o)
\end{aligned} \tag{18}$$

where we have defined

$$\begin{aligned}
A_{J(S)}^{(\alpha)}(k_o) &\equiv \frac{2}{\pi} \sigma_\alpha^{(J,S)} r_\alpha \int_0^\infty dk P \left(\frac{[g_J^{(\alpha)}(k_o)]^2}{k^2 - k_o^2} \right) \\
&- \frac{1}{\sqrt{2}} \delta_{J,0} \delta_{\alpha,\beta} \delta_{\alpha,T}, \quad \alpha = C, T,
\end{aligned} \tag{19}$$

$$A_J^{(C,T)}(k_o) \equiv \frac{2}{\pi} \sigma_c^{(J,S)} r_c \int_0^\infty dk P \left(\frac{g_J^{(c)}(k) g_J^{(T)}(k)}{k^2 - k_o^2} \right). \tag{20}$$

Of course, $\sigma_T^{J,1} \equiv \sigma_T^{(J)}$, and we have suppressed the S dependence in the functions $B_T^{(J)}(k_o)$ and $A_J^{(C,T)}(k_o)$ since $S = 1$ only in these cases. The integrals in Eqs. (19) and (20) are evaluated in Appendices 4, 6, and 7.

The solution to the linear equations in Eqs. (17) and (18) is straightforward, with the result that

$$B_c^{(J,S)}(k_o) = \left\{ g_J^{(c)}(k_o) \left[1 - \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1}(1 - \delta_{J,0}) A_J^{(T)}(k_o) \right] + g_J^{(T)}(k_o) \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1}(1 - \delta_{J,0}) \frac{\sigma_T^{(J)} r_T}{\sigma_c^{(J,S)} r_c} A_J^{(C,T)}(k_o) \right\} / \Delta, \quad (21)$$

$$B_T^{(J)}(k_o) = \left\{ g_J^{(c)}(k_o) A_J^{(C,T)}(k_o) + g_J^{(T)}(k_o) (1 - A_{J,S}^{(c)}(k_o)) \right\} / \Delta, \quad (22)$$

where

$$\Delta = \Delta_{J,S}(k_o) = \left[1 - A_{J,S}^{(c)}(k_o) \right] \left[1 - \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1}(1 - \delta_{J,0}) A_J^{(T)}(k_o) \right] - \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1}(1 - \delta_{J,0}) \left(\frac{\sigma_T^{(J)} r_T}{\sigma_c^{(J,S)} r_c} \right) \left(A_J^{(C,T)}(k_o) \right)^2. \quad (23)$$

Upon using Eqs. (14) and (16), the singlet momentum wave

function $v_{Jk_o}^{(s)}(\ell)$ becomes

$$v_{Jk_0}^{(S)}(\ell) = N_V^{(J,S)}(k_0)$$

$$\left\{ \begin{aligned} & \delta(k_0 - \ell) + \frac{2}{\pi} \sigma_c^{(J,S)} r_c g_J^{(c)}(\ell) B_c^{(J,S)}(k_0) P\left(\frac{1}{\ell^2 - k_0^2}\right) \\ & + \frac{2}{\pi} \frac{2J+1}{3\sqrt{J(J+1)}} \delta_{S,1} (1 - \delta_{J,0}) \sigma_T^{(J)} r_T g_J^{(T)}(\ell) B_T^{(J)}(k_0) P\left(\frac{1}{\ell^2 - k_0^2}\right) \end{aligned} \right\},$$

(24)

where $B_c^{(J,S)}(k_0)$ and $B_T^{(J)}(k_0)$ are given by Eqs. (21) through (23).

(The normalization constant $N_V^{(J,S)}(k_0)$ is determined in Sec. I.3.)

Upon substituting Eq. (24) into Eq. (7), we obtain

$$v_{J,k_0}^{(S)}(r) =$$

$$N_V^{(J,S)}(k_0) \left\{ \begin{aligned} & F_J'(k_0 r) + \frac{2}{\pi} \sigma_c^{(J,S)} r_c B_c^{(J,S)}(k_0) F_J^{(c)}(k_0, r) \\ & + \frac{2}{\pi} \frac{2J+1}{3\sqrt{J(J+1)}} \sigma_T^{(J)} r_T \delta_{S,1} B_T^{(J)}(k_0) (1 - \delta_{J,0}) F_J^{(T)}(k_0, r) \end{aligned} \right\},$$

(25)

where

$$F_J^{(\alpha)}(k_0, r) = \int_0^\infty dk P\left(\frac{g_J^{(\alpha)}(k) F_J'(kr)}{k^2 - k_0^2}\right), \quad \alpha = (C, T). \quad (26)$$

Now, as r approaches infinity,*

$$v_{J,k_0}^{(S)}(r) \xrightarrow{r \rightarrow \infty} N_V^{(J,S)}(k_0)$$

$$\left\{ \begin{aligned} & F_J'(kr) \\ & + \frac{1}{k_0} G_J'(k_0 r) \left\{ \begin{aligned} & \sigma_c^{(J,S)} r_c g_J^{(c)}(k_0) B_c^{(J,S)}(k_0) \\ & + \frac{2J+1}{3\sqrt{J(J+1)}} \sigma_T^{(J)} r_T \delta_{S,1} (1-\delta_{J,0}) g_J^T(k_0) B_T^{(J)}(k_0) \end{aligned} \right\} \end{aligned} \right\} \quad (27)$$

*From Appendix 11, $F_\ell^{(\alpha)}(k_0 r) = \frac{\pi}{2k_0} [G_\ell'(k_0 r) g_\ell^\alpha(k_0 r) -$

$\Delta_\ell(k_0 r) e g_\ell^\alpha(k_0 r)]$ where $g_\ell^\alpha(k_0 r)$ is given by Eq. (36) in Appendix 11.

Then since $H_\ell^{(-)}(\tau) \xrightarrow{\tau \rightarrow \infty} e^{i\ell\pi/2} e^{-i\tau}$, and

$Z_\ell(\mu, \tau) \xrightarrow{\tau \rightarrow \infty} \frac{e^{-(\mu-i\tau)}}{\mu-i\tau} \tau^\ell e^{-i\ell\pi/2}$, with $\mu = \lambda a$ and $\tau = ka$, it follows that

$$F_\ell^{(\alpha)}(k_0 r) \xrightarrow{r \rightarrow \infty} \frac{\pi}{2k_0} g_\ell^\alpha(k_0) G_\ell'(k_0 r) + \mathcal{O}(e^{-\lambda \alpha r}).$$

Upon using the asymptotic forms for $F_J'(\rho)$ and $G_J'(\rho)$, we then obtain from Eq. (27),

$$\begin{aligned} v_{J,k_0}^{(S)}(r) \xrightarrow{r \rightarrow \infty} N_v^{(J,S)}(k_0) \left(\sin(k_0 r - \frac{J\pi}{2} - \delta_J'(k_0)) \right. \\ \left. + \tan \delta_{J,S}^{(\sigma)}(k_0) \cos(k_0 r - \frac{J\pi}{2} + \delta_J'(k_0)) \right), \end{aligned} \quad (28)$$

where

$$\begin{aligned} \tan \delta_{J,S}^{(\sigma)}(k_0) &= \frac{1}{k_0} \left[\sigma_c^{(J,S)} r_c g_J^{(c)}(k_0) B_c^{(J,S)}(k_0) \right. \\ &\quad \left. + (1 - \delta_{J,0}) \delta_{S,1} \frac{2J+1}{3\sqrt{J(J+1)}} \sigma_T^{(J)} r_T g_J^{(T)}(k_0) B_T^{(J)}(k_0) \right] \\ &= \frac{\pi}{2k_0} \langle k_0 | B_v^{J,S} | k_0 \rangle. \end{aligned} \quad (29)$$

The hard-core phase shift $\delta_J'(k_0)$ is given by

$$\tan \delta_J'(k_0) \equiv -F_J(ka)/G_J(ka), \quad \delta_0'(k_0) \equiv -k_0 a. \quad (30)$$

The triplet $J = 0$ solution can be deduced from Eq. (25), after comparing Eqs. (4) and (6), by the replacement:

$$v^{(S)} \rightarrow w \text{ and } \frac{(2J+1)}{3\sqrt{J(J+1)}} \delta_{S,1} (1 - \delta_{J,0}) \rightarrow \delta_{J,0}.$$

Triplet State Solutions

Next we determine the solution to the coupled equations for the triplet states from Eqs. (5) and (6). We define the momentum-transformed wave functions $u_{Jk_0}(k)$ and $w_{Jk_0}(k)$ by

$$u_{Jk_0}(r) \equiv \int_0^\infty dk F'_{J-1}(kr) u_{Jk_0}(k) \quad (31)$$

$$w_{Jk_0}(r) \equiv \int_0^\infty dk F'_{J+1}(kr) w_{Jk_0}(k). \quad (32)$$

Upon substituting Eqs. (31) and (32) into Eqs. (5) and (6) and using Eq. (12) for the momentum transformed potentials, we obtain
(for $J > 0$)

$$\int_0^\infty dk (k_0^2 - k^2) F'_{J-1}(kr) u_{Jk_0}(k) \\ = \left\{ \begin{aligned} & -\sigma_c^{(J)} r_c f_c(r) \int_0^\infty dk g_{J-1}^{(c)}(k) u_{Jk_0}(k) \\ & -\sigma_T^{(J)} r_T f_T(r) \int_0^\infty dk [g_{J+1}^{(T)}(k) w_{Jk_0}(k) - \frac{(J-1)}{3\sqrt{J(J+1)}} g_{J+1}^{(T)}(k) u_{J,k_0}(k)] \end{aligned} \right\}, \quad (33)$$

and

$$\int_0^{\infty} dk (k_0^2 - k^2) F'_{J+1}(kr) w_{Jk_0}(k)$$

$$= \left\{ \begin{aligned} & - \sigma_c^{(J)} r_c f_c(r) \int_0^{\infty} dk g_{J+1}^{(c)}(k) w_{Jk_0}(k) \\ & - \sigma_c^{(J)} r_T f_T(r) \int_0^{\infty} dk [g_{J-1}^{(T)}(k) u_{Jk_0}(k) - \frac{(J+2)}{3\sqrt{J(J+1)}} g_{J+1}^{(T)}(k) w_{Jk_0}(k)] \end{aligned} \right\} \quad (34)$$

Upon multiplying Eq. (33) by $\frac{2}{\pi} F'_{J-1}(\ell r)$, Eq. (34) by $\frac{2}{\pi} F'_{J+1}(\ell r)$,

integrating over the range $a \leq r \leq \infty$, and using Eqs. (11) and (12),

we obtain

$$(k_0^2 - \ell^2) u_{Jk_0}(\ell)$$

$$= \left\{ \begin{aligned} & - \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J-1}^{(c)}(\ell) \int_0^{\infty} dk g_{J-1}^{(c)}(k) u_{Jk_0}(k) \\ & - \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J-1}^T(\ell) \int_0^{\infty} dk [g_{J+1}^{(T)}(k) w_{Jk_0}(k) - \frac{(J-1)}{3\sqrt{J(J+1)}} g_{J-1}^{(T)}(k) u_{Jk_0}(k)] \end{aligned} \right\},$$

and

$$(k_o^2 - \ell^2) w_{Jk_o}(\ell)$$

$$= \left\{ \begin{aligned} & - \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J+1}^{(c)}(\ell) \int_0^\infty dk g_{J+1}^{(c)}(k) w_{Jk_o}(k) \\ & - \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J+1}^T(\ell) \int_0^\infty dk [g_{J-1}^{(T)}(k) u_{Jk_o}(k) - \frac{(J+2)}{3\sqrt{J(J+1)}} g_{J+1}^{(T)}(k) w_{Jk_o}(k)] \end{aligned} \right\}.$$

We next assume a solution to Eqs. (35) and (36) of the form

$$u_{Jk_o}(\ell) \equiv N_u^{(J)}(k_o) \left\{ \delta(k_o - \ell) + P\left(\frac{\langle \ell | B_u^{(J)} | k_o \rangle}{\ell^2 - k_o^2} \right) \right\}, \quad (37)$$

$$w_{Jk_o}(\ell) \equiv N_w^{(J)}(k_o) \left\{ \delta(k_o - \ell) + P\left(\frac{\langle \ell | B_w^{(J)} | k_o \rangle}{\ell^2 - k_o^2} \right) \right\}, \quad (38)$$

and obtain

$$\begin{aligned} & N_u^{(J)}(k_o) \langle \ell | B_u^{(J)} | k_o \rangle \\ &= \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J-1}^{(c)}(\ell) N_u^{(J)}(k_o) \left[g_{J-1}^{(c)}(k_o) + \int_0^\infty dk g_{J-1}^{(c)}(k) P\left(\frac{\langle k | B_u^{(J)} | k_o \rangle}{k^2 - k_o^2} \right) \right] \\ &+ \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J-1}^T(\ell) \end{aligned}$$

$$\left\{ \begin{aligned} & N_w^{(J)}(k_o) \left[g_{J+1}^{(T)}(k_o) + \int_0^\infty dk g_{J+1}^{(T)}(k) P\left(\frac{\langle k | B_w^{(J)} | k_o \rangle}{k^2 - k_o^2}\right) \right] \\ & - \frac{N_u^{(J)}(k_o)}{6\sqrt{J(J+1)}} \left[g_{J-1}^{(T)}(k_o) + \int_0^\infty dk g_{J-1}^{(T)}(k) P\left(\frac{\langle k | B_u^{(J)} | k_o \rangle}{k^2 - k_o^2}\right) \right] \end{aligned} \right\}, \quad (39)$$

and

$$N_w^{(J)}(k_o) < \ell | B_w^{(J)} | k_o >$$

$$= \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J+1}^{(c)}(\ell) N_w^{(J)}(k_o) \left[g_{J+1}^{(c)}(k_o) + \int_0^\infty dk g_{J+1}^{(c)}(k) P\left(\frac{\langle k | B_w^{(J)} | k_o \rangle}{k^2 - k_o^2}\right) \right]$$

$$+ \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J+1}^{(T)}(\ell)$$

$$\left\{ \begin{aligned} & N_u^{(J)}(k_o) \left[g_{J-1}^{(T)}(k_o) + \int_0^\infty dk g_{J-1}^{(T)}(k) P\left(\frac{\langle k | B_u^{(J)} | k_o \rangle}{k^2 - k_o^2}\right) \right] \\ & - \frac{(J+2)}{3\sqrt{J(J+1)}} N_w^{(J)}(k_o) \left[g_{J+1}^{(T)}(k_o) + \int_0^\infty dk g_{J+1}^{(T)}(k) P\left(\frac{\langle k | B_w^{(J)} | k_o \rangle}{k^2 - k_o^2}\right) \right] \end{aligned} \right\}. \quad (40)$$

Examination of Eqs. (39) and (40) suggests that we define the functions $B_w^{J(c)}(k_o)$, $B_u^{J(T)}(k_o)$, $B_w^{J(c)}(k_o)$, and $B_w^{J(T)}(k_o)$ by the relationships,

$$\langle \ell | B_u^{(J)} | k_o \rangle$$

$$= \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J-1}^{(c)}(\ell) B_u^{J(c)}(k_o) + \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J-1}^{(T)}(\ell) B_u^{J(T)}(k_o), \quad (41)$$

and

$$\langle \ell | B_w^{(J)} | k_o \rangle$$

$$= \frac{2}{\pi} \sigma_c^{(J)} r_c g_{J+1}^{(c)}(\ell) B_w^{J(c)}(k_o) + \frac{2}{\pi} \sigma_T^{(J)} r_T g_{J+1}^{(T)}(\ell) B_w^{J(T)}(k_o). \quad (42)$$

Then upon substituting Eqs. (41) and (42) into Eqs. (39) and (40), we obtain four linear coupled equations which may be written in matrix form as

$$Q \begin{bmatrix} B_u^{J(c)}(k_o) \\ B_u^{J(T)}(k_o) \\ B_w^{J(c)}(k_o) \\ B_w^{J(T)}(k_o) \end{bmatrix} = \begin{bmatrix} g_{J-1}^{(c)}(k_o) \\ \eta^{(J)}(k_o) g_{J+1}^{(T)}(k_o) - \beta^{(J)} g_{J-1}^{(T)}(k_o) \\ g_{J+1}^{(c)}(k_o) \\ g_{J-1}^{(T)}(k_o) - \gamma^{(J)} \eta^{(J)}(k_o) g_{J+1}^{(T)}(k_o) \end{bmatrix}, \quad (43)$$

where

$$\begin{aligned}
\alpha \equiv & \begin{bmatrix} 1-A_{J-1}^{(c)}(k_0) & -\alpha^{(J)} A_{J-1}^{(c,T)}(k_0) \\ \beta^{(J)} A_{J-1}^{(c,T)}(k_0) & 1+\beta^{(J)} A_{J-1}^{(T)}(k_0) \\ 0 & 0 \\ -A_{J-1}^{(c,T)}(k_0) & -A_{J-1}^{(T)}(k_0) \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 \\ -\eta^{(J)}(k_0) A_{J+1}^{(c,T)}(k_0) & -\eta^{(J)}(k_0) A_{J+1}^{(T)}(k_0) \\ 1-A_{J+1}^{(c)}(k_0) & -\alpha^{(J)} A_{J+1}^{(c,T)}(k_0) \\ \gamma^{(J)} \eta^{(J)}(k_0) A_{J+1}^{(c,T)}(k_0) & \eta^{(J)}(k_0) \left(1+\gamma^{(J)} A_{J+1}^{(T)}(k_0) \right) \end{bmatrix},
\end{aligned}
\tag{44}$$

and we have defined

$$\begin{aligned}
\alpha^{(J)} & \equiv \sigma_T^{(J)} r_{T'} / \sigma_c^{(J)} r_c, \\
\beta^{(J)} & \equiv (J-1) / 3\sqrt{J(J+1)}, \\
\gamma^{(J)} & \equiv (J+2) / 3\sqrt{J(J+1)}, \\
\eta^{(J)}(k_0) & \equiv N_w^{(J)}(k_0) / N_u^{(J)}(k_0).
\end{aligned}
\tag{45}$$

The functions $A_L^{(\alpha,\beta)}(k_0)$ with $L = J \pm 1$ and $\alpha, \beta = C$ or T are defined by

$$A_L^{(C)}(k_0) \equiv \frac{2}{\pi} \sigma_C^{(J)} r_C \int_0^\infty dkP \left[\frac{(g_L^{(C)}(k))^2}{k^2 - k_0^2} \right], \quad (46)$$

$$A_L^{(T)}(k_0) \equiv \frac{2}{\pi} \sigma_T^{(J)} r_T \int_0^\infty dkP \left[\frac{(g_L^{(T)}(k))^2}{k^2 - k_0^2} \right] - \frac{1}{\sqrt{2}} \delta_{L,0} \quad (47)$$

$$A_L^{(C,T)}(k_0) \equiv \frac{2}{\pi} \sigma_C^{(J)} r_C \int_0^\infty dkP \left[\frac{g_L^{(C)}(k) g_L^{(T)}(k)}{k^2 - k_0^2} \right]. \quad (48)$$

Further, if we define the quantities

$$D_{J\pm 1}(k_0) \equiv A_{J\pm 1}^{(T)}(k_0) + \alpha^{(J)} \left(A_{J\pm 1}^{(C,T)}(k_0) \right)^2 / (1 - A_{J\pm 1}^{(C)}(k_0)), \quad (49)$$

and

$$\bar{g}_{J\pm 1}(k_0) \equiv g_{J\pm 1}^{(T)}(k_0) + g_{J\pm 1}^{(C)}(k_0) A_{J\pm 1}^{(C,T)}(k_0) / (1 - A_{J\pm 1}^{(C)}(k_0)), \quad (50)$$

then the solution to Eq. (43) can be obtained by elementary row

operations* and takes the form

$$B_u^{J(c)}(k_o) = \left\{ \begin{aligned} & [\beta^{(J)}((1+\gamma^{(J)}D_{J+1}(k_o))^{-D_{J+1}}(k_o))] \\ & [A_{J-1}^{(T)}(k_o)g_{J-1}^{(c)}(k_o) - \alpha^{(J)}A_{J-1}^{(c,T)}(k_o)g_{J-1}^{(T)}(k_o)] \\ & + \frac{(1+\gamma^{(J)}D_{J+1}(k_o))g_{J-1}^{(c)}(k_o) + \eta^{(J)}(k_o)\alpha^{(J)}A_{J-1}^{(c,T)}(k_o)\bar{g}_{J+1}(k_o)}{(1-A_{J-1}^{(c)}(k_o))((1+\beta^{(J)}D_{J-1}(k_o))(1+\gamma^{(J)}D_{J+1}(k_o))^{-D_{J-1}}(k_o)D_{J+1}(k_o))} \end{aligned} \right\},$$

$$B_u^{J(T)}(k_o) = \frac{\eta^{(J)}(k_o)\bar{g}_{J+1}(k_o) - (\beta^{(J)}(1+\gamma^{(J)}D_{J+1}(k_o))^{-D_{J+1}}(k_o))\bar{g}_{J-1}(k_o)}{(1+\beta^{(J)}D_{J-1}(k_o))(1+\gamma^{(J)}D_{J+1}(k_o))^{-D_{J-1}}(k_o)D_{J+1}(k_o)},$$

* Denote the rows by Roman numerals. The requisite operations are

- (1) $\beta^{(J)} \cdot IV + II \rightarrow II$; (2) $I/(1-A_{J-1}^{(c)}(k_o)) \rightarrow I$;
- (3) $III/(1-A_{J+1}^{(c)}(k_o)) \rightarrow III$; (4) $A_{J-1}^{(c,T)} \cdot I + IV \rightarrow IV$;
- (5) $(1-\beta^{(J)})\gamma^{(J)}\eta^{(J)}(k_o)A_{J+1}^{(c,T)}(k_o) \cdot III + II \rightarrow II$;
- (6) $-\gamma^{(J)}\eta^{(J)}(k_o)A_{J+1}^{(c,T)}(k_o) \cdot III + IV \rightarrow IV$; (7) $D_{J-1}(k_o) \cdot II + IV \rightarrow IV$;
- (8) $< \eta^{(J)}(k_o)((1+\beta^{(J)}D_{J-1}(k_o))(1+\gamma^{(J)}D_{J+1}(k_o))^{-D_{J+1}}(k_o)D_{J-1}(k_o)) >^{-1}$
 $\cdot IV \rightarrow IV$; (9) $-\eta^{(J)}(k_o)[\beta^{(J)} - (1-\beta^{(J)})\gamma^{(J)}D_{J+1}(k_o)] \cdot IV + II \rightarrow II$;
- (10) $\alpha^{(J)}A_{J-1}^{(c,T)}(k_o)/(1-A_{J-1}^{(c)}(k_o)) \cdot II + I \rightarrow I$.

$$B_W^{J(c)}(k_0)$$

$$= \left\{ \begin{aligned} & \eta^{(J)}(k_0) \bar{g}_{J+1}(k_0) - (\beta^{(J)}(1 + \gamma^{(J)} D_{J+1}(k_0)) - D_{J+1}(k_0)) \bar{g}_{J-1}(k_0) \\ & + \frac{\eta^{(J)}(k_0) (1 + \beta^{(J)} D_{J-1}(k_0) g_{J+1}^{(c)}(k_0)) + \alpha^{(J)} A_{J+1}^{(c,T)}(k_0) \bar{g}_{J-1}(k_0)}{\eta^{(J)}(k_0) ((1 + \beta^{(J)} D_{J-1}(k_0)) (1 + \gamma^{(J)} D_{J+1}(k_0))} \end{aligned} \right\},$$

$$-D_{J-1}(k_0) D_{J+1}(k_0) (1 - A_{J+1}^{(c)}(k_0))$$

$$B_W^{J(T)}(k_0)$$

(51)

$$= \frac{\bar{g}_{J-1}(k_0) - \eta^{(J)}(k_0) [\gamma^{(J)} (1 + \beta^{(J)} D_{J-1}(k_0)) - D_{J-1}(k_0)] \bar{g}_{J+1}(k_0)}{\eta^{(J)}(k_0) ((1 + \beta^{(J)} D_{J-1}(k_0)) (1 + \gamma^{(J)} D_{J+1}(k_0)) - D_{J-1}(k_0) D_{J+1}(k_0))},$$

as may also be verified by direct substitution of Eqs. (51) into Eq. (43).

Upon substituting Eqs. (37), (38), (41) and (42) into Eqs. (31) and (32), and using Eq. (26), we obtain

$$\left\{ \begin{aligned} u_{Jk_0}(r) &= N_u^{(J)}(k_0) \{ F'_{J-1}(k_0 r) + \frac{2}{\pi} \sigma_c^{(J)} r_c B_u^{J(c)}(k_0) F_{J-1}^{(c)}(k_0 r) + \\ &+ \frac{2}{\pi} \sigma_T^{(J)} r_T B_u^{J(T)}(k_0) F_{J-1}^{(T)}(k_0 r) \} \\ w_{Jk_0}(r) &= N_w^{(J)}(k_0) \{ F'_{J+1}(k_0 r) + \frac{2}{\pi} \sigma_c^{(J)} r_c B_w^{J(c)}(k_0) F_{J+1}^{(c)}(k_0 r) + \\ &+ \frac{2}{\pi} \sigma_T^{(J)} r_T B_w^{J(T)}(k_0) F_{J+1}^{(T)}(k_0 r) \} \end{aligned} \right\} \cdot \quad (52)$$

As r approaches infinity, Eqs. (51) and (52) become [see footnote below Eq. (26)]

$$\begin{aligned} u_{Jk_0}(r) \xrightarrow{r \rightarrow \infty} N_u^{(J)}(k_0) \{ F'_{J-1}(k_0 r) + \frac{1}{k_0} [\sigma_c^{(J)} r_c B_u^{J(c)}(k_0) g_{J-1}^{(c)}(k_0) \\ + \sigma_T^{(J)} r_T B_u^{J(T)}(k_0) g_{J-1}^{(T)}(k_0)] G'_{J-1}(k_0) \}, \end{aligned} \quad (53)$$

$$\begin{aligned} w_{Jk_0}(r) \xrightarrow{r \rightarrow \infty} N_w^{(J)}(k_0) \{ F'_{J+1}(k_0 r) + \frac{1}{k_0} [\sigma_c^{(J)} r_c B_w^{J(c)}(k_0) g_{J+1}^{(c)}(k_0) \\ + \sigma_T^{(J)} r_T B_w^{J(T)}(k_0) g_{J+1}^{(T)}(k_0)] G'_{J+1}(k_0 r) \}. \end{aligned} \quad (54)$$

Then, upon using the asymptotic forms for $F_L'(kr)$ and $G_L'(kr)$, Eqs.

(53) and (54) become

$$u_{Jk_0}(r) \xrightarrow{r \rightarrow \infty} N_u^{(J)}(k_0) [\sin(k_0 r - (J-1)\pi/2 + \delta_{J-1}'(k_0)) + \tan\delta_{J-1}^{(\sigma)}(k_0) \cos(k_0 r - (J-1)\frac{\pi}{2} + \delta_{J-1}'(k_0))], \quad (55)$$

$$w_{Jk_0}(r) \xrightarrow{r \rightarrow \infty} N_w^{(J)}(k_0) [\sin(k_0 r - (J+1)\pi/2 + \delta_{J+1}'(k_0)) + \tan\delta_{J+1}^{(\sigma)}(k_0) \cos(k_0 r - (J+1)\pi/2 + \delta_{J+1}'(k_0))], \quad (56)$$

where we have defined

$$\begin{aligned} \tan\delta_{J-1}^{(\sigma)}(k_0) &= \frac{1}{k_0} [\sigma_c^{(J)} r_c B_u^{J(c)}(k_0) g_{J-1}^{(c)}(k_0) + \sigma_T^{(J)} r_T B_u^{J(T)}(k_0) g_{J-1}^{(T)}(k_0)] \\ &= \frac{\pi}{2k_0} \langle k_0 | B_u^{(J)} | k_0 \rangle, \end{aligned} \quad (57)$$

and

$$\begin{aligned} \tan\delta_{J+1}^{(\sigma)}(k_0) &= \frac{1}{k_0} [\sigma_c^{(J)} r_c B_w^{J(c)}(k_0) g_{J+1}^{(c)}(k_0) + \sigma_T^{(J)} r_T B_w^{J(T)}(k_0) g_{J+1}^{(T)}(k_0)] \\ &= \frac{\pi}{2k_0} \langle k_0 | B_w^{(J)} | k_0 \rangle. \end{aligned} \quad (58)$$

The hard-core phase shifts $\delta_L'(k_0)$ are defined by Eq. (30). The functions $g_L^\alpha(k_0)$, $A_L^{\alpha\beta}(k_0)$, and $F_L^\alpha(k_0 r)$ are determined explicitly in Appendices 1, 4, and 11, respectively. The wave function normalization and the connection with Stapp phase-shift parameters is treated in Sec. I.3. With the results of Sec. I.3 for $N_u^{(J)}(k_0)$ and $N_w^{(J)}(k_0)$, the wave functions are then determined completely.

I.3 Relation of the Solution to Scattering Parameters

In order to complete the solution to the two-body problem developed in Section I.2, the normalization constants must be determined. As part of this calculation, the connection must be established between the phase shifts used in Section I.2 and the Blatt-Biedenharn and Stapp "bar" phase shifts. The so-called "mixing parameters" must also be identified.

For the uncoupled states, the asymptotic wave-function (Section I.2, Eqs. (28) and (29)), is given by

$$v_{Jk_0}^{(S)}(r) \xrightarrow{k_0 r \gg 1} N_v^{(J,S)}(k_0) \{ \sin[k_0 r - \frac{J\pi}{2} + \delta_J'(k_0)] + \tan \delta_{J,S}^{(\sigma)}(k_0) \cos[k_0 r - \frac{J\pi}{2} + \delta_J'(k_0)] \}, \quad (1)$$

where

$$\begin{aligned} \tan \delta_{J,S}^{(\sigma)}(k_0) &= \frac{\pi}{2k_0} \langle k_0 | B_v^{(J,S)} | k_0 \rangle \\ &= \frac{1}{k_0} [\sigma_c^{(J,S)} r_c g_J^{(c)}(k_0) B_c^{(J,S)}(k_0) \\ &\quad + (1 - \delta_{J,0}) \delta_{S,1} \frac{(2J+1)}{3\sqrt{J(J+1)}} \sigma_T^{(\sigma)} r_T g_J^{(T)}(k_0) B_T^{(J)}(k_0)], \quad (2) \end{aligned}$$

and the phase shift, $\delta_J'(k_0)$, is defined by

$$\tan \delta_J'(k_0) = -F_J(ka)/G_J(ka); \quad \delta_0'(k_0) = -k_0 a. \quad (3)$$

If we define

$$\delta_{JJ}^{(S)}(k_0) = \delta_J^{(S)}(k_0) + \delta_{J,S}^{(\sigma)}(k_0), \quad (4)$$

then Eq. (1) may be rewritten as

$$v_{Jk_0}^{(S)}(r) \xrightarrow{k_0 r \gg 1} \frac{N_v^{(J,S)}(k_0) e^{-i\delta_{JJ}^{(S)}(k_0)}}{2i \cos \delta_{J,S}^{(J)}(k_0)} \{-e^{-i[k_0 r - \frac{J\pi}{2}]} + e^{2i\delta_{JJ}^{(S)}(k_0)} e^{i[k_0 r - \frac{J\pi}{2}]} \}. \quad (5)$$

But from Section I.1, Eqs. (21) through (23), we have for the outgoing wave case ($\mu = +$), that

$$v_{Jk_0}^{(S)}(r) \xrightarrow{k_0 r \gg 1} \frac{1}{2i} \{-e^{-i[k_0 r - \frac{J\pi}{2}]} + e^{2i\delta_{JJ}^{(S)}(k_0)} e^{i[k_0 r - \frac{J\pi}{2}]} \}. \quad (6)$$

Comparison of Eqs. (5) and (6) shows that the singlet-state normalization is given by

$$\begin{aligned} N_v^{(J,S)}(k_0) &= \cos \delta_{J,S}^{(J)}(k_0) e^{i\delta_{JJ}^{(\sigma)}(k_0)} \\ &= \frac{e^{i\delta_{JJ}^{(S)}(k_0)}}{\sqrt{1 + \tan^2 \delta_{J,S}^{(J)}(k_0)}}, \end{aligned} \quad (7)$$

where

$$e^{i\delta_{JJ}^{(S)}(k_0)} = \left\{ \frac{[1 - \tan\delta_J'(k_0) \tan\delta_{J,S}^{(\sigma)}(k_0)] + i[\tan\delta_J'(k_0) + \tan\delta_{J,S}^{(\sigma)}(k_0)]}{[1 - \tan\delta_J'(k_0) \tan\delta_{J,S}^{(\sigma)}(k_0)] - i[\tan\delta_J'(k_0) + \tan\delta_{J,S}^{(\sigma)}(k_0)]} \right\}^{\frac{1}{2}}. \quad (8)$$

The substitutions indicated below Eq. (30) can be made in the above equations to obtain the corresponding equations for the special triplet ($S = 1$), $L = 1$, $J = 0$ case.

Treatment of Coupled Triplet States

The triplet analysis is somewhat more involved. The asymptotic wave functions (Section I.2, Eqs. (55) and (56)), are given by

$$u_{Jk_0}(r) \underset{k_0 r \gg 1}{\sim} \frac{N_u^{(J)}(k_0)}{\cos\delta_{J-1}^{(J)}(k_0)} \sin[k_0 r - (J-1)\pi/2 + \delta_{J-1}'(k_0) + \delta_{J-1}^{(\sigma)}(k_0)]$$

$$w_{Jk_0}(r) \underset{k_0 r \gg 1}{\sim} \frac{N_w^{(J)}(k_0)}{\cos\delta_{J+1}^{(\sigma)}(k_0)} \sin[k_0 r - (J+1)\pi/2 + \delta_{J+1}'(k_0) + \delta_{J+1}^{(\sigma)}(k_0)] \quad (9)$$

According to Section I.1, Eq. (51), there are two solutions of the coupled equations; namely,

$$\begin{aligned} \frac{1}{r} \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}^{\alpha}(\vec{r}, \xi_1, \xi_2) &= \frac{1}{r} u_{\mathbf{J} \mathbf{k}_0 \alpha}(r) \phi_{\mathbf{J}-1,1,\mathbf{J}}^{\mathbf{M}}(n_r, \xi_1, \xi_2) \\ \frac{1}{r} \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}^{\beta}(\vec{r}, \xi_1, \xi_2) &= \frac{1}{r} u_{\mathbf{J} \mathbf{k}_0 \beta}(r) \phi_{\mathbf{J}+1,1,\mathbf{J}}^{\mathbf{M}}(n_r, \xi_1, \xi_2) \\ &\quad + \left. \begin{aligned} &+ \frac{1}{r} w_{\mathbf{J} \mathbf{k}_0 \alpha}(r) \phi_{\mathbf{J}+1,1,\mathbf{J}}^{\mathbf{M}}(n_r, \xi_1, \xi_2) \\ &+ \frac{1}{r} w_{\mathbf{J} \mathbf{k}_0 \beta}(r) \phi_{\mathbf{J}-1,1,\mathbf{J}}^{\mathbf{M}}(n_r, \xi_1, \xi_2) \end{aligned} \right\}, \end{aligned} \quad (10)$$

where we can also include the "outgoing" functions of Eq. (50) with the definitions

$$\left. \begin{aligned} \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}^{\alpha}(\vec{r}, \xi_1, \xi_2) &\equiv \langle \vec{r}, \xi_1, \xi_2 | k_0, \mathbf{S} \mathbf{J} \mathbf{M}; \mathbf{J}-1 \rangle_{\mathbf{S}=1}^{(+)} \\ \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}^{\beta}(\vec{r}, \xi_1, \xi_2) &\equiv \langle \vec{r}, \xi_1, \xi_2 | k_0, \mathbf{S} \mathbf{J} \mathbf{M}; \mathbf{J}+1 \rangle_{\mathbf{S}=1}^{(+)} \end{aligned} \right\}. \quad (11)$$

The wave-function normalization condition,

$$\sum_{\xi_1, \xi_2} \int d^3r \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}^{\dagger}(\vec{r}, \xi_1, \xi_2) \psi_{\mathbf{J} \mathbf{M} \mathbf{k}_0}(\vec{r}, \xi_1, \xi_2) = \frac{\pi}{2} \delta(k_0 - k_0), \quad (12)$$

requires that

$$|N_u^{(J)}(k_o)|^2 \left[\frac{1}{\cos^2 \delta_{J-1}^{(J)}(k_o)} + \frac{|\eta^{(J)}(k_o)|^2}{\cos^2 \delta_{J+1}^{(J)}(k_o)} \right]^2 = 1, \quad (13)$$

or

$$|N_u^{(J)}(k)| = \frac{1}{[1 + \tan^2 \delta_{J-1}^{(J)}(k_o) + [1 + \tan^2 \delta_{J+1}^{(J)}(k_o)] |\eta^{(J)}(k_o)|^2]^{\frac{1}{2}}}, \quad (14)$$

where

$$\eta^{(J)}(k_o) = N_w^{(J)}(k_o) / N_u^{(J)}(k_o). \quad (15)$$

These expressions hold for both α and β states (whose labels were omitted as superscripts in Eqs. (12) through (15)).

The orthogonality of α and β states requires

$$\begin{aligned} \sum_{\xi_1, \xi_2} \int d^3r \psi_{J\mathbf{M}k_o}^{\alpha\dagger}(\vec{r}, \xi_1, \xi_2) \psi_{J\mathbf{M}k_o}^{\beta}(\vec{r}, \xi_1, \xi_2) = \\ = \sum_{\xi_1, \xi_2} \int d^3r \psi_{J\mathbf{M}k_o}^{\beta\dagger}(\vec{r}, \xi_1, \xi_2) \psi_{J\mathbf{M}k_o}^{\alpha}(\vec{r}, \xi_1, \xi_2) = 0. \end{aligned} \quad (16)$$

Upon substituting Eqs. (9) and (10) into (16) we find the condition

$$\begin{aligned} & \eta^{(J)\alpha^\dagger}(k_0) \eta^{(J)\beta}(k_0) [1 + \tan \delta_{J+1,\alpha}^{(\sigma)\dagger}(k_0) \tan \delta_{J+1,\beta}^{(\sigma)}(k_0)] \\ & = - [1 + \tan \delta_{J-1,\alpha}^{(\sigma)\dagger}(k_0) \tan \delta_{J-1,\beta}^{(\sigma)}(k_0)], \end{aligned} \quad (17)$$

and similarly for $\alpha \neq \beta$. Now from the results obtained in Section 1.2, Eqs. (51), (57) and (58), we may define

$$\begin{aligned} \tan \delta_{J-1}^{(\sigma)}(k_0) &= \eta^{(J)}(k_0) T_1(k_0) + T_2(k_0), \\ \tan \delta_{J+1}^{(\sigma)}(k_0) &= \frac{1}{\eta^{(J)}(k_0)} T_4(k_0) + T_3(k_0), \end{aligned} \quad (18)$$

for both α and β states, where (suppressing J -dependence in the notation for $T_i(k_0)$)

$$\begin{aligned} & T_1(k) \\ & = T_4(k) = \frac{1}{k} \frac{\sigma_T^{(J)} r_T \bar{g}_{J-1}(k) \bar{g}_{J+1}(k)}{[1+\beta^{(J)} D_{J-1}(k)] [1+\gamma^{(J)} D_{J+1}(k)] - D_{J-1}(k) D_{J+1}(k)}, \end{aligned}$$

$$\begin{aligned}
T_2(k) &= \frac{1}{k} \left\{ \frac{\sigma_c^{(J)} r_c (g_{J-1}^{(c)}(k))^2}{1 - A_{J-1}^{(c)}(k)} \right. \\
&\quad \left. - \frac{\sigma_T^{(J)} r_T (\bar{g}_{J-1}(k))^2 [\beta^{(J)} (1 + \gamma^{(J)} D_{J+1}(k)) - D_{J-1}(k)]}{[1 + \beta^{(J)} D_{J-1}(k)] [1 + \gamma^{(J)} D_{J+1}(k)] - D_{J-1}(k) D_{J+1}(k)} \right\}, \\
T_3(k) &= \frac{1}{k} \left\{ \frac{\sigma_c^{(J)} r_c (g_{J+1}^{(c)}(k))^2}{1 - A_{J+1}^{(c)}(k)} \right. \\
&\quad \left. - \frac{\sigma_T^{(J)} r_T (\bar{g}_{J+1}(k))^2 [\gamma^{(J)} (1 + \beta^{(J)} D_{J-1}(k)) - D_{J+1}(k)]}{[1 + \beta^{(J)} D_{J-1}(k)] [1 + \gamma^{(J)} D_{J+1}(k)] - D_{J-1}(k) D_{J+1}(k)} \right\}. \quad (19)
\end{aligned}$$

Upon substituting Eqs. (18) into Eq. (17), we obtain for the orthogonality condition the result

$$\begin{aligned}
&\eta^{(J)\beta}(k_0) \\
&= - \frac{[1 + T_4^2(k_0) + T_2^2(k_0)] + [T_1(k_0)T_2(k_0) + T_3(k_0)T_4(k_0)] \eta^{(J)\alpha\dagger}(k_0)}{[T_1(k_0)T_2(k_0) + T_3(k_0)T_4(k_0)] + [1 + T_1^2(k_0) + T_3^2(k_0)] \eta^{(J)\alpha\dagger}(k_0)}. \quad (20)
\end{aligned}$$

Although we have allowed them to be complex quantities, in general, we shall show explicitly that the phase shifts and normalization factors $\eta^{(J)}(k_0)$, in Eqs. (17) through (20), are all real quantities in the Blatt-Biedenharn case.

Blatt-Biedenharn Parameters

The Blatt-Biedenharn states¹ are determined by defining both u_{Jk_0} and w_{Jk_0} to have the same total phase shift in Eqs. (9), i.e., we write

$$\begin{aligned}\delta_\alpha(k_0) &= \delta'_{J-1}(k_0) + \delta_{J-1,\alpha}^{(\sigma)}(k_0) = \delta'_{J+1}(k_0) + \delta_{J+1,\alpha}^{(\sigma)}(k_0) \\ \delta_\beta(k_0) &= \delta'_{J-1}(k_0) + \delta_{J-1,\beta}^{(\sigma)}(k_0) = \delta'_{J+1}(k_0) + \delta_{J+1,\beta}^{(\sigma)}(k_0).\end{aligned}\quad (21)$$

From Eq. (21) we then obtain for either α or β states

$$\delta_{J-1}^{(\sigma)}(k_0) - \delta_{J+1}^{(\sigma)}(k_0) = \delta'_{J+1}(k_0) - \delta'_{J-1}(k_0), \quad (22)$$

and thus we are led to define a function (independent of σ)

$$\begin{aligned}C_J(k_0 a) &= \tan[\delta'_{J+1}(k_0 a) - \delta'_{J-1}(k_0 a)] \\ &= \frac{\tan \delta_{J-1}^{(\sigma)}(k_0) - \tan \delta_{J+1}^{(\sigma)}(k_0)}{1 + \tan \delta_{J-1}^{(\sigma)}(k_0) \tan \delta_{J+1}^{(\sigma)}(k_0)}.\end{aligned}\quad (23)$$

¹Blatt, J. M. and Biedenharn, L. C., Rev. Mod. Phys., **24**, 1952, p. 258.

Upon substituting Eqs. (18) into Eq. (23), we obtain a quadratic expression in $\eta^{(J)}(k_0)$ (for either α or β states), given by

$$(\eta^{(J)})^2_{T_1} [1 - C_J T_3] + \eta^{(J)} [T_2 - T_3 - C_J (1 + T_2 T_3 + T_1 T_4)] - T_4 (1 + C_J T_2) = 0, \quad (24)$$

where we have suppressed the k_0 -dependence momentarily. If we assume $\eta^{(J)}$ is complex and of the form

$$\eta^{(J)} = \eta_r^{(J)} + \eta_i^{(J)}, \quad (25)$$

then we readily find from Eq. (24) that

$$\begin{aligned} \eta_r^{(J)} &= \frac{T_3 - T_2 + C_J (1 + T_2 T_3 + T_1 T_4)}{2T_1 (1 - C_J T_3)} \equiv a, \\ \eta_i^{(J)} &= \pm [(\eta_r^{(J)})^2 + \frac{T_4 (1 + C_J T_2)}{T_1 (1 - C_J T_3)}]^{1/2} \equiv \pm b, \end{aligned} \quad (26)$$

where we choose the state α to correspond to the minus sign, for $\eta_i^{(J)}$, and the state β to correspond to the plus sign [see also Eqs. (31)]. Then

$$\begin{aligned} \eta^{(J)\alpha^\dagger}(k_0) &= a - b^*, \\ \eta^{(J)\beta}(k_0) &= a + b. \end{aligned} \quad (27)$$

We note that these normalization factors will be real if

$b^2 = a^2 + \frac{T_4}{T_1} \frac{1+C_J T_2}{1-C_J T_3} > 0$, which condition is readily shown to be true quite generally. Substitution of Eqs. (27) into Eq. (20) yields, further, the result that the orthogonality condition is satisfied if, and only if, b is real and

$$(T_1 - T_4)\{C_J(T_2 - T_3) + (1 - T_1 T_4 + T_2 T_3)\} = 0. \quad (28)$$

Since our solution has the property $T_1 = T_4$, [see Eqs. (19)], the orthogonality condition is satisfied precisely.

The phase shifts are readily determined by substituting Eqs. (26) and (27) into Eqs. (18). With $b = \text{real}$, we obtain the functions

$$\tan \delta_{J-1,\alpha}^{(\sigma)}(k_0)$$

$$= \frac{T_3 + C_J(1+T_1^2) + T_2(1-C_J T_3) - \sqrt{[T_3 + C_J(1-T_1^2) - T_2(1-C_J T_3)]^2 + 4T_1^2(1+C_J^2)}}{2(1-C_J T_3)}$$

$$\tan \delta_{J-1,\beta}^{(\sigma)}(k_0)$$

$$= \frac{T_3 + C_J(1+T_1^2) + T_2(1-C_J T_3) + \sqrt{[T_3 + C_J(1-T_1^2) - T_2(1-C_J T_3)]^2 + 4T_1^2(1+C_J^2)}}{2(1-C_J T_3)}$$

$$\tan \delta_{J+1,\alpha}^{(\sigma)}(k_0)$$

$$= \frac{T_2 - C_J(1+T_1^2) + T_3(1+C_J T_2) - \sqrt{[T_2 - C_J(1-T_1^2) - T_3(1+C_J T_2)]^2 + 4T_1^2(1+C_J^2)}}{2(1+C_J T_2)}$$

$$\tan \delta_{J+1,\beta}^{(\sigma)}(k_0)$$

$$= \frac{T_2 - C_J(1+T_1^2) + T_3(1+C_J T_2) + \sqrt{[T_2 - C_J(1+T_1^2) - T_3(1+C_J T_2)]^2 + 4T_1^2(1+C_J^2)}}{2(1+C_J T_2)} \quad (29)$$

In the limit of zero attraction, $\sigma = T_1 = T_2 = T_3 = 0$, we obtain

$$\tan \delta_{J-1,\alpha}^{(\sigma)}(k_0) = 0,$$

$$\tan \delta_{J-1,\beta}^{(\sigma)}(k_0) = C_J(k_0 a),$$

$$\tan \delta_{J+1,\alpha}^{(\sigma)}(k_0) = -C_J(k_0 a),$$

$$\tan \delta_{J+1,\beta}^{(\sigma)}(k_0) = 0, \quad (30)$$

$$\eta^{(J),\alpha}(k_0) = [\eta^{(J),\beta}(k_0)]^{-1} = 0.$$

Upon using Eqs. (21), (23), and (30), we may then make the consistent identification in the limit of zero attraction that

$$\delta_{\alpha}(k_0) \xrightarrow{\sigma \rightarrow 0} \delta_{J-1}^{\alpha}(k_0),$$

$$\delta_{\beta}(k_0) \xrightarrow{\sigma \rightarrow 0} \delta_{J+1}^{\beta}(k_0), \quad (31)$$

which is as it should be for our labeling of the α and β states in Eq. (11).

The determination of the Blatt-Biedenharn mixing parameter $\epsilon_J(k_0)$ is straightforward from the definition,

$$\tan \epsilon_J(k_0) \equiv \lim_{r \rightarrow \infty} \left[\frac{w_{Jk_0\alpha}(r)}{u_{Jk_0\alpha}(r)} \right] = - \lim_{r \rightarrow \infty} \left[\frac{u_{Jk_0\beta}(r)}{w_{Jk_0\beta}(r)} \right]. \quad (32)$$

Upon using Eqs. (9) and (21), we find

$$\tan \epsilon_J(k_0) = \frac{\left[\frac{N_w^{(J)}(k_0) \cos \delta_{J-1}^{(\sigma)}(k_0)}{N_u^{(J)}(k_0) \cos \delta_{J+1}^{(\sigma)}(k_0)} \right]_{\alpha}}{\left[\frac{N_u^{(J)}(k_0) \cos \delta_{J+1}^{(\sigma)}(k_0)}{N_w^{(J)}(k_0) \cos \delta_{J-1}^{(\sigma)}(k_0)} \right]_{\beta}} = - \frac{\left[\frac{N_u^{(J)}(k_0) \cos \delta_{J+1}^{(\sigma)}(k_0)}{N_w^{(J)}(k_0) \cos \delta_{J-1}^{(\sigma)}(k_0)} \right]_{\beta}}{\left[\frac{N_w^{(J)}(k_0) \cos \delta_{J-1}^{(\sigma)}(k_0)}{N_u^{(J)}(k_0) \cos \delta_{J+1}^{(\sigma)}(k_0)} \right]_{\alpha}}, \quad (33)$$

and upon substituting Eqs. (18) into (33), we obtain

$$\begin{aligned} \tan^2 \epsilon_J(k_0) &= \frac{(\eta^{(J)\alpha})^2(1+T_3^2)+2T_1T_3\eta^{(J)\alpha}+T_1^2}{(\eta^{(J)\alpha})^2T_1^2+2T_1T_2\eta^{(J)\alpha}+T_2^2} \\ &= \frac{(\eta^{(J)\beta})^2T_1^2+2T_1T_2\eta^{(J)\beta}+T_2^2}{(\eta^{(J)\beta})^2(1+T_3^2)+2T_1T_3\eta^{(J)\beta}+T_1^2}. \end{aligned} \quad (34)$$

In order that the two equalities of Eq. (34) both hold, we must have the condition,

$$\begin{aligned} &\{ \eta^{(J)\alpha} \eta^{(J)\beta} (1+T_3^2+T_1^2) + T_1(T_3+T_2)(\eta^{(J)\alpha} + \eta^{(J)\beta}) + (T_1^2 + 1 + T_2^2) \} \\ &\times \{ \eta^{(J)\alpha} \eta^{(J)\beta} (1+T_3^2-T_1^2) + T_1(T_3-T_2)(\eta^{(J)\alpha} + \eta^{(J)\beta}) \\ &\quad + (T_1^2 - 1 - T_2^2) \} = 0. \end{aligned} \quad (35)$$

But use of Eqs. (26) and (27), which are equivalent to the orthogonality condition (17) or (20), yields that the first bracket in this last expression is zero identically. Therefore, both equalities in Eqs. (32) and (33) are valid (as they must be in general). In the limit that the tensor force is zero, i.e., when $\sigma_T = T_1$
 $= \eta^{(J)\alpha} = [\eta^{(J)\beta}]^{-1} = 0$, we obtain that there is no mixing,

$$\tan \epsilon_J(k_0) \xrightarrow{T_1 \rightarrow 0} 0, \quad (36)$$

Similarly,* since $T_1(k) \xrightarrow{k \rightarrow 0} O(k^{2JH})$, we find that there is no mixing for small wave numbers, i.e.,

$$\lim_{k_0 \rightarrow 0} \epsilon_J(k_0) = 0. \quad (37)$$

The normalization, orthogonality condition, phases shifts and mixing parameters have now all been explicitly determined for the Blatt-Bidenharn states and the connection with the phase shifts of the general solution, Eqs. (18) and (19), established.

*From Appendices 1 and 4 it follows that $g_\ell(k) \rightarrow O(k^{\ell+1})$, and $A_\ell^\alpha(k)$ is even in k . [See also Eqs. (49) and (50) in Section I.2.] Using these results in the first of Eqs. (19), it follows that

$$T_1(k) \xrightarrow{k \rightarrow 0} O(k^{2J+1}), \quad T_2(k) \xrightarrow{k \rightarrow 0} O(k^{2J-1}), \quad \text{and} \quad T_3(k) \xrightarrow{k \rightarrow 0} O(k^{2J+3}).$$

Stapp "Bar" Parameters

We next make the connection of our solution with the Stapp "bar" phase shifts. We observe first that the complication which occurs throughout this analysis is due entirely to the presence of tensor forces. Thus, although J is a good quantum number, where $\underline{J} = \underline{L} + \underline{S}$, the ordinary angular momentum L is not when $L = J \pm 1$. There is a mixing of the two triplet states $L = J - 1$ and $L = J + 1$. The Blatt-Biedenharn choice, Eqs. (21), for this mixing may be aesthetically pleasing from a theoretical point of view, but is not a convenient choice for analysis of experimental scattering data. The Stapp choice, which turns out to be convenient from all practical considerations, is defined by writing the "outgoing" solution of the coupled differential equations, in the asymptotic region, according to the following convention:

$$\begin{aligned}
 u_{J(J-1)k_o}(r) &\xrightarrow{k_o r \gg 1} \frac{1}{2i} \{-e^{-i[k_o r - (J-1)\pi/2]} \\
 &\quad + \alpha_{J(J-1)}^{(+)}(k_o) e^{i[k_o r - (J-1)\pi/2]}\} \\
 w_{J(J-1)k_o}(r) &\xrightarrow{k_o r \gg 1} \frac{1}{2i} \alpha_{JJ}^{(+)}(k_o) e^{i[k_o r - (J+1)\pi/2]} \\
 u_{J(J+1)k_o}(r) &\xrightarrow{k_o r \gg 1} \frac{1}{2i} \alpha_{JJ}^{(+)}(k_o) e^{i[k_o r - (J-1)\pi/2]} \\
 w_{J(J+1)k_o}(r) &\xrightarrow{k_o r \gg 1} \frac{1}{2i} \{-e^{-i[k_o r - (J+1)\pi/2]} \\
 &\quad + \alpha_{J(J+1)}^{(+)}(k_o) e^{i[k_o r - (J+1)\pi/2]}\}. \tag{38}
 \end{aligned}$$

Here, we may regard the first (second) set of states to be those labeled by $\alpha(\beta)$ in Eqs. (10). The fact that the "mixing function" $\alpha_{JJ}^{(+)}(k_o)$ must be the same for both $w_{J(J-1)k_o}$ follows from a time-reversal argument applied to the reaction matrices $G^{(+)}$ and $G^{(-)}$ on the energy shell. The α 's are related to the bar scattering parameters of Stapp, Ypsilantis and Metropolis² by the definitions

$$\begin{aligned}\alpha_{J(J+1)}^{(+)}(k_o) &= \cos 2\bar{\epsilon}_J(k_o) e^{2i\bar{\delta}_{J(J+1)}(k_o)} \\ \alpha_{JJ}^{(+)}(k_o) &= i \sin 2\bar{\epsilon}_J(k_o) e^{i[\bar{\delta}_{J(J-1)}(k_o) + \bar{\delta}_{J(J+1)}(k_o)]}\end{aligned}\quad (39)$$

In order to establish the relationship between the parameters of Eqs. (38) and (39), and those of our general analytic solution, Eqs. (9), we rewrite the latter equations in the form,

$$\begin{aligned}u_{Jk_o}(r) &\xrightarrow{k_o r \gg 1} \\ \frac{N_u^{(J)}(k_o)}{2i} &\left\{ \begin{array}{l} -e^{-i[k_o r - (J-1)\pi/2 + \delta'_{J-1}(k_o)]} [1 - i \tan \delta_{J-1}^{(\sigma)}(k_o)] \\ +e^{i[k_o r - (J-1)\pi/2 + \delta'_{J-1}(k_o)]} [1 + i \tan \delta_{J-1}^{(\sigma)}(k_o)] \end{array} \right\}, \\ w_{Jk_o}(r) &\xrightarrow{k_o r \gg 1} \\ \frac{N_w^{(J)}(k_o)}{2i} &\left\{ \begin{array}{l} -e^{-i[k_o r - (J+1)\pi/2 + \delta'_{J+1}(k_o)]} [1 - i \tan \delta_{J+1}^{(\sigma)}(k_o)] \\ +e^{i[k_o r - (J+1)\pi/2 + \delta'_{J+1}(k_o)]} [1 + i \tan \delta_{J+1}^{(\sigma)}(k_o)] \end{array} \right\}.\end{aligned}\quad (40)$$

²Stapp, H. P., Ypsilantis, T. J., and Metropolis, N., Phys. Rev., **105**, 1957, p. 302.

Comparison of Eqs. (38) and (40) then yields the conclusion that:

$$1 - i \tan \delta_{J+1}^{(\sigma)}(k_0) = 0 \quad \text{for the } (J-1), \text{ or } \alpha, \text{ solution,}$$

$$1 - i \tan \delta_{J-1}^{(\sigma)}(k_0) = 0 \quad \text{for the } (J+1), \text{ or } \beta, \text{ solution.} \quad (41)$$

Upon substituting Eqs. (18) into (41), we find the Stapp choice of mixing (with $\eta = N_w/N_u$) gives

$$\begin{aligned} \eta^{J(J-1)}(k_0) &= \frac{iT_1(k_0)}{1-iT_3(k_0)}, \\ \eta^{J(J+1)}(k_0) &= \frac{1-iT_2(k_0)}{iT_1(k_0)}. \end{aligned} \quad (42)$$

The orthogonality condition, Eq. (20), is satisfied by these expressions. We then define

$$\begin{aligned} A_{J(J-1)}^{J\pm}(k_0) &= 1 \pm i T_2(k_0) \mp \frac{T_1^2(k_0)}{1-iT_3(k_0)}, \\ B_{J(J+1)}^{J\pm}(k_0) &= 1 \pm i T_3(k_0) \mp \frac{T_1^2(k_0)}{1-iT_3(k_0)}, \end{aligned} \quad (43)$$

so that Eqs. (40) may be written, using Eqs. (18) and (41) through (43), as

$$u_{J(J-1)k_0}^{(r)} \xrightarrow{k_0 r \gg 1} \frac{N_u^{J(J-1)}(k_0)}{2i} \left\{ \begin{aligned} &-A_{J(J-1)}^{J-}(k_0) e^{-i[k_0 r - (J-1)\pi/2 + \delta_{J-1}'(k_0)]} \\ &+A_{J(J-1)}^{J+}(k_0) e^{i[k_0 r - (J-1)\pi/2 + \delta_{J-1}'(k_0)]} \end{aligned} \right\}$$

$$w_{J(J-1)k_0}(r) \xrightarrow{k_0 r \gg 1} \frac{N_w^{J(J-1)}(k_0)}{i} e^{i[k_0 r - (J+1)\pi/2 + \delta'_{J+1}(k_0)]}$$

$$u_{J(J+1)k_0}(r) \xrightarrow{k_0 r \gg 1} \frac{N_u^{J(J+1)}(k_0)}{i} e^{i[k_0 r - (J-1)\pi/2 + \delta'_{J-1}(k_0)]}$$

$$w_{J(J+1)k_0}(r) \xrightarrow{k_0 r \gg 1}$$

$$\frac{N_w^{J(J+1)}(k_0)}{2i} \left\{ \begin{array}{l} -B_{J(J+1)}^{J-}(k_0) e^{-i[k_0 r - (J+1)\pi/2 + \delta'_{J+1}(k_0)]} \\ +B_{J(J+1)}^{J+}(k_0) e^{i[k_0 r - (J+1)\pi/2 + \delta'_{J+1}(k_0)]} \end{array} \right\} . \quad (44)$$

Let us now define

$$\delta_{J(J\pm 1)}(k_0) = \bar{\delta}_{J(J\pm 1)}(k_0) - \delta'_{J\pm 1}(k_0). \quad (45)$$

Comparison of Eqs. (44) with (38) and (39) then allows us to determine the unknown coefficients in these equations. In particular, we find the results (omitting the k_0 dependence of the $T_i(k_0)$ functions) for the normalization constants,

$$N_u^{J(J-1)}(k_0) = \frac{(1-iT_3)e^{i\delta'_{J-1}(k_0)}}{(1-iT_2)(1-iT_3)+T_1^2},$$

$$N_w^{J(J-1)}(k_0) = \frac{iT_1 e^{i\delta'_{J-1}(k_0)}}{(1-iT_2)(1-iT_3)+T_1^2},$$

$$\begin{aligned}
N_u^{J(J+1)}(k_o) &= \frac{iT_1 e^{i\delta'_{J+1}(k_o)}}{(1-iT_2)(1-iT_3)+T_1^2}, \\
N_w^{J(J+1)}(k_o) &= \frac{(1-iT_2)e^{i\delta'_{J+1}(k_o)}}{(1-iT_2)(1-iT_3)+T_1^2}; \quad (46)
\end{aligned}$$

for the mixing parameters $\alpha_{JJ'}^{(+)}(k_o)$,

$$\begin{aligned}
\alpha_{J(J\pm 1)}^{(+)}(k_o) &= \frac{[(1-iT_2)(1+iT_3)-T_1^2]}{[(1-iT_2)(1-iT_3)+T_1^2]} e^{2i\delta'_{J\pm 1}(k_o)}, \\
\alpha_{JJ}^{(+)}(k_o) &= \frac{2iT_1}{(1-iT_2)(1-iT_3)+T_1^2} e^{i[\delta'_{J+1}(k_o)+\delta'_{J-1}(k_o)]}; \quad (47)
\end{aligned}$$

and for the phase shifts and mixing parameter $\epsilon_J(k_o)$,

$$\begin{aligned}
\tan 2\delta_{J(J-1)}(k_o) &= \frac{2[T_2(1+T_3^2)-T_3T_1^2]}{1+T_3^2-T_2^2-(T_1^2-T_2T_3)^2}, \\
\tan 2\delta_{J(J+1)}(k_o) &= \frac{2[T_3(1+T_2^2)-T_2T_1^2]}{1+T_2^2-T_3^2-(T_1^2-T_2T_3)^2}, \\
\tan 2\bar{\epsilon}_J(k_o) &= \frac{2T_1}{[(1+T_2T_3-T_1^2)^2+(T_2-T_3)^2]^{\frac{1}{2}}}. \quad (48)
\end{aligned}$$

The $\delta_{J(J\pm 1)}(k_o)$ are related to the $\bar{\delta}_{J(J\pm 1)}(k_o)$ by Eq. (45) and the functions $T_i(k_o)$, $i = 1, 2, 3$ are defined by Eqs. (19).

With this explicit determination of the phase shifts, mixing parameters and normalization constants in terms of known functions, the solution to the proposed two-body problem (Eqs. (1) through (6) in Chapter I.2) can be said to be completed.

We remark that the connection between the Blatt-Biedenharn and Stapp parameters is given by

$$\begin{aligned}\bar{\delta}_{J(J+1)}(k_o) + \bar{\delta}_{J(J-1)}(k_o) &= \delta_{J\alpha} + \delta_{J\beta}, \\ \sin [\bar{\delta}_{J(J-1)}(k_o) - \bar{\delta}_{J(J+1)}(k_o)] &= \frac{\tan 2\bar{\epsilon}_J(k_o)}{\tan 2\epsilon_J(k_o)} < 1, \\ \sin [\delta_{J\alpha} - \delta_{J\beta}] &= \frac{\sin 2\bar{\epsilon}_J(k_o)}{\sin 2\epsilon_J(k_o)},\end{aligned}\tag{49}$$

and

$$\begin{aligned}\alpha_{J(J-1)}^{(+)}(k_o) &= e^{2i\delta_{J\alpha}(k_o)} \cos^2 \epsilon_J(k_o) + e^{2i\delta_{J\beta}(k_o)} \sin^2 \epsilon_J(k_o), \\ \alpha_{JJ}^{(+)}(k_o) &= [e^{2i\delta_{J\alpha}(k_o)} - e^{2i\delta_{J\beta}(k_o)}] \sin \epsilon_J(k_o) \cos \epsilon_J(k_o), \\ \alpha_{J(J+1)}^{(+)}(k_o) &= e^{2i\delta_{J\alpha}(k_o)} \sin^2 \epsilon_J(k_o) + e^{2i\delta_{J\beta}(k_o)} \cos^2 \epsilon_J(k_o).\end{aligned}\tag{50}$$

I.4 Bound-State Solution and the Deuteron RMS Radius.

The bound-state deuteron wave function may be obtained by solving the equations of Section I.1 with $k_0^2 \rightarrow -\gamma^2$. We define the momentum-space wave function

$$\phi_L^{(\gamma J, ST)}(k) = \langle kL, JST | \gamma \rangle, \quad (1)$$

and note, for the states with $L = J \pm 1$, that

$$\begin{aligned} \langle r | \gamma \rangle &= \langle \vec{r}, \xi_1, \xi_2, \zeta_1, \zeta_2 | \gamma J M S T_M \rangle \\ &= \frac{2}{\pi} \frac{1}{r} \int_0^\infty d\ell \sum_L \phi_L^{(\gamma J, ST)}(\ell) F_L(\ell r) \phi_{LSJ}^M(\hat{n}_r, \xi_1, \xi_2) \chi_{TM_T}(\zeta_1, \zeta_2) \end{aligned} \quad (2)$$

may be written (similarly to Eqs. (50) and (51) of Section I.1) as

$$\begin{aligned} \langle r | \gamma \rangle &= \frac{1}{r} [u_J^B(r) \phi_{J-1,1,J}^M(\hat{n}_r, \xi_1, \xi_2) + w_J^B(r) \phi_{J+1,1,J}^M(\hat{n}_r, \xi_1, \xi_2)] \chi_{TM_T}(\zeta_1, \zeta_2), \end{aligned} \quad (3)$$

where

$$\sum_{\xi_1, \xi_2, \zeta_1, \zeta_2} \int d^3r \langle \gamma | r \rangle \langle r | \gamma \rangle = 1. \quad (4)$$

From Eqs. (2) and (3) we see that

$$\begin{aligned} u_J^B(r) &= \frac{2}{\pi} \int_0^\infty d\ell \phi_{J-1}^{(\gamma J, ST)}(\ell) F_{J-1}(\ell r), \\ w_J^B(r) &= \frac{2}{\pi} \int_0^\infty d\ell \phi_{J+1}^{(\gamma J, ST)}(\ell) F_{J+1}(\ell r). \end{aligned} \quad (5)$$

Upon inverting Eq. (5), we obtain

$$\begin{aligned}
\langle k, J-1; JST | \gamma \rangle &\equiv \phi_{J-1}^{(\gamma J, ST)}(k) = \int_0^\infty dr F_{J-1}(kr) u_J^B(r), \\
\langle k, J+1; JST | \gamma \rangle &\equiv \phi_{J+1}^{(\gamma J, ST)}(k) = \int_0^\infty dr F_{J+1}(kr) w_J^B(r),
\end{aligned} \tag{6}$$

where we are interested only in a γ which refers to the deuteron state. The normalization condition, Eq. (4), becomes

$$\int_0^\infty dr [(u_J^B(r))^2 + (w_J^B(r))^2] = 1. \tag{7}$$

The coupled equations for the deuteron state ($J = 1$, $S = 1$) are given by Eq. (53) in Section I.1, with $k_0^2 = -\gamma^2$, as

$$\begin{aligned}
\left(\frac{d^2}{dr^2} - \gamma^2 \right) u_1^B(r) &= -\sigma_c r_c f_c(r) \int_a^\infty dr' f_c(r') u_1^B(r') \\
&\quad - \sigma_T r_T f_T(r) \int_a^\infty dr' f_T(r') w_1^B(r'), \\
\left(\frac{d^2}{dr^2} - \gamma^2 - \frac{6}{r^2} \right) w_1^B(r) &= -\sigma_T r_T f_T(r) \int_a^\infty dr' f_T(r') [u_1^B(r') - \frac{1}{\sqrt{2}} w_1^B(r')] \\
&\quad - \sigma_c r_c f_c(r) \int_a^\infty dr' f_c(r') w_1^B(r').
\end{aligned} \tag{8}$$

As in Section I.2, we now introduce

$$u_1^B(r) = \int_0^\infty dk F_0'(kr) u_B(k),$$

$$w_1^B(r) = \int_0^\infty dk F_2'(kr) w_B(k), \quad (9)$$

where $u_1^B(r)$ and $w_1^B(r)$ vanish for $r \leq a$ because of the assumed hard-sphere interaction. Upon defining $g_J^{(\alpha)}(k)$ as in Eq. (12) of Section I.2 and substituting Eq. (9) into Eq. (8), we find (proceeding as in Section I.2)

$$(\ell^2 + \gamma^2) u_B(\ell) = \frac{2\sigma_c r_c}{\pi} g_0^{(c)}(\ell) \int_0^\infty dk g_0^{(c)}(k) u_B(k)$$

$$+ \frac{2\sigma_T r_T}{\pi} g_0^{(T)}(\ell) \int_0^\infty dk g_0^{(T)}(k) w_B(k)$$

$$(\ell^2 + \gamma^2) w_B(\ell) = \frac{2\sigma_T r_T}{\pi} g_2^{(T)}(\ell) \int_0^\infty dk [g_0^{(T)}(k) u_B(k) - \frac{1}{\sqrt{2}} g_2^{(T)}(k) w_B(k)]$$

$$+ \frac{2\sigma_c r_c}{\pi} g_2^{(c)}(\ell) \int_0^\infty dk g_2^{(c)}(k) w_B(k). \quad (10)$$

These coupled integral equations can be rewritten as

$$u_B(\ell) = \frac{1}{\ell^2 + \gamma^2} [g_0^{(c)}(\ell) b_0^{(c)} + g_0^{(T)}(\ell) d_2^{(T)}]$$

$$w_B(\ell) = \frac{1}{\ell^2 + \gamma^2} [g_2^{(T)}(\ell) (b_0^{(T)} - \frac{1}{\sqrt{2}} d_2^{(T)}) + g_2^{(c)}(\ell) d_2^{(c)}], \quad (11)$$

where, with $\alpha = C$ or T ,

$$\begin{aligned} b_L^{(\alpha)} &= \frac{2\sigma_{\alpha} r_{\alpha}}{\pi} \int_0^{\infty} dk g_L^{(\alpha)}(k) u_B(k), \\ d_L^{(\alpha)} &= \frac{2\sigma_{\alpha} r_{\alpha}}{\pi} \int_0^{\infty} dk g_L^{(\alpha)}(k) w_B(k). \end{aligned} \quad (12)$$

In order to determine the quantities $b_L^{(\alpha)}$ and $d_L^{(\alpha)}$, we substitute Eqs. (11) into Eqs. (12), and, after defining (see Eqs. (46) - (48) in Section I.2),

$$\begin{aligned} B_L^{(c)}(\gamma) &= \frac{2\sigma_c r_c}{\pi} \int_0^{\infty} dk \frac{[g_L^{(c)}(k)]^2}{k^2 + \gamma^2} = A_L^{(c)}(i\gamma), \\ B_L^{(T)}(\gamma) &= \frac{2\sigma_T r_T}{\pi} \int_0^{\infty} dk \frac{[g_L^{(T)}(k)]^2}{k^2 + \gamma^2} - \frac{1}{\sqrt{2}} \delta_{L,0} = A_L^{(T)}(i\gamma), \\ B_L^{(c,T)}(\gamma) &= \frac{2\sigma_c r_c}{\pi} \int_0^{\infty} dk \frac{g_L^{(c)}(k) g_L^{(T)}(k)}{k^2 + \gamma^2} = A_L^{(c,T)}(i\gamma). \end{aligned} \quad (13)$$

we obtain the four linear equations

$$\begin{pmatrix} 1 - B_0^{(c)}(\gamma) & 0 & 0 & -B^{(c,T)}(\gamma) \\ -\alpha' B_0^{(c,T)}(\gamma) & 1 & 0 & -(B_0^{(T)}(\gamma) + \frac{1}{\sqrt{2}}) \\ 0 & -B_2^{(c,T)}(\gamma) & 1 - B_2^{(c)}(\gamma) & \frac{1}{\sqrt{2}} B_2^{(c,T)}(\gamma) \\ 0 & -B_2^{(T)}(\gamma) & -\alpha' B_2^{(L,T)}(\gamma) & 1 + \frac{1}{\sqrt{2}} B_2^{(T)}(\gamma) \end{pmatrix} \begin{pmatrix} b_0^{(c)} \\ b_0^{(T)} \\ d_2^{(c)} \\ d_2^{(T)} \end{pmatrix} = 0, \quad (14)$$

in which $\alpha' = \frac{\sigma_T r_T}{\sigma_C r_C}$. The determinant, D, of the coefficients in

Eq. (14) must vanish identically if there are to be non-trivial solutions. This eigenvalue condition determines γ ; namely, one obtains

$$D(\gamma) = (1 - B_0^{(c)}(\gamma))(1 - B_2^{(c)}(\gamma)) - B_0'(\gamma)B_2'(\gamma) = 0, \quad (15)$$

where

$$B_L'(\gamma) = \alpha'(B_L^{(c,T)}(\gamma))^2 + B_L^{(T)}(\gamma)[1 - B_L^{(c)}(\gamma)]. \quad (16)$$

Then, upon solving Eq. (14) for $b_0^{(T)}$, $d_2^{(c)}$ and $d_2^{(T)}$ in terms of $b_0^{(c)}$ (which will be determined below), we find the results

$$\begin{aligned} b_0^{(T)} &= \frac{b_0^{(c)}}{B_0^{(c,T)}(\gamma)} \left\{ B_0'(\gamma) + \frac{1}{\sqrt{2}} [1 - B_0^{(c)}(\gamma)] \right\}, \\ d_2^{(c)} &= \frac{b_0^{(c)}}{B_0^{(c,T)}(\gamma)} \frac{B_2^{(c,T)}(\gamma)B_0'(\gamma)}{[1 - B_2^{(c)}(\gamma)]}, \\ d_2^{(T)} &= \frac{b_0^{(c)}}{B_0^{(c,T)}(\gamma)} (1 - B_0^{(c)}(\gamma)). \end{aligned} \quad (17)$$

Finally, substitution of Eqs. (17) into Eqs. (11) gives

$$\begin{aligned}
 u_B(\ell) &= \frac{1}{\ell^2 + \gamma^2} \left[g_0^{(c)}(\ell) + \frac{(1 - B_0^{(c)}(\gamma))}{B_0^{(c,T)}(\gamma)} g_0^{(T)}(\ell) \right] b_0^{(c)} \\
 w_B(\ell) &= \frac{1}{\ell^2 + \gamma^2} \left[B_0'(\gamma) g_2^{(T)}(\ell) + \frac{B_2^{(c,T)}(\gamma)}{B_2'(\gamma)} (1 - B_0^{(c)}(\gamma)) g_2^{(c)}(\ell) \right] \\
 &\quad \frac{b_0^{(c)}}{B_0^{(c,T)}(\gamma)}. \tag{18}
 \end{aligned}$$

Now, according to Eqs. (7) and (9) the normalization condition can be written in the form

$$\int_0^\infty dk (u_B^2(k) + w_B^2(k)) = 2/\pi, \tag{19}$$

which by Eq. (5) is equivalent to $\int_0^\infty dk [\phi_0^2(k) + \phi_2^2(k)] = \pi/2$.

Substituting Eqs. (18) into (19), and solving for $b_0^{(c)}$, then yields

$$(b_0^{(c)})^2 = \frac{2}{\pi} (B_2'(\gamma))^2 (B_0^{(c,T)}(\gamma))^2 / D', \tag{20}$$

where

$$\begin{aligned}
 D' &\equiv (1 - B_0^{(c)}(\gamma))^2 [(B_2^{(c,T)}(\gamma))^2 c_2^{(c)}(\gamma) + (B_2'(\gamma))^2 c_0^{(T)}(\gamma)] \\
 &\quad + (B_2'(\gamma))^2 [(B_0^{(c,T)}(\gamma))^2 c_0^{(c)}(\gamma) + (B_0'(\gamma))^2 c_2^{(T)}(\gamma)] \\
 &\quad + 2B_2'(\gamma) [1 - B_0^{(c)}(\gamma)] [B_2'(\gamma) B_0^{(c,T)}(\gamma) c_0^{(c,T)}(\gamma) \\
 &\quad + B_0'(\gamma) B_2^{(c,T)}(\gamma) c_2^{(c,T)}(\gamma)], \tag{21}
 \end{aligned}$$

and

$$C_L^{(\alpha\beta)}(\gamma) = \int_0^\infty d\ell \frac{g_L^\alpha(\ell) g_L^\beta(\ell)}{(\ell^2 + \gamma^2)^2}$$

$$\rightarrow C_L^{(\alpha)}(\gamma) \quad \text{if } \alpha = \beta.$$
(22)

The integrals $B_i^{(\alpha\beta)}(\gamma)$ of Eqs. (13) and $C_L^{(\alpha\beta)}(\gamma)$ are evaluated explicitly in Appendix 10. The explicit value of γ to be used is determined from the eigenvalue condition of Eq. (15), or alternatively, given γ , a condition between $\sigma_T r_T$ and $\sigma_C r_C$ is implied.

Deuteron RMS Radius

The deuteron rms radius is defined by

$$\langle r^2 \rangle = \int_a^\infty dr \, r^2 [(u_1^B(r))^2 + (w_1^B(r))^2].$$
(23)

Now, upon defining

$$F_\ell^{(\alpha)}\left(\frac{r}{a}, i\gamma a\right) = \int_0^\infty d\tau \frac{F_\ell'(\tau x) a g_\ell^{(\alpha)}(\tau)}{\tau^2 + (\gamma a)^2},$$
(24)

where $x = r/a$ and $\tau = k/a$, and substituting Eq. (18) into (9),

we find

$$\begin{aligned}
u_1^B(r) &= b_0^{(c)}(\gamma) [F_0^{(c)}(x, i\gamma a) + \frac{(1-B_0^{(c)}(\gamma))}{B_0^{(c,T)}(\gamma)} F_0^{(T)}(x, i\gamma a)] \\
w_1^B(r) &= b_0^{(c)}(\gamma) \left[\frac{B_0'(\gamma)}{B_0^{(c,T)}(\gamma)} F_2^{(T)}(x, i\gamma a) \right. \\
&\quad \left. + \frac{B_2^{(c,T)}(\gamma)(1-B_0^{(c)}(\gamma))}{B_2'(\gamma)B_0^{(c,T)}(\gamma)} F_2^{(c)}(x, i\gamma a) \right]
\end{aligned} \tag{25}$$

in which $B_0'(\gamma)$ is related to $B_2'(\gamma)$ by Eq. (15).

Defining further,

$$\begin{aligned}
Q_{00}^{(\alpha\beta)}(\gamma) &= \int_1^\infty dx \, x^2 F_0^{(\alpha)}(x, i\gamma a) F_0^{(\beta)}(x, i\gamma a) \\
Q_{22}^{(\alpha\beta)}(\gamma) &= \int_1^\infty dx \, x^2 F_2^{(\alpha)}(x, i\gamma a) F_2^{(\beta)}(x, i\gamma a)
\end{aligned} \tag{26}$$

and substituting Eq. (25) into (23), we obtain

$$\langle r^2 \rangle = a^3 (b_o^{(c)}(\gamma))^2$$

$$\left[\begin{aligned} & Q_{0,0}^{(c,c)}(\gamma) + \frac{2(1-B_o^{(c)}(\gamma))}{B_o^{(c,T)}(\gamma)} Q_{0,0}^{(c,T)}(\gamma) \\ & + \left(\frac{-B_o^{(c)}(\gamma)}{B_o^{(c,T)}(\gamma)} \right)^2 Q_{0,0}^{(T,T)}(\gamma) \\ & + \left(\frac{B_o'(\gamma)}{B_o^{(c,T)}(\gamma)} \right) Q_{2,2}^{(T,T)}(\gamma) + \frac{2B_o'(\gamma)B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))Q_{22}^{(c)}(\gamma)}{B_2'(\gamma)(B_o^{(c,T)}(\gamma))^2} \\ & + \left(\frac{B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))}{B_2'(\gamma)B_o^{(c,T)}(\gamma)} \right)^2 Q_{2,2}^{(c,c)}(\gamma) \end{aligned} \right] \quad (27)$$

The second integral, $Q_{22}^{(\alpha\beta)}(\gamma)$, is determined explicitly in Section I.5 on the deuteron quadrupole moment.

Since (from Appendix 11)

$$F_o^{(\alpha)}(x, i\gamma a) = \frac{\pi}{2} \frac{1}{(1-\gamma^2/\mu_\alpha^2)} [e^{-\gamma(x-1)} - e^{-\mu_\alpha(x-1)}], \quad \begin{aligned} \gamma &\rightarrow \gamma_a \\ \mu_\alpha &= \lambda_\alpha a \end{aligned} \quad (28)$$

it is readily determined that

$$\begin{aligned}
Q_{00}^{(\alpha\beta)}(\gamma) &= \frac{1}{2} c_{00}^{(\alpha\beta)}(\gamma) [e^{2\gamma} \alpha_2(2\gamma) - e^{\gamma+\mu_\alpha} \alpha_2(\gamma+\mu_\alpha) - e^{\gamma+\mu_\beta} \alpha_2(\gamma+\mu_\beta) \\
&\quad + e^{\mu_\alpha+\mu_\beta} \alpha_2(\mu_\alpha+\mu_\beta)] \\
&= c_{00}^{(\alpha\beta)}(\gamma) \left[\frac{e_2(2\gamma)}{\gamma^3} - \frac{e_2(\gamma+\mu_\alpha)}{\gamma^3 \mu_\alpha^3} - \frac{e_2(\gamma+\mu_\beta)}{\gamma^3 \mu_\beta^3} + \frac{e_2(\mu_\alpha+\mu_\beta)}{\mu_\alpha^3 \mu_\beta^3} \right], \quad \begin{array}{l} \gamma \rightarrow \gamma_a \\ \mu_\alpha = \lambda_{\alpha a} \end{array} \\
&\hspace{15em} (29)
\end{aligned}$$

where

$$c_{00}^{(\alpha\beta)}(\gamma) = \frac{\pi^2}{2} \frac{1}{(1-\gamma^2/\mu_\alpha^2)(1-\gamma^2/\mu_\beta^2)}, \quad \begin{array}{l} \gamma \rightarrow \gamma_a \\ \mu_\alpha = \lambda_{\alpha a} \end{array} \quad (30)$$

$$\alpha_n(z) = \int_1^\infty dt \, t^n e^{-zt}, \quad \text{Re}(z) > 0, \quad (n = 0, 1, 2)$$

$$= \frac{n!}{z^{n+1}} e^{-z} e_n(z), \quad (31)$$

and

$$e_n(z) = \sum_{k=0}^n \frac{(z)^k}{k!}. \quad (32)$$

Equation (29), together with the results of Section I.5 for $Q_{22}^{(\alpha\beta)}(\gamma)$, completes the analytic determination of the deuteron rms radius.

Finally, we determine the momentum-space wave functions for the deuteron, as introduced at the outset of this section and defined explicitly by Eqs. (6). Upon substituting Eqs. (9) and (18), we obtain

$$\phi_0(k) = \frac{\pi}{2} b_0^{(c)}(\gamma) \frac{\cos ka}{(k^2 + \gamma^2)} [g_0^{(c)}(k) + \frac{(1 - B_0^{(c)}(\gamma))}{B_0^{(c,T)}(\gamma)} g_0^{(T)}(k)],$$

$$\phi_2(k) = \frac{\pi}{2} \frac{b_0^{(c)}(\gamma)}{B_0^{(c,T)}(\gamma)} \cdot \frac{G_2(ka)}{\Delta_2(ka)} \frac{1}{(k^2 + \gamma^2)} [B_0'(\gamma) g_2^{(T)}(k)$$

$$+ \frac{B_2^{(c,T)}(\gamma)}{B_2'(\gamma)} (1 - B_0^{(c)}(\gamma)) g_2^{(c)}(\gamma)].$$

1.5 Deuteron Quadrupole Moment

The deuteron quadrupole moment is given by¹

$$Q_d = \frac{1}{5\sqrt{2}} \int_0^\infty dr r^2 u_1^B(r) w_1^B(r) - \frac{1}{20} \int_0^\infty dr r^2 (w_1^B(r))^2, \quad (1)$$

where the bound state wave functions are defined as

$$u_1^B(r) = \int_0^\infty dk F'_0(kr) u_B(k),$$

$$w_1^B(r) = \int_0^\infty dk F'_2(kr) w_B(k). \quad (2)$$

The bound-state momentum wave functions were determined by Eqs. (15) and (18) of Section I.4 to be

$$u_B(k) = \frac{b_0^C(\gamma)}{k^2 + \gamma^2} \left[g_0^C(k) + \frac{1 - B_0^C(\gamma)}{B_0^{C,T}(\gamma)} g_0^T(k) \right]$$

$$w_B(k) = \frac{b_0^C(\gamma)}{k^2 + \gamma^2} \left[(1 - B_2^C(\gamma)) g_2^T(k) + B_2^{C,T}(\gamma) g_2^C(k) \right] \frac{(1 - B_0^C(\gamma))}{B_2^C(\gamma) B_0^{C,T}(\gamma)}. \quad (3)$$

The normalization constant $b_0^C(\gamma)$ and the coefficients $B_l^{\alpha\beta}(\gamma)$ ($\alpha, \beta = C$ or T) are defined in Section I.4 and evaluated explicitly in Appendix 10.

Upon substituting Eqs. (3) into Eqs. (2), and defining

$$F_l^{\alpha}\left(\frac{r}{a}, i\gamma a\right) = \int_0^\infty \frac{d\tau F'_l(\tau x) a g_l^{\alpha}(\tau)}{\tau^2 + (\gamma a)^2}, \quad (4)$$

¹Blatt, J. M. and Weisskopf, V. F., THEORETICAL NUCLEAR PHYSICS, Wiley & Sons, New York, 1952, p. 105.

where $x = r/a$, and $\tau = ka$, we obtain

$$u_1^B(r) = b_0^c(\gamma) \left[F_0^{(c)}(x, i\gamma a) + \frac{(1 - B_0^c(\gamma))}{B_0^{c,T}(\gamma)} F_0^{(T)}(x, i\gamma a) \right],$$

$$w_1^B(r) = b_0^c(\gamma) \left[\frac{B_0'(\gamma)}{B_0^{c,T}(\gamma)} F_2^{(T)}(x, i\gamma a) + \frac{B_2^{(c,T)}(\gamma)(1 - B_0^c(\gamma))}{B_2^1(\gamma)B_0^{c,T}(\gamma)} F_2^{(c)}(x, i\gamma a) \right], \quad (5)$$

as with Eqs. (24) and (25) in the previous section, where according to Eq. (15) of Section I.4,

$$B_0^1(\gamma) = (1 - B_0^c(\gamma))(1 - B_2^c(\gamma))/B_2'(\gamma). \quad (6)$$

If next we define the two basic integrals involved in the evaluation of Eq. (1), and henceforth write γ for γa , as

$$Q_{02}^{\alpha\beta}(\gamma) = \int_1^\infty dx x^2 F_0^{(\alpha)}(x, i\gamma) F_2^{(\beta)}(x, i\gamma),$$

$$Q_{22}^{\alpha\beta}(\gamma) = \int_1^\infty dx x^2 F_2^{(\alpha)}(x, i\gamma) F_2^{(\beta)}(x, i\gamma), \quad (7)$$

then, upon substituting Eqs. (5) into (1) and using Eqs. (7), the expression for the deuteron quadrupole moment takes the following form:

$$Q_d = a^3 [b_o^{(c)}(\gamma)]^2$$

$$\left[\begin{aligned} & \left\{ \frac{B_o'(\gamma)}{B_o^{(c,T)}(\gamma)} Q_{02}^{(c,T)}(\gamma) + \frac{(1-B_o^{(c)}(\gamma))B_o'(\gamma)}{(B_o^{(c,T)}(\gamma))^2} Q_{02}^{(TT)}(\gamma) \right. \\ & \left. + \frac{B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))}{B_2'(\gamma)B_o^{(c,T)}(\gamma)} Q_{02}^{(cc)}(\gamma) + \frac{B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))^2}{B_2'(\gamma)(B_o^{(c,T)}(\gamma))^2} Q_{02}^{(Tc)}(\gamma) \right\} \\ & - \frac{1}{20} \left\{ \left(\frac{B_o'(\gamma)}{B_o^{(c,T)}(\gamma)} \right)^2 Q_{22}^{(TT)}(\gamma) + \frac{2B_o'(\gamma)B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))}{B_2'(\gamma)(B_o^{(c,T)}(\gamma))^2} Q_{22}^{(c,T)}(\gamma) \right. \\ & \left. + \left(\frac{B_2^{(c,T)}(\gamma)(1-B_o^{(c)}(\gamma))}{B_2'(\gamma)B_o^{(c,T)}(\gamma)} \right)^2 Q_{22}^{(cc)}(\gamma) \right\} \end{aligned} \right] \quad (8)$$

Examination of Eq. (8) indicates that the complete determination of Q_d depends only upon the evaluation of the integrals $Q_{02}^{\alpha\beta}(\gamma)$ and $Q_{22}^{\alpha\beta}(\gamma)$, since all the coefficients are previously-encountered and explicitly-known functions.

We consider the evaluation of $Q_{02}^{\alpha\beta}(\gamma)$ first. According to Appendix 11, we may write Eq. (4) in the form, with $\mu_\alpha = \lambda_\alpha a$,

$$F_{\ell}^{\alpha}(x, i\gamma) = (-)^{\ell} \frac{\pi}{4} \frac{\mu_{\alpha}^2 e^{\mu_{\alpha}}}{\gamma^{\ell+1}} \left\{ \begin{aligned} & \frac{-B_{\ell}^{(-)}(\gamma x)}{B_{\ell}^{(-)}(\gamma)} [B_{\ell}^{(+)}(\gamma) U_{\ell}^{\alpha}(\gamma) - B_{\ell}^{(-)}(\gamma) U_{\ell}^{\alpha}(-\gamma)] \\ & + \frac{1}{x^{\ell-1}} [B_{\ell}^{(+)}(\gamma x) U_{\ell}^{\alpha}(\gamma x) - B_{\ell}^{(-)}(\gamma x) U_{\ell}^{\alpha}(-\gamma x)] \end{aligned} \right\}, \quad (9)$$

$$\text{where } F_0^{\alpha}(x, i\gamma) = \frac{\pi}{2} \left(\frac{1}{1-\gamma^2/\mu_{\alpha}^2} \right) \begin{bmatrix} e^{-\gamma(x-1)} & -e^{-\mu_{\alpha}(x-1)} \end{bmatrix} \text{ for the}$$

case $\ell = 0$, and

where

$$B_{\ell}^{(\pm)}(\gamma x) \equiv i^{\ell} H_{\ell}^{(\mp)}(i\gamma x) = \frac{e^{\pm\gamma x}}{(\gamma x)^{\ell}} [A_{\ell}^{(\pm)}(\gamma x)], \quad (10)$$

$$U_{\ell}^{\alpha}(\gamma x) \equiv \sum_{s=0}^{\ell} \Gamma_{\ell s}(\gamma x) E_s((\mu_{\alpha} + \gamma)x), \quad (11)$$

and the real function $A_{\ell}^{(\pm)}(\gamma x)$ is defined by

$$A_{\ell}^{(\pm)}(\gamma x) \equiv \hat{\alpha}_{\ell}(i\gamma x) \pm i\hat{\beta}_{\ell}(i\gamma x) = \sum_{s=0}^{\ell} \Gamma_{\ell s}(\mp\gamma x). \quad (12)$$

New functions introduced by these definitions are all given

explicitly in Appendix 1 and Sec. I.6. We note that Eq. (9) is

well-defined even for $\gamma > \mu_{\alpha}$, since in this case we may write for

the apparently divergence factors [see Eq. (5) in Appendix 1]

$$U_{\ell}^{\alpha}(-\gamma) - \frac{1}{x^{\ell-1}} U_{\ell}^{\alpha}(-\gamma x) = \sum_{s=0}^{\ell} \Gamma_{\ell s}(-\gamma) \int_1^x \frac{dt}{t^s} e^{(\gamma-\mu_{\alpha})t}, \quad (13)$$

which is a convergent expression for $x = \text{finite}$.

We next define

$$\begin{aligned} c_{02}^{\alpha\beta}(\gamma) &= \frac{\pi^2}{8} \frac{\mu_{\beta}^2 e^{\mu_{\beta}}}{\gamma^5 (1-\gamma^2/\mu_{\alpha}^2)}, \\ r_2^{\beta}(\gamma) &= \gamma^2 [B_2^{(+)}(\gamma) U_2^{\beta}(\gamma) - B_2^{(-)}(\gamma) U_2^{\beta}(-\gamma)] \\ &= [e^{\gamma A_2^{(+)}}(\gamma) U_2^{\beta}(\gamma) - e^{-\gamma A_2^{(-)}}(\gamma) U_2^{\beta}(-\gamma)], \end{aligned} \quad (14)$$

where Eq. (10) has been used, and the integrals

$$\begin{aligned} D_{n,s}^{(1)}(\Lambda) &\equiv \int_1^{\infty} dx x^n E_s(\Lambda x), \quad \text{Re}(\Lambda) > 0, \\ D_{n,s}^{(2)}(\Lambda_1, \Lambda_2) &\equiv \int_1^{\infty} dx x^n e^{-\Lambda_1 x} E_s(\Lambda_2 x), \quad \text{Re}(\Lambda_1 + \Lambda_2) > 0, \end{aligned} \quad (15)$$

$$\rightarrow_{s=1, n \leq 0} J_{|n|,1}(\Lambda_1, \Lambda_2) = D_{-n}(\Lambda_1, \Lambda_2),$$

(which are evaluated explicitly in Appendices 12 and 13). Upon substituting Eq. (9) into the first of Eqs. (7) and then using definitions (10) - (12), (14) and (15), we obtain

$$Q_{02}^{\alpha\beta}(\gamma) =$$

$$\left[C_{02}^{\alpha\beta}(\gamma) \frac{f_2^\beta(\gamma)}{A_2^{(-)}(\gamma)} \sum_{s=0}^2 \Gamma_{2s}(\gamma) [-\alpha_{2-s}(2\gamma)e^{2\gamma} + \alpha_{2-s}(\mu_\alpha + \gamma)e^{\mu_\alpha + \gamma}] \right. \\ \left. + C_{02}^{\alpha\beta}(\gamma) \sum_{s,t=0}^2 \left\{ \begin{array}{l} \Gamma_{2s}(\gamma)\Gamma_{2t}(-\gamma)D_{3-s-t,s}^{(1)}(\mu_\alpha + \gamma)e^\gamma \\ -\Gamma_{2s}(\gamma)\Gamma_{2t}(-\gamma)D_{3-s-t,s}^{(2)}(\mu_\alpha - \gamma, \mu_\beta + \gamma)e^{\mu_\alpha} \\ -\Gamma_{2s}(-\gamma)\Gamma_{2t}(\gamma)D_{3-s-t,s}^{(2)}(2\gamma, \mu_\beta - \gamma)e^\gamma \\ +\Gamma_{2s}(-\gamma)\Gamma_{2t}(\gamma)D_{3-s-t,s}^{(2)}(\mu_\alpha + \gamma, \mu_\beta - \gamma)e^{\mu_\alpha} \end{array} \right\} \right], \quad (16)$$

where $\alpha_n(z)$, for $n \geq 0$, is defined by Eqs. (31) and (32) in Section I.4.

Now we evaluate $Q_{22}^{\alpha\beta}(\gamma)$, which is somewhat more complicated algebraically. Upon substituting Eq. (9) into the second of Eqs. (7), we obtain

$$Q_{22}^{\alpha\beta}(\gamma) = C_{22}^{\alpha\beta}(\gamma) \int_1^{\infty} dx$$

$$\left[\begin{aligned} & x^2 \left(\frac{B_2^{(-)}(\gamma x)}{B_2^{(-)}(\gamma)} \right)^2 f_2^{\alpha}(\gamma) f_2^{\beta}(\gamma) \\ & - \gamma^2 x \frac{B_2^{(-)}(\gamma x)}{B_2^{(-)}(\gamma)} \left\{ f_2^{\beta}(\gamma) [B_2^{(+)}(\gamma x) U_2^{(\alpha)}(\gamma x) - B_2^{(-)}(\gamma x) U_2^{(\alpha)}(-\gamma x)] \right. \\ & \quad \left. + f_2^{\alpha}(\gamma) [B_2^{(+)}(\gamma x) U_2^{(\beta)}(\gamma x) - B_2^{(-)}(\gamma x) U_2^{(\beta)}(-\gamma x)] \right\} \\ & + \gamma^4 [B_2^{(+)}(\gamma x) U_2^{\alpha}(\gamma x) - B_2^{(-)}(\gamma x) U_2^{\alpha}(-\gamma x)] \\ & \quad \times [B_2^{(+)}(\gamma x) U_2^{\beta}(\gamma x) - B_2^{(-)}(\gamma x) U_2^{\beta}(-\gamma x)] \end{aligned} \right] , \quad (17)$$

where we used Eq. (14) for $f_2^{\beta}(\gamma)$ and have also defined

$$C_{22}^{\alpha\beta}(\gamma) = \frac{\pi^2}{16} \frac{\mu_{\alpha}^2 \mu_{\beta}^2}{\gamma^{10}} e^{\mu_{\alpha} + \mu_{\beta}}. \quad (18)$$

The first term of Eq. (17) integrates easily to yield

$$\int_1^{\infty} dx x^2 \left(\frac{B_2^{(-)}(\gamma x)}{B_2^{(-)}(\gamma)} \right)^2 f_2^{\alpha}(\gamma) f_2^{\beta}(\gamma) =$$

$$\frac{f_2^{\alpha}(\gamma) f_2^{\beta}(\gamma)}{(A_2^{(-)}(\gamma))^2} \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) \alpha_{2-s-t}(2\gamma) e^{2\gamma}. \quad (19)$$

The second term of Eq. (17) integrates to yield

$$\begin{aligned}
& -\gamma^2 \int_1^\infty dx \, x \frac{B_2^{(-)}(\gamma x)}{B_2^{(-)}(\gamma)} \left\{ \begin{aligned} & f_2^\beta(\gamma) [B_2^{(+)}(\gamma x) U_2^\alpha(\gamma x) - B_2^{(-)}(\gamma x) U_2^\alpha(-\gamma x)] \\ & + f_2^\alpha(\gamma) [B_2^{(+)}(\gamma x) U_2^\beta(\gamma x) - B_2^{(-)}(\gamma x) U_2^\beta(-\gamma x)] \end{aligned} \right\} \\
& = - \frac{f_2^\beta(\gamma) e^\gamma}{A_2^{(-)}(\gamma)} \sum_{r,s,t=0}^2 \left\{ \begin{aligned} & \Gamma_{2s}(\gamma) \Gamma_{2t}(-\gamma) \Gamma_{2r}(\gamma) D_{3-s-t-r,r}^{(1)}(\mu_\alpha + \gamma) \\ & - \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) \Gamma_{2r}(-\gamma) D_{3-s-t-r,r}^{(2)}(2\gamma, \mu_\alpha - \gamma) \end{aligned} \right\} \\
& - \frac{f_2^\alpha(\gamma) e^\gamma}{A_2^{(-)}(\gamma)} \sum_{r,s,t=0}^2 \left\{ \begin{aligned} & \Gamma_{2s}(\gamma) \Gamma_{2t}(-\gamma) \Gamma_{2r}(\gamma) D_{3-s-t-r,r}^{(1)}(\mu_\beta + \gamma) \\ & - \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) \Gamma_{2r}(-\gamma) D_{3-s-t-r,r}^{(2)}(2\gamma, \mu_\beta - \gamma) \end{aligned} \right\}.
\end{aligned} \tag{20}$$

The third term of Eq. (17) is somewhat more complicated. We define the integrals

$$\begin{aligned}
D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) & \equiv \int_1^\infty dx \, x^n e^{\Lambda_1 x} E_1(\Lambda_2 x) E_1(\Lambda_3 x), \quad \text{Re}(-\Lambda_1 + \Lambda_2 + \Lambda_3) > 0, \\
D_n^{(4)}(\Lambda_1, \Lambda_2) & \equiv \int_1^\infty dx \, x^n E_1(\Lambda_1 x) E_1(\Lambda_2 x), \quad \text{Re}(\Lambda_1 + \Lambda_2) > 0,
\end{aligned} \tag{21}$$

which are evaluated explicitly in Appendix 13. Then upon substituting Eqs. (10) through (12) into the third term of Eq. (17), and using the reduction formula for the exponential integral $E_n(z)$ [see Eq. (4) of Appendix 6]

$$E_n(z) = \frac{\theta(n-1)}{(n-1)!} e^{-z} \sum_{k=1}^{n-1} (n-1-k)! (-z)^{k-1} + \frac{(-z)^{n-1}}{(n-1)!} E_1(z), \quad (22)$$

we find after considerable manipulation that

$$\begin{aligned} & \gamma^4 \int_1^\infty dx [B_2^{(+)}(\gamma x) U_2^\alpha(\gamma x) - B_2^{(-)}(\gamma x) U_2^\alpha(-\gamma x)] \\ & \quad [B_2^{(+)}(\gamma x) U_2^\beta(\gamma x) - B_2^{(-)}(\gamma x) U_2^\beta(-\gamma x)] \\ &= \gamma^4 \sum_{s,t=0}^2 \alpha_{2-s-t} (\mu_\alpha + \mu_\beta) \left[\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right] \left[\frac{\Gamma_{2t}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2t}(\gamma)}{\mu_\beta - \gamma} \right] \\ & \quad + 3\gamma^2 \sum_{s,t=0}^2 \alpha_{1-s-t} (\mu_\alpha + \mu_\beta) \left\{ \left[\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right] \left[\Gamma_{2t}(-\gamma) - \Gamma_{2t}(\gamma) \right] \right. \\ & \quad \left. + \left[\frac{\Gamma_{2t}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2t}(\gamma)}{\mu_\beta - \gamma} \right] \left[\Gamma_{2s}(-\gamma) - \Gamma_{2s}(\gamma) \right] \right\} \\ & \quad + 9 \sum_{s,t=0}^2 \alpha_{-s-t} (\mu_\alpha + \mu_\beta) [\Gamma_{2s}(-\gamma) - \Gamma_{2s}(\gamma)] [\Gamma_{2t}(-\gamma) - \Gamma_{2t}(\gamma)] \\ & \quad - 3\gamma^2 \mu_\beta \sum_{s,t=0}^2 \left[\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right] \left[\Gamma_{2t}(-\gamma) D_{2-s-t,1}^{(2)}(\mu_\alpha - \gamma, \mu_\beta + \gamma) \right. \\ & \quad \left. - \Gamma_{2t}(\gamma) D_{2-s-t,1}^{(2)}(\mu_\alpha + \gamma, \mu_\beta - \gamma) \right] \\ & \quad - 3\gamma^2 \mu_\alpha \sum_{s,t=0}^2 \left[\frac{\Gamma_{2t}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2t}(\gamma)}{\mu_\beta - \gamma} \right] \left[\Gamma_{2s}(-\gamma) D_{2-s-t,1}^{(2)}(\mu_\beta - \gamma, \mu_\alpha + \gamma) \right. \\ & \quad \left. - \Gamma_{2s}(\gamma) D_{2-s-t,1}^{(2)}(\mu_\beta + \gamma, \mu_\alpha - \gamma) \right] \\ & \quad - 9\mu_\beta \sum_{s,t=0}^2 [\Gamma_{2s}(-\gamma) - \Gamma_{2s}(\gamma)] \left[\Gamma_{2t}(-\gamma) D_{1-s-t,1}^{(2)}(\mu_\alpha - \gamma, \mu_\beta + \gamma) \right. \\ & \quad \left. - \Gamma_{2t}(\gamma) D_{1-s-t,1}^{(2)}(\mu_\alpha + \gamma, \mu_\beta - \gamma) \right] \end{aligned}$$

$$\begin{aligned}
& -9\mu_\alpha \sum_{s,t=0}^2 [\Gamma_{2t}(-\gamma) - \Gamma_{2t}(\gamma)] \left[\begin{array}{l} \Gamma_{2s}(-\gamma) D_{1-s-t,1}^{(2)}(\mu_\beta - \gamma, \mu_\alpha + \gamma) \\ -\Gamma_{2s}(\gamma) D_{1-s-t,1}^{(2)}(\mu_\beta + \gamma, \mu_\alpha - \gamma) \end{array} \right] \\
& + 9\mu_\alpha \mu_\beta \sum_{s,t=0}^2 \left\{ \begin{array}{l} \Gamma_{2s}(-\gamma) \Gamma_{2t}(-\gamma) D_{2-s-t}^{(3)}(2\gamma, \mu_\alpha + \gamma, \mu_\beta + \gamma) \\ \quad + \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) D_{2-s-t}^{(3)}(-2\gamma, \mu_\alpha - \gamma, \mu_\beta - \gamma) \\ -\Gamma_{2s}(\gamma) \Gamma_{2t}(-\gamma) D_{2-s-t}^{(4)}(\mu_\alpha - \gamma, \mu_\beta + \gamma) \\ \quad - \Gamma_{2s}(-\gamma) \Gamma_{2t}(\gamma) D_{2-s-t}^{(4)}(\mu_\alpha + \gamma, \mu_\beta - \gamma) \end{array} \right\} \quad (23)
\end{aligned}$$

Upon collecting together Eqs. (19), (20), and (23), we obtain [with

$$\Gamma_{2,0}(\gamma) = \gamma^2, \Gamma_{2,1}(\gamma) = 3\gamma, \Gamma_{2,2}(\gamma) = 3 \text{ and } \alpha_{-|n|}(\Lambda) = E_{|n|}(\Lambda)]$$

$$\frac{Q_{22}^{\alpha\beta}(\gamma)}{C_{22}^{\alpha\beta}(\gamma)} = \frac{f_2^\alpha(\gamma) f_2^\beta(\gamma)}{(A_2^{(-)}(\gamma))^2} \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) \alpha_{2-s-t}(2\gamma) e^{2\gamma}$$

$$- \frac{f_2^\beta(\gamma) e^\gamma}{A_2^{(-)}(\gamma)}$$

$$\left\{ \begin{array}{l} \gamma^2 \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) \left[\frac{\Gamma_{st}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2t}(+\gamma)}{\mu_\alpha - \gamma} \right] \alpha_{2-s-t}(\mu_\alpha + \gamma) \\ - 18\gamma \sum_{s=0}^2 \Gamma_{2s}(\gamma) E_s(\mu_\alpha + \gamma) \\ - 3\mu_\alpha \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) [\Gamma_{2t}(-\gamma) D_{2-s-t,1}^{(1)}(\mu_\alpha + \gamma) - \Gamma_{2t}(\gamma) D_{2-s-t,1}^{(2)}(2\gamma, \mu_\alpha - \gamma)] \end{array} \right\}$$

$$-\frac{f_2^\alpha(\gamma)e^\gamma}{A_2^{(-)}(\gamma)}$$

$$\left\{ \begin{aligned} & \gamma^2 \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) \left[\frac{\Gamma_{2t}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2t}(\gamma)}{\mu_\beta - \gamma} \right] \alpha_{2-s-t}(\mu_\beta + \gamma) \\ & - 18\gamma \sum_{s=0}^2 \Gamma_{2s}(\gamma) E_s(\mu_\beta + \gamma) \\ & - 3\mu_\beta \sum_{s,t=0}^2 \Gamma_{2s}(\gamma) [\Gamma_{2t}(-\gamma) D_{2-s-t,1}^{(1)}(\mu_\beta + \gamma) - \Gamma_{2t}(\gamma) D_{2-s-t,1}^{(2)}(2\gamma, \mu_\beta - \gamma)] \end{aligned} \right\}$$

$$+ \gamma^4 \sum_{s,t=0}^2 \left[\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right] \left[\frac{\Gamma_{2t}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2t}(\gamma)}{\mu_\beta - \gamma} \right] \alpha_{2-s-t}(\mu_\alpha + \mu_\beta)$$

$$- 18\gamma^3 \sum_{s,t=0}^2 \left[\left(\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right) + \left(\frac{\Gamma_{2s}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\beta - \gamma} \right) \right] E_s(\mu_\alpha + \mu_\beta)$$

$$+ 324\gamma^2 E_2(\mu_\alpha + \mu_\beta)$$

$$- 3\gamma^2 \mu_\beta \sum_{s,t=0}^2$$

$$\left(\frac{\Gamma_{2s}(-\gamma)}{\mu_\alpha + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\alpha - \gamma} \right) [\Gamma_{2t}(-\gamma) D_{2-s-t,1}^{(2)}(\mu_\alpha - \gamma, \mu_\beta + \gamma)$$

$$- \Gamma_{2t}(\gamma) D_{2-s-t,1}^{(2)}(\mu_\alpha + \gamma, \mu_\beta - \gamma)]$$

$$\begin{aligned}
& - 3\gamma^2 \mu_\alpha \sum_{s,t=0}^2 \\
& \left(\frac{\Gamma_{2s}(-\gamma)}{\mu_\beta + \gamma} - \frac{\Gamma_{2s}(\gamma)}{\mu_\beta - \gamma} \right) [\Gamma_{2t}(-\gamma) D_{2-s-t,1}^{(2)}(\mu_\beta - \gamma, \mu_\alpha + \gamma) \\
& \quad - \Gamma_{2t}(\gamma) D_{2-s-t,1}^{(2)}(\mu_\beta + \gamma, \mu_\alpha - \gamma)] \\
& + 54\gamma \mu_\beta \sum_{s=0}^2 [\Gamma_{2s}(-\gamma) D_{-s,1}^{(2)}(\mu_\alpha - \gamma, \mu_\beta + \gamma) - \Gamma_{2s}(\gamma) D_{-s,1}^{(2)}(\mu_\alpha + \gamma, \mu_\beta - \gamma)] \\
& + 54\gamma \mu_\alpha \sum_{s=0}^2 [\Gamma_{2s}(-\gamma) D_{-s,1}^{(2)}(\mu_\beta - \gamma, \mu_\alpha + \gamma) - \Gamma_{2s}(\gamma) D_{-s,1}^{(2)}(\mu_\beta + \gamma, \mu_\alpha - \gamma)] \\
& + 9\mu_\alpha \mu_\beta \sum_{s,t=0}^2 (s+t \neq 3) \\
& \left\{ \Gamma_{2s}(-\gamma) \Gamma_{2t}(-\gamma) D_{2-s-t}^{(3)}(2\gamma, \mu_\alpha + \gamma, \mu_\beta + \gamma) + \Gamma_{2s}(\gamma) \Gamma_{2t}(\gamma) D_{2-s-t}^{(3)}(-2\gamma, \mu_\alpha - \gamma, \mu_\beta - \gamma) \right. \\
& \quad \left. - \Gamma_{2s}(\gamma) \Gamma_{2t}(-\gamma) D_{2-s-t}^{(4)}(\mu_\alpha - \gamma, \mu_\beta + \gamma) - \Gamma_{2s}(-\gamma) \Gamma_{2t}(\gamma) D_{2-s-t}^{(4)}(\mu_\alpha + \gamma, \mu_\beta - \gamma) \right\}
\end{aligned} \tag{24}$$

When $s=t=2$, replace $D_{-2}^{(3)}(n_1, n_2, n_3)$ by $e^{n_1} E_1(n_2) E_1(n_3)$ and $D_{-2}^{(4)}(n_2, n_3)$ by $E_1(n_2) E_1(n_3)$.

Further algebraic manipulation of these results, Eqs. (16) and (24), appears unwarranted inasmuch as the answers are expressed in terms of explicitly defined and calculable functions. Several

remarks are in order, however.

First, the answers are well-defined, albeit somewhat obtrusely, and can be expressed [see the appropriate Appendices] in terms of polynomials, exponentials, logarithms, exponential integrals, and summable infinite series. Second, the possible condition $\gamma > \mu$ (such as might occur in a parameter-searching process) causes no difficulty in evaluation. In the case of Eq. (16) one may explicitly exhibit the fact that no terms involving $E_1(\mu - \gamma)$ survive the summing process. For certain terms one may use the analytic continuation condition (see Appendix 12)

$$D_1(\alpha, \bar{\beta}) + D_1(\bar{\beta}, \alpha) = E_1(\alpha)E_1(\bar{\beta}), \quad (25)$$

where $\alpha \equiv \mu_\alpha + i\tau$, $\bar{\beta} \equiv \mu_\beta - i\tau$, in the limits $\tau \rightarrow \pm i\gamma$ to specifically eliminate $E_1(\mu - \gamma)$. Thus, each of the terms on the l.h.s of Eq. (25) is explicitly determined and well-defined in this limit.

1. 6 Explicit Expressions for the Solution Integrals.

In this section we summarize the explicit expressions occurring in Sec. I.2 - I.5, as determined in the Appendices. The notation is described in the referenced Appendix.

1) Sec. I.2, Eq. (12): Appendix 1, Eq. (16)

$$\begin{aligned} \mathcal{E}_\ell(k) &\equiv \int_a^\infty dr f(r) F'_\ell(kr) , \\ &= \frac{\lambda \mu e^\mu}{\tau^\ell \Delta_\ell(\tau)} \operatorname{Im}[H_\ell^{(-)}(\tau) Z_\ell(\mu, \tau)] \end{aligned}$$

where $\mu = \lambda a$, $\tau = ka$, $\Delta_\ell(\tau) = [F_\ell^2(\tau) + G_\ell^2(\tau)]^{1/2}$,

$H_\ell^\pm(\tau) = G_\ell(\tau) + iF_\ell(\tau)$, and $Z_\ell(\mu, \tau)$ is defined by Eqs. (5), (6),

(9), and (13) in Appendix 1*

*In particular, the exponential integral is defined by

$$E_n(z) \equiv \int_1^\infty \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots$$

The frequently occurring function $\Gamma_{\ell s}(\tau)$ is defined by

$$\Gamma_{\ell s}(\tau) = \frac{(\ell + \frac{1}{2}, s)}{2^s} \tau^{\ell-s},$$

with

$$(\ell + \frac{1}{2}, s) = \frac{(\ell + s)!}{s!(\ell - s)!} \quad \text{for } 0 \leq s \leq \ell$$

(2) Sec. I.2, Eqs. (19), (20), (46), (47), and (48): Appendices 4-7, 12.

The integrals of the $A_J^{(\alpha\beta)}(k_0)$ differ only by simple multiplicative factors from the integral of Appendix 4, Eq. (1); namely,

$$K_{\ell}^{\alpha\beta}(k_0) \equiv \int_0^{\infty} dk g_{\ell}^{\alpha}(k) g_{\ell}^{\beta}(k) P\left(\frac{1}{k^2 - k_0^2}\right), \quad (2)$$

which has the solution given by Eqs. (65) and (75) in Appendix 4

$$aK_{\ell}^{\alpha\beta}(\tau_0) = -\frac{\pi}{4} \mu_{\alpha}^2 \mu_{\beta}^2 e^{\mu_{\alpha} + \mu_{\beta}} \text{Im} \left\{ \begin{aligned} & \frac{H_{\ell}^{(+)}(\tau_0)}{H_{\ell}^{(-)}(\tau_0)} \bar{Z}_{\ell}(\mu_{\alpha}, \tau_0) \bar{Z}_{\ell}(\mu_{\beta}, \tau_0) \\ & + \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) e^{-i(s-t)\pi/2} J_{s,t}^{(\alpha, \bar{\beta})} + J_{s,t}^{(\beta, \bar{\alpha})} \end{aligned} \right\} \quad (3)$$

$$+ \frac{\pi}{4} \mu_{\alpha}^2 \mu_{\beta}^2 e^{\mu_{\alpha} + \mu_{\beta}} \sum_{s,t=0}^{\ell} \sum_{p=0}^{\frac{s+t-1}{2}} \frac{\gamma_{\ell s} \gamma_{\ell t} (-)^p \Gamma_{\ell, s, t, p}^{\alpha\beta}}{(s+t-1-2p)! (\tau_0^2)^{p+1}},$$

where (Appendix 6, Eq. (14))

$$J_{s,t}(\alpha, \bar{\beta})$$

$$= \delta_{t,0} \frac{1}{\bar{\beta}} E_s(\alpha + \bar{\beta}) + \delta_{s,0}(1 - \delta_{t,0}) \left\{ \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{t-k}(\alpha + \bar{\beta}) \right. \\ \left. + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{1}{\alpha} [e^{-\alpha} E_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})] \right\}$$

$$+ (1 - \delta_{s,0})(1 - \delta_{t,0})$$

$$\cdot \left\{ \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{s+t-k}(\alpha + \bar{\beta}) \right. \\ \left. + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{(-\alpha)^{s-1}}{(s-1)!} D_1(\alpha, \bar{\beta}) \right. \\ \left. + \frac{\theta(s-1)}{(s-1)!} \frac{(-\beta)^{t-1}}{(t-1)!} \sum_{k=1}^{s-1} (-\alpha)^{s-1-k} (k-1)! [e^{-\alpha} E_1(\bar{\beta}) - E_{k+1}(\alpha + \bar{\beta})] \right\},$$

(4)

and with $\Theta(0) = 0$, (see Eqs. (4) and (21) of Appendix 7)

$$I_{\ell, s, t, p}^{\alpha\beta} = (-)^{\ell+s+t} \left(L_{s, t, p}^{\alpha\beta(+)} + L_{s, t, p}^{\beta\alpha(+)} \right) + (-)^{t-1} \left(L_{s, t, p}^{\alpha\beta(-)} + L_{s, t, p}^{\beta\alpha(-)} \right),$$

with

$$L_{s, t, p}^{\alpha\beta(\pm)} = \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q$$

$$\left\{ \begin{aligned} & \theta(s-2p-q) \frac{(s-1-2p-q)!}{(\mu_\beta)^{s-2p-q}} \sum_{r=0}^{s-1-2p-q} \frac{(\mu_\beta)^r}{r!} E_{s-q-r}(\mu_\alpha + \mu_\beta) \\ & + \theta(q+2p+1-s) \left\{ \begin{aligned} & \frac{\theta(q+2p-s)}{(q+2p-s)!} \sum_{r=1}^{q+2p-s} (q+2p-s-r)! (-\mu_\beta)^{r-1} \\ & \left\{ \begin{aligned} & \theta(r-2p)\alpha_{r-1-2p}(\mu_\alpha + \mu_\beta) \\ & + \theta(2p+1-r)E_{2p+1-r}(\mu_\alpha + \mu_\beta) \end{aligned} \right\} \end{aligned} \right\} \\ & + \frac{(-\mu_\beta)^{q+2p-s}}{(q+2p-s)!} \left\{ \begin{aligned} & \theta(q-s+1)D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta) \\ & + \theta(s-q)D_{s-q,1}^{(2)}(\mu_\alpha, \mu_\beta) \end{aligned} \right\} \end{aligned} \right\} \quad (6)$$

The function $D_n(\alpha, \bar{\beta})$ is given by Eqs. (20) and (23) in Appendix 12, for the case $n = 1$, as

$$D_1(\alpha, \bar{\beta})$$

$$= \left\{ \begin{array}{l} \left\{ -\frac{1}{2} \left[\frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right] - (\gamma + \ln \bar{\beta}) E_1(\alpha) \right\} \\ \left\{ - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) \right\} \end{array} \right\} \text{ if } |\bar{\beta}/\alpha| < 1$$

$$\left\{ \begin{array}{l} \left\{ \frac{1}{2} \left[\frac{\pi^2}{6} + (\gamma + \ln \bar{\beta})^2 \right] + (\gamma + \ln \alpha + E_1(\alpha)) E_1(\bar{\beta}) \right\} \\ \left\{ + \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}) \right\} \end{array} \right\} \text{ if } |\alpha/\bar{\beta}| < 1$$

(7)

and for $n \geq 2$ as

$$D_n(\alpha, \bar{\beta}) = \frac{(-\alpha)^{n-1}}{(n-1)!} D_1(\alpha, \bar{\beta})$$

$$+ \theta(n-1) \sum_{p=1}^{n-1} \frac{(\alpha)^{n-1-p} (p-1)!}{(n-1)!} [e^{-\alpha} E_1(\bar{\beta}) - E_{p+1}(\alpha + \bar{\beta})] .$$

Also, according to Eq. (4) in Appendix 12, $D_0(\alpha, \bar{\beta}) = \frac{1}{\alpha} [e^{-\alpha} e_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})]$ [$= D_{s-q}(\mu_\alpha, \mu_\beta)$ when $s \geq q$]*. Finally, the function

$D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta)$ is determined by Eq. (27) in Appendix 12 to be

$$D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta) = \alpha_{q-s}(\mu_\alpha) E_1(\mu_\beta) - \frac{(q-s)!}{(\mu_\alpha)^{q-s+1}} E_1(\mu_\alpha + \mu_\beta) - \sum_{r=1}^{q-s} \frac{(q-s)! e^{-(\mu_\alpha + \mu_\beta)} e_{r-1}(\mu_\alpha + \mu_\beta)}{r(\mu_\alpha)^{q-s+1-r} (\mu_\alpha + \mu_\beta)^r}, \quad (8)$$

where $\alpha_{-1}(z) = E_1(z)$ and

$$\begin{aligned} \alpha_n(z) &\equiv \int_1^\infty dt e^{-zt} t^n, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \\ &= \frac{n!}{z^{n+1}} e^{-z} e_n(z), \\ e_n(z) &\equiv \sum_{p=0}^n \frac{z^p}{p!}. \end{aligned} \quad (9)$$

*See Eqs. (2) in Appendix 13 and (1) in Appendix 12.

(3) Sec. I.2, Eq. (26): Appendix 11, Eq. (28)

$$\begin{aligned}
 F_l^\alpha(r) &= \int_0^\infty dk \, g_l^\alpha(k) F_l'(kr) P\left(\frac{1}{k^2 - k_0^2}\right) \\
 &= \frac{\pi}{2} \frac{a}{\tau_0} [\varepsilon_l^\alpha(\tau_0) G_l'(\tau_0 x) - x \Delta_l(\tau_0 x) g_l^\alpha(\tau_0 x)] .
 \end{aligned}
 \tag{10}$$

with $x = r/a$,

$$G_l'(\tau x) = \frac{G_l(\tau x) G_l(\tau) + F_l(\tau x) F_l(\tau)}{\Delta_l(\tau)}
 \tag{11}$$

and $\alpha = C, T$.

(4) Sec. I.3

All the functions are defined in terms of previously determined functions.

(5) Sec. I.4, Eq. (13): Appendix 10.

The functions

$$B_l^{\alpha\beta}(\gamma) \equiv \frac{2\sigma_{\alpha}^r}{\pi} K_l^{\alpha\beta}(i\gamma) - \frac{1}{\sqrt{2}} \delta_{l,0} \delta_{\alpha\beta} \delta_{\alpha,T} \quad (12)$$

and

$$C_l^{\alpha\beta}(\gamma) \equiv - \frac{\pi}{4\gamma\sigma_{\alpha}^r} \frac{\partial}{\partial\gamma} (B_l^{\alpha\beta}(\gamma)),$$

are given by Eq. (9) - (25) of Appendix 10.

(6) Sec. I.5, Eq. (21): Appendix 13, Eqs. (30), (24), (33), (37), (38)

The functions $D_{n,s}^{(1)}(\Lambda)$, $D_{n,s}^{(2)}(\Lambda_1, \Lambda_2)$, $D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3)$, and $D_n^{(4)}(\Lambda_1, \Lambda_2)$ are displayed as integrals in Appendix 13, Eq. (1) - (4) and as solutions in Eq. (13), (15) - (17), (21) - (24), (30), (36), and (37). It should be noted that the coefficient of the

function $D_{-1}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3)$ in the deuteron quadrupole moment is zero and that function is not evaluated.

(7) Sec II.1, Eq. (12): Appendix 5.

The coefficients $a_n^{(\alpha)}(L)$ for $n = 0, 2, 4$ and $L = 0, 1, 2$ are given by Eq. (4), (5), (28), (41), (38), (37), and (40) in Appendix 5. For $L = 0$, the results are summarized in Eqs. (44)-(46).

PART II: PARAMETER FITTING

II.1 Beginning Series for the Scattering Length, Effective Range, and Shape-Dependent Parameter

The low-energy expansion of $k \cot \bar{\delta}(k)$ for the triplet "S" states is given, and considering the Stapp phase shift convention, writing $\bar{\delta}_{1,0}(k) = \bar{\delta}_0(k)$, by

$$k \cot \bar{\delta}_0(k) = -\frac{1}{3} \frac{1}{a} + \frac{1}{2} {}^3r_e k^2 - {}^3P({}^3r_e)^3 k^4 + O(k^6) \quad (1)$$

where 3a (the triplet scattering length), 3r_e (the triplet effective range), and 3P (the triplet shape-dependent parameter) may be determined from a low- k expansion of the functions derived in Sec. I.3.

There we determined with Eq. (45) that

$$\bar{\delta}_0(k) = \delta_0^1(k) + \delta_0(k) \quad (2)$$

where $\delta_0^1(k) = -ka$. Now upon using Eq. (2) we get

$$k \cot \bar{\delta}_0(k) = \frac{1 - \tan \delta_0^1(k) \tan \delta_0(k)}{k^{-1} \tan \delta_0^1(k) + k^{-1} \tan \delta_0(k)} \quad (3)$$

where (see Eq. (48) in Sec. I.3)

$$\frac{1}{2} \tan 2\delta_0(k) = \frac{T_2(k) [1 + T_3^2(k)] - T_1^2(k) T_3(k)}{1 + T_3^2(k) - T_2^2(k) - [T_1^2(k) - T_2(k)T_3(k)]^2}, \quad (4)$$

and (see footnote below Eq. 137) in Sec. I.3) for $J = 1$

$$T_1(k) \xrightarrow[k \rightarrow 0]{} O(k^3),$$

$$T_2(k) \xrightarrow[k \rightarrow 0]{} O(k), \quad (5)$$

$$T_3(k) \xrightarrow[k \rightarrow 0]{} O(k^5).$$

Using the limiting forms of Eq. (5) in Eq. (4), we find

$$\tan \delta_0(k) \xrightarrow[k \rightarrow 0]{} T_2(k) [1 + O(k^{10})], \quad (6)$$

and Eq. (3) becomes

$$k \cot \bar{\delta}_0(k) \cong \frac{1 + ka \left[1 + \frac{1}{3} (ka)^2 \right] T_2(k)}{k^{-1} T_2(k) - a \left[1 + \frac{1}{3} (ka)^2 + \frac{2}{15} (ka)^4 \right]} \quad (7)$$

Now, according to the second of Eqs. (19) of Sec. I.3, for the case

$J = 1$ and writing $\sigma_\alpha^J = \sigma_\alpha$,

$$T_2(k) = \frac{1}{k} \left\{ \frac{\sigma_c r_c}{1 - A_0^{(c)}(k)} \left(g_0^{(c)}(k) \right)^2 + \frac{\sigma_T r_T \left(\bar{g}_0(k) \right)^2 D_0(k)}{1 - D_2(k) \left(D_0(k) - \frac{1}{\sqrt{2}} \right)} \right\}, \quad (8)$$

where the second and third of Eqs. (45) in Sec. I.2 have also been used. To proceed we need Eq. (49) and (50) of Sec. I.2, which are

$$D_L(k) = A_L^{(T)}(k) + \frac{\sigma_T r_T}{\sigma_c r_c} \left(A_L^{(C,T)}(k) \right)^2 / \left(1 - A_L^{(c)}(k) \right), \quad (9)$$

and

$$\bar{g}_L(k) = g_L^{(T)}(k) + g_L^{(C)}(k) A_L^{(C,T)}(k) / [1 - A_L^{(C)}(k)]. \quad (10)$$

We also require Eq. (18) from Appendix 1, which can be expanded for low wave numbers as follows, with $\lambda_\alpha = r_\alpha^{-1}$,

$$g_0^{(\alpha)}(k) = \frac{k}{1 + k^2 r_\alpha^2} = k [1 - k^2 r_\alpha^2 + k^4 r_\alpha^4 + O(k^6)] \quad (11)$$

We next define the low-k expansion of $A_L^{(\alpha)}(k)$ by

$$A_L^{(\alpha)}(k) = A_0^\alpha(L) + A_2^\alpha(L)k^2 + A_4^\alpha(L)k^4 + O(k^6) \quad (12)$$

where $\alpha = C, T$, or (C, T) . Upon substituting Eq. (12) into (9),

$$\text{we obtain } D_L(k) = d_0^{(L)} + d_2^{(L)}k^2 + d_4^{(L)}k^4 + O(k^6), \quad (13)$$

where

$$d_0^{(L)} = a_0^{(T)}(L) + \frac{\sigma_T r_T}{\sigma_C r_C} \frac{[a_0^{(C,T)}(L)]^2}{1 - a_0^{(C)}(L)}$$

$$d_2^{(L)} = a_2^{(T)}(L) + \frac{\sigma_T r_T}{\sigma_C r_C} \frac{[a_0^{(C,T)}(L)]^2}{1 - a_0^{(C)}(L)} \left(2 \frac{a_2^{(C,T)}(L)}{a_0^{(C,T)}(L)} + \frac{a_2^{(C)}(L)}{1 - a_0^{(C,T)}(L)} \right)^2$$

$$d_4^{(L)} = a_4^{(T)}(L) + \frac{\sigma_T r_T}{\sigma_C r_C} \frac{[a_0^{(C,T)}(L)]^2}{1 - a_0^{(C)}(L)} \times$$

$$\times \left\{ \frac{2 a_4^{(C,T)}(L)}{a_0^{(C)}(L)} + \frac{a_4^{(C)}(L)}{1 - a_0^{(C)}(L)} + \left[\left(\frac{a_2^{(C)}(L)}{1 - a_0^{(C)}(L)} \right)^2 + \left(\frac{a_2^{(C,T)}(L)}{a_0^{(C)}(L)} \right)^2 \right] \right\}$$

$$\left. + 2 \frac{a_2^{(c,t)}(L)}{a_0^{(c)}(L)} \cdot \frac{a_2^{(c)}(L)}{1-a_0^{(c)}(L)} \right]^2 \Bigg\}. \quad (14)$$

Using the notation of Eq. (13) we may express the second denominator of Eq. (12) as

$$\frac{1}{1 - D_2(k) (D_0(k) - 1/\sqrt{2})} = \frac{1}{1 - d_0^{(2)} d_0^{(c)'} } \left\{ 1 + q_2 k^2 + (q_4 + q_2^2) k^4 + O(k^6) \right\} \quad (15)$$

where $d_0^{(o)'} \equiv (d_0^{(o)} - 1/\sqrt{2})$ and

$$q_2 \equiv d_2^{(2)} d_0^{(o)'} + d_2^{(o)} d_0^{(2)} / (1 - d_0^{(2)} d_0^{(o)'}),$$

$$q_4 \equiv (d_4^{(2)} d_0^{(o)'} + d_4^{(o)} d_0^{(2)} + d_2^{(2)} d_2^{(o)}) / (1 - d_0^{(2)} d_0^{(o)'}). \quad (16)$$

Then

$$\frac{D_0(k)}{1 - D_2(k) (D_0(k) - 1/2)} = \frac{d_0^{(o)}}{1 - d_0^{(2)} d_0^{(o)'}} \left\{ 1 + \left(q_2 + \frac{d_2^{(o)}}{d_0^{(o)'}} \right) k^2 + \left(q_4 + q_2^2 + \frac{d_4^{(o)}}{d_0^{(o)'}} + q_2 \frac{d_2^{(o)}}{d_0^{(o)'}} \right) k^4 + O(k^6) \right\}. \quad (17)$$

Upon substituting Eqs. (11) and (12), we obtain

$$\frac{1}{k} \bar{g}_0(k) = \frac{(1 + a_0^{(c,T)}(o))}{1 - a_0^{(o)}(o)} (1 + g^{(2)} k^2 + g^{(4)} k^4) \quad (18)$$

where

$$g^{(2)} = \left[\frac{a_o^{(c,T)}(o)}{1-a_o^{(c)}(o)} \left(\frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} + \frac{a_2^{(c,T)}(o)}{a_o^{(c,T)}(o)} - r_c^2 \right) - r_T^2 \right] \left(1 + \frac{a_o^{(c,T)}(o)}{1-a_o^{(c)}(o)} \right)^{-1}$$

$$g^{(4)} = \left[r_T^4 + \frac{a_o^{(c,T)}(o)}{1-a_o^{(c)}(o)} \left\{ \frac{a_2^{(c,T)}(o)}{a_o^{(c,T)}(o)} \cdot \frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} \right. \right.$$

$$\left. \frac{a_4^{(c,T)}(o)}{d_o^{(c,T)}(o)} + \frac{a_4^{(c)}(o)}{1-a_o^{(c)}(o)} + \left(\frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} \right)^2 \right.$$

$$\left. \left. - r_c^2 \left(\frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} + \frac{a_2^{(c,T)}(o)}{a_o^{(c,T)}(o)} \right) \right] \left(1 - \frac{a_o^{(c,T)}(o)}{1-a_o^{(c)}(o)} \right)^{-1}$$

(19)

Finally, upon substituting Eqs. (11), (12), (17) and (19) into Eq. (8) we obtain for the triplet function $T_2(k)$, the low- k expansion

$$\frac{1}{k} T_2(k) = T_o^{(2)} + T_2^{(2)} k^2 + T_4^{(2)} k^4 + O(k^6), \quad (20)$$

where

$$T_0^{(2)} \equiv \frac{\sigma_c r_c}{1-a_o^{(c)}(o)} + \frac{\sigma_{T T d_o^{(o)}}}{1-d_o^{(2)} d_o^{(o)}} \left(1 + \frac{a_o^{c, (o)}}{1-a_o^{(c)}(o)} \right)^2,$$

$$T_2^{(2)} \equiv \frac{\sigma_c r_c}{1-a_o^{(c)}(o)} \left(\frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} - 2r_c^2 \right)$$

$$+ \frac{\sigma_{T T d_o^{(o)}}}{1-d_o^{(2)} d_o^{(o)}} \left(\frac{1+a_o^{(c, \tau)}(o)}{1-a_o^{(c)}(o)} \right)^2 \left(q_2 + \frac{d_2^{(o)}}{d_o^{(o)}} + 2g^{(2)} \right)$$

$$T_4^{(2)} \equiv \frac{\sigma_c r_c}{1-a_o^{(c)}(o)} \left(\frac{a_4^{(o)}(o)}{1-a_o^{(c)}(o)} + \frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} \right)^2$$

$$- 2r_c^2 \frac{a_2^{(c)}(o)}{1-a_o^{(c)}(o)} + 3r_c^4$$

$$+ \frac{\sigma_{T T d_o^{(o)}}}{1-d_o^{(2)} d_o^{(o)}} \left(1 + \frac{a_o^{(c, \tau)}(o)}{1-a_o^{(c)}(o)} \right)^2 \times$$

$$\left(2g^{(4)} + (g^{(2)})^2 + q_4 + (q_2)^2 + d_4^{(o)}/d_o^{(o)} + \right. \\ \left. q_2 d_2^{(o)}/d_o^{(o)} + g^{(2)} (q_2 + d_2^{(o)}/d_o^{(o)}) \right)$$

Substitution of Eq. (20) into Eq. (7) then gives Eq. (1),

$$k \cot \delta_o(k) = -\frac{1}{3_a} + \frac{1}{2} {}^3r_e k^2 - {}^3p({}^3r_e) {}^3k^4 + O(k^6), \quad (22)$$

where the effective range parameters are given explicitly by

$${}^3a/a \equiv 1 - T_o^{(2)}/a,$$

$${}^3r_e/a \equiv 2 [-T_2^{(2)}/a^3 + 1/3 - (1 - {}^3a/a) {}^3a/a] / ({}^3a/a)^2,$$

$$3p({}^3r_e/a)^3 \equiv (T_4^{(2)}/a^5 - 2/15) / ({}^3a/a)^2 + 1/2 ({}^3r_e/a)$$

$$(1 + 1/2 \frac{{}^3a}{a} \frac{{}^3r_e}{a} - 2 \frac{{}^3a}{a} + \frac{2}{3} \frac{{}^3a}{a} + {}^3a/a - 4/3) \quad (23)$$

In an analogous manner, the low-energy expansion for the singlet S states is given, writing $\delta_{o,o}^{(o)}(k) = \delta_o^{(o)}(k)$, by

$$k \cot \delta_o^{(o)}(k) = -\frac{1}{1_a} + 1/2 {}^1r_e k^2 - {}^1p({}^1r_e) {}^3k^4 + O(k^6), \quad (24)$$

where 1a (the singlet scattering length), 1r_e (the singlet effective range), and 1p (the singlet shape-dependent parameter) may be determined from the low-momentum expansion of functions derived in Section I.3. There we determined by Eq. (4) that

$$\delta_o^{(0)}(k) = \delta_o^{(1)}(k) + \delta_o^{(0)}(k) \quad (25)$$

where $\delta_o^{(1)}(k) = -ka$ and for the singlet $J = 0$ state (see Eqs. (21), (23), and (29) of Section 1.2)

$$\begin{aligned} \tan \delta_o^{(\sigma)}(k) &= \frac{\sigma_c r_c}{k} g_o^{(c)}(k) B_c^{(0,0)}(k) \\ &= \frac{\sigma_c r_c}{k} (g_o^{(c)}(k))^2 / (1 - A_o^{(c)}(k)). \end{aligned} \quad (26)$$

Upon using Eqs. (11) and (12) we find for the low momentum expansion of Eq. (26) the result

$$\tan \delta_o^{(\sigma)}(k) = k [T_o^{(0)} + T_2^{(0)} k^2 + T_4^{(0)} k^4 + O(k^6)] , \quad (27)$$

where

$$\begin{aligned} T_o^{(0)} &\equiv \frac{\sigma_c r_c}{1 - a_o^{(c)}(0)} , \\ T_2^{(0)} &\equiv \frac{\sigma_c r_c}{1 - a_o^{(c)}(0)} \left(\frac{a_2^{(c)}(0)}{1 - a_o^{(c)}(0)} - 2 r_c^2 \right) , \\ T_4^{(0)} &\equiv \frac{\sigma_c r_c}{1 - a_o^{(c)}(0)} \left(\frac{a_4^{(c)}(0)}{1 - a_o^{(c)}(0)} + \left(\frac{a_2^{(c)}(0)}{1 - a_o^{(c)}(0)} \right)^2 \right. \\ &\quad \left. - 2 r_c^2 \frac{a_2^{(c)}(0)}{1 - a_o^{(c)}(0)} + 3 r_c^4 \right) . \end{aligned} \quad (28)$$

Since Eq. (7) applies exactly for the singlet case, with $T_2(k) \rightarrow \tan \delta_0^{(\sigma)}(k)$, we obtain immediately from Eq. (23) the results

$$1_{a/a} \equiv 1 - T_0^{(o)}/a \quad ,$$

$$1_{r_e/a} \equiv 2 \left[-T_2^{(o)}/a^3 + 1/3 - (1 - 1_{a/a}) 1_{a/a} \right] / (1_{a/a})^2 \quad ,$$

$$1_P (1_{r_e/a})^3 \equiv (T_4^{(o)}/a^5 - 2/15) / (1_{a/a})^3$$

$$+ 1/2 \frac{1_r}{a} \left(1 + 1/2 \frac{1_a}{a} \frac{1_{r_e}}{a} - 2 \frac{1_a}{a} \right.$$

$$\left. + \frac{2}{3} \frac{1_a}{a} + 1_{a/a} - 4/3 \right.$$

The only unknown functions in this analysis of the low-energy expansion of $k \cot \delta(k)$, for the singlet and triplet scattering length, effective range, and shape-dependent parameters, are the $a_n^{(\alpha)}(L)$ of Eq. (12) with $n = 0, 2, 4$. These functions have been determined explicitly in Appendix 5.

II.2 Function Evaluation by Continued Fractions

Several functions and infinite series of a complex variable occur in the results of this paper. These may all be evaluated readily on a computer by using continued fractions in the domain of convergence and with standard methods.^{1,2}

As an illustration of the techniques involved,¹ we demonstrate explicitly the construction of the two basic types of formulation required for digital computation: (1) The formulation of contracted continued fractions; and (2) The computation of an analytic (convergent) infinite series by continued fractions. The application of these two techniques to functions in this thesis is straightforward.

Contracted Continued Fractions - The Exponential Integral

The exponential integral, $E_n(z)$, is defined by

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \quad (1)$$

and is a single-valued function in the complex z -plane cut along the negative real axis.

The continued fraction representation of Eq. (1) may be written as²

$$e^z E_n(z) = \frac{1}{z+} \frac{n}{1+} \frac{1}{z+} \frac{n+1}{1+} \frac{2}{z+} \dots + \frac{p}{z+} \frac{n+p}{1+} \dots \quad (2)$$

¹Khovanskii, A. N., THE APPLICATION OF CONTINUED FRACTIONS AND THEIR GENERALIZATIONS TO PROBLEMS IN APPROXIMATION THEORY, P. Nordhoff Press, Groningen, Holland, 1965.

²M. Abramowitz and I. A. Stegun, eds., HANDBOOK OF MATHEMATICAL FUNCTIONS WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES, Dover Publications, New York, 1965.

Equation (2) may be expressed in standard form as¹

$$e^z E_n(z) = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_p}{b_p} + \frac{a_{p+1}}{b_{p+1}} + \dots \quad (3)$$

Comparison of Eqs. (2) and (3) yields the coefficient identification

$$\begin{aligned} a_1 &= 1; a_{2p} = n+p-1; a_{2p+1} = p \text{ for } p > 0, \\ b_0 &= 0; b_{2p} = 1; b_{2p+1} = z \text{ for } p \geq 0. \end{aligned} \quad (4)$$

The p^{th} approximant (convergent) of Eq. (2) to Eq. (1) for the exponential integral would coincide with the first p terms of a series. However, the contracted (condensed) form¹ has the property that the p^{th} approximant coincides with the first $2p$ terms of the same series and converges monotonically (not oscillatorily) to the function, provided, of course, the standard form is convergent. These properties have definite computational advantages in some instances. In the case of the exponential integral a reduction in computing time by as much as a factor 30 can be realized with the contracted form.

The p^{th} convergent of a continued fraction is given as the ratio of two polynomials, p_p and Q_p , say*

*Equation (5) is sometimes written in the form

$$\frac{p_p}{Q_p} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

$$\frac{P_p}{Q_p} \equiv b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \frac{a_{p-1}}{b_{p-1} +} \frac{a_p}{b_p} , \quad (5)$$

where the polynomials P_p and Q_p satisfy the recursion relations¹

$$P_p = b_p P_{p-1} + a_p P_{p-2} , \quad (6)$$

$$Q_p = b_p Q_{p-1} + a_p Q_{p-2} ,$$

with the coefficients a_p and b_p determined from Eq. (3). Using Eqs. (3) and (6), the first few terms take the form

$$P_{-1} = 1; Q_{-1} = 0; P_0 = \frac{b_0}{1}; Q_0 = 1; \frac{P_1}{Q_1} = \frac{b_0 b_1 + a_1}{b_1}; \text{etc.} \quad (7)$$

We now follow the derivation of reference (1) and construct the contracted form of Eq. (3). Let

$$K_p = P_{2p}/Q_{2p} , \quad (8)$$

and use the first of Eqs. (6) (rewritten for the indices $2p$, $2p-1$, and $2p-2$) to form the quantity

$$b_{2p-2} P_{2p} + b_{2p} b_{2p-2} P_{2p-1} - a_{2p-1} b_{2p} P_{2p-2},$$

which results in the recursion relation

$$b_{2p-2}^p P_{2p} = ((a_{2p} + b_{2p-1} b_{2p}) b_{2p-2} + a_{2p-1} b_{2p})^p P_{2p-2} - a_{2p-2} a_{2p-1} b_{2p}^p P_{2p-4} . \quad (9)$$

Upon comparing Eqs. (6) and (9) it is clear that the coefficients of the contracted form are given (for $p > 1$) by

$$\left. \begin{aligned} a'_p &\equiv \text{coefficient of } P_{2p-4} = - \frac{a_{2p-2} a_{2p-1} b_{2p}}{b_{2p-2}} \\ b'_p &\equiv \text{coefficient of } P_{2p-2} = \frac{(a_{2p} + b_{2p-1} b_{2p}) b_{2p-2} + a_{2p-1} b_{2p}}{b_{2p-2}} \end{aligned} \right\} , \quad (10)$$

and therefore Eq. (8) takes for form (with $a'_1 = a_1 b_2$ and $b'_1 = b_1 b_2 + a_2$)

$$K_p = b_0 + \frac{a_1 b_2}{b_1 b_2 + a_2} - \frac{a_2 a_3 b_4}{((b_2 b_3 + a_3) b_4 + b_2 a_4)} - \dots \quad (11)$$

$$- \frac{a_{2p-2} a_{2p-1} b_{2p}}{((b_{2p-2} b_{2p-1} + a_{2p-1}) b_{2p} + b_{2p-2} a_{2p})} ,$$

or

$$K_p = b_0 + \frac{a'_1}{b'_1} + \frac{a'_2}{b'_2} + \dots + \frac{a'_p}{b'_p} . \quad (12)$$

Upon applying this contraction process to the exponential-integral

representation of Eq. (4), we obtain with the aid of Eq. (10)

$$\left. \begin{aligned} b_0 &= 0; \quad a'_1 = 1; \quad b'_1 = z + n \\ a'_p &= -(n+p-2)(p-1) \\ b'_p &= z+n+2(p-1) \end{aligned} \right\} \text{ for } p > 1 \quad . \quad (13)$$

We now drop the prime notation in Eq. (13) and use the contracted form exclusively for the exponential integral.

The formulation for continued fractions developed thus far has the following properties: (a) The p^{th} convergent (approximate) corresponds exactly to $2p$ terms of the original continued fraction; and, (b) The convergents monotonically approach a limit (provided, of course, the original continued fraction was convergent).

Now we need to develop an efficient evaluation technique for use on digital computers. An easy way to evaluate the sum of an infinite sequence of terms until the remainder is less than some preselected small quantity is as follows: Perform the sum in such a manner that each successively contributing term is related to a previous term recursively, and start at the beginning of the series.

It can be deduced that¹

$$\frac{P_p}{Q_p} - \frac{P_{p-1}}{Q_{p-1}} = \frac{(-)^{p+1} \prod_{q=2}^p a_q}{Q_{p-1}Q_p} \quad . \quad (14)$$

using the trivial identity

$$\frac{P_p}{Q_p} = \sum_{r=2}^p \left(\frac{P_r}{Q_r} - \frac{P_{r-1}}{Q_{r-1}} \right) + \frac{P_1}{Q_1} \quad (15)$$

in Eq. (14), we have (upon noting that $P_1/Q_1 = \frac{a_1}{b_1} = 1/Q_1$) that

$$\frac{P_p}{Q_p} = \frac{1}{Q_1} + \sum_{r=2}^p \frac{(-)^{r+1} \left(\prod_{q=2}^r a_q \right) \left(\prod_{q=0}^{r-2} Q_q \right)}{\left(\prod_{q=1}^r Q_q \right)}, \quad (16)$$

where $Q_0 = 1$.

Upon defining the quantity T_r by

$$T_r = -T_{r-1} a_r \frac{Q_{r-2}}{Q_r} \quad (17)$$

and using an induction argument, we find

$$T_r = (-)^{s-1} T_{r-s-1} \frac{\left(\prod_{q=0}^s a_{r-q} \right) \left(\prod_{q=2}^{s+2} Q_{r-q} \right)}{\left(\prod_{q=0}^s Q_{r-q} \right)},$$

which can be written for $s = r-2$ as

$$T_r = (-)^{r-1} T_1 \frac{\left(\prod_{q=0}^{r-2} a_{r-q} \right) \left(\prod_{q=2}^r Q_{r-q} \right)}{\left(\prod_{q=0}^{r-2} Q_{r-q} \right)}. \quad (18)$$

After noting that $T_1 = 1/Q_1$, we change indices in Eq. (18) and obtain the result,

$$T_r = (-)^{r+1} \frac{\left(\prod_{q=2}^r a_q \right) \left(\prod_{q=0}^{r-2} Q_q \right)}{\left(\prod_{q=1}^r Q_q \right)} . \quad (19)$$

Using Eq. (19) we may rewrite Eq. (16) as

$$\frac{P}{Q_P} = \sum_{r=1}^P T_r , \quad (20)$$

where

$$\left. \begin{aligned} T_1 &= 1/Q_1 \\ T_r &= -T_{r-1} a_r \frac{Q_{r-2}}{Q_r} \text{ for } r > 1 \end{aligned} \right\} \quad (21)$$

for the p^{th} convergent.

One further algebraic transformation is sometimes desirable (as is the case for the exponential integral) in order that loss of significance is not suffered in the computation of Eq. (21). The term $a_r Q_{r-2}/Q_r$ can become very large as r goes to infinity (for the exponential integral $|a_r| \xrightarrow{r \rightarrow \infty} r^2$). This difficulty can be circumvented by the following algebraic transformation. Using the second of Eqs. (6), we may write

$$\frac{a_r Q_{r-2}}{Q_r} = \frac{1}{\frac{Q_r}{a_r Q_{r-2}}} = \frac{1}{1 + \frac{b_r Q_{r-1}}{\frac{a_r Q_{r-2}}{a_r Q_{r-2}}}} \quad . \quad (22)$$

We then define the coefficients q_r by

$$\begin{aligned} q_1 &= Q_1, \\ q_r &\equiv \frac{Q_r}{Q_{r-1}} = b_r + \frac{a_r}{q_{r-1}} \quad \text{for } r > 1, \end{aligned} \quad (23)$$

in terms of which Eq. (22) becomes

$$\frac{a_r Q_{r-2}}{Q_{r-1}} = \frac{1}{1 + \frac{b_r}{\frac{a_r}{q_{r-1}}}} \quad . \quad (24)$$

Computational Procedures

Equations (20), (21), (23), and (24) constitute the computational technique for evaluation of a contracted continued fraction. The specific application to the exponential integral can be accomplished as follows: ℓ is a counter to guarantee that the convergence criteria has been met three times

"Initialization"

$$a_1 = 1; b_1 = z+n; q_1 = z+n; T_1 = 1/q_1; T_1 \rightarrow \text{Sum}; l = 1$$

"Computation Loop"

for $p = 2, 3, \dots, \text{infinity}$

$$a_p = -(n+p-2)(p-1)$$

$$b_p = z+n+2(p-1)$$

$$T_p = -T_{p-1}/(1+b_p q_{p-1}/a_p)$$

$$q_p = b_p + a_p/q_{p-1}$$

$$T_p + \text{Sum} \rightarrow \text{Sum}$$

"Convergence Criteria"

$$\text{if } \text{Re}(T_p/\text{Sum}) < \epsilon \text{ and } \text{Im}(T_p/\text{Sum}) < \epsilon$$

$$l + 1 \rightarrow l$$

if $l \geq 4$: exit from loop

"Exiting Computation"

$$E_n(z) = e^{-z} \text{Sum}.$$

(25)

Typically, we set $\epsilon \sim 10^{-3}$ to 10^{-10} on the MANIAC computer.

Finally we review the conditions for convergence of continued fractions, with specific emphasis on the exponential integral.

Since the contracted form of the continued fraction reproduces the standard form which has all positive coefficients, we use¹

"Theorem: The divergence of at least one of the series

$$\sum_{n=1}^{\infty} \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_4 \dots a_{2n}} b_{2n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{a_2 a_4 \dots a_{2n}}{a_1 a_3 \dots a_{2n+1}} b_{2n+1} \quad (26)$$

is a NASC for the convergence of a continued fraction with positive coefficients.

From Eq. (4), we have

$$\begin{aligned} a_1 a_3 \dots a_{2p-1} &= (p-1)! , \\ a_2 a_4 \dots a_{2p} &= (n+p-1)!/(n-1)! . \end{aligned} \quad (27)$$

Since $b_{2p} = 1$, the first series of Eq. (26) becomes

$$\sum_{n=1}^{\infty} \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_4 \dots a_{2n}} b_{2n} = (n-1)! \sum_{p=1}^{\infty} \frac{(p-1)!}{(n+p-1)!} , \quad (28)$$

and the second series if

$$\sum_{n=1}^{\infty} \frac{a_2 a_4 \dots a_{2n}}{a_1 a_3 \dots a_{2n+1}} b_{2n+1} = \frac{z}{(n-1)!} \sum_{p=1}^{\infty} \frac{(n+p-1)!}{p!} , \quad (29)$$

where $b_{2p+1} = z$. If we denote the p^{th} term in the series (29) by

$$c_p = (n+p-1)!/p! \quad (30)$$

and use the ratio test,

$$\left| \frac{c_{p+1}}{c_p} \right| = \left| 1 + \frac{n-1}{p+1} \right| \cong 1, \quad (31)$$

then we see that this series is divergent, thereby assuring the convergence of the continued fraction for the exponential integral.

The domain of convergence may be investigated using a theorem by Khovanskii:¹

"Theorem: The condition $\lim_{v \rightarrow \infty} c_v = c \neq 0$ is sufficient for the uniform convergence of the continued fraction

$$\left[\frac{c_1}{1}; \frac{c_v}{1} \right]_2^\infty \equiv \frac{c_1}{1+} \frac{c_2}{1+} \frac{c_3}{1+} \dots \quad (32)$$

with the exception of a finite number of points of inessential divergence, in any domain $T \subset \bar{T}$, where $\bar{T}$ is the complex z -plane cut along that segment of the line $\arg(z) = -\arg(c)$ which joins the point $-(4c)^{-1}$ and the point at infinity and which does not pass through the origin."

A continued fraction of the form of Eq. (3) can be transformed¹ to the form of Eq. (32) (when $b_0 = 0$) by setting

$$c_1 = \frac{a_1}{b_1} \quad \text{and} \quad c_p = \frac{a_p}{b_{p-1} b_p} \quad \text{for } p \geq 2. \quad (33)$$

Upon substituting Eq. (13) for the contracted coefficients of the exponential integral into Eq. (33), we find

$$c_p = \frac{+(n+p-2)(p-1)}{(z+n+2(p-2))(z+n+2(p-1))}, \quad (34)$$

which has the limit

$$\lim_{p \rightarrow \infty} c_p \equiv c = \frac{1}{4}. \quad (35)$$

Therefore, the condensed continued fraction representation for the exponential integral converges everywhere in the complex plane except along the strip

$$-\infty < z < -\frac{1}{4c} (= -1) \quad (36)$$

which is the same domain as the definition of the exponential integral given in Eq. (1).

Infinite Convergent Series

Consider the infinite convergent series

$$S(z) = \sum_{n=0}^{\infty} c_n z^n, \quad (37)$$

where we shall need to construct a continued fraction representation which reproduces Eq. (37) to a specified predetermined accuracy.

Using the Euler identity,¹ we may write

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} c_n z^n \\ &= c_0 + \frac{c_2 z}{1 - \frac{c_1}{1 + \frac{c_2 z}{c_1}} - \frac{c_3 z}{1 + \frac{c_2 z}{c_1}} - \dots - \frac{c_n z}{1 + \frac{c_2 z}{c_1}} - \dots} \end{aligned} \quad (38)$$

which also can be written as

$$S(z) = c_0 + \frac{c_1 z}{1} - \frac{c_2 z}{c_1 + c_2 z} - \frac{c_1 c_3 z}{c_2 + c_3 z} - \dots - \frac{c_{n-2} c_n z}{c_{n-1} + c_n z} - \dots \quad (39)$$

For the series of interest in this thesis $c_0 = 0$, and using the notation of Eq. (3), Eq. (38) has the coefficient identification given by

$$\left. \begin{aligned} b_0 &= c_0 = 0 \\ a_1 &= c_1 z; \quad a_p = -\frac{c_p z}{c_{p-1}} \quad \text{for } p \geq 2 \\ b_1 &= 1; \quad b_p = 1 + \frac{c_p z}{c_{p-1}} \quad \text{for } p \geq 2 \end{aligned} \right\} . \quad (40)$$

Similarly, Eq. (39) leads to

$$\left. \begin{aligned} b_0 &= c_0 = 0 \\ a_1 &= c_1 z; \quad a_2 = -c_2 z; \quad a_p = -c_{p-2} c_p z \quad \text{for } p > 2 \\ b_1 &= 1; \quad b_p = c_{p-1} + c_p z \quad \text{for } p > 2 \end{aligned} \right\} . \quad (41)$$

The choice of which representation is most convenient of Eqs. (38) or (39) depends on the form of the coefficients c_n .

If we consider the series

$$S(z) \equiv \sum_{n=1}^{\infty} \frac{(-z)^n}{n^2 n!} \quad (42)$$

and identify $c_n = (-)^n / n^2 n!$, it is convenient to use Eqs. (38) and

(40). Upon using Eqs. (40), we find quite generally, from the second of Eqs. (6), that $Q_n = 1$ for all n . For the special case of series (42), we obtain

$$a_p = -\frac{c_p z}{c_{p-1}} = +(1 - 1/n)^2 z/n \quad . \quad (43)$$

Referring to the general analysis beginning with Eq. (14), we then obtain, in analogy with Eq. (25), the computational procedure:

<pre> "Initialization" T₁ = -z; T₁ → Sum; l = 1 "Computational Loop" for p = 2, 3, ..., infinity T_p = -T_{p-1} × z × (1-1/p)²/p T_p + Sum → Sum "Convergence Criteria" if p > 4 define s,θ by polar (Sum) define t,φ by polar (T_p) if t/s < ε: l + 1 → l if l > 4: exit from loop "Exiting Computation" S(z) = Sum.</pre>	}	(44)
--	---	------

Similarly the series (for $|z/w| < 1$)

$$T(z, w) \equiv \sum_{p=1}^{\infty} \frac{(-z/w)^p}{p^2} e^{-w} e_{p-1}(w) \quad (45)$$

where

$$e_n(w) \equiv \sum_{r=0}^n \frac{w^r}{r!} \quad (46)$$

may be evaluated by the computational procedure (the analysis is left as an exercise to the reader):

"Initialization"

$$u = -z/w; s_1 = w; q_1 = 1; v_1 = 1 + s_1; T_1 = 1; T_1 \rightarrow S; l = 1$$

"Computation Loop"

for $p = 2, 3, \dots, \text{infinity}$

$$T_p = T_{p-1} u (1 - 1/p)^2 v_{p-1}/q_{p-1}$$

$$q_p = v_{p-1}; s_p = w s_{p-1}/p; s_p + v_{p-1} \rightarrow v_p$$

$$T_p + S \rightarrow S$$

"Convergence Criteria"

if $p > 4$

define r, θ by polar (S)

define t, ϕ by polar (T_p)

if $|t/r| < \epsilon$: $l + 1 \rightarrow l$

if $l > 4$: exit from loop

"Exiting Computation"

$$T(z, w) = u e^{-w} S.$$

(47)

Equations (25), (44), and (47) give representative computational procedures for the evaluation of certain continued fractions. The notation is essentially that of MADCAP-VI³ for the MANIAC computer.

³Sammet, J. E., PROGRAMMING LANGUAGES: HISTORY AND FUNDAMENTALS, Prentice-Hall, New York, 1968.

II.3 χ^2 Minimization Techniques*

The adjustment of model parameters to obtain a "good" fit to the phase shifts is usually treated in a least squares sense, wherein χ^2 , the goodness of fit parameter, is given by

$$\chi^2 = \sum \left(\frac{\delta_n - \delta_n(\vec{p})}{\epsilon_n} \right)^2$$

where δ_n are the phase shifts to be fitted (taken as experimental data) with associated uncertainties ϵ_n , and $\delta_n(\vec{p})$ are the model predictions dependent upon the parameters p .

Numerous methods and techniques¹⁻³ have been described in the literature for the minimization of χ^2 . They usually are reported to be rapid for a particular class of problems and involve determining $\partial\chi^2/\partial p$ in the process of minimizing χ^2 . If the parametric dependence of the phase shifts is reasonably simple, forming the gradients is a straightforward task. If the starting point for the guessing of the equilibrium values is "near" the answer, then these methods work well.

*The analysis leading to the minimization method presented in this section was developed by the author and Dr. K. F. Famularo of the Los Alamos Scientific Laboratory. A detailed report is in progress.

¹Cziffra, P. and Moravcsik, M. J., "A Practical Guide to the Method of Least Squares," LASL Report UCRL-8523, unpublished.

²Signell, P., Yoder, N. R., and Matos, J. E., Phys. Rev., **135**, 1964, p. B1128.

³Arndt, A. and MacGregor, J. H., "Nucleon-Nucleon Phase Shift Analyses by Chi-Squared Minimization," in METHODS IN COMPUTATIONAL PHYSICS, B. Alder, S. Fernback, and M. Rotenberg, eds., Academic Press, New York, 1966.

However, if one contemplates differentiating the derived expressions obtained in this thesis with respect to core diameter, strength and range of the central and tensor interaction, it is readily recognized that such a formidable task, even if performed, would greatly increase the computation time, and accuracy requirements upon the basic functions. Further, most derivative methods develop difficulties if a parametric derivative occurs with a discontinuity.

For these reasons an optimization technique was developed utilizing only function values. The fundamental notion, "ridge running", due to Bellman can be described as follows: Given a function $f(\vec{v}) \equiv (v_1, v_2, \dots, v_n)$ to be optimized, use the dynamic programming algorithm (Extrem $\equiv$ Max or Min).

$$\text{Extrem}_{\{v_n\}} [f(\vec{v})] = \text{Extrem}_{\{v_n\}} \left(\text{Extrem}_{\{v_{n-1}\}} [f(\vec{v})] \right)$$

In words, Eq. (2) says that in order to optimize over an n-dimensional space, first find the extremum over an n-1 dimensional space, etc. The process uses direct function evaluation and quadratic interpolation for extremal position (with careful checking for false extremals due to significance.) Careful record is kept of the optimal value ("pick the winners" technique) at any stage in any dimensional subset, an n-dimensional directional tangent is developed and used for the next optimal guess, and utilization of second differences is made to reduce the search volume and to adjust step sizes. The ordering of v_n is permissible for increased efficiency.

II.4 Preliminary Numerical Calculations

The 1S_0 low energy scattering parameters derived in Sec. II.1 can be used to determine our model parameters (strength, σ_c , range, r_c , and core diameter, a). The 1S_0 low-energy scattering parameters (scattering length and effective range) used, as determined by Noyes¹ (hereafter, N), were

$$\begin{aligned}^1a &= -23.7 F \\^1r_e &= 2.7 F\end{aligned}\tag{1}$$

In Figure 1 we exhibit the model parameters, σ_c and μ_c ($= a/r_c$), as well as the shape-dependent parameter, 1p , as a function of core-diameter, a , using Eq. (1) and Sec. II.1, Eq. 1).

In Figure 2 we show the comparison of the exact model phase shifts and the low-energy phase-shift expansion ($\sigma = 1.91$, $\mu = .72$ and $a = .3F$), where

$$\Delta\delta = |\delta_{\text{exact}} - \delta_{\text{low-energy}}|\tag{2}$$

using Sec. II.1, Eq. (29) and Sec. I.2, Eq. (30).

¹Noyes, H. P., Phys. Rev., 130, 1963, p. 2025; also Noyes, H. P., Nucl. Phys., 74, 1965, p. 508.

In Figure 3 we compare the phase-shifts using our model parameters with the phase-shift analysis of MacGregor, Arndt, and Wright (hereafter, MAW), for various core diameters, a .

Finally, in Figure 4, the phenomenological phase-shift calculations of Hamada-Johnston ($^1a = -17$, $^1r_e = 2.83$, $^1p = +.016$) are compared with our model.

From these preliminary calculations we conclude that our model can give a reasonable fit to the 1S_0 phase-shifts for n-p scattering. The negative shape-dependent parameter, 1p , determination is opposite that predicted by Noyes¹ and Hamada-Johnston³.

A complete model-parameter determination by least-squares fitting of the phase-shifts calculated by this model to those of the experimental phase-shift analysis of MacGregor, Arndt and Wright³ was instigated in a MADCAP program for the MANIAC, a Los Alamos computer. Although individual phase-shifts were evaluatable, the complete model-parameter determination was not feasible to perform because of the availability, computer speed, and number significance required.

²MacGregor, M. H., Arndt, R. A., and Wright, R. M., Phys. Rev., **182**, 1969, p. 1714.

³Hamada, T. and Johnston, I. D., Nucl. Phys., **34**, 1962, p. 382.

A faster (40-80 times), more number significance computer, like a CDC-7600, needs to be used in order to achieve a meaningfully significant numerical solution in a reasonable time. The funding for these calculations on the CDC-7600 is not presently available.

Some of the functions occurring in this thesis are displayed graphically as CAL COMP plots⁴ in Figures 5-44, for representative argument values. The displayed functions are summarized in Table I.

All the figures have been rotated so as to best display the surfaces involved. The function origin is not necessarily at the apparent axis origin. For example, in Fig. 5, the "logarithmic singularity" is actually at $x = .05$, $y \approx 0$. Figure 41, $\text{Re}(D_1(\beta, \alpha))$, illustrates the result of "lost" significance in a calculation (the edges of the graph exhibiting an unreal apparent discontinuity). This particular graph was included for that purpose only. Had we restricted x and y to be infinitionally away from 1, the graph would have not exhibited that edge effect. This type of behavior on a long (5-10 hour) calculation is, of course, an utter disaster.

Figure 43, on the other hand, is the graph of a function used in the deuteron quadrupole moment expressions. The discontinuity exhibited there is exactly as would be predicted and presents no calculational difficulty.

⁴The slight jitter in the graphs is due to the CAL COMP mechanical pen motion.

Table I

<u>Function</u>	<u>Definition</u>	<u>Figures</u>
$\ln z$	Standard	5,6
$E_{\ell}(z)$	Exponential Integral $\ell = -2, -1, 1, 2, 3$	7,14
$Z_{\ell}(\mu, \tau)$	Appendix 2 $\ell = 0, 1, 2, 3$	15,24
$H_{\ell}(z)$	Spherical Bessel Function $\ell = 0, 1,$	25,28
$T(z, w)$	Appendix 12 ⁵	29,34
$S(z)$	Appendix 12 ⁶	35,36
$A(z)$	Abel's function Appendix 12 ⁷	37,38
$D_1(\beta, \alpha)$	Appendix 12	39,42
$D_1(\mu_{\alpha} - \gamma \mu_{\beta} + \gamma)$	Appendix 7, Eq. (19)	43
$g_{\ell}(\tau, \mu)$	Appendix 1, $\ell = 2$	44

$${}^5T(z, w) \equiv \sum_{p=1}^{\infty} \frac{(-z/w)^p}{p^2} e^{-w} e_{p-1}(w)$$

$${}^6(z) \equiv \sum_{p=1}^{\infty} \frac{(-z)^p}{p^2 p}$$

$${}^7A(z) \equiv \sum_{p=1}^{\infty} \frac{(-z)^p}{p^2}$$

Figure 5

Real $\ln(z)$, $z = x+iy$, $0.05 < x < 5$, $-4 < y < 4$

Figure 6

$\text{Imag } \ln(Z), z = x+iy, 0.05 < x < 5, -4 < y < 4$

Figure 7

Real $E_{\ell}(z)$, $z = x+iy$, $.2 < x < 2$, $.2 < y < 2$, $\ell = -2$

Figure 8

$\text{Imag } E_l(z), z = x+iy, .2 < x < 2, .2 < y < 2, l = -2$

Figure 9

Real $E_{\ell}(Z)$, $Z = x+iy$, $.2 < x < 2$, $.2 < y < 2$, $\ell = -1$

Figure 10

$\text{Imag } E_l(Z), Z = x+iy, .2 < x < 2, .2 < y < 2, l=-1$

Figure 11

Real $E_l(z)$, $z = x+iy$, $.2 < x < 2$, $.2 < y < 2$, $l = 1$

Figure 12

$\text{Imag } E_\ell(z), z = x+iy, .2 < x < 2, .2 < y < 2, \ell = 1$

Figure 13

Real $E_l(z)$, $z = x+iy$, $.2 < x < 2$, $.2 < y < 2$, $l = 2$

Figure 14

$\text{Imag } E_k(z), z = x+iy, -2 < x < 2, -2 < y < 2, k = 2$

Figure 15

Real $Z_{\ell}(\mu, \tau), \mu = x, \tau = -y, .2 < x < 2, .2 < y < 2, \ell = 0$

Figure 16

$\text{Imag } Z_\ell(\mu, \tau), \mu=x, \tau=-y, .2 < x < 2, .2 < y < 2, \ell = 0$

Figure 17

Real $Z_l(\mu, \tau)$, $\mu=x, \tau=-y$, $-0.2 < x < 0.2$, $-0.2 < y < 0.2$, $l=1$

Figure 18

$\text{Imag } Z_l(u, \tau), u = x, \tau = -y, -0.2 < x < 0.2, -0.2 < y < 0.2, l = 1$

Figure 19

Real $Z_\ell(\mu, \tau), \mu=x, \tau=-y, .1 < x < 10, .1 < y < 10, \ell = 2$

Figure 20

$\text{Imag } Z_l(\mu, \tau), \mu=x, \tau=-y, .1 < x < 10, .1 < y < 10, l = 2$

Figure 21

Real $z_l(\mu, \tau)$, $\mu=x$, $\tau=-y$, $.2 < x < 2$, $.2 < y < 2$, $l = 2$

Figure 22

$\text{Imag } Z_l(\mu, \tau), \mu=x, \tau=-y, .2 < x < 2, .2 < y < 2, l = 2$

Figure 23

Real $Z_l(\mu, \tau), \mu=x, \tau=-y, .5 < x < 2, .5 < y < 2, l = 3$

Figure 24

$\text{Imag } Z_\ell(\mu, \tau), \mu = x, \tau = -y, .5 < x < 2, .5 < y < 2, \ell = 3$

Figure 25

$$\text{Real } H_l(z) = G_l(z) + iF_l(z), \quad z = x+iy, \quad .2 < x < 2, \\ -2 < y < 2, \quad l = 0$$

Figure 26

$$\text{Imag } H_{\ell}(z) = G_{\ell}(z) + iF_{\ell}(z), \quad z = x+iy, \quad -2 < x < 2, \\ -2 < y < 2, \quad \ell = 0$$

Figure 27

$$\text{Real } H_l(z) = G_l(z) + iF_l(z), \quad z = x+iy, \quad .2 < x < 2, \\ + 2 < y < 2, \quad l = 1$$

Figure 28

$$\text{Imag } H_{\ell}(z) = G_{\ell}(z) + iF_{\ell}(z), \quad z = x+iy, \quad .2 < x < 2, \\ +2 < y < 2, \quad \ell = 1$$

Figure 29

Real $T(z, w)$, $z = x+iy$, $w = u+iy$, $s = x/u = 1$
 $.2 < x < 2$, $.2 < y < 2$

Figure 30

$\text{Imag } T(z, w), z = x+iy, w = u+iy, s = x/u = 1$
 $.2 < x < 2, .2 < y < 2$

Figure 31

Real $T(z,w)$, $z = x+iy$, $w = u+iy$, $s = x/u = .75$
 $.2 < x < 2$, $.2 < y < 2$

Figure 32

$\text{Imag } T(z, w), z = x+iy, w = u+iy, s = x/u = .75$
 $.2 < x < 2, .2 < y < 2$

Figure 33

Real $T(z, w)$, $z = x+iy$, $w = u+iy$, $s = x/u = .75$,
 $.2 < x < 2$, $-2 < y < 2$

Figure 34

$\text{Imag } T(z,w), z = x+iy, w = u+iy, s = x/u = .75,$
 $.2 < x < 2, -2 < y < 2$

Figure 35

Real $S(z)$, $z = x+iy$, $-.2 < x < .2$, $-.2 < y < .2$

Figure 36

$\text{Imag } S(z), z = x+iy, .2 < x < 2, .2 < y < 2$

Figure 37

Real $A(z)$, $z = xe^{iy\pi/2}$, $0.01 < x < .99$, $0 < y < 10$

Figure 38

$\text{Imag } A(z), z = xe^{iy\pi/2}, 0.01 < x < .99, 0 < y < 10$

Figure 39

Real $D_1(\beta, \alpha)$, $\beta = x - iy$, $\alpha = x/s + iy$, $s = 2$,
 $.1 < x < 1$, $.1 < y < 1$

Figure 40

$\text{Imag } D_1(\beta, \alpha), \beta = x-iy, \alpha = x/s + iy, s = 2,$
 $.1 < x < 1, .1 < y < 1$

Figure 41

Real $D_1(\beta, \alpha)$, $\beta = x - iy$, $\alpha = x/s + iy$, $s = 1$
 $.1 < x < 1$, $.1 < y < 1$

Figure 42

$\text{Imag } D_1(\beta, \alpha), \beta = x-iy, \alpha = x/s + iy, s = 1$
 $.1 < x < 1, .1 < y < 1$

Figure 43

$$D_1(\mu_\alpha - \gamma, \mu_\beta + \gamma), \quad x = \mu_\alpha - \gamma, \quad y = \mu_\beta + \gamma, \quad \gamma = .01$$

$$.1 < x < 3, \quad .1 < y < 2$$

Figure 44

$$G_l(\tau, \mu), \tau = y, \mu = x, .4 < x < 2, .4 < y < 5, l = 2$$

PART III: NUCLEAR MATTER APPLICATIONS

III.1 Review of Scattering Theory with Tensor Forces Using Bar Phase Shifts

In this section we examine and review the theory of partial-wave reaction matrices (A matrices) when a tensor interaction is present. We explicitly derive the closure relationship after developing the connection with the bar phase shifts.

From Sec. I.1, Eq. (6) we may write the $G^{(\mu)}$ matrix, for a local potential, as,

$$\langle k | G^{(\mu)} | k_0 \rangle = \sum_{\xi, \zeta} \int d^3r \langle k | r \rangle_0 V(r) \langle r | k_0 \rangle^{(\mu)} \quad (1)$$

For convenience, we change the normalization of the reaction matrix, calling the new matrix $A^{(\mu)}$, defining

$$\langle k | G^{(\mu)} | k_0 \rangle \equiv - \frac{4\pi}{k_0} \frac{\hbar^2}{M} \langle k | A^{(\mu)} | k_0 \rangle, \quad (2)$$

where

$$U \equiv \frac{M}{\hbar^2} V.$$

Then we may express the A matrix in terms of the wavefunctions as

$$\langle k | A^{(\mu)} | k_0 \rangle = - \frac{k_0}{4\pi} \sum_{\xi, \zeta} \int d^3r \langle k | r \rangle_0 U(r) \langle r | k_0 \rangle^{(\mu)} \quad (3)$$

Upon using the free-particle and two-nucleon wavefunctions given by Sec. I.1, Eqs. (19) and (20), respectively, Eq. (3) can be rewritten as

$$\langle k|A^{(\mu)}|k_0\rangle = -\frac{4\pi}{k} \sum_{\substack{LM_L \\ JM \\ L'M'_L}} e^{i(L'-L)\pi/2} Y_{L'M'_L}^*(\hat{n}_{k_0}) Y_{LM_L}(\hat{n}_k)$$

$$(L'SM_L M_S | JM) (LSM_L M_S | JM) \langle LSJM; TM_T; k | U | L'SJM; TM_T; k_0 \rangle^{(\mu)}, \quad (4)$$

where

$$\langle LSJM; TM_T; k | U | L'SJM; TM_T; k_0 \rangle^{(\mu)} \equiv$$

$$\sum_{L''} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \chi_{TM_T}^\dagger(\zeta_1 \zeta_2) F_L(kr) U(r) \zeta_1 \zeta_2$$

$$\times \langle r | k_0 L'' L' \rangle^{(\mu)} \phi_{L'' SJ}^M(\hat{n}_r \xi_1 \xi_2) \chi_{TM_T}(\zeta_1 \zeta_2) \equiv -k \langle k_L | A_{ST,J}^{(\mu)} | k_0 L' \rangle.$$

(5)

With the second equality we have defined, quite generally, the partial-wave reaction matrix, which is not a function of either M or M_T . Using Eq. (31) of Sec. I.1, Eq. (5) may be rewritten as

$$\begin{aligned} \langle k_L | A_{ST,J}^{(\mu)} | L' \rangle = & \\ & -\frac{1}{k} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \chi_{TM_T}^\dagger(\zeta_1 \zeta_2) F_L(kr) U(r) \zeta_1 \zeta_2 \\ & \langle r \xi_1 \xi_2 | LSJM; L' \rangle^{(\mu)} \chi_{TM_T}(\zeta_1 \zeta_2). \end{aligned}$$

(6)

This expression can now easily be generalized to the case of a non-local interaction, Eq. (36) of Sec. I.1.

In order to relate the A matrices to scattering parameters, we next define four linearly independent free-particle wavefunctions (of momentum $\hbar k$), which involve the Blatt-Biedenharn mixing parameters $\epsilon_J(k_0)$ of Eq. (24), Sec. I.1 or Eq. (32) of Sec. I.3.

$$\begin{aligned}
 v_{k_0, J}^{\mu}(kr; \hat{n}_r \xi_1 \xi_2) &\equiv \cos \epsilon_J(k_0) F_{J-1}(kr) \phi_{J-1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) \\
 &\quad + \sin \epsilon_J(k_0) F_{J+1}(kr) \phi_{J+1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) , \\
 v_{k_0, J\beta}^M(kr; \hat{n}_r \xi_1 \xi_2) &\equiv - \sin \epsilon_J(k_0) F_{J-1}(kr) \phi_{J-1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) \\
 &\quad + \cos \epsilon_J(k_0) F_{J+1}(kr) \phi_{J+1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) , \\
 G_{k_0, J\alpha}^M(kr; \hat{n}_r \xi_1 \xi_2) &\equiv \cos \epsilon_J(k_0) G_{J-1}(kr) \phi_{J-1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) \\
 &\quad + \sin \epsilon_J(k_0) G_{J+1}(kr) \phi_{J+1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) , \\
 G_{k_0, J\beta}^M(kr; \hat{n}_r \xi_1 \xi_2) &\equiv - \sin \epsilon_J(k_0) G_{J-1}(kr) \phi_{J-1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) \\
 &\quad + \cos \epsilon_J(k_0) G_{J+1}(kr) \phi_{J+1, 1, J}^M(\hat{n}_r \xi_1 \xi_2) . \quad (7)
 \end{aligned}$$

These functions (times $\frac{1}{r}(\frac{\hat{z}}{r})^{\frac{1}{2}}$) form a complete* orthonormal set over a

* Complete functions of total spin $S = 1$, total angular momentum J and parity $(-)^{J+1}$.

large spherical box of radius R. For regions including the origin only the first two of Eqs. (7) are regular. Upon inverting the first two equations, we obtain

$$\begin{aligned}
 F_{J-1}(kr) \phi_{J+1,1,J}^M(\hat{n}_r \xi_1 \xi_2) &= \cos \epsilon_J(k_0) y_{k_0 J \alpha}^M(kr; \hat{n}_r \xi_1 \xi_2) \\
 &\quad - \sin \epsilon_J(k_0) y_{k_0 J \beta}^M(kr; \hat{n}_r \xi_1 \xi_2) \quad , \\
 F_{J+1}(kr) \phi_{J+1,1,J}^M(\hat{n}_r \xi_1 \xi_2) &= \sin \epsilon_J(k_0) y_{k_0 J \alpha}^M(kr; \hat{n}_r \xi_1 \xi_2) \\
 &\quad + \cos \epsilon_J(k_0) y_{k_0 J \beta}^M(kr; \hat{n}_r \xi_1 \xi_2) \quad (8)
 \end{aligned}$$

We also define four real reaction matrices using the Blatt-Biedenharn states, Eqs. (51) of Sec. I.1.

$$\begin{aligned}
 \cos \delta_{J\alpha}^{(k)} \langle k\alpha | A_{ST,J} | k_0\alpha \rangle &\equiv -\frac{1}{k} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} y_{k_0 J \alpha}^{M\dagger}(kr; \hat{n}_r \xi_1 \xi_2) \\
 &\quad \zeta_1 \zeta_2 \\
 &\quad \times y_{TM_T}^{\dagger}(\zeta_1 \zeta_2) U(\vec{r}, \sigma_1 \sigma_2 \tau_1 \tau_2) y_{TM_T}(\zeta_1 \zeta_2) \langle \vec{r} \xi_1 \xi_2 | k_0 J M_j \alpha \rangle \quad , \\
 \cos \delta_{J\alpha}^{(k_0)} \langle k\alpha | A_{ST,J} | k_0\alpha \rangle &\equiv -\frac{1}{k} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} y_{k_0 J \beta}^{M\dagger}(kr; \hat{n}_r \xi_1 \xi_2) \\
 &\quad \zeta_1 \zeta_2 \\
 &\quad \times y_{TM_T}^{\dagger}(\zeta_1 \zeta_2) U(r, \sigma_1 \sigma_2 \tau_1 \tau_2) y_{TM_T}(\zeta_1 \zeta_2) \langle \vec{r} \xi_1 \xi_2 | k_0 J M_j \rangle \quad ,
 \end{aligned}$$

$$\begin{aligned}
\cos \delta_{J3}(k_0) \langle k_0 | A_{ST,J} | k_0 \rangle &\equiv - \frac{1}{k} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} v_{k_0 J \alpha}^{M+}(kr; \hat{n}_r \xi_1 \xi_2) \\
&\quad \zeta_1 \zeta_2 \\
&\times v_{TM_T}^+(\zeta_1 \zeta_2) U(\vec{\tau} \sigma_1 \sigma_2 \tau_1 \tau_2) v_{TM_T}(\zeta_1 \zeta_2) \langle \vec{\tau} \xi_1 \xi_2 | k_0 J M_J \rangle , \\
\cos \delta_{J3}(k_0) \langle k_0 | A_{ST,J} | k_0 \rangle &\equiv - \frac{1}{k} \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} v_{k_0 J 3}^{M+}(kr; \hat{n}_r \xi_1 \xi_2) \\
&\quad \zeta_1 \zeta_2 \\
&\times v_{TM_T}^+(\zeta_1 \zeta_2) U(\vec{\tau} \sigma_1 \sigma_2 \tau_1 \tau_2) v_{TM_T}(\zeta_1 \zeta_2) \langle \vec{\tau} \xi_1 \xi_2 | k_0 J M_J \rangle , \quad (9)
\end{aligned}$$

We note that k times the right side of each of Eqs. (9) represents a momentum transform [using the first two of Eqs. (7)] of a radial Schrodinger equation of the type of Eqs. (52) of Sec. I.1, for the Blatt-Biedenharn states (24) of Sec. I.1. These equations can be solved formally in momentum space, with homogeneous solutions of the form

$$\frac{\pi}{2} \cos \delta_{J\alpha} \delta(k-k_0) \delta_{\alpha, \alpha'}$$

After returning to position space and using Eqs. (1) and (16') of Appendix 8, for the Bessel functions (F_ℓ, G_ℓ) instead of (F'_ℓ, G'_ℓ) , then the asymptotic form of the solutions for these equations can be identified as linear combinations of all four free-particle functions (7) and the diagonal A-matrix elements. Finally, comparison with Eqs. (26) of Sec. I.1 leads to the conclusion that

$$\langle k\alpha | A_{ST,J} | k\alpha \rangle = \tan \delta_{J\alpha}(k_0)$$

$$\langle k\beta | A_{ST,J} | k\beta \rangle = \tan \delta_{J\beta}(k_0)$$

$$\langle k\alpha | A_{ST,J} | k\beta \rangle = \langle k\beta | A_{ST,J} | k\alpha \rangle = 0 \quad (10)$$

Having dealt rather easily with A-matrices for Blatt-Biedenharn states, we turn now to the A-matrices defined by Eq. (6). Upon substituting Eqs. (8) into Eq. (6) and using Eqs. (9) and I.1. (32), we obtain

$$\langle k, J-1 | A_{ST,J}^{(\mu)} | l, J-1 \rangle$$

$$= \left\{ \begin{array}{l} B_{J\alpha}^{(\mu)}(l) \cos \delta_{J\alpha}(l) \cos \epsilon_J(l) \begin{bmatrix} \cos \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\alpha \rangle \\ -\sin \epsilon_J(l) \langle k\beta | A_{ST,J} | l\alpha \rangle \end{bmatrix} \\ -B_{J\beta}^{(\mu)}(l) \cos \delta_{J\beta}(l) \sin \epsilon_J(l) \begin{bmatrix} \cos \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\beta \rangle \\ -\sin \epsilon_J(l) \langle k\beta | A_{ST,J} | l\beta \rangle \end{bmatrix} \end{array} \right\},$$

$$\langle k, J-1 | A_{ST,J}^{(\mu)} | l, J+1 \rangle$$

$$= \left\{ \begin{array}{l} B_{J\alpha}^{(\mu)}(l) \cos \delta_{J\alpha}(l) \sin \epsilon_J(l) \begin{bmatrix} \cos \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\alpha \rangle \\ -\sin \epsilon_J(l) \langle k\beta | A_{ST,J} | l\alpha \rangle \end{bmatrix} \\ +B_{J\beta}^{(\mu)}(l) \cos \delta_{J\beta}(l) \cos \epsilon_J(l) \begin{bmatrix} \cos \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\beta \rangle \\ -\sin \epsilon_J(l) \langle k\beta | A_{ST,J} | l\beta \rangle \end{bmatrix} \end{array} \right\},$$

$$\langle k, J+1 | A_{ST,J}^{(\mu)} | l, J-1 \rangle$$

$$= \left\{ \begin{aligned} & B_{J\alpha}^{(\mu)}(l) \cos \delta_{J\alpha}(l) \cos \epsilon_J(l) \begin{bmatrix} \sin \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\alpha \rangle \\ + \cos \epsilon_J(l) \langle k\beta | A_{ST,J} | l\alpha \rangle \end{bmatrix} \\ & - B_{J\beta}^{(\mu)}(l) \cos \delta_{J\beta}(l) \sin \epsilon_J(l) \begin{bmatrix} \sin \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\beta \rangle \\ + \cos \epsilon_J(l) \langle k\beta | A_{ST,J} | l\beta \rangle \end{bmatrix} \end{aligned} \right\},$$

$$\langle k, J+1 | A_{ST,J}^{(\mu)} | l, J+1 \rangle$$

$$= \left\{ \begin{aligned} & B_{J\alpha}^{(\mu)}(l) \cos \delta_{J\alpha}(l) \sin \epsilon_J(l) \begin{bmatrix} \sin \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\beta \rangle \\ + \cos \epsilon_J(l) \langle k\beta | A_{ST,J} | l\alpha \rangle \end{bmatrix} \\ & + B_{J\beta}^{(\mu)}(l) \cos \delta_{J\beta}(l) \cos \epsilon_J(l) \begin{bmatrix} \sin \epsilon_J(l) \langle k\alpha | A_{ST,J} | l\beta \rangle \\ + \cos \epsilon_J(l) \langle k\beta | A_{ST,J} | l\beta \rangle \end{bmatrix} \end{aligned} \right\}.$$

(11)

With the aid of Eqs. (10), the diagonal elements of (11), then become

$$\langle k, J-1 | A_{ST,J}^{(\mu)} | k, J-1 \rangle$$

$$= B_{J\alpha}^{(\mu)}(k) \sin \delta_{J\alpha}(k) \cos^2 \epsilon_J(k) + B_{J\beta}^{(\mu)}(k) \sin \delta_{J\beta}(k) \sin^2 \epsilon_J(k),$$

$$\langle k, J-1 | A_{ST,J}^{(\mu)} | k, J+1 \rangle$$

$$[B_{J\alpha}^{(\mu)}(k) \sin \delta_{J\alpha}(k) - B_{J\beta}^{(\mu)}(k) \sin \delta_{J\beta}(k)] \sin \epsilon_J(k) \cos \epsilon_J(k) ,$$

$$\langle k, J+1 | A_{ST,J}^{(\mu)} | k, J-1 \rangle = \langle k, J-1 | A_{ST,J}^{(\mu)} | k, J+1 \rangle ,$$

$$\langle k, J+1 | A_{ST,J}^{(\mu)} | k, J+1 \rangle$$

$$= B_{J\alpha}^{(\mu)}(k) \sin \delta_{J\alpha}(k) \sin^2 \epsilon_J(k) + B_{J\beta}^{(\mu)}(k) \sin \delta_{J\beta}(k) \cos^2 \epsilon_J(k) , \quad (12)$$

where according to Eqs. (30) of Sec. I.1,

$$B_{J\alpha}^{(+)}(k) \equiv e^{+i\delta_{J\alpha}(k)} , \quad B_{J\beta}^{(+)}(k) \equiv e^{+i\delta_{J\beta}(k)} ,$$

$$B_{J\alpha}^{(0)}(k) \equiv 1/\cos \delta_{J\alpha}(k) , \quad B_{J\beta}^{(0)}(k) \equiv 1/\cos \delta_{J\beta}(k) . \quad (13)$$

Now the Stapp parameters are defined by¹ (see also Eqs. (50)

of Sec. I.3 for the case $\mu = +$].

$$\alpha_{J(J-1)}^{(+)}(k)$$

$$\equiv \left[e^{+2i\delta_{J\alpha}(k)} \cos^2 \epsilon_J(k) + e^{+2i\delta_{J\beta}(k)} \sin^2 \epsilon_J(k) \right] = \left[\alpha_{J(J-1)}^{(+)}(k) \right]^* ,$$

¹Stapp, H. P., Ypsilantis, T. J., and Metropolis, N., Phys. Rev., 105, 1957, p. 302.

$$\alpha_{JJ}^{(+)}(k)$$

$$= \left[e^{+2i\delta_{J\alpha}(k)} - e^{+2i\delta_{J\beta}(k)} \right] \sin \epsilon_J(k) \cos \epsilon_J(k) = \left[\alpha_{JJ}^{(+)}(k) \right]^* ,$$

$$\alpha_{J(J+1)}^{(+)}(k)$$

$$= e^{+2i\delta_{J\alpha}(k)} \sin^2 \epsilon_J(k) + e^{+2i\delta_{J\beta}(k)} \cos^2 \epsilon_J(k) = \left[\alpha_{J(J+1)}^{(+)}(k) \right]^* ,$$

and

$$\alpha_{J(J-1)}^{(o)}(k) \equiv \tan \delta_{J\alpha}(k) \cos^2 \epsilon_J(k) + \tan \delta_{J\beta}(k) \sin^2 \epsilon_J(k) ,$$

$$\alpha_{JJ}^{(o)}(k) \equiv [\tan \delta_{J\alpha}(k) - \tan \delta_{J\beta}(k)] \sin \epsilon_J(k) \cos \epsilon_J(k) ,$$

$$\alpha_{J(J+1)}^{(o)}(k) \equiv \tan \delta_{J\alpha}(k) \sin^2 \epsilon_J(k) + \tan \delta_{J\beta}(k) \cos^2 \epsilon_J(k) , \quad (14)$$

Upon substituting Eqs. (14) into Eqs. (12), we find the results

$$\underline{\mu = 0} \quad \langle k, J-1 | A_{ST,J}^{(o)} | k, J-1 \rangle = \alpha_{J(J-1)}^{(o)}(k)$$

$$\langle k, J-1 | A_{ST,J}^{(o)} | k, J+1 \rangle = \alpha_{JJ}^{(o)}(k)$$

$$\langle k, J+1 | A_{ST,J}^o | k, J-1 \rangle = \alpha_{JJ}^{(o)}(k)$$

$$\langle k, J+1 | A_{ST,J}^o | k, J+1 \rangle = \alpha_{J(J+1)}^o(k)$$

$$\begin{aligned}
\mu = + \quad \langle k, J-1 | A_{ST,J}^{(+)} | k, J-1 \rangle &= \frac{1}{2i} [\alpha_{J(J-1)}^{(+)}(k) - 1] \\
\langle k, J-1 | A_{ST,J}^{(+)} | k, J+1 \rangle &= \frac{1}{2i} \alpha_{JJ}^{(+)}(k) \\
\langle k, J+1 | A_{ST,J}^{(+)} | k, J-1 \rangle &= \frac{1}{2i} \alpha_{JJ}^{(+)}(k) \\
\langle k, J+1 | A_{ST,J}^{(+)} | k, J+1 \rangle &= \frac{1}{2i} [\alpha_{J(J+1)}^{(+)}(k) - 1]
\end{aligned}$$

and

$$\begin{aligned}
\mu = - \quad \langle k, J-1 | A_{ST,J}^{(-)} | k, J-1 \rangle &= \frac{1}{2i} [1 - \alpha_{J(J-1)}^{(-)}(k)] \\
\langle k, J-1 | A_{ST,J}^{(-)} | k, J+1 \rangle &= -\frac{1}{2i} \alpha_{JJ}^{(-)}(k) \\
\langle k, J+1 | A_{ST,J}^{(-)} | k, J-1 \rangle &= -\frac{1}{2i} \alpha_{JJ}^{(-)}(k) \\
\langle k, J+1 | A_{ST,J}^{(-)} | k, J+1 \rangle &= \frac{1}{2i} [1 - \alpha_{J(J+1)}^{(-)}(k)] \quad (15)
\end{aligned}$$

From Eqs. (15) it is clear that

$$\langle k_L | A_{ST,J}^{(-)} | k_L' \rangle = \langle k_L | A_{ST,J}^{(+)} | k_L' \rangle^* \quad (16)$$

Moreover, upon referring to Sec. I.3, Eqs. (19), (39), and (47), it can be seen that we have now determined explicitly the diagonal $A_{ST,J}^{(+)}$ matrix elements in terms of the "bar" phase shifts and (in turn) in terms of the exact solution to the two-body problem of this thesis.

A most convenient feature of the Blatt-Biedenharn reaction matrices, Eqs. (9), is that they are real. Upon using this fact and Eqs. (11), it is a matter of simple substitution to prove that

$$\sum_{L''} \langle kL | A_{ST,J}^{(\mu)} | k_0 L'' \rangle \langle lL' | A_{ST,J}^{(\mu)} | k_0 L'' \rangle^* = \text{real} \quad (17)$$

and

$$\begin{aligned} \sum_{L''} \langle kL | A_{ST,J}^{(+)} | k_0 L'' \rangle \langle lL'' | A_{ST,J}^{(+)} | k_0 L'' \rangle^* = \\ \sum_{L''} \langle kL | A_{ST,J}^{(-)} | k_0 L'' \rangle \langle lL' | A_{ST,J}^{(-)} | k_0 L'' \rangle^* \end{aligned} \quad (18)$$

[These relations are also clearly true for the uncoupled cases $L = L' = J$]. Upon setting $k = k_0$, we also obtain with the aid of Eqs. (10) and (11)

$$i \sum_{L''} \langle kL | A_{ST,J}^{(+)} | kL'' \rangle \langle lL' | A_{ST,J}^{(+)} | kL'' \rangle^* = -i \text{Im} \langle lL' | A_{ST,J}^{(+)} | kL \rangle^* \quad (19)$$

Finally, use of Eq. (15) and Sec. I.3, Eq. (39), leads to the conclusion that

$$\begin{aligned} \sum_{L=J+1} \text{Re} \langle kL | A_{ST,J}^{(+)} | kL \rangle &= \frac{1}{2} \cos 2\bar{\epsilon}_J(k) [\sin 2\delta_{J(J-1)}(k) + \sin 2\delta_{J(J+1)}(k)] \\ &= \frac{1}{2} [\sin 2\delta_{J\alpha}(k) + \sin 2\delta_{J\beta}(k)] \end{aligned} \quad (20)$$

and

$$\begin{aligned} \sum_{L, L''} |\langle kL | A_{ST,J}^{(+)} | k_0 L'' \rangle|^2 &= \cos^2 \delta_{J\alpha}(k_0) [\langle k\alpha | A_{ST,J} | k_0 \alpha \rangle^2 + \langle k\alpha | A_{ST,J} | k_0 \alpha \rangle^2] \\ &\quad + \cos^2 \delta_{J\beta}(k_0) [\langle k\alpha | A_{ST,J} | k_0 \beta \rangle^2 + \langle k\beta | A_{ST,J} | k\beta \rangle^2] \end{aligned} \quad (21)$$

Properties of the Reaction Matrices

We now consider the partial-wave expansion, and other properties of the reaction matrices as a prelude to a study of the unitarity conditions satisfied by these matrices. From Eqs. (4) and (5) we have

$$\begin{aligned} \langle k|A^{(\mu)}|k_0\rangle = & 4\pi \sum_{\substack{LM_L \\ JM \\ L'M'_L}} e^{i(L'-L)\pi/2} Y_{L'M'_L}^*(\hat{n}_{k_L}) Y_{LM_L}(\hat{n}_k) (L'SM_L M'_S | JM) \\ & \times (LSM_L M_S | JM) \langle kL|A_{ST,J}^{(\mu)}|k_0 L'\rangle . \end{aligned} \quad (22)$$

Using Eq. (2) and Sec. I.1, Eq. (11), we next write the wavefunction $\langle r|k_0\rangle^{(\mu)}$ as

$$\langle r|k_0\rangle^{(\mu)} = \sum_{M'_S} \int d^3k \langle r|k\rangle_0 \left\{ \delta_{k,k_0} - \frac{1}{2\pi^2 k_0} \left[\frac{\langle k|A^{(\mu)}|k_0\rangle}{k_0^2 - k^2 + i\mu\epsilon} \right] \right\} , \quad (23)$$

where

$$\delta_{k,k_0} \equiv \delta^{(3)}(\vec{k}-\vec{k}_0) \delta_{M_S M'_S} . \quad (24)$$

After substituting the partial-wave expansions for given by Sec. I.1, Eq. (18), and (22) into Eq. (23) we obtain

$$\langle r|k_0 \rangle^{(\mu)} = \frac{4\pi}{k_0 r} \sum_{\substack{LM \\ JM}} e^{iL\pi/2} Y_{LM}^*(\hat{n}_{k_0}) (LSM_L M_L | JM) \chi_{TM_T}(\zeta_1 \zeta_2) \times$$

$$\left\{ \begin{aligned} & F_L(k_0 r) \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \\ & - \frac{2}{\pi} \int_0^\infty k dk \sum_{L'} F_{L'}(kr) \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) \left[\frac{\langle kL' | A_{ST,J}^{(\mu)} | k_0 L \rangle}{k^2 - k_0^2 + i\mu\epsilon} \right] \end{aligned} \right\}. \quad (25)$$

And upon comparing Eq. (25) with Eqs. (34), Sec. I.1, we find

$$\langle \vec{r} \xi_1 \xi_2 | k_0 \text{STML} \rangle^{(\mu)} = \int_0^\infty dk \sum_{L'} F_{L'}(kr) \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) \left\{ \delta(k_0 - k) \delta_{LL'} + \frac{2}{\pi} k \left[\frac{\langle kL' | A_{ST,J}^{(\mu)} | k_0 L \rangle}{k^2 - k_0^2 - i\mu\epsilon} \right] \right\}, \quad (26)$$

which is a generalization for tensor forces of the result usually written down.

We now investigate the asymptotic forms of Eq. (26). First, we rewrite Eq. (26) as

$$\begin{aligned} \langle \vec{r} \xi_1 \xi_2 | k_0 \text{STML} \rangle^{(\mu)} &= F_L(k_0 r) \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \\ &+ \left(\frac{2}{\pi}\right)^2 \sum_{L'} \int_0^\infty dk \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) F_{L'}(kr) \int_0^\infty dr' \frac{F_{L'}(kr') F_{L'}(r')}{k^2 - k_0^2 + i\mu\epsilon} \end{aligned}$$

$$\times \mathcal{L} \langle F_L' | A_{ST,J}^{(\mu)} | k_0 L \rangle \quad (27)$$

Now

$$\frac{1}{k^2 - k_0^2 - i\mu\epsilon} \equiv P \left(\frac{1}{k^2 - k_0^2} \right) + i\mu\pi\delta(k^2 - k_0^2) \quad , \quad (28)$$

and therefore we can write

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty dk \frac{F_L'(kr) F_L'(kr')}{k^2 - k_0^2 - i\mu\epsilon} \\ &= \frac{2}{\pi} P \int_0^\infty dk \frac{F_L'(kr) F_L'(kr')}{k^2 - k_0^2} + \frac{i\mu}{k_0} \int_0^\infty dk \delta(k - k_0) F_L'(kr) F_L'(kr') \end{aligned} \quad (29)$$

The first integral of Eq. (29) is derived in Appendix 8*, with the result

$$\begin{aligned} & \frac{2}{\pi} \int_0^\infty dk \frac{F_L'(kr) F_L'(kr')}{k^2 - k_0^2 - i\mu\epsilon} \\ &= \frac{1}{k_0} \left[F_L'(k_0 r) [G_L'(k_0 r') + i\mu F_L'(k_0 r')] \theta(r' - r) \right. \\ & \quad \left. + [G_L'(k_0 r) + i\mu F_L'(k_0 r)] F_L'(k_0 r') \theta(r - r') \right] \end{aligned} \quad (30)$$

* This Green's Function is obtained from the more general function $G_L(k_0 r, k_0 r')$ of Appendix 8, by taking the limit $a \rightarrow 0$ and using definitions (33) and (34) of Appendix 11 which give $F_L'(kr) \xrightarrow{a \rightarrow 0} F_L(kr)$, $G_L'(kr) \xrightarrow{a \rightarrow 0} G_L(kr)$.

Then, using Eq. (30) and the asymptotic form for $F_\ell(0)$ and $G_\ell(0)$ in Eq. (27) we find for the case $\mu = 0$ the result

$$\begin{aligned} \langle \vec{r} \xi_1 \xi_2 | k_{0\text{STML}} \rangle^{(0)} &\xrightarrow[kr \gg 1]{} \sin(kr - L\pi/2) \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \\ &+ \sum_{L'} \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) \cos(k_0 r - L\pi/2) \langle k_0 L' | A_{ST,J}^{(0)} | k_0 L \rangle. \end{aligned} \quad (31)$$

For the cases $\mu = \pm$, we use both Eqs. (29) and (30) to obtain

$$\begin{aligned} \langle \vec{r} \xi_1 \xi_2 | k_{0\text{SJML}} \rangle^{(\mu)} &\xrightarrow[k_0 r \gg 1]{} \sin(k_0 r - L\pi/2) \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \\ &+ \sum_{L'} \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) \langle k_0 L' | A_{ST,J}^{(\mu)} | k_0 L \rangle e^{i\mu(k_0 r - \frac{L'\pi}{2})}. \end{aligned} \quad (32)$$

Next we consider the specific case $\mu = +$ and write Eq. (32) in the form

$$\begin{aligned} 2i \langle \vec{r} \xi_1 \xi_2 | k_{0\text{SJML}} \rangle^{(+)} &\xrightarrow{} -e^{-i(k_0 r - L\pi/2)} \phi_{LSJ}^M(\hat{n}_r \xi_1 \xi_2) \\ &+ \sum_{L'} \phi_{L'SJ}^M(\hat{n}_r \xi_1 \xi_2) \beta_{LL'}^{(+)}(k_0) e^{i(k_0 r - L\pi/2)}, \end{aligned} \quad (33)$$

where from Eqs. (15) we can make the identifications

$$\begin{aligned} \beta_{(J-1)(J-1)}^{(+)}(k_0) &= \alpha_{J(J-1)}^{(+)}(k_0); \quad \beta_{(J-1)(J+1)}^{(+)}(k_0) = \alpha_{J(J+1)}^{(+)}(k_0); \\ \beta_{(J-1)(J+1)}^{(+)}(k_0) &= \beta_{(J+1)(J-1)}^{(+)}(k_0) = \alpha_{JJ}^{(+)}(k_0). \end{aligned} \quad (34)$$

Conservation of probability and wave-function orthogonality require that the conditions

$$|\beta_{(J-1)(J-1)}^{(+)}(k_o)|^2 + |\beta_{(J-1)(J+1)}^{(+)}(k_o)|^2 = 1 ,$$

$$|\beta_{(J+1)(J+1)}^{(+)}(k_o)|^2 + |\beta_{(J+1)(J-1)}^{(+)}(k_o)|^2 = 1 ,$$

$$\beta_{(J-1)(J-1)}^*(k_o)\beta_{(J+1)(J-1)}(k_o) + \beta_{(J+1)(J-1)}^*(k_o)\beta_{(J-1)(J+1)}(k_o) = 0 . \quad (35)$$

be satisfied. The conditions of Eq. (35), which are indeed satisfied by definitions (3a) in Sec. I.3, are just those necessary to insure that the matrix U, defined by

$$U \equiv \begin{pmatrix} \beta_{(J-1)(J-1)}^{(+)}(k_o) & \beta_{(J-1)(J+1)}^{(+)}(k_o) \\ \beta_{(J+1)(J-1)}^{(+)}(k_o) & \beta_{(J+1)(J+1)}^{(+)}(k_o) \end{pmatrix} , \quad (36)$$

is unitary. Thus, amongst the β_{LL} , there can be at most four independent parameters, and we may write the most general $\beta_{LL}(k_o)$ in terms of the $\alpha^{(*)}$ parameters of Eqs. (34) [see also Eqs. (39) in Sec. I.3] as

$$\begin{aligned}
\beta_{(J-1)(J-1)}^{(+)}(k_0) &= \alpha_{J(J-1)}^{(+)}(k_0) \quad , \\
\beta_{(J+1)(J+1)}^{(+)}(k_0) &= \alpha_{J(J+1)}^{(+)}(k_0) \quad , \\
\beta_{(J-1)(J+1)}^{(+)}(k_0) &= \alpha_{JJ}^{(+)}(k_0) e^{i\rho_J(k_0)} \quad , \\
\beta_{(J+1)(J-1)}^{(+)}(k_0) &= \alpha_{JJ}^{(+)}(k_0) e^{-i\rho_J(k_0)} \quad ,
\end{aligned} \tag{37}$$

where the real quantity $\rho_J(k_0)$ is undetermined by the unitarity condition. Normally we choose $\rho_J(k_0) = 0$, but it is interesting to determine the rationale for that choice.

First we consider the consequences of time-reversal invariance. According to Sec. I.1, Eq. (11), the most general form of the "outgoing" scattered wavefunction is

$$\begin{aligned}
\langle \vec{r} \xi_1 \xi_2 | k_0 S M_S \rangle^{(+)} &= \langle \vec{r} \xi_1 \xi_2 | \vec{k}_0 S M_S \rangle_0 + \frac{1}{(2\pi)^3} \sum_{M'_S} \\
&\int \frac{d^3 k \langle \vec{r} \xi_1 \xi_2 | \vec{k} S M'_S \rangle_0 \langle \vec{k} S M'_S | G^{(+)} | k_0 S M_S \rangle}{E(k_0) - E(k) + i\epsilon}
\end{aligned} \tag{38}$$

where, under time reversal, $k_o \rightarrow k_o$, $M_S \rightarrow -M_S$ and the scattered wave has a distorted incoming part. Thus*

$$\begin{aligned}
 & T \langle \vec{r} \xi_1 \xi_2 | k_o S M_S \rangle^{(+)} \\
 &= (-1)^{M_S} \langle r \xi_1 \xi_2 | -\vec{k}_o, S, -M_S \rangle_o \\
 &+ \frac{1}{(2\pi)^3} \sum_{M'_S} \int d^3k \frac{\langle \vec{r} \xi_1 \xi_2 | -\vec{k}, S, -M'_S \rangle_o \langle -\vec{k}, S, -M'_S | G^{(-)} | -k_o, S, -M_S \rangle}{E(k_o) - E(k) - i\epsilon} \quad (39)
 \end{aligned}$$

Now, the time reversal operator² can be written as

$$T = UK, \quad (40)$$

* We use the Condon-Shortly representation for spin-1 states for which

$$M_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad M_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad M_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{where } |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } |1, 1\rangle = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\text{Then: } U = -e^{-i\pi M_y/\hbar} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } U |1, 1\rangle = -|1, 1\rangle,$$

$$U |1, 0\rangle = |1, 0\rangle; \quad U |1, -1\rangle = -|1, -1\rangle.$$

²Wick, G.C., Am. Rev. Nucl. Sci., 8, 1958, p.1.

where K is the complex conjugation operator and U is a unitary operator such that when applied to the angular momentum operator M , upon applying Eq. (40) directly to Eq. (38), we obtain

$$T(\vec{r}\xi_1\xi_2|\vec{k}_0S_M_S)^{(+)} = (-1)^{M_S} \left\{ \langle r\xi_1\xi_2|-\vec{k}_0,S,-M_S\rangle_0 + \frac{1}{(2\pi)^3} \sum_{M'_S} \frac{\int d^3k \langle \vec{r}\xi_1\xi_2|-\vec{k},S,-M'_S\rangle_0 (-)^{M'_S-M_S} \langle kSM'_S|G^{(+)}|kSM_S\rangle^*}{E(k) - E(k_0) - i\epsilon} \right\} \quad (42)$$

Then by comparing Eqs. (39) and (42) we conclude that

$$\langle kSM'_S|G^{(+)}|\vec{k}_0S_M_S\rangle = (-)^{M_S-M'_S} \langle -\vec{k},S,-M'_S|G^{(-)}|-\vec{k}_0,S,-M_S\rangle^* \quad (43)$$

or, using Eq. (2)

$$\langle kSM'_S|A^{(+)}|k_0SM_S\rangle = \langle -\vec{k},S,-M'_S|A^{(-)}|-\vec{k}_0,S,-M_S\rangle^* (-)^{M'_S-M_S} \quad (44)$$

We next substitute the partial-wave expansion (22) into both sides of Eq. (44), and use symmetry properties to deduce the relation of the Clebsch-Gordon coefficients and spherical harmonics,*

$$\langle kL|A_{ST,J}^{(-)}|k_0L'\rangle^* = \langle kL|A_{ST,J}^{(+)}|k_0L'\rangle, \quad (45)$$

$$^*Y_{L,-M_L}^*(-\hat{n}_i) = (-)^{L+M_L} Y_{LM_L}(\hat{n}_k)$$

$$(LS-M_L-M_S|J,-M) = (-)^{L+S+J+2M} (LSM_LM_S|JM)$$

where $(-)^{L-L'} = 1$. For the special case $k = k_0$, Eq. (45) is satisfied if we define [see Eqs. (15)]

$$\begin{aligned}\langle k_0 L | A_{ST,J}^{(+)} | k_0 L' \rangle &\equiv \frac{1}{2i} [\mathfrak{z}_{LL'}^{(+)}(k_0) - \delta_{LL'}] , \\ \langle k_0 L | A_{ST,J}^{(-)} | k_0 L' \rangle &\equiv \frac{1}{2i} [\delta_{LL'} - \mathfrak{z}_{LL'}^{(-)}(k_0)] ,\end{aligned}\quad (46)$$

provided we require

$$[\mathfrak{z}_{LL'}^{(-)}(k_0)]^* = \mathfrak{z}_{LL'}^{(+)}(k_0) . \quad (47)$$

Thus, Eqs. (37) represent a legitimate solution in that time-reversal invariance does not preclude $\rho_J(k_0) \neq 0$.

Next we check that the total probability for incoming and outgoing parts of the wavefunction $\langle r | k_0 \rangle^{(+)}$ is conserved. Thus with the aid of Eq. (34) of Sec. I.1, we find

$$\begin{aligned}& \sum_{\xi_1 \xi_2} \sum_{\xi_1 \xi_2} \int d^3 r \langle \vec{k}_0 S M_S | \vec{r} \xi_1 \xi_2 \rangle \langle \vec{r} \xi_1 \xi_2 | \vec{k}'_0 S M'_S \rangle \\ &= \frac{(4\pi)^2}{k_0 k'_0} \sum_{LJ} \sum_{L'J'} e^{i(L'-L)\pi/2} \int \frac{d^3 r}{r^2} Y_{L'M_L}^*(\hat{n}_{k'_0}) Y_{LM_L}(\hat{n}_{k_0}) (L' S M'_L M'_S | J' M') \\ & \times \sum_{\xi_1 \xi_2} (L S M_L M_S | J M) \langle k_0 S J M, L | \vec{r} \xi_1 \xi_2 \rangle^{(+)} \langle \vec{r} \xi_1 \xi_2 | k'_0 S J' M', L \rangle^{(+)}\end{aligned}\quad (48)$$

Then, from Eqs. (33) and (35) we have

$$\begin{aligned}
 & \sum_{\xi_1 \xi_2} \int \frac{d^3 r}{r^2} \langle k_0 S J M; L | \vec{r} \xi_1 \xi_2 \rangle^{(+)} \langle \vec{r} \xi_1 \xi_2 | \vec{k}' S J' M'; L' \rangle^{(+)} \\
 &= \frac{\pi}{4} \delta_{LL'} \delta_{JJ'} \delta_{MM'} \delta(k_0 - k'_0) + \frac{\pi}{4} \delta_{JJ'} \delta_{MM'} \delta(k_0 - k'_0) \sum_{L''} \left(\delta_{LL''}^{(+)}(k_0) \delta_{L'L''}^{(+)}(k_0) \right) \\
 &= \frac{\pi}{2} \delta_{LL'} \delta_{JJ'} \delta_{MM'} \delta(k_0 - k'_0) ,
 \end{aligned} \tag{49}$$

which guarantees total probability conservation.

If now we return to Eq. (44) (which is a consequence of time reversal invariance) and rewrite it on the energy shell ($E(k) = E(k_0)$) only, then we obtain the result

$$\langle \vec{k} S M_S | A^{(+)} | \vec{k}_0 S M'_S \rangle = (-)^{M'_S - M_S} \langle -\vec{k}_0, S, -M'_S | A^{(+)} | -\vec{k}, S, -M_S \rangle , \tag{50}$$

Since from Sec. I.1, Eq. (14),

$$G^{(\mu)} = V + V \frac{1}{E - H_0 + i\mu\epsilon} G^{(\mu)} , \tag{51}$$

where $E(k_0)$ is associated with the right-hand momentum state $\vec{k}_0$ and V is assumed to be Hermitian. Moreover, Eq. (51) shows that on the energy shell $E(k) = E(k_0)$,

$$(G^{(+)})^+ = G^{(-)} , \tag{52}$$

which implies

$$(A^{(+)})^+ = A^{(-)} . \quad (53)$$

When this last relation is substituted into Eq. (50), we obtain the important result that on the energy shell, $E(k) = E(k_0)$,

$$\langle \vec{k}_{SM_S} | A^{(+)} | \vec{k}_0^{SM_S} \rangle = (-)^{M_S, -M_S} \langle -\vec{k}_0, S, -M_S | A^{(+)} | -\vec{k}, S, -M_S \rangle . \quad (54)$$

Upon substituting the partial-wave expansion (22) into this result, we obtain finally the identity, with $(-1)^{L-L'} = 1$,

$$\langle k_0 L' | A_{ST,J}^{(+)} | k_0 L \rangle = \langle k_0 L | A_{ST,J}^{(+)} | k_0 L' \rangle . \quad (55)$$

Therefore, the reaction matrix is symmetric on the energy shell, Equations (55), (46), and (37) show finally that we must require.

$$n_J(k_0) = 0. \quad (56)$$

We conclude that the definitions (39) in Sec. I.3 are indeed quite generally and consistent.

General Unitarity Conditions

We now derive the unitarity conditions satisfied by the reaction matrices $\langle kL' | A_{ST,J}^{(\mu)} | k_0 L \rangle$ for $\mu = \pm$. First we consider the orthogonality condition. From

$$\int \frac{d^3 r}{r^2} \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; L \rangle^{(\mu)\dagger} \langle \vec{r} \xi_1 \xi_2 | \ell_0 S J M; L' \rangle^{(\mu)} = \frac{\pi}{2} \delta_{LL'} \delta(k_0 - \ell_0) \quad (57)$$

[see Eq. (49) for the case $\mu = +1$, we find, upon substituting

Eq. (26),

$$\begin{aligned}
 & \int \frac{d^3 r}{r^2} \langle r \tilde{\epsilon}_1 \tilde{\epsilon}_2 | k_o S_{JM}; L \rangle^{(\mu)*} \langle r \tilde{\epsilon}_1 \tilde{\epsilon}_2 | \ell_o S_{JM}; L' \rangle^{(\mu)} \\
 &= \int \frac{d^3 r}{r^2} \int_0^\infty dk d\ell \sum_{L'' L'''} F_{L''}^{(\mu)}(kr) \phi_{L'' S_J}^{M*}(\hat{n}_r \tilde{\epsilon}_1 \tilde{\epsilon}_2) F_{L'''}^{(\mu)}(\ell r) \phi_{L''' S_J}^M(\hat{n}_r \tilde{\epsilon}_1 \tilde{\epsilon}_2) \\
 &\times \left\{ \delta(k_o - k) \delta_{LL''} + \frac{2}{\pi} k \left[\frac{\langle k L'' | A_{ST,J}^{(\mu)} | k_o L \rangle^*}{k^2 - k_o^2 + i\mu\epsilon} \right] \right\} \\
 &\times \left\{ \delta(\ell_o - \ell) \delta_{L' L'''} + \frac{2}{\pi} \ell \left[\frac{\langle \ell L''' | A_{ST,J}^{(\mu)} | \ell_o L' \rangle}{\ell^2 - \ell_o^2 - i\mu\epsilon} \right] \right\} = \frac{\pi}{2} \delta_{LL'} \delta(k_o - \ell_o) \\
 &+ \frac{[\ell_o \langle \ell_o L' | A_{ST,J}^{(\mu)} | k_o L \rangle^* - k_o \langle k_o L | A_{ST,J}^{(2)} | \ell_o L' \rangle]}{\ell_o^2 - k_o^2 + i\mu\epsilon} \\
 &+ \frac{2}{\pi} \int_0^\infty k^2 dk \sum_{L''} \left[\frac{\langle k L'' | A_{ST,J}^{(\mu)} | k_o L \rangle^*}{k^2 - k_o^2 + i\mu\epsilon} \right] \left[\frac{\langle k L'' | A_{ST,J}^{(\mu)} | \ell_o L' \rangle}{k^2 - \ell_o^2 - i\mu\epsilon} \right]. \quad (58)
 \end{aligned}$$

Using Eqs. (19), (28), and (57), and the identity

$$P\left(\frac{1}{x}\right) P\left(\frac{1}{y}\right) = \left(\frac{1}{y-x}\right) \left[P\left(\frac{1}{x}\right) - P\left(\frac{1}{y}\right)\right] + \pi \delta(x) \delta(y), \quad (59)$$

Eqs. (58) give for the orthogonality conditions, when $\lambda = (+)$

$$\begin{aligned}
 0 &= \frac{1}{2} P\left(\frac{1}{\ell_o^2 - k_o^2}\right) [\ell_o \langle \ell_o L' | A_{ST,J}^{(\mu)} | k_o L \rangle^* - k_o \langle k_o L | A_{ST,J}^{(\mu)} | \ell_o L' \rangle] \\
 &+ \frac{1}{2} P\left(\frac{1}{\ell_o^2 - k_o^2}\right) \sum_{L''} \left[\begin{aligned} &\ell_o \langle \ell_o L'' | A_{ST,J}^{(\mu)} | k_o L \rangle^* \beta_{L'' L}^{(\mu)}(k_o) \\ &- k_o \langle k_o L'' | A_{ST,J}^{(\mu)} | \ell_o L' \rangle \beta_{L'' L}^{(\mu)}(k_o) \end{aligned} \right]
 \end{aligned}$$

$$+ \frac{2}{\pi} \left(\frac{1}{2-k_o^2} \right) \int_0^\infty k^2 dk \sum_{L''} \langle kL'' | A_{ST,J}^{(\mu)} | k_o L \rangle^*$$

$$\langle kL'' | A_{ST,J}^{(\mu)} | l_o L' \rangle \left[P \left(\frac{1}{k^2 - l_o^2} \right) - P \left(\frac{1}{k^2 - k_o^2} \right) \right], \quad (60)$$

where the $\beta_{LL'}^{(\mu)}(k_o)$ are defined by Eq. (37), with $\rho_J(k_o) = 0$, for both cases $\mu = +$ and $\mu = -$.

The completeness condition, for $\mu = (+)$, is now used finally to obtain the closure relation for the partial-wave reaction matrices. Completeness requires

$$\begin{aligned} \int d^3 k_o \sum_{\substack{SM_S \\ TM_T}} \langle \vec{r} \xi_1 \xi_2 \zeta_1 \zeta_2 | \vec{k}_o SM_S TM_T \rangle^{(\mu)} \langle \vec{k}_o SM_S TM_T | \vec{r}' \xi'_1 \xi'_2 \zeta'_1 \zeta'_2 \rangle^{(\mu)} \\ + (2\pi)^3 \sum_{JM_S TM_T} \langle \vec{r} \xi_1 \xi_2 \zeta_1 \zeta_2 | \gamma JM_S TM_T \rangle \langle \gamma JM_S TM_T | \vec{r}' \xi'_1 \xi'_2 \zeta'_1 \zeta'_2 \rangle \\ = (2\pi)^3 \delta^{(3)}(\vec{r} - \vec{r}') \delta_{\xi_1 \xi'_1} \delta_{\xi_2 \xi'_2} \delta_{\zeta_1 \zeta'_1} \delta_{\zeta_2 \zeta'_2}, \end{aligned} \quad (61)$$

where the sum over γ in the second term on the left side is a sum over bound states. Upon using Sec. I.1, Eq. (34), and Sec. I.4, Eq. (2), Eq. (61) becomes

$$\int_0^\infty dk_o \sum_{LSJM} \langle \vec{r} \xi_1 \xi_2 | k_o SJM; L \rangle^{(\mu)} \langle k_o SJM; L | \vec{r}' \xi'_1 \xi'_2 \rangle^{(\mu)}$$

$$+ \left(\frac{2}{\pi}\right) \sum_{\substack{\gamma \text{JSM} \\ \text{LL}'}} \int_0^\infty d\ell dk \phi_L^{(\gamma \text{J})}(\ell) F_L(\ell r) \phi_L^{(\gamma \text{J})*}$$

$$(k) F_{L'}(kr') \phi_{\text{LSJ}}^M(\hat{n}_r \xi_1 \xi_2) \phi_{\text{L'SJ}}^{M\dagger}(\hat{n}_r' \xi_1' \xi_2')$$

$$= \frac{\pi}{2} \delta(r-r') \delta(\phi-\phi') \delta(\cos \theta - \cos \theta') \delta_{\xi_1 \xi_1'} \delta_{\xi_2 \xi_2'} \quad (62)$$

Upon substituting Eq. (26) into Eq. (68), we obtain

$$\begin{aligned} & \sum_{\substack{\text{SJM} \\ \text{LL}'}} \int_0^\infty dk d\ell F_L(\ell r) F_{L'}(kr') \phi_{\text{LSJ}}^M(\hat{n}_r \xi_1 \xi_2) \phi_{\text{L'SJ}}^{M\dagger}(\hat{n}_r' \xi_1' \xi_2') \\ & \times \left\{ \sum_{L''} \int_0^\infty dk_o \left(\delta(k_o - \ell) \delta_{LL''} + \frac{2}{\pi} \ell \left[\frac{\langle \ell L | A_{\text{ST},J}^{(\mu)} | k_o L'' \rangle}{\ell^2 - k_o^2 - i\mu\epsilon} \right] \right) \right. \\ & \times \left. \left(\delta(k_o - k) \delta_{L'L''} + \frac{2}{\pi} k \left[\frac{\langle k L' | A_{\text{ST},J}^{(\mu)} | k_o L'' \rangle^*}{k^2 - k_o^2 + i\mu\epsilon} \right] \right) \right. \\ & \left. + \frac{2}{\pi} \sum_{\gamma} \phi_L^{(\gamma \text{J})}(\ell) \phi_L^{(\gamma \text{J})*}(k) \right\} \\ & = \frac{\pi}{2} \delta(r-r') \delta(\phi-\phi') \delta(\cos \theta - \cos \theta') \delta_{\xi_1 \xi_1'} \delta_{\xi_2 \xi_2'} \quad (63) \end{aligned}$$

The completeness condition (63) will indeed be satisfied,

provided that

$$\begin{aligned} & \sum_{L''} \int_0^\infty dk_o \left\{ \delta(k_o - \ell) \delta_{LL''} + \frac{2}{\pi} \ell \left[\frac{\langle \ell L | A_{\text{ST},J}^{(\mu)} | k_o L'' \rangle}{\ell^2 - k_o^2 - i\mu\epsilon} \right] \right\} \\ & \times \left\{ \delta(k_o - k) \delta_{L'L''} + \frac{2}{\pi} k \left[\frac{\langle k L' | A_{\text{ST},J}^{(\mu)} | k_o L'' \rangle^*}{k^2 - k_o^2 + i\mu\epsilon} \right] \right\} \end{aligned}$$

$$+ \frac{2}{\pi} \sum_Y \phi_L^{(YJ)}(\ell) \phi_L^{(YJ)*}(k) = \delta(k-\ell) \delta_{LL'} \quad (64)$$

Upon using Eqs. (19), (28), and (59), we may rewrite this closure condition, for $\mu = +$ and $\mu = -$, as

$$\begin{aligned} & P \left(\frac{1}{k^2 - \ell^2} \right) \text{Re} [k \langle kL' | A_{ST,J}^{(\mu)} | \ell L \rangle^* - \ell \langle \ell L | A_{ST,J}^{(\mu)} | kL' \rangle] \\ & + \frac{2}{\pi} \left(\frac{1}{k^2 - \ell^2} \right) k \ell \int_0^\infty dk_0 \left[\sum_{L''} \langle \ell L | A_{ST,J}^{(\mu)} | k_0 L'' \rangle \langle kL' | A_{ST,J}^{(\mu)} | k_0 L'' \rangle^* \right. \\ & \times \left. \left[P \left(\frac{1}{\ell^2 - k_0^2} \right) - P \left(\frac{1}{k^2 - k_0^2} \right) \right] \right. \\ & \left. + \sum_Y \phi_L^{(YJ)}(\ell) \left[\phi_{L'}^{(YJ)}(k) \right]^* = 0 \quad , \right. \end{aligned} \quad (65)$$

where we note from Eq. (17) that

$$\sum_{L''} \langle \ell L | A_{ST,J}^{(\mu)} | k_0 L'' \rangle \langle kL' | A_{ST,J}^{(\mu)} | k_0 L'' \rangle^* = \text{real.}$$

III.2 Review of the Calculations of $\langle k_1 k_2 | R(t_2, t_1) | k_3 k_4 \rangle$ with Tensor Forces.

We wish to determine the general form of the matrix elements of the two-body operator

$$R(t_2, t_1) = - \frac{\partial}{\partial t_1} \left\{ e^{t_2 H_0^{(2)}} e^{-(t_2 - t_1) H^{(2)}} e^{-t_1 H_0^{(2)}} \right\}, \quad (1)$$

which appears in Mohling's formulation of Quantum Statistical Mechanics ¹.

We shall then identify the functions $f_1(k_1 k_2 | k_3 k_4)$ and $f_2(k_1, k_2 | k_3 k_4 | k_3 k_4)$ which occur in the definition of certain two-body reaction matrix elements, $g_1(k_1 k_2 | k_3 k_4)$, which occur conspicuously in Mohling's Theory of Nuclear Matter ².

Using the method and notation of Section V of Reference (2), one can show that the pair function, which is the symmetrized matrix element of $R(t_2, t_1)$ is given by

$$\begin{aligned} t_2 \begin{bmatrix} k_1 k_2 \\ k_3 k_4 \end{bmatrix}_{t_1} &= \langle k_1 k_2 | R(t_2, t_1) | \ell_1 \ell_2 \rangle + \epsilon \langle k_1 k_2 | R(t_2, t_1) | \ell_2 \ell_1 \rangle \\ &= \left[\frac{(2\pi)^3}{\Omega^2} \right] \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{\ell}_1 - \vec{\ell}_2) \end{aligned}$$

¹Mohling, F., Phys. Rev., 122, 1961, p. 1043.

²Mohling, F., Phys. Rev., 124, 1961, p. 583.

$$\begin{aligned}
& \sum_{\substack{ST \\ M_S M_{S1} M_T}} \left(\frac{1}{2} \frac{1}{2} m_1 m_2 \middle| \begin{smallmatrix} SM_S \\ SM_{S1} \end{smallmatrix} \right) \left(\frac{1}{2} \frac{1}{2} n_1 n_2 \middle| \begin{smallmatrix} SM_S \\ SM_{S1} \end{smallmatrix} \right) \\
& \times \left(\frac{1}{2} \frac{1}{2} p_1 p_2 \middle| \begin{smallmatrix} TM_T \\ TM_T \end{smallmatrix} \right) \left(\frac{1}{2} \frac{1}{2} q_1 q_2 \middle| \begin{smallmatrix} TM_T \\ TM_T \end{smallmatrix} \right) \\
& \left\{ \begin{aligned}
& e^{t_1 [\omega(k_{12}) - \omega(\ell_{12})]} \left[\begin{aligned}
& - \langle \ell_{12} | G^{(+)} | k_{12} \rangle^* \\
& + i \pi \delta_{k_0} [\omega(k_0) - \omega(k_{12})] \langle k_{12} | G^{(+)} | k_0 \rangle \langle \ell_{12} | G^{(+)} | k_0 \rangle^*
\end{aligned} \right] \\
& + \sum_{k_0} \langle k_{12} | G^{(+)} | k_0 \rangle \langle \ell_{12} | G^{(+)} | k_0 \rangle^* P \left[\frac{1}{\omega(k_{12}) - \omega(k_0)} \right] \\
& \quad \times e^{t_2 [\omega(k_{12}) - \omega(k_0)]} e^{t_1 [\omega(k_0) - \omega(\ell_{12})]} \\
& + \sum_{\gamma} [\omega(\ell_{12}) - \omega(\gamma)] \langle k_{12} | \gamma \rangle \langle \gamma | \ell_{12} \rangle e^{t_2 [\omega(k_{12}) - \omega(\gamma)]} e^{t_1 [\omega(\gamma) - \omega(\ell_{12})]} \\
& + \varepsilon(-)^{S+T} \times \left\{ \text{corresponding expression with } \vec{\ell}_{12} \rightarrow \vec{\ell}_{21} = -\vec{\ell}_{12} \right\}
\end{aligned} \right\} \quad (2)
\end{aligned}$$

In Eq. (2), $G^{(+)}$ is the outgoing reaction matrix defined in Section I.1, Eq. (12) or (14), γ refers to the bound states of the Hamiltonian $H^{(2)}$, $k = (\vec{k}; \frac{1}{2}, m; \frac{1}{2}, p)$, and $k_0 = (\vec{k}_0; SM_S''; TM_T)$ refers to the continuum states of $H^{(2)}$. We have also used the condensed notation $k_{12} = (\vec{k}_{12}; SM_S; T_1 M_T)$ and $\ell_{12} = (k_{12}, SM_S'; TM_T)$. In addition we have used the fact that for nuclear forces $S = \text{spin}$ and $(T_1 M_T) = \text{isotopic spin}$ are all good quantum numbers. As usual

the symbol $(s_1 s_2 m_1 m_2 | SM_s)$ is the Clebsch-Gordon coefficient and $\epsilon = -1$ for Fermi statistics. Finally, we have

$$\sum_{\mathbf{k}_0} \rightarrow \frac{1}{(2\pi)^3} \int d^3 k_0 \sum_{SM_s''} \sum_{T M_T}$$

for continuum states.

Upon substituting the partial-wave expansion for $G^{(+)}$; Section III.1, Eqs. (2) and (22), into Eq. (4) one can show after some manipulations that

$$\begin{aligned} t_2 \begin{bmatrix} k_1 k_2 \\ k_3 k_4 \end{bmatrix}_{t_1} &= \frac{(2\pi)^6}{2} \left(\frac{2\hbar^2}{\pi M} \right) \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{\ell}_1 - \vec{\ell}_2) \\ &\times \sum_{\substack{ST \\ M_S M_S', M_T}} \left(\frac{1}{2} \frac{1}{2} m_1 m_2 | SM_s \right) \left(\frac{1}{2} \frac{1}{2} n_1 n_2 | SM_s' \right) \left(\frac{1}{2} \frac{1}{2} p_1 p_2 | T M_T \right) \\ &\quad \left(\frac{1}{2} \frac{1}{2} q_1 q_2 | T M_T \right) \\ &\times \sum_{\substack{LL'J \\ M_L M_L' M}} e^{i(L'-L)\pi/2} [1 + \epsilon(-)]^{S+T+L'} Y_{L'M'}(n_\ell) Y_{LM_L}(n_k) \times \\ &\quad \times (LSM_L M_S | JM) (L'SM_L' M_S' | JM) \\ &k_{12}^{-1} e^{t_1[\omega(k_{12}) - \omega(\ell_{12})]} \langle \ell_{12} L' | A_{ST,J}^{(+)} | k_{12} L \rangle^* \\ &\quad + i \sum_{L''} \langle k_{12} L | A_{ST,J}^{(+)} | k_{12} L'' \rangle \langle \ell_{12} L' | A_{ST,J}^{(+)} | k_{12} L'' \rangle^* \\ &+ \frac{2}{\pi} \int d\mathbf{k}_0 \sum_{L''} \langle k_{12} L | A_{ST,J}^{(+)} | k_0 L'' \rangle \langle \ell_{12} L' | A_{ST,J}^{(+)} | k_0 L'' \rangle^* P\left(\frac{1}{k_{12}^2 - k_0^2}\right) \end{aligned}$$

$$\begin{aligned}
& \times e^{t_2[\omega(k_{12}) - \omega(k_0)]} e^{t_1[\omega(k_0) - \omega(\ell_{12})]} \\
& + \frac{1}{k_{12} \ell_{12}} \sum_{\gamma} (\ell^2 + \gamma^2) \phi_L^{(J,ST)*}(\ell_{12}) \phi_L^{(\gamma J,ST)}(k_{12}) \\
& \times e^{t_2[\omega(k_{12}) - \omega(\gamma)]} e^{t_1[\omega(\gamma) - \omega(\ell_{12})]} \quad (3)
\end{aligned}$$

where $\langle kL | A_{ST,J}^{(+)} | k_0 L' \rangle$ is defined by Section III.1, Eq. (6) as

$$\begin{aligned}
\langle kL | A_{ST,J}^{(+)} | k_0 L' \rangle &= -\frac{1}{k} \sum_{\xi_1 \xi_2} \frac{d^3 r}{r} \phi^{M^+}(n_r, \xi_1, \xi_2) {}^+ TM_T \\
&\quad \zeta^1 \zeta^2 \\
&\times F_L(kr) U(r) \langle r \xi_1 \xi_2 | SJM; L \rangle^{(\mu)} \chi_{TM_T}(\zeta_1 \zeta_2) \quad (4)
\end{aligned}$$

The bound-state function $\phi_L^{(\gamma J,ST)}(k)$ are defined by Eq. (2) in Section I.4 and can be related readily to the $\langle k | \gamma \rangle$ of Eq. (2) by using Eq. (14) of Section I.1.

We next rewrite Eq. (3) with the aid of Eqs. (17) and (19) in Section III.1, as

$$\begin{aligned}
& t_2 \begin{bmatrix} k_1 k_2 \\ k_3 k_4 \end{bmatrix}_{t_1} = \frac{(2\pi)^6}{\pi^2} \left(\frac{2M^2}{\pi M} \right) \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{\ell}_1 - \vec{\ell}_2) \\
& \times \sum_{\substack{ST \\ M_S M'_S M_T}} \left(\frac{1}{2} \frac{1}{2} m_1 m_2 | SM_S \right) \left(\frac{1}{2} \frac{1}{2} n_1 n_2 | SM'_S \right) \left(\frac{1}{2} \frac{1}{2} p_1 p_2 | TM_T \right) \left(\frac{1}{2} \frac{1}{2} q_1 q_2 | TM_T \right)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{LL'J \\ M_L M_L' M}} e^{i(L'-L)\pi/2} [1 + \epsilon(-)^{S+T+L'}] Y_{L'M_L'}^* (\hat{n}_\ell) Y_{LM_L} (\hat{n}_k) \\
& \times (LSM_L M_S | JM) (L'SM_L' M_S' | JM) \\
& \times \langle k_{12} L | R_{ST,J} (t_2 t_1) | \ell_{12} L' \rangle, \quad (5)
\end{aligned}$$

where

$$\begin{aligned}
& \langle kL | R_{ST,J}(t_2 t_1) | \ell L' \rangle = \\
& \frac{1}{k} e^{t_1[\omega(k) - \omega(\ell)]} \text{Re}(\langle \ell L' | A_{ST,J}^{(+)} | kL \rangle^*) \\
& + \frac{2}{\pi} \int dk_0 \sum_{L''} \langle kL | A_{ST,J}^{(+)} | k_0 L'' \rangle \langle \ell L' | A_{ST,J}^{(+)} | k_0 L'' \rangle P\left(\frac{1}{k^2 - k_0^2}\right) \\
& \times e^{t_2[\omega(k) - \omega(k_0)]} e^{t_1[\omega(k_0) - \omega(\ell)]} \\
& + \frac{1}{k\ell} \sum_{\gamma} (\ell^2 + k^2) \phi_L^{(\gamma J, ST)}(\ell) \phi_L^{(\gamma J, ST)}(k) \\
& \times e^{t_2[\omega(k) - \omega(\gamma)]} e^{t_1[\omega(\gamma) - \omega(\ell)]} \\
& = \text{real}, \quad (6)
\end{aligned}$$

$$\text{and } \omega(\gamma) \equiv -\frac{\hbar^2}{M} \gamma^2.$$

We also write down the sum over spin states of the diagonal matrix elements of Eq. (5). Thus one may verify, for $\epsilon = -1$, that

$$\begin{aligned}
\sum_{m_2 q_2}^{t_2} \begin{bmatrix} k_1 k_2 \\ k_1 k_2 \end{bmatrix}_{t_1} &= \frac{1}{4} \sum_{m_1 m_2 q_1 q_2}^{t_2} \begin{bmatrix} k_1 k_2 \\ k_1 k_2 \end{bmatrix}_{t_1} \\
&= \frac{(2\pi)^3}{\Omega} \left(\frac{\hbar^2}{M} (8\pi^2)^{-1} \sum_{JLST} (2T+1)(2J+1) [1 - (-)^{S+T+L}] \right. \\
&\quad \times \langle k_{12}^L | R_{JST}(t_2, t_1) | k_{12}^L \rangle, \quad (7)
\end{aligned}$$

where, using Eq. (6) and Section II.2, Eq. (20),

$$\begin{aligned}
\langle k_L | R_{JST}(t_2, t_1) | k_L \rangle &= (2k)^{-1} \cos 2\bar{\epsilon}_J(k) \sin 2\bar{\delta}_{JL}(k) \\
&+ \frac{2}{\pi} \sum_{L''} \int_0^\infty dk_0 \langle k_L | A_{JST}^{(+)} | k_0^{L''} \rangle^2 P\left(\frac{1}{k^2 - k_0^2}\right) e^{(t_2 - t_1)[\omega(k) - \omega(k_0)]} \\
&+ \frac{1}{k^2} \sum_{\gamma} (\gamma^2 + k^2) [\phi_L^{(\gamma J, ST)}(k)]^2 e^{(t_2 - t_1)[\omega(k) - \omega(\gamma)]} \quad (8)
\end{aligned}$$

The functions $f_1(k_1 k_2 | \ell_1 \ell_2)$ and $f_2(k_1 k_2 | k_5 k_6 | \ell_1 \ell_2)$ of reference (2) are given by

$$f_1(k_1 k_2 | \ell_1 \ell_2) = \left[\frac{(2\pi)^3}{\Omega} \right]^2 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{\ell}_1 - \vec{\ell}_2) \tilde{f}_1(k_{12} | \ell_{12}) \quad (9)$$

where

$$\begin{aligned}
 \tilde{f}_1(k_{12} | \ell_{12}) &= \left(\frac{2\hbar^2}{\pi M k} \right) \sum_{S M_S M_S' M_T} \left(\frac{1}{2} \frac{1}{2} m_1 m_2 | S M_S \right) \left(\frac{1}{2} \frac{1}{2} n_1 n_2 | S M_S' \right) \\
 &\times \left(\frac{1}{2} \frac{1}{2} p_1 p_2 | T M_T \right) \left(\frac{1}{2} \frac{1}{2} q_1 q_2 | T M_T \right) \\
 &\times \sum_{L L' J M} e^{i(L'-L)\pi/2} [1 - (-)^{S+T+L}] Y_{L'M_L}(\hat{\ell}_{12}) Y_{LM_L}(\hat{k}_{12}) \\
 &\times (L S M_S | J M) (L' S M_S' | J M) \operatorname{Re} \langle \ell L' | A_{JST}^{(+)} | k L \rangle^*, \quad (10)
 \end{aligned}$$

and

$$\begin{aligned}
 f_2(k_1 k_2 | k_5 k_6 | \ell_1 \ell_2) &= \left[\frac{(2\pi)^3}{\Omega} \right]^2 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{\ell}_1 - \vec{\ell}_2) \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{k}_5 - \vec{k}_6) \\
 &\times \tilde{f}_2(k_{12} | k_{56} | \ell_{12}), \quad (11)
 \end{aligned}$$

where

$$\begin{aligned}
 \sum_{k_0} \tilde{f}_2(k_{12} | k_0 | \ell_{12}) e^{-(t_2 - t_1)\omega(k_0)} &= \\
 \left(\frac{2\hbar^2}{\pi M} \right) \sum_{S M_S M_S' M_T} &\left(\frac{1}{2} \frac{1}{2} m_1 m_2 | S M_S \right) \left(\frac{1}{2} \frac{1}{2} n_1 n_2 | S M_S' \right) \left(\frac{1}{2} \frac{1}{2} p_1 p_2 | T M_T \right) \left(\frac{1}{2} \frac{1}{2} q_1 q_2 | T M_T \right) \\
 \sum_{L L' J M} e^{i(L'-L)\pi/2} &[1 - (-)^{S+T+L}] Y_{L'M_L}^*(\hat{\ell}_{12}) Y_{LM_L}(\hat{k}_{12}) \\
 &\times (L S M_S | J M) (L' S M_S' | J M)
 \end{aligned}$$

$$\left\{ \begin{aligned} & \frac{2}{\pi} \sum_{L''} \int_0^{\infty} dk_0 \langle kL | A_{JST}^{(+)} | k_0 L'' \rangle \langle \ell L' | A_{JST}^{(+)} | k_0 L' \rangle^* \\ & P\left(-\frac{1}{k^2 - k_0^2}\right) e^{-(t_2 - t_1)\omega(k_0)} \\ & + \frac{1}{k\ell} \sum_{\gamma} (\ell^2 + \delta^2) \phi_L^{(\gamma J, ST)}(k) \phi_{L'}^{(\gamma J, ST)}(\ell) e^{-(t_2 - t_1)\omega(\gamma)} \end{aligned} \right\} \quad (12)$$

Finally, we may write out the reaction matrix² $g_1(k_1 k_2 | k_3 k_4)$,

$$\begin{aligned} g(k_1 k_2 | k_3 k_4) &= \left[\frac{(2\pi)^3}{\Omega} \right]^2 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \left(\frac{2\pi^2}{\pi M} \right) \\ &\times \sum_{S M_S M_S' M_T} \left(\frac{1}{2} \frac{1}{2} m_1 m_2 | S M_S \right) \left(\frac{1}{2} \frac{1}{2} r_1 n_2 | S M_S' \right) \left(\frac{1}{2} \frac{1}{2} p_1 p_2 | T M_T \right) \\ &\quad \left(\frac{1}{2} \frac{1}{2} q_1 q_2 | T M_T \right) \\ &\times \sum_{L L' J M} e^{i(L' - L)\pi/2} [1 - (-)^{S+T+L}] Y_{L' M'}^* (\hat{\ell}_{12}) Y_{L M} (\hat{k}_{12}) \\ &\quad \times (L S M_L M_S | J M) (L' S' M_L' M_S' | J M) g_1(k_{12} | k_{34})_{Q, JST, LL'}, \end{aligned} \quad (13)$$

where (with $Q = \frac{1}{2} (\vec{k}_1 + \vec{k}_2)$, $\vec{k} = \frac{1}{2} (\vec{k}_2 - \vec{k}_1)$, and $\vec{\ell} = \frac{1}{2} (\vec{k}_4 - \vec{k}_3)$)

$$\begin{aligned}
\tilde{g}_1(k, \ell)_{Q, JST, LL'} &\equiv \frac{1}{k} \operatorname{Re} \langle \ell L' | A_{ST, J}^{(+)} | kL \rangle \\
&+ \frac{2}{\pi} \sum_{L''} \int_0^\infty dk_0 \langle kL | A_{ST, J}^{(+)} | k_0 L'' \rangle \langle \ell L' | A_{ST, J}^{(+)} | k_0 L'' \rangle * \\
&\quad \times \left[P\left(\frac{1}{k^2 - k_0^2}\right) + P\left(\frac{1}{k_0^2 + \epsilon_k}\right) \right] \\
&+ \frac{1}{k\ell} \sum_Y (\ell^2 + \gamma^2)(k^2 + \delta^2) \phi_L^{(\gamma J, ST)}(k) \phi_{L'}^{(\gamma J, ST)}(\ell) \left[\frac{1}{k^2 + \gamma^2} + \frac{1}{\epsilon_\ell - \gamma^2} \right] \\
&+ g_{HC}(k, \ell)_{Q, JST, LL'} \quad , \tag{14}
\end{aligned}$$

where

$$\epsilon_\ell \equiv Q^2 - \frac{M}{2} [\omega^{\vec{Q}-\vec{\ell}} + \omega^{\vec{Q}+\vec{\ell}}] - \ell^2. \tag{15}$$

$\omega = \omega$

In Eq. (15), $\omega^{\vec{Q}}$ is the quasi-particle energy-momentum relation given in reference (2). By setting $\epsilon = -\ell^2$ in Eq. (15) we must get all but one of the terms in the closure relation, Eq. (65) of Section III.1. The hard-core correction term, g_{HC} , is derived in the Appendix of reference (2). It accounts for the change in the "size" of the Hilbert space for two nucleons when one uses the mathematical idealization of an infinite repulsive core in the two-particle interaction. One finds

$$g_{\text{HC}}(k|\ell)_{\text{Q,JST,LL}'} = (k^2 + \epsilon_k) \delta_{L,L'} \delta_{M_S,M_S'} \left[\frac{1}{2} < k | U_L(0,0) | \ell > \right]$$

where $\frac{1}{2\pi} < k | U_L(0,0) | \ell >$ is given by Eq. (A11) of reference 2.

III.3 Explicit Expressions for the A Matrix Integrals

From Eq. (6) of Sec. III.1, the partial-wave reaction matrix may be written as

$$\begin{aligned} \langle kL | A_{JST}^{(+)} | k_0 L' \rangle = k^{-1} \sum_{\substack{\xi_1 \xi_2 \\ \xi_1 \xi_2}} \int \frac{d^3 r}{r^2} [\phi_{LST}^M(\hat{r}, \xi_1, \xi_2)]^\dagger F_L(kr) \chi_{TM_T}^+(\xi_1, \xi_2) \\ U(\vec{r}, \vec{\sigma}_1, \vec{\sigma}_2, \tau_1, \tau_2) \chi_{TM_T}(\xi_1, \xi_2) \langle \vec{r} \xi_1 \xi_2 | k_0 S J M; L' \rangle^{(+)} \end{aligned} \quad (1)$$

where $F_L(kr)$ represents the free-particle state for an incoming plane-wave, and $\frac{\hbar^2 U}{M}$ is the nuclear interaction. The wave-functions are given explicitly in Sec. I.1, Eq. (50).

Using Sec. I.1, Eqs. (35)-(49) we obtain

$$\langle kJ | A_{J0T}^{(+)} | k_0 J \rangle = -k^{-1} \int_a^\infty dr F_J(kr) U_c^{(1)}(r) u_J^{(1)}(r) \quad (2)$$

for the singlet $S = 0$ states, and for the triplet $S = 1$ states we find

$$\begin{aligned} \langle kL | A_{JIT}^{(+)} | k_0 J \rangle = -k^{-1} \int_a^\infty dr F_L(kr) U_c^{(3)}(r) \left[u_J^{(-)}(r) \delta_{L,J-1} \right. \\ \left. + w_J^{(-)}(r) \delta_{L,J+1} \right] \\ -k^{-1} \int_a^\infty dr F_L(kr) U_T^{(3)}(r) \left[\frac{2(J-1)}{2J+1} \delta_{L,J-1} + \frac{6\sqrt{J(J+1)}}{2J+1} \delta_{L,J+1} \right] u_J^{(-)}(r) \\ -k^{-1} \int_a^\infty dr F_L(kr) U_T^{(3)}(r) \left[\frac{6\sqrt{J(J+1)}}{2J+1} \delta_{L,J-1} - \frac{2(J+2)}{2J+1} \delta_{L,J+1} \right] w_J^{(-)}(r) \\ \langle kJ | A_{JIT}^{(+)} | k_0 J \rangle = -k^{-1} \int_a^\infty dr F_J(kr) \left[U_c^{(3)}(r) u_J^{(3)}(r) + 2 U_T^{(3)}(r) u_J^{(3)}(r) \right] \\ \langle kL | A_{JIT}^{+} | k_0, J-1 \rangle = -k^{-1} \int_a^\infty dr F_L(kr) U_c^{(3)}(r) \left[u_J^{(+)}(r) \delta_{L,J-1} + w_J^{(+)}(r) \delta_{L,J+1} \right] \end{aligned} \quad (3)$$

Equations (2) and (3) would also have spin-orbit contributions except that we have not included them elsewhere in the interaction.

If we define the real integral

$$h_J^{(\alpha)}(k) \equiv k^{-1} \int_a^{\infty} dr F_J(kr) f^{\alpha}(r), \quad \alpha = c, T \quad (4)$$

which is evaluated explicitly in Appendix 14, then Eqs. (2) and (3) can be evaluated in the form

$$\begin{aligned} \langle kL' | A_{J1T}^{(+)} | k_0 L \rangle &= \sum_{\alpha} h_{L'}^{(\alpha)}(k) \psi_{JLL}^{(\alpha)}(k_0) \quad \text{for } S = 1 \text{ only} \\ \langle kJ | A_{J0T}^{(+)} | k_0 J \rangle &= h_J^{(c,1)}(k) v_J^{(1,c)}(k_0) \quad \text{for } S = 0 \text{ only.} \end{aligned} \quad (5)$$

where $\psi_{JJJ}^{\alpha}(k_0) \equiv v_J^{(3,c)}(k_0)$.

The functions $\psi_{JJ-1L}^{(\alpha)}(k_0)$ and $\psi_{JJ+1L}^{(\alpha)}(k_0)$ involve the coupled triplet solutions. Writing out the explicit integrals involving these terms it can be shown that ($\alpha = c, T$)

$$\begin{aligned} \psi_{JJ-1L}^{(\alpha)}(k_0) &= \sigma_{\alpha} r_{\alpha} N_u^{(L)}(k_0) B_u^{J(\alpha)}(k_0) \\ \psi_{JJ+1L}^{(\alpha)}(k_0) &= \sigma_{\alpha} r_{\alpha} N_w^{(L)}(k_0) B_w^{J(\alpha)}(k_0) \end{aligned} \quad (6)$$

where the normalization factors are evaluated in Sec. I.3, Eq. (46) and the functions $B_u^{J(\alpha)}(k_0)$ and $B_w^{J(\alpha)}(k_0)$ are given explicitly in Sec. I.2, Eq. (51).

The determination of the expressions for $v_J^{(1,c)}(k_o)$ and $v_J^{(3,c)}(k_o)$ is straightforward using Eq. (2)-(4) and Sec. I.1, Eq. (25) and (26), along with Appendices 1,4,5, and 11, with the result that

$$v_J^{(1,c)}(k_o) = \sigma_c r_c N_v^{(J,1)} \left\{ g_J^{(c)1}(k_o) + \frac{2}{\pi} \sigma_c r_c B_c^{(J,o)}(k_o) K_J^{(c,c)}(k_o) \right\}$$

$$v_J^{(3,c)}(k_o) = \sigma_c r_c N_v^{(J,3)} \left\{ g_J^{(c),3}(k_o) + \frac{2}{\pi} \sigma_c r_c B_c^{(J,1)}(k_o) K_J^{(c,c)}(k_o) \right.$$

$$\left. + \frac{2}{\pi} \frac{2J+1}{3\sqrt{J(J+1)}} \sigma_T^{(J)} r_T \delta_{S,1} (1-\delta_{J,o}) B_T^{(J)}(k_o) K_J^{(c,T)}(k_o) \right\}$$

$$v_J^{(3,T)}(k_o) = \sigma_T^{(J)} r_T N_v^{(J,3)} \left[\frac{(2J+1) (1-\delta_{J,o})}{6 \sqrt{J(J+1)}} + \frac{1}{4} \delta_{J,o} \right]$$

$$\left\{ g_J^{(T)}(k_o) + \frac{2}{\pi} \sigma_c r_c B_T^{(J)}(k_o) K_J^{(c,T)}(k_o) \right. \\ \left. + \frac{2}{\pi} \sigma_T^{(J)} r_T \frac{(2J+1) (1-\delta_{J,o})}{2 \sqrt{J(J+1)}} \delta_{S,1} B_T^{(J)}(k_o) K_J^{(T,T)}(k_o) \right\} \quad (7)$$

where $B_c^{(J,S)}(k_o)$ and $B_T^{(J)}(k_o)$ are given explicitly in Sec. I.1, Eq. (21)-(22) and $g_J^\alpha(K)$ and $K^{(\alpha,\beta)}(k_o)$ are given explicitly in Appendices 1 and 4.

Hard-Core Contribution to the $A^{(+)}$ Matrix

The coupled triplet state wave-equations can be written (see Sec. I.1, Eq. (53) in the form

$$\left[\frac{d^2}{dr^2} + k_0^2 - \frac{J(J-1)}{r^2} \right] u_J(r) = \left[U_c^{(3)}(r) - \frac{2(J-1)}{(2J+1)} U_T^{(3)}(r) \right] u_J(r) + \frac{6\sqrt{J(J+1)}}{(2J+1)} U_T^{(3)}(r) w_J(r)$$

$$\left[\frac{d^2}{dr^2} + k_0^2 - \frac{(J+1)(J+2)}{r^2} \right] w_J(r) = \frac{6\sqrt{J(J+1)}}{(2J+1)} U_T^{(3)}(r) u_J(r) + \left[U_c^{(3)}(r) - \frac{2(J+2)}{2J+1} U_T^{(3)}(r) \right] w_J(r) \quad (8)$$

where $U_c^{(3)}(r)$ includes an infinite repulsive core. If we solve these coupled equations for a finite repulsive core and go to the limit of an infinite repulsive core, then we will find that for $r < a$

$$U_c^{(3)}(r) u_J(r) = U_c^{(3)}(r) w_J(r) = 0$$

because $u_J(r)$ and $w_J(r)$ decay exponentially under a repulsive barrier. Also, by definition, $U_T^{(3)}(r) = 0$ for $r < a$.

We assume now that

$$U_c^{(3)}(r) u_J(r) = D_J^{(c)}(k_0) \delta(r-a)$$

$$U_c^{(3)}(r) w_J(r) = D_J^T(k_0) \delta(r-a) \quad (9)$$

In order to determine $D_J^{(c)}(k_0)$ and $D_J^{(T)}(k_0)$, which could be zero, we integrate the coupled equations, Eq. (8), from $(a-\delta)$ to $(a+\delta)$ using the assumed expressions for $U_c^{(3)}(r)$ $u_J(r)$ and $U_c^{(3)}(r)$ $w_J(r)$. Then, taking the limit $\delta \rightarrow 0$, we find

$$D_J^{(c)}(k_0) = \lim_{\delta \rightarrow 0} \int_{a-\delta}^{a+\delta} dr \left(\frac{d^2}{dr^2} + k_0^2 - \frac{J(J-1)}{r^2} \right) u_J(r) \\ = \frac{d}{dr} u_J(r) \Big|_{r=a^+} - \frac{du_J(r)}{dr} \Big|_{r=a^-} \quad (10)$$

where $\frac{du_J(r)}{dr} \Big|_{r=a^-} = 0$ in the limit of an infinite core.

We conclude that

$$D_J^{(c)}(k_0) = \hat{u}_J(a^+) , \\ D_J^{(T)}(k_0) = \hat{w}_J(a^+) . \quad (11)$$

Therefore, by comparing the expressions in Eqs. (2), (3) and (5) we see that the $A^{(+)}$ matrices of Eq. (5) must be corrected. This can be readily done by letting the α sum in Eq. (5) go over $\alpha = (c, T, D)$ where D represents the derivative term to be included for the hard core.

Defining

$$h_J^{(D)}(k_0) = -k^{-1} F_J(ka) \quad (12)$$

and noting that

$$\left. \frac{d F_L^{\sim}(kr)}{dr} \right|_{r=a} = \frac{k}{\Delta_L(ka)} \quad , \quad (13)$$

upon differentiating Sec. I.2, Eq. (52) we obtain

$$\begin{aligned} \psi_{J J-1 L}^{(D)}(k_0) &= N_u^{(J)}(k_0) \left\{ \frac{k_0}{\Delta_{J-1}(k_0 a)} + \frac{2}{\pi} \sigma_c r_c B_u^{(J,c)}(k_0) \right. \\ &\quad \left. + \frac{2}{\pi} \sigma_T r_T B_u^{(J,T)}(k_0) B_{J-1}^{(T)}(k_0) \right\} \\ \psi_{J J+1 L}^{(D)}(k_0) &= N_w^J(k_0) \left\{ \frac{k_0}{\Delta_{J+1}(k_0 a)} - \frac{2}{\pi} \sigma_c r_c B_w^{(J,c)}(k_0) B_{J+1}^{(c)}(k_0) \right. \\ &\quad \left. + \frac{2}{\pi} \sigma_T r_T B_w^{(J,T)}(k_0) B_{J+1}^{(T)}(k_0) \right\} \end{aligned} \quad (14)$$

where the integral

$$B_l^\alpha(k_0) \equiv \int_0^\infty dk \frac{k g^{(\alpha)}(k)}{\Delta_l(ka)} P\left(\frac{1}{k^2 - k_0^2}\right) \quad (15)$$

is explicitly evaluated in Appendix 9.

Finally, differentiating Eq. (25) of Sec. I.2 we obtain

$$v_{Jk_0}^{(S)}(a) = N_V^{(J,S)}(k_0) \left\{ \frac{k_0}{\Delta_J(k_0 a)} + \frac{2}{\pi} \sigma_c(k_0) B_J^{(c)}(k_0) + \frac{2}{\pi} \frac{2J+1}{3\sqrt{J(J+1)}} (1-\delta_{J,0}) \delta_{S,1} \sigma_T^{(J)} r_T B_T^{(J)}(k_0) B_J^{(T)}(k_0) \right\} \quad (16)$$

which, upon identifying $\psi_{JJJ}^D(k_0) = v_{Jk_0}^{(1)}(a)$, and $v_J^{(1,c)}(k_0) = v_J^{(1,c)}(k_0) = v_{Jk_0}^{(o)}(a)$, completes the evaluation of the hard-core terms to the $A^{(+)}$ matrix.

Matrix Elements for Nuclear Matter Calculations

In Sec. III.2, Eq. (8), (12), and (14), the terms occur

$$\sum_L \langle kL | A_{JST}^{(+)} | k_0 L'' \rangle \langle \ell L | A_{JST}^{(+)} | k_0 L'' \rangle$$

in the determination of the matrix elements of $R(t_2, t_1)$. We may now write this expression in terms of known, explicitly determined functions, as

$$\langle kJ | A_{JOT}^{(+)} | k_0 J \rangle \langle \ell J | A_{JOT}^{(+)} | k_0 J \rangle^* = h_J^{(c,1)}(k) h_J^{*(c,1)}(\ell) |v_J^{(1,c)}(k_0)|^2 \quad (17)$$

$$\langle kJ | A_{J1T}^{(+)} | k_0 J \rangle \langle \ell J | A_{J1T}^{(+)} | k_0 J \rangle^* = \sum_{\alpha, \beta} h_J^{(\alpha)}(k) h_J^{*(\beta)}(\ell) \psi_{JJJ}^{(\alpha)}(k_0) \psi_{JJJ}^{(\beta)}(k_0) \quad (18)$$

$$\sum_{L''} \langle k, J-1 | A_{J1T}^{(+)} | k_0 L'' \rangle \langle \ell, J-1 | A_{J1T}^{(+)} | k_0 L'' \rangle^* =$$

$$\sum_{\alpha, \beta} h_{J-1}^{(\alpha)}(k) h_{J-1}^{(\beta)}(\ell) \left[\psi_{J, J-1, J-1}^{(\alpha)}(k_0) \psi_{J, J-1, J-1}^{(\beta)}(k_0) + \psi_{J, J-1, J+1}^{(\alpha)}(k_0) \psi_{J, J-1, J+1}^{(\beta)}(k_0) \right] \quad (19)$$

$$\begin{aligned}
& \sum_{L''} \langle k, J-1 | A_{J1T}^{(+)} | k_0 L \rangle \langle \ell, J+1 | A_{J1T}^{(+)} | k_0 L'' \rangle^* = \\
& \sum_{\alpha, \beta} k_{J-1}^{(\alpha)}(k) h_{J+1}^{(\beta)}(\ell) \left[\psi_{J, J-1, J-1}^{(\alpha)}(k_0) \psi_{J, J-1, J-1}^{(\beta)*}(k_0) + \psi_{J, J+1, J}^{(\alpha)}(k_0) \right. \\
& \qquad \qquad \qquad \left. + \psi_{J, J+1, J+1}^{(\beta)*}(k_0) \right] \\
& \sum_{L''} \langle k, J+1 | A_{J1T}^{(+)} | k_0 L'' \rangle \langle \ell, J-1 | A_{J1T}^{(+)} | k_0 L'' \rangle^* = \\
& \sum_{\alpha, \beta} h_{J+1}^{(\alpha)}(k_0) h_{J-1}^{(\beta)*}(\ell) \left[\psi_{J, J+1, J-1}^{(\alpha)}(k_0) \psi_{J, J-1, J-1}^{(\beta)*}(k_0) \right. \\
& \qquad \qquad \qquad \left. + \psi_{J, J+1, J+1}^{(\alpha)}(k_0) \psi_{J, J-1, J+1}^{(\beta)*}(k_0) \right] \quad (21)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{L''} \langle k, J+1 | A_{J1T}^{(+)} | k_0 L'' \rangle \langle \ell, J+1 | A_{J1T}^{(+)} | k_0 L'' \rangle^* = \\
& \sum_{\alpha, \beta} h_{J+1}^{(\alpha)}(k) h_{J+1}^{(\beta)*}(\ell) \left[\psi_{J, J+1, J+1}^{(\alpha)*}(k_0) \psi_{J, J+1, J+1}^{(\beta)*}(k_0) \right. \\
& \qquad \qquad \qquad \left. + \psi_{J, J+1, J+1}^{(\alpha)}(k_0) \psi_{J, J+1, J+1}^{(\beta)*}(k_0) \right] \quad (22)
\end{aligned}$$

where the α, β summations range over c, T , and D .

With this result, Eq. (18)-(22) and Eq. (5), the reaction matrix $g_1(k, \ell)_{Q, JST, LL'}$ can be investigated since we have completely determined all the constituent terms explicitly in terms of known evaluable functions.

PART IV: CONCLUSIONS

In this thesis we have obtained the exact analytical solutions to the Schrödinger wave equation for all angular momentum states for a two-nucleon, non-relativistic, spin-dependent and non-local interaction with a repulsive hard core (of diameter a).

Using contour integration techniques and a complete orthonormal set of functions on the interval (a, ∞) the spatial and momentum wave-functions were explicitly determined.

All the phase-shift parameters were exhibited in terms of known explicitly determined functions by making identification with the Blatt-Biedenharn and Stapp "bar" phase shifts.

Using the wave-function solutions, an explicit analytical determination of the deuteron bound-state conditions, RMS radius, and quadrupole moment was performed by further contour integration techniques.

The scattering length, effective range, and shape-dependent parameter were explicitly determined from a low-energy expansion of the S-wave phase shift expressions in terms of the strength and ranges of the central and tensor interaction, and the hard core diameter.

Rapid computational methods using continued fractions to evaluate exponential integrals and series expressions with complex arguments have been developed. Examples of the function evaluations are shown graphically.

Preliminary numerical calculations of the parameters of this exact analytical model were constructed and compared to the work of other authors. The results suggest that a more detailed numerical study on a very fast computer would be feasible and worthwhile.

In perspective to the major goal under consideration, this thesis represents an intermediate summary of results. We originally sought to investigate the analytical calculation of the thermodynamical potential and the ground-state energy of nuclear matter using our non-local central and tensor interaction model with the model parameters (hard core diameter, the strength and ranges of the central and tensor interaction) adjusted to give a reasonable fit to the scattering data and deuteron properties.

Although in this thesis we addressed ourselves primarily to the determination of the two-body wave functions with a central and tensor interaction, we outlined their use in the determination and properties of the reaction matrices. In particular, the reality conditions, the unitarity condition, and the closure condition are explicitly demonstrated for the tensor interaction.

The analysis and review of the two-body reaction matrix elements was carried to the point of determining explicit expressions for nuclear matter calculation matrix elements, so that an investigation of the kernel of the reaction matrix $g_1(k,l)_{Q,JST,LL'}$ is possible now for the tensor interaction of this thesis.

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APPENDIX 1

ON THE MOMENTUM TRANSFORMED POTENTIAL $g_\ell(k)$

The momentum-transformed potential for a spatially-separable Yukawa potential and hard core of diameter a is defined to be

$$g_\ell(k) \equiv \int_a^\infty dr f(r) F'_\ell(kr) , \quad (1)$$

where the separable potential $f(r)$ is (for $r > a$)

$$f(r) \equiv \lambda^2 e^{-\lambda(r-a)} , \quad (2)$$

and $F'_\ell(kr)$, which forms a complete orthonormal set over the interval $a \leq r \leq \infty$, is given by

$$\left. \begin{aligned} F'_\ell(kr) &\equiv \frac{F_\ell(kr)G_\ell(ka) - F_\ell(ka)G_\ell(kr)}{\Delta_\ell(ka)} \\ \Delta_\ell^2(ka) &\equiv F_\ell^2(ka) + G_\ell^2(ka) \\ F_\ell(\rho) &\equiv \rho j_\ell(\rho) \\ G_\ell(\rho) &\equiv -\rho \eta_\ell(\rho) \end{aligned} \right\} . \quad (3)$$

Here, $j_\ell(\rho)$ and $\eta_\ell(\rho)$ are the regular and irregular spherical Bessel functions.

The functions $F_\ell(\rho)$ and $G_\ell(\rho)$ can be written

$$\begin{aligned}\rho^\ell F_\ell(\rho) &= \hat{\alpha}_\ell(\rho) \sin \rho - \hat{\beta}_\ell(\rho) \cos \rho = \text{Im}[e^{i\rho}(\hat{\alpha}_\ell(\rho) - i\hat{\beta}_\ell(\rho))] , \\ \rho^\ell G_\ell(\rho) &= \hat{\beta}_\ell(\rho) \sin \rho + \hat{\alpha}_\ell(\rho) \cos \rho = \text{Re}[e^{i\rho}(\hat{\alpha}_\ell(\rho) - i\hat{\beta}_\ell(\rho))] ,\end{aligned}\quad (4)$$

where

$$\left. \begin{aligned}\hat{\alpha}_\ell(\rho) - i\hat{\beta}_\ell(\rho) &\equiv \sum_{s=0}^{\ell} \Gamma_{\ell s}(\rho) e^{-i(\ell-s)\pi/2} \\ \Gamma_{\ell s}(\rho) &\equiv \frac{(\ell + \frac{1}{2}, s)}{2^s} \rho^{\ell-s}\end{aligned}\right\} , \quad (5)$$

and the Hankel symbol $(\ell + \frac{1}{2}, s)$ is defined by

$$(\ell + \frac{1}{2}, s) \equiv \frac{(\ell + s)!}{(\ell - s)! s!} , \quad 0 \leq s \leq \ell . \quad (6)$$

Upon inserting Eq. (3) into (1) we obtain

$$\begin{aligned}g_\ell(k) &= \frac{\mu \lambda e^\mu}{\tau^\ell \Delta_\ell(\tau)} [\tau^{\ell-1} G_\ell(\tau) \int_\tau^\infty d\rho e^{-\gamma \rho} F_\ell(\rho) \\ &\quad - \tau^{\ell-1} F_\ell(\tau) \int_\tau^\infty d\rho e^{-\gamma \rho} G_\ell(\rho)] ,\end{aligned}\quad (7)$$

where we have defined $\rho = kr$, $\tau = ka$, $\mu = \lambda a$, and $\gamma = \mu/\tau = \lambda/k$.

Upon substituting Eq. (4) into (7) and using Eq. (5), we find the following form for the integrals occurring in Eq. (7):

$$\begin{aligned}
& \int_{\tau}^{\infty} d\rho e^{-\gamma\rho} F_l(\rho) \\
&= \sum_{s=0}^l \frac{(l+\frac{1}{2}, s)}{2^s} \int_{\tau}^{\infty} \frac{d\rho e^{-\gamma\rho}}{\rho} [\sin \rho \cos (l-s)\pi/2 - \cos \rho \sin (l-s)\pi/2] \\
& \int_{\tau}^{\infty} d\rho e^{-\gamma\rho} G_l(\rho) \\
&= \sum_{s=0}^l \frac{(l+\frac{1}{2}, s)}{2^s} \int_{\tau}^{\infty} \frac{d\rho e^{-\gamma\rho}}{\rho} [\sin \rho \sin (l-s)\pi/2 + \cos \rho \cos (l-s)\pi/2].
\end{aligned} \tag{8}$$

Integrals of the type appearing in Eq. (8) are related to the general exponential integral $E_n(z)$, of order n , defined by

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \tag{9}$$

This may be seen more clearly by noting that the integrals of Eq. (8) can be written as

$$\left. \begin{aligned} \int_{\tau}^{\infty} \frac{d\rho e^{-\gamma\rho}}{\rho^s} \cos \rho &= \operatorname{Re} \left\{ \int_{\tau}^{\infty} \frac{d\rho e^{-(\gamma-i)\rho}}{\rho^s} \right\} \\ \int_{\tau}^{\infty} \frac{d\rho e^{-\gamma\rho}}{\rho^s} \sin \rho &= \operatorname{Im} \left\{ \int_{\tau}^{\infty} \frac{d\rho e^{-(\gamma-i)\rho}}{\rho^s} \right\} \end{aligned} \right\}. \tag{10}$$

Upon making the change of variable $\rho = \tau x$ and using Eq. (9), we find for the integrals of Eq. (10) the solution

$$\int_{\tau}^{\infty} \frac{d\rho e^{-(\gamma-i)\rho}}{\rho^s} = \frac{1}{\tau^{s-1}} \int_1^{\infty} \frac{dx e^{-(\mu-i\tau)x}}{x^s} = \frac{1}{\tau^{s-1}} E_s(\mu - i\tau) . \quad (11)$$

Upon inserting Eqs. (10) and (11) into Eqs. (8) and (7) and using Eq. (5) we obtain

$$g_l(k) = \frac{\lambda \mu e^{\mu}}{\tau^l \Delta_l(\tau)} \left\{ \begin{array}{l} G_l(\tau) \sum_{s=0}^l \Gamma_{ls}(\tau) \operatorname{Im}\{e^{-i(l-s)\pi/2} E_s(\mu-i\tau)\} \\ - F_l(\tau) \sum_{s=0}^l \Gamma_{ls}(\tau) \operatorname{Re}\{e^{-i(l-s)\pi/2} E_s(\mu-i\tau)\} \end{array} \right\} . \quad (12)$$

It is now convenient to define two new functions $W_l(\mu, \tau)$ and $Z_l(\mu, \tau)$ by

$$\left. \begin{array}{l} W_l(\mu, \tau) \equiv e^{\mu-i\tau} [\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)] \\ Z_l(\mu, \tau) \equiv \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} E_s(\mu-i\tau) \end{array} \right\} . \quad (13)$$

Using Eqs. (4) and (13), it is easily shown that

$$W_l(\mu, \tau) = e^{\mu} \tau^l [G_l(\tau) - iF_l(\tau)] . \quad (14)$$

We next define

$$H_l^{(\pm)}(\tau) \equiv G_l(\tau) \pm iF_l(\tau) , \quad (15)$$

and use Eqs. (13) through (15) to write the final expression for the momentum transformed potential, $g_\ell(k)$, as

$$g_\ell(k) = \frac{\lambda \mu e^\mu}{\tau^\ell \Delta_\ell(\tau)} \operatorname{Im}[H_\ell^{(-)}(\tau) Z_\ell(\mu, \tau)] \quad (16)$$

$$= \frac{\lambda \mu}{\tau^\ell \Delta_\ell(\tau)} \operatorname{Im}\left[\frac{W_\ell(\mu, \tau) Z_\ell(\mu, \tau)}{\tau^\ell}\right], \quad (17)$$

where $\tau = ka$ and $\mu = \lambda a$. We note that the singular behavior of $g_\ell(k)$, when extended to the complex k -plane, depends only upon the zeros of $\tau^\ell \Delta_\ell(\tau)$, τ^ℓ , and the poles of $Z_\ell(\mu, \tau)$, since $W_\ell(\mu, \tau)$ is without poles.

For general $\ell > 0$ the above expressions cannot be written in simple functional form because of the occurrence of the exponential integrals in the function $Z_\ell(\mu, \tau)$. However, for $\ell = 0$ we obtain the simple result that

$$g_0(k) = \frac{\lambda \mu \tau}{\mu^2 + \tau^2} = \frac{k}{1 + k^2/\lambda^2}, \quad (18)$$

which is independent of the core diameter a . This result can be derived directly from Eq. (1).

Low-Momentum Limit

It is of interest to compare the low-momentum limit of the expression given in Eq. (16) for $g_\ell(k)$, obtained by letting $k \rightarrow 0$, with two other direct calculational results: (1) Finite hard core diameter $a \neq 0$, $k \rightarrow 0$; and (2) zero hard core diameter $a = 0$, $k \rightarrow 0$. First we obtain the low-momentum behavior of the function $Z_\ell(\mu, \tau)$,

In Appendix 2, Eqs. (42), the low momentum behavior of the function $Z_\ell(\mu, \tau)$, defined by Eq. (13) is determined to be

$$\left. \begin{aligned} \text{Re}(Z_\ell(\mu, \tau)) &\xrightarrow{\tau \rightarrow 0} \sum_{p=0}^{\leq \ell/2} \frac{(2\ell-2p-1)!!}{2^p p!} E_{\ell-2p}(\mu) \tau^{2p} \\ &+ \sum_{p=0}^{\leq \frac{\ell-1}{2}} \frac{(2p-1)!!}{2^{\ell-p} (\ell-p)!} \alpha_{\ell-2p}(\mu) \tau^{2\ell-2p} \\ &+ \sum_{p=1}^{\infty} \frac{(-)^p \alpha_{\ell+2p}(\mu) \tau^{2\ell+2p}}{(2p-1)!! 2^{\ell+p} (\ell+p)!} \end{aligned} \right\} , \quad (19)$$

and

$$\text{Im}(Z_\ell(\mu, \tau)) \xrightarrow{\tau \rightarrow 0} \sum_{p=0}^{\infty} \frac{(-)^p \alpha_{\ell+2p+1}(\mu) \tau^{2\ell+2p+1}}{2^p p! (2\ell+2p+1)!!} , \quad (20)$$

where

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots, \quad (21)$$

$$\alpha_n(z) \equiv \int_1^{\infty} dt e^{-zt} t^n, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \quad (22)$$

$$= \frac{n!}{z^{n+1}} e^{-z} e_n(z),$$

$$e_n(z) \equiv \sum_{p=0}^n \frac{z^p}{p!}, \quad (23)$$

and $(2l+1)!! \equiv (2l+1)(2l-1)(2l-3)\dots 5.3.1$ with $(-1)!! = 1$ and

$$(2l)!! = 2^l l!$$

In order to determine the low-momentum behavior of $g_l(k)$ to $O(\tau^{l+1})$ we note that

$$\left. \begin{aligned} j_l(\tau) &\rightarrow \frac{\tau^l}{(2l+1)!!} \\ \eta_l(\tau) &\rightarrow -\frac{(2l-1)!!}{\tau^{l+1}} \end{aligned} \right\}, \quad (24)$$

which, when substituted into Eq. (3), implies

$$\left. \begin{aligned} F_l(\tau) &\rightarrow \frac{\tau^{l+1}}{(2l+1)!!} \\ G_l(\tau) &\rightarrow \frac{(2l-1)!!}{\tau^l} \\ \tau^l \Delta_l(\tau) &\rightarrow (2l-1)!! \end{aligned} \right\}. \quad (25)$$

Upon substituting Eqs. (19) through (25) into Eq. (16) and keeping only the leading terms, we find that

$$g_l(k) \xrightarrow[\tau \rightarrow 0]{} \frac{\lambda \mu e^\mu}{(2l+1)!!} [\alpha_{l+1}(\mu) - E_l(\mu)] \tau^{l+1} \quad (26)$$

$$\xrightarrow[\tau \rightarrow 0]{} \frac{\lambda(l+1)!}{(2l+1)!!} \left[\sum_{n=0}^{l+1} \frac{\mu^n}{n!} - \frac{\mu^{l+2} e^\mu E_l(\mu)}{(l+1)!} \right] (\tau/\mu)^{l+1},$$

where $\tau = ka$ and $\mu = \lambda a$. Now

$$\mu^{l+2} e^\mu E_l(\mu) \xrightarrow[\mu \rightarrow 0]{} \begin{cases} \mathcal{O}(\mu) & \text{if } l = 0 \\ \mathcal{O}(\mu^3 \ln \mu) & \text{if } l = 1 \\ \mathcal{O}(\mu^{l+2}) & \text{if } l \geq 2 \end{cases} \quad (27)$$

Using Eq. (27) in (26) we immediately see that as $a \rightarrow 0$, a finite contribution to $g_l(k)$ remains; namely,

$$g_l(k) \xrightarrow[k \rightarrow 0, a \rightarrow 0]{} \frac{\lambda(l+1)!}{(2l+1)!!} (k/\lambda)^{l+1}. \quad (28)$$

We may check the results obtained in Eqs. (26) and (28) quite simply. Assuming small k , we substitute Eq. (25) into (7) and obtain the immediate result that

$$g_l(k) \xrightarrow[k \rightarrow 0]{} \frac{\lambda \mu e^\mu}{\tau^l \Delta_l(\tau)} \left[\frac{\tau^{l-1} G_l(\tau)}{(2l+1)!!} \int_{\tau}^{\infty} d\rho e^{-\gamma \rho} \rho^{l+1} \right. \quad (29)$$

$$\left. - \tau^{l-1} (2l-1)!! F_l(\tau) \int_{\tau}^{\infty} \frac{d\rho e^{-\gamma \rho}}{\rho^l} \right].$$

Making the change of variable $\rho = \tau x$ and using Eqs. (21) and (22), we obtain

$$g_l(k) \xrightarrow{k \rightarrow 0} \frac{\lambda \mu e^\mu}{\tau^l \Delta_l(\tau)} \left[\frac{\tau^{2l+1} G_l(\tau) \alpha_{l+1}(\mu)}{(2l+1)!!} - (2l-1)!! F_l(\tau) E_l(\mu) \right] \quad (30)$$

Upon inserting Eq. (25) into (30) we then obtain the result

$$g_l(k) \xrightarrow{k \rightarrow 0} \frac{\lambda(l+1)!}{(2l+1)!!} \left[\sum_{n=0}^{l+1} \frac{\mu^n}{n!} - \frac{\mu^{l+2} e^\mu E_l(\mu)}{(l+1)!} \right] (\tau/\mu)^{l+1} \quad (31)$$

which is in complete agreement with Eq. (26).

Finally we evaluate the case of zero hard core diameter in the limit of small k by setting $a = 0$ in Eq. (1) and evaluating Eq. (1) directly. From Eq. (3) it follows that

$$F'_l(\rho) \xrightarrow{a \rightarrow 0} F_l(\rho). \quad (32)$$

Then the integral to be solved, $\bar{g}(k)$, is given by

$$\bar{g}_l(k) = \int_0^\infty dr f(r) F_l(kr) = \gamma \int_0^\infty d\rho e^{-\gamma\rho} \rho j_l(\rho) \quad (33)$$

where $\gamma \equiv \lambda/k$. But from¹

$$\int_0^\infty \rho e^{-\gamma \rho} \rho^{\mu-1} J_\nu(\beta \rho) d\rho = \frac{(\beta/2)^\nu \Gamma(\nu+\mu)}{(\gamma^2+\beta^2)^{\frac{\mu}{2}(\mu+\nu)} \Gamma(\nu+1)} {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}, \nu+1; \frac{\beta^2}{\gamma^2+\beta^2}\right), \quad (34)$$

it follows that (with $\mu = 3/2$, $\nu = \ell + 1/2$, $\gamma = \lambda/k$, $\beta = 1$)

$$\bar{g}_\ell(k) = \frac{\lambda\sqrt{\pi}}{2^{\ell+1}} \frac{\Gamma(\ell+2)}{\Gamma(\ell+3/2)} \frac{(k/\lambda)^{\ell+1}}{(1+k^2/\lambda^2)^{\frac{\ell+2}{2}}} {}_2F_1\left(\frac{\ell+2}{2}, \frac{\ell}{2}, \ell+3/2; \frac{k^2/\lambda^2}{1+k^2/\lambda^2}\right). \quad (35)$$

Now

$${}_2F_1(a, b, c, z) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \quad (36)$$

is convergent on the unit circle $z = 1$. Hence, $\bar{g}_\ell(k)$ is convergent for all k in the interval $0 \leq k \leq \infty$, and is given by

$$\bar{g}_\ell(k) = \frac{\lambda\sqrt{\pi}}{2^{\ell+1}} \frac{\Gamma(\ell+2)}{\Gamma(\ell/2)\Gamma(\frac{\ell+2}{2})} \frac{(k/\lambda)^{\ell+2}}{(1+k^2/\lambda^2)^{\frac{\ell+2}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\ell+2}{2}+n)\Gamma(\frac{\ell}{2}+n)}{n!\Gamma(\ell+\frac{3}{2}+n)} \frac{(k^2/\lambda^2)^n}{(1+k^2/\lambda^2)^n}. \quad (37)$$

As $k \rightarrow 0$, we obtain the low-momentum limit of $\bar{g}_\ell(k)$ for zero hard core radius; namely,

¹Watson, G. N., A TREATISE ON THE THEORY OF BESSEL FUNCTIONS, Cambridge Univ. Press, Cambridge, 1952, p. 385.

$$\bar{g}_\ell(k) \xrightarrow[k \rightarrow 0]{} \frac{\lambda(\ell+1)!}{(2\ell+1)!!} (k/\lambda)^{\ell+1}, \quad (38)$$

in complete agreement with the result obtained in Eq. (28).

In conclusion, we have derived the general form, Eqs. (16) and (17), for the momentum transformed potential $g_\ell(k)$ in the case of a spatially-separable Yukawa plus hard-core potential, Eq. (2), and have shown that the order of taking the limits $a \rightarrow 0$ and $k \rightarrow 0$ are independent.

APPENDIX 2

ON THE LOW-MOMENTUM BEHAVIOR OF $Z_\ell(\mu, \tau)$

We define the complex function $Z_\ell(\mu, \tau)$ of exponential integrals by

$$Z_\ell(\mu, \tau) = \sum_{s=0}^{\ell} \Gamma_{\ell s}(\tau) e^{-i(\ell-s)\pi/2} E_s(\mu - i\tau), \quad (1)$$

where

$$\Gamma_{\ell s}(\tau) = \frac{(\ell + \frac{1}{2}, s)}{2^s} \tau^{\ell-s}, \quad (2)$$

and the Hankel symbol, $(\ell + 1/2, s)$ is defined by

$$(\ell + \frac{1}{2}, s) = \frac{(\ell + s)!}{s!(\ell - s)!}. \quad (3)$$

The exponential integral, $E_s(\mu - i\tau)$, of order s , is defined by

$$E_s(\mu - i\tau) = \int_1^{\infty} \frac{dx e^{-(\mu - i\tau)x}}{x^s},$$

$$\operatorname{Re}(\mu - i\tau) > 0, \quad s = 0, 1, 2, \dots, \quad (4)$$

and, in the notation of the text, $\tau = ka$, $\mu = \lambda a$.

For a finite hard-core diameter $a \neq 0$, we are interested in the low-momentum limit, $\tau \rightarrow 0$ ($k \rightarrow 0$), of $Z_\ell(\mu, \tau)$ with μ held fixed. Upon expanding $e^{i\tau x}$ in the expression for $E_s(\mu - i\tau)$, Eq. (4), we obtain

$$E_s(\mu - i\tau) = \int_1^\infty \frac{dx e^{-(\mu - i\tau)x}}{x^s} = \int_1^\infty \frac{dx e^{-\mu x} e^{i\tau x}}{x^s}$$

$$\xrightarrow{\tau \rightarrow 0} \sum_{k=0}^\infty \frac{(i\tau)^k}{k!} \int_1^\infty \frac{dx e^{-\mu x} x^k}{x^s} . \quad (5)$$

The summation of Eq. (5) is now divided into two intervals according to whether $k \leq s$ or $k \geq s$. Upon utilizing the definitions

$$\alpha_l(\mu) \equiv \int_1^\infty dx e^{-\mu x} x^l = \frac{l!}{\mu^{l+1}} e^{-\mu} e_l(\mu);$$

$$\operatorname{Re}(\mu) > 0; \quad l = 0, 1, 2, \dots, \quad (6)$$

$$e_l(\mu) \equiv \sum_{n=0}^l \frac{\mu^n}{n!}, \quad (7)$$

Equation (5) becomes

$$E_s(\mu - i\tau) \xrightarrow{\tau \rightarrow 0} \sum_{k=0}^s \frac{(i\tau)^{s-k}}{(s-k)!} E_k(\mu) + \sum_{k=1}^\infty \frac{(i\tau)^{s+k}}{(s+k)!} \alpha_k(\mu). \quad (8)$$

Upon inserting Eq. (8) into Eq. (1), we obtain

$$Z_l(\mu, \tau) \xrightarrow{\tau \rightarrow 0} \sum_{s=0}^l \sum_{k=0}^s \frac{(\frac{l+1}{2}, s)(-)^{l-s}}{2^s(s-k)!} E_k(\mu) (i\tau)^{l-k}$$

$$+ \sum_{s=0}^l \sum_{k=1}^\infty \frac{(\frac{l+1}{2}, s)(-)^{l-s}}{2^s(s+k)!} \alpha_k(\mu) (i\tau)^{l+k}. \quad (9)$$

If we designate

$$Z_l(\mu, \tau) = Z_l^{(1)}(\mu, \tau) + iZ_l^{(2)}(\mu, \tau),$$

corresponding to the two summations of Eq. (9), and consider the real and imaginary parts of $Z_l^{(i)}(\mu, \tau)$ for $i = 1, 2$, then we see that we now need to examine the low-momentum behavior of the following functions:

$$\left. \begin{aligned} X_l^{(1)}(\mu, \tau) &\equiv \operatorname{Re} Z_l^{(1)}(\mu, \tau) = \operatorname{Re} \sum_{s=0}^l \sum_{k=0}^s \frac{(l+\frac{1}{2}, s)(-)^{l-s}}{2^s(s-k)!} E_k(\mu)(i\tau)^{l-k} \\ X_l^{(2)}(\mu, \tau) &\equiv \operatorname{Re} Z_l^{(2)}(\mu, \tau) = \operatorname{Re} \sum_{s=0}^l \sum_{k=1}^{\infty} \frac{(l+\frac{1}{2}, s)(-)^{l-s}}{2^s(s+k)!} \alpha_k(\mu)(i\tau)^{l+k} \\ Y_l^{(1)}(\mu, \tau) &\equiv \operatorname{Im} Z_l^{(1)}(\mu, \tau) = \operatorname{Im} \sum_{s=0}^l \sum_{k=0}^s \frac{(l+\frac{1}{2}, s)(-)^{l-s}}{2^s(s-k)!} E_k(\mu)(i\tau)^{l-k} \\ Y_l^{(2)}(\mu, \tau) &\equiv \operatorname{Im} Z_l^{(2)}(\mu, \tau) = \operatorname{Im} \sum_{s=0}^l \sum_{k=0}^s \frac{(l+\frac{1}{2}, s)(-)^{l-s}}{2^s(s+k)!} \alpha_k(\mu)(i\tau)^{l+k} \end{aligned} \right\} \quad (10)$$

Clearly, because of the i^{l+k} factors, terms in $X_l^{(1)}(\mu, \tau)$ and $X_l^{(2)}(\mu, \tau)$ do not contribute unless $l \pm k$ is an even integer. Similarly, terms in $Y_l^{(1)}(\mu, \tau)$ and $Y_l^{(2)}(\mu, \tau)$ do not contribute unless $l \pm k$ is an odd integer.

Upon defining

$$\begin{aligned} Q_{n,m}(-\frac{1}{2}) &\equiv \frac{1}{n!} \sum_{s=m}^n \binom{n}{s} \left(-\frac{1}{2}\right)^s \frac{(s+n)!}{(s-m)!}, \\ R_{n,m}(-\frac{1}{2}) &\equiv \frac{1}{n!} \sum_{s=0}^n \binom{n}{s} \left(-\frac{1}{2}\right)^s \frac{(s+n)!}{(s+m)!}, \end{aligned} \quad (11)$$

Equations (10) may be rewritten as

$$\left. \begin{aligned} X_{\ell}^{(1)}(\mu, \tau) &= \operatorname{Re} \sum_{k=0}^{\ell} (i\tau)^{\ell+k} (-)^{\ell} E_k(\mu) Q_{\ell,k}(-\frac{1}{2}) \\ Y_{\ell}^{(1)}(\mu, \tau) &= \operatorname{Im} \sum_{k=0}^{\ell} (i\tau)^{\ell-k} (-)^{\ell} E_k(\mu) Q_{\ell,k}(-\frac{1}{2}) \\ X_{\ell}^{(2)}(\mu, \tau) &= \operatorname{Re} \sum_{k=1}^{\infty} (i\tau)^{\ell+k} (-)^{\ell} \alpha_k(\mu) R_{\ell,k}(-\frac{1}{2}) \\ Y_{\ell}^{(2)}(\mu, \tau) &= \operatorname{Im} \sum_{k=1}^{\infty} (i\tau)^{\ell+k} (-)^{\ell} \alpha_k(\mu) R_{\ell,k}(-\frac{1}{2}) \end{aligned} \right\} \quad (10')$$

After performing the mapping $\ell \rightarrow 2\ell$ if ℓ is even and $\ell \rightarrow 2\ell+1$ if ℓ is odd, and keeping only the k indices corresponding to the evenness and oddness conditions mentioned above, Eqs. (10') may be written in the form

l even

$$X_{2l}^{(1)}(\mu, \tau) = (-)^{2l} \sum_{k=0}^l (-)^{l-k} E_{2k}(\mu) \tau^{2(l-k)} Q_{2l, 2k}(-\frac{1}{2})$$

$$X_{2l}^{(2)}(\mu, \tau) = (-)^{2l} \sum_{k=1}^{\infty} (-)^{l+k} \alpha_{2k}(\mu) \tau^{2(l+k)} R_{2l, 2k}(-\frac{1}{2})$$

$$Y_{2l}^{(1)}(\mu, \tau) = (-)^{2l} \sum_{k=0}^{l-1} (-)^{l-k+1} E_{2k+1}(\mu) \tau^{2(l-k)-1} Q_{2l, 2k+1}(-\frac{1}{2})$$

$$Y_{2l}^{(2)}(\mu, \tau) = (-)^{2l} \sum_{k=0}^{\infty} (-)^{l+k} \alpha_{2k+1}(\mu) \tau^{2l+2k+1} R_{2l, 2k+1}(-\frac{1}{2})$$

l odd

$$X_{2l+1}^{(1)}(\mu, \tau) = (-)^{2l+1} \sum_{k=0}^l (-)^{l-k} E_{2k+1}(\mu) \tau^{2(l-k)} Q_{2l+1, 2k+1}(-\frac{1}{2})$$

$$X_{2l+1}^{(2)}(\mu, \tau) = (-)^{2l+1} \sum_{k=0}^{\infty} (-)^{l+k+1} \alpha_{2k+1}(\mu) \tau^{2l+2k+2} R_{2l+1, 2k+1}(-\frac{1}{2})$$

$$Y_{2l+1}^{(1)}(\mu, \tau) = (-)^{2l+1} \sum_{k=0}^l (-)^{l-k} E_{2k}(\mu) \tau^{2l-2k+1} Q_{2l+1, 2k}(-\frac{1}{2})$$

$$Y_{2l+1}^{(2)}(\mu, \tau) = (-)^{2l+1} \sum_{k=1}^{\infty} (-)^{l+k} \alpha_{2k}(\mu) \tau^{2l+2k+1} R_{2l+1, 2k}(-\frac{1}{2})$$

(12)

Evaluation of Eqs. (12) and hence the low-momentum behavior of

$Z_\rho(\mu, \tau)$ has thus been reduced to the determination of the coefficient values $Q_{n,m}$ and $R_{n,m}$ for the explicit (n,m) pairs $(2l, 2k)$, $(2l+1, 2k)$, $(2l, 2k+1)$, and $(2l+1, 2k+1)$. For the $Q_{n,m}$ coefficients the restriction $m \leq n$ applies, but for the $R_{n,m}$ coefficients both the intervals $m \leq n$ and $m > n$ must be considered.

The Coefficients $Q_{n,m}$

We examine $Q_{n,m}$ first. From the binomial theorem we have

$$[a(a+1)]^n = \sum_{s=0}^n \binom{n}{s} a^{s+n}. \quad (13)$$

Differentiating Eq. (13) $(n+m)$ times and comparing the answer with Eq. (11) yields the result that

$$Q_{n,m}(a) = \frac{a^m}{n!} \frac{d^{n+m}}{da^{n+m}} ((a(a+1))^n) = \frac{1}{n!} \sum_{s=m}^n \binom{n}{s} a^s \frac{(s+n)!}{(s-m)!}, \quad (14)$$

where $m \leq n$. We shall be interested specifically in $a = -1/2$.

The largest m value for which $Q_{n,m}$ is non-vanishing is $m = n$, which yields the value

$$Q_{n,n}(a) = (2a)^n (2n-1)!! , \quad (15)$$

where

$$(2n+1)!! \equiv (2n+1)(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1, \quad (-1)!! \equiv 1.$$

Letting $m = n - 2p - 1$ for the interval $0 \leq p \leq \left(\frac{n-1}{2}\right)$ in Eq. (14),

we obtain

$$Q_{n,n-2p-1}\left(-\frac{1}{2}\right) = \frac{1}{n!} a^{n-2p-1} \frac{d^{2n-2p-1}}{da^{2n-2p-1}} ((a(a+1))^n) \Big|_{a=-\frac{1}{2}}. \quad (16)$$

Upon noting that (with $q < r$)

$$\begin{aligned}
 S_{q,r}(-\tfrac{1}{2}) &= \frac{d^{2q+1}}{da^{2q+1}} ((a(a+1))^r) \Big|_{a=-\frac{1}{2}} \\
 &= \frac{d^{2q}}{da^{2q}} (r(1+2a)(a(a+1))^{r-1}) \Big|_{a=-\frac{1}{2}} \\
 &= r \sum_{t=0}^{2q} \binom{2q}{t} \frac{d^t}{da^t} (1+2a) \Big|_{a=-\frac{1}{2}} \frac{d^{2q-t}}{da^{2q-t}} ((a(a+1))^{r-1}) \Big|_{a=-\frac{1}{2}} \\
 &= (2r)(2q) S_{q-1,r-1}(-\tfrac{1}{2}), \tag{17}
 \end{aligned}$$

It is easily deduced that

$$S_{q,r}(-\tfrac{1}{2}) = 2^{2k} \frac{r!}{(r-k)!} \frac{q!}{(q-k)!} S_{q-k,r-k}(-\tfrac{1}{2}) \tag{18}$$

for the interval $0 \leq k \leq q$. Upon substituting Eq. (18) into Eq. (16) we obtain

$$\begin{aligned}
 Q_{n,n-2p-1}(-\tfrac{1}{2}) &= \frac{1}{n!} (-\tfrac{1}{2})^{n-2p-1} S_{n-p-1,n}(-\tfrac{1}{2}) \\
 &= \frac{(-)^{n-2p-1} 2^{2k-n+2p+1} (n-p-1)!}{(n-k)!(n-p-1-k)!} S_{n-p-1-k,n-k}(-\tfrac{1}{2}). \tag{19}
 \end{aligned}$$

Letting $k = n-p-1$ in Eq. (19) and noting that

$$S_{0,p+1}(-\tfrac{1}{2}) = \frac{d}{da} ((a(a+1))^{p+1}) \Big|_{a=-\frac{1}{2}} = 0, \tag{20}$$

we conclude that

$$Q_{n,n-2p-1}(-\frac{1}{2}) = 0 \quad (21)$$

for the interval $0 \leq p \leq \left(\frac{n-1}{2}\right)$.

Next, we let $m = n-2p$ for the interval $0 \leq p \leq \left(\frac{n}{2}\right)$ in Eq. (14) and obtain

$$Q_{n,n-2p}(-\frac{1}{2}) = \frac{a^{n-2p}}{n!} \frac{d^{2n-2p}}{da^{2n-2p}} [(a(a+1))^n] \Big|_{a=-\frac{1}{2}}. \quad (22)$$

Upon noting that (with $q \leq r$)

$$\begin{aligned} T_{q,r}(-\frac{1}{2}) &= \frac{d^{2q}}{da^{2q}} ((a(a+1))^r) \Big|_{a=-\frac{1}{2}} \\ &= \frac{d^{2q-1}}{da^{2q-1}} (r(1+2a)(a(a+1))^{r-1}) \Big|_{a=-\frac{1}{2}} \\ &= r \sum_{t=0}^{2q-1} \binom{2q-1}{t} \frac{d^t}{da^t} (1+2a) \Big|_{a=-\frac{1}{2}} \frac{d^{2q-1-t}}{da^{2q-1-t}} ((a(a+1))^{r-1}) \Big|_{a=-\frac{1}{2}} \\ &= 2r (2q-1) T_{q-1,r-1}(-\frac{1}{2}), \end{aligned} \quad (23)$$

It is easily deduced that

$$T_{q,r}(-\frac{1}{2}) = 2^k \frac{r!}{(r-k)!} \frac{(2q-1)!!}{(2q-2k-1)!!} T_{q-k,r-k}(-\frac{1}{2}) \quad (24)$$

for the interval $0 \leq k \leq q$.

Upon substituting Eq. (24) into Eq. (22) with $q = n, p=k$,
we obtain

$$\begin{aligned}
 Q_{n,n-2p}(-\tfrac{1}{2}) &= \frac{(-)^{n-2p}}{n!} \frac{1}{2^{n-2p}} T_{n-p,n}(-\tfrac{1}{2}) \\
 &= \frac{(-)^{n-2p}}{(n-k)!} \frac{2^{k-n+2p} (2n-2p-1)!!}{(2n-2p-2k-1)!!} T_{n-p-k,n-k}(-\tfrac{1}{2}) \\
 &= \frac{(-)^{n-p}}{2^p p!} (2n - 2p - 1)!!
 \end{aligned} \tag{25}$$

for the interval $0 \leq p \leq (n/2)$ and where we have used the result that $T_{0,p}(-1/2) = (-)^p / 2^p$. When $p = 0$, Eq. (25) agrees with the earlier result in Eq. (15).

The Coefficients $R_{n,m}$

Next we consider the coefficients $R_{n,m}$ defined in Eq. (11). First we consider the case $m \leq n$, and in complete analogy with Eq. (14) we write

$$R_{n,m}(a) = \frac{a^{-m}}{n!} \frac{d^{n-m}}{da^{n-m}} ((a(a+1))^n) = \frac{1}{n!} \sum_{s=0}^n \binom{n}{s} a^s \frac{(s+n)!}{(s+m)!}, \tag{26}$$

which is valid for the interval $0 \leq m \leq n$. We need to consider the coefficients

$$\begin{aligned}
 &R_{n,n-2p-1}(-\tfrac{1}{2}) \quad \text{for } 0 \leq p \leq \left(\frac{n-1}{2}\right), \\
 &R_{n,n-2p}(-\tfrac{1}{2}) \quad \text{for } 0 \leq p \leq (n/2).
 \end{aligned} \tag{27}$$

It is clear from Eq. (11) that $Q_{n,n-2p-1} \rightarrow R_{n,n-2p-1}$ and $Q_{n,n-2p} \rightarrow R_{n,n-2p}$ under the mapping $p \rightarrow n-p-1$ and $p \rightarrow n-p$, respectively.

Applying this mapping to Eqs. (21) and (25) we obtain

$$R_{n,n-2p-1}(-\frac{1}{2}) = 0, \quad 0 \leq p \leq \left(\frac{n-1}{2}\right), \quad (28)$$

$$R_{n,n-2p}(-\frac{1}{2}) = \frac{(-)^p (2p-1)!!}{2^{n-p} (n-p)!} \quad 0 \leq p \leq (n/2),$$

which can also be verified directly by a calculation analogous to that performed on the $Q_{n,m}$'s. We note that $R_{n,m}(-1/2) = [2^n n!]^{-1}$.

Whenever $m > n$ the negative exponent of the differential operator for $R_{n,m}(a)$ in Eq. (26) suggests the use of multiple integration. If we let $m = n + k$, $n \geq 0$, $m \geq 0$ in Eq. (26), then the apparent form is suggested after defining

$$U_{n,n+k}(a) = n! a^{n+k} R_{n,n+k}(a)$$

$$R_{n,n+k}(a) = \frac{1}{n!} \sum_{s=0}^n \binom{n}{s} a^s \frac{(s+n)!}{(s+n+k)!}. \quad (29)$$

Upon integrating Eq. (29), it is clear that

$$U_{n,n+k+1}(a) = \int_0^a dx U_{n,n+k}(x) \quad (30)$$

and hence, by induction for $k \geq 1$ we may form

$$U_{n,n+k}(a) = \int_0^a dx_{k-1} \int_0^{x_{k-1}} dx_{k-2} \dots \int_0^{x_1} dx U_{n,n}(x) , \quad (31)$$

where $a = x_k \geq x_{k-1} \geq \dots \geq x_1 \geq 0$. Equation (31) can be rewritten as

$$U_{n,n+k}(a) = \int_0^a dx U_{n,n}(x) \phi_k(x, a) , \quad (32)$$

where

$$\begin{aligned} \phi_k(x, a) &= \int_x^a dx_{k-1} \dots \int_x^{x_3} dx_2 \int_x^{x_2} dx_1 \\ &= \int_x^a dx_{k-1} \dots \int_x^{x_3} dx_2 (x_2 - x) \\ &= \frac{(a-x)^{k-1}}{(k-1)!} . \end{aligned} \quad (33)$$

Thus,

$$U_{n,n+k}(x) = \frac{1}{(k-1)!} \int_0^a dx (a-x)^{k-1} U_{n,n}(x) , \quad (34)$$

where $U_{n,n}(x)$ is continuous on the interval $0 \leq x \leq a$.

Utilizing Eq. (34) in Eq. (29), where $U_{n,n}(x) = (1+x)^n x^n$, and making the change of variable $x = a(1 - \sin \theta)$, we have, for $a = -1/2$, the result that

$$R_{n,n+k}(-\frac{1}{2}) = \frac{1}{(k-1)!!2^n n!} \int_0^{\pi/2} d\theta \sin^{k-1}(\theta) \cos^{2n+1}(\theta) \quad (35)$$

for $n \geq 0$, $k \geq 1$. Assuming

$$\int_0^{\pi/2} d\theta \sin^{k-1}(\theta) \cos^{2n+1}(\theta) = \frac{(2n)!!(k-2)!!}{(2n+k)!!} , \quad (36)$$

which is satisfied for $k = 1, 2$, and 3 , we have by induction on k that

$$\begin{aligned} & \int_0^{\pi/2} d\theta \sin^{k+1}(\theta) \cos^{2n+1}(\theta) \\ &= \int_0^{\pi/2} d\theta (\sin^{k-1}(\theta) \cos^{2n+1}(\theta) - \sin^{k-1}(\theta) \cos^{2n+3}(\theta)) \\ &= (k-2)!! \left[\frac{(2n)!!}{(2n+k)!!} - \frac{(2n+2)!!}{(2n+k+2)!!} \right] \\ &= \frac{k!!(2n)!!}{(2n+k+2)!!} , \end{aligned} \quad (37)$$

so that Eq. (36) is indeed correct. Substituting Eq. (36) into Eq. (35) then yields the result

$$R_{n,n+k}(-\frac{1}{2}) = \frac{1}{(k-1)!!(2n+k)!!} \quad \text{for } n \geq 0, k \geq 0 . \quad (38)$$

Summary of Results

Summarizing the results obtained for the $Q_{n,m}$ and $R_{n,m}$ coefficients we have

$$\left. \begin{aligned} Q_{n,n-2p-1}(-\tfrac{1}{2}) &= 0, & 0 \leq p \leq \left(\tfrac{n-1}{2}\right) \\ Q_{n,n-2p}(-\tfrac{1}{2}) &= \frac{(-)^{n-p}(2n-2p-1)!!}{2^p p!}, & 0 \leq p \leq (n/2) \\ R_{n,n-2p-1}(-\tfrac{1}{2}) &= 0, & 0 \leq p \leq \left(\tfrac{n-1}{2}\right) \\ R_{n,n-2p}(-\tfrac{1}{2}) &= \frac{(-)^p(2p-1)!!}{2^{n-p}(n-p)!}, & 0 \leq p \leq (n/2) \\ R_{n,n+k+1}(-\tfrac{1}{2}) &= \frac{1}{k!!(2n+k+1)!!}, & k \geq 0 \end{aligned} \right\} \quad (39)$$

Substituting Eqs. (39) into Eqs. (12) we find

l even

$$X_{2l}^{(1)}(\mu, \tau) = \sum_{k=0}^l \frac{\tau^{2(l-k)} E_{2k}(\mu) (2l+2k-1)!!}{2^{l-k} (l-k)!}$$

$$X_{2l}^{(2)}(\mu, \tau) = \sum_{k=1}^l \frac{\tau^{2(l+k)} \alpha_{2k}(\mu) (2l-2k-1)!!}{2^{l+k} (l+k)!}$$

$$+ \sum_{k=0}^{\infty} \frac{(-)^{k+1} \tau^{4l+2k+2} \alpha_{2k+2l+2}(\mu)}{(2k+1)!! 2^{l+k+1} (k+2l+1)!}$$

$$Y_{2l}^{(1)}(\mu, \tau) = 0$$

$$Y_{2l}^{(2)}(\mu, \tau) = \sum_{k=0}^{\infty} \frac{(-)^k \tau^{4l+2k+1} \alpha_{2l+2k+1}(\mu)}{2^k k! (2l+4l+1)!!}$$

l odd

$$X_{2l+1}^{(1)}(\mu, \tau) = \sum_{k=0}^l \frac{\tau^{2(l-k)} E_{2k+1}(\mu) (2l+2k+1)!!}{2^{l-k} (l-k)!}$$

$$X_{2l+1}^{(2)}(\mu, \tau) = \sum_{k=0}^l \frac{\tau^{2l+2k+2} \alpha_{2k+1}(\mu) (2l-2k-1)!!}{2^{l+k+1} (l+k+1)!}$$

$$+ \sum_{k=0}^{\infty} \frac{(-)^{k+1} \tau^{4l+2k+4} \alpha_{2k+2l+3}(\mu)}{(2k+1)!! 2^{l+k+2} (2l+k+2)!}$$

$$Y_{2l+1}^{(1)}(\mu, \tau) = 0$$

$$Y_{2l+1}^{(2)}(\mu, \tau) = \sum_{k=0}^{\infty} \frac{(-)^k \tau^{4l+2k+3} \alpha_{2k+2l+2}(\mu)}{2^k k! (4l+2k+3)!}$$

(40)

Transforming the indices of Eqs. (40) by $2\ell \rightarrow \ell_{\text{even}}$ and $2\ell + 1 \rightarrow \ell_{\text{odd}}$, we obtain

$$\left. \begin{aligned}
 X_{\ell}^{(1)}(\mu, \tau) &= \sum_{p=0}^{\leq (\ell/2)} \frac{(2\ell-2p-1)!!}{2^p p!} E_{\ell-2p}(\mu) \tau^{2p} \\
 X_{\ell}^{(2)}(\mu, \tau) &= \sum_{p=0}^{\leq (\frac{\ell-1}{2})} \frac{(2p-1)!!}{2^{\ell-p} (\ell-p)!} \alpha_{\ell-2p}(\mu) \tau^{2\ell-2p} \\
 &+ \sum_{k=1}^{\infty} \frac{(-)^k \alpha_{\ell+2k}(\mu) \tau^{2\ell+2k}}{(2k-1)!! 2^{k+\ell} (k+\ell)!} \\
 Y_{\ell}^{(1)}(\mu, \tau) &= 0 \\
 Y_{\ell}^{(2)}(\mu, \tau) &= \sum_{k=0}^{\infty} \frac{(-)^k \alpha_{\ell+2k+1}(\mu) \tau^{2\ell+2k+1}}{2^k k! (2k+2\ell+1)!!}
 \end{aligned} \right\} \cdot \quad (41)$$

Hence, as $\tau \rightarrow 0$, we obtain the low momentum behavior of $Z_{\ell}(\mu, \tau)$ as

$$\left. \begin{aligned}
 \text{Re} Z_{\ell}(\mu, \tau) &\xrightarrow{\tau \rightarrow 0} \sum_{p=0}^{\leq (\ell/2)} \frac{(2\ell-2p-1)!!}{2^p p!} E_{\ell-2p}(\mu) \tau^{2p} \\
 &+ \sum_{p=0}^{\leq (\frac{\ell-1}{2})} \frac{(2p-1)!! \alpha_{\ell-2p}(\mu) \tau^{2\ell-2p}}{2^{\ell-p} (\ell-p)!} \\
 &+ \sum_{k=1}^{\infty} \frac{(-)^k \alpha_{\ell+2k}(\mu) \tau^{2\ell+2k}}{(2k-1)!! 2^{k+\ell} (k+\ell)!} \\
 \text{Im} Z_{\ell}(\mu, \tau) &\xrightarrow{\tau \rightarrow 0} \sum_{k=0}^{\infty} \frac{(-)^k \alpha_{\ell+2k+1}(\mu) \tau^{2\ell+2k+1}}{2^k k! (2k+2\ell+1)!!}
 \end{aligned} \right\} , \quad (42)$$

where the functions in Eq. (42) are defined as

$$\left. \begin{aligned} E_n(z) &\equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re} z > 0; \quad n = 0, 1, 2, \dots \\ \alpha_n(z) &\equiv \int_1^{\infty} dt t^n e^{-zt}, \quad \operatorname{Re} z > 0; \quad n = 0, 1, 2, \dots \\ (2l+1)!! &\equiv (2l+1)(2l-1)(2l-3) \dots 5.3.1 \\ (n/2) &= \begin{cases} n/2 & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases} \end{aligned} \right\} \quad (43)$$

APPENDIX 3

ON THE ZEROS OF THE FUNCTION $\tau^{2\ell} \Delta_\ell^2(\tau)$

The function $\tau^{2\ell} \Delta_\ell^2(\tau)$ occurs in the denominator of certain integrands of principle-value integrals to which the Cauchy Residue theorem is applied. The zeros of this function being poles of those integrands, we investigate their detailed structure in general, and in particular, make an explicit evaluation of all the zeros in the index range $0 \leq \ell \leq 10$.

The function $\Delta_\ell^2(\tau)$ is defined as

$$\Delta_\ell^2(\tau) \equiv G_\ell^2(\tau) + F_\ell^2(\tau) = H_\ell^{(+)}(\tau) H_\ell^{(-)}(\tau) , \quad (1)$$

where

$$\begin{aligned} H_\ell^{(+)}(\tau) &\equiv G_\ell(\tau) + iF_\ell(\tau) , \\ \tau^\ell G_\ell(\tau) &= \operatorname{Re}([\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)]e^{i\tau}) , \\ \tau^\ell F_\ell(\tau) &= \operatorname{Im}([\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)]e^{i\tau}) , \end{aligned} \quad (2)$$

and

$$\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau) \equiv \sum_{s=0}^{\ell} \frac{(\ell+\frac{1}{2}, s)}{2^s} e^{-i(\ell-s)\pi/2} \tau^{\ell-s} . \quad (3)$$

The Hankel symbol, $(\ell+\frac{1}{2}, s)$, is defined as

$$(\ell+\frac{1}{2}, s) \equiv \frac{(\ell+s)!}{(\ell-s)!s!} , \quad 0 \leq s \leq \ell . \quad (4)$$

Thus, after substituting Eq. (2) into (1) we see that we must examine the zeros of

$$\tau^{2l} \Delta_l^2(\tau) = \alpha_l^2(\tau) + \hat{\beta}_l^2(\tau) = [\hat{\alpha}_l(\tau) + i\beta_l(\tau)][\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)] \quad (5)$$

with regard to their distribution and value throughout the complex plane. These zeros will be very closely related to those of the spherical Hankel functions of the first and second kind, $h_l^{(1)}(\tau)$ and $h_l^{(2)}(\tau)$, which can be written as

$$h_l^{(1)}(\tau) = -\frac{ie^{i\tau}}{\tau^{l+1}} [\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)] , \quad (6)$$

$$h_l^{(2)}(\tau) = \frac{ie^{-i\tau}}{\tau^{l+1}} [\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)] .$$

It is known that $h_l^{(1)}(\tau)$ has distinct zeros located symmetrically about the imaginary axis and that if l is odd, one zero is on the imaginary axis.¹⁻³ The symmetry can be seen as follows: If $\alpha_l(\tau) + i\beta_l(\tau)$ has a complex root at $\tau = \gamma$, then $-\gamma$, the negative complex conjugate (using a bar to denote complex conjugation), is also a root (since $\alpha_l(\tau)$ is an even function and $\beta_l(\tau)$ is an odd function of (τ) because

¹Falkenberg, H. and Hilb, E., Göttinger Nachrichten, 1916, pp. 190-196.

²Hilb, E., Math. Z., 15, 1922, pp. 274-279.

³Falkenberg, H., Math. Z., 35, 1932, pp. 457-463.

$$\hat{\alpha}_l(-\bar{\gamma}) + i\hat{\beta}_l(-\bar{\gamma}) = \hat{\alpha}_l(\bar{\gamma}) - i\hat{\beta}_l(\bar{\gamma}) = \overline{[\hat{\alpha}_l(\gamma) + i\hat{\beta}_l(\gamma)]} = 0. \quad (7)$$

And for l odd the only way to maintain the symmetry $\gamma = -\bar{\gamma}$ is for one root to lie on the imaginary axis.

In order to obtain accurate values for the complex zeros of $\alpha_l(\gamma) + i\beta_l(\gamma)$, we determine the zeros of the polynomial $D_l(Z)$ defined by

$$D_l(Z) \equiv \sum_{s=0}^l (l + \frac{1}{2} - s) Z^{l-s}. \quad (8)$$

Hence, for γ a root of Eq. (5) we have

$$2^{-l} D_l(2i\gamma) = \hat{\alpha}_l(\gamma) + i\hat{\beta}_l(\gamma) = 0, \quad (9)$$

$$2^{-l} D_l(-2i\gamma) = \hat{\alpha}_l(\gamma) - i\hat{\beta}_l(\gamma) = 0.$$

For small values of l , algebraic solutions of Eq. (8) are readily obtained. If $l = 0$, there are no zeros because $D_0(Z) = 1$. If $l = 1$, $D_1(Z) = Z + 2$, and $\gamma = i$ is the root of $\alpha_l(\gamma) + i\beta_l(\gamma) = 0$. Similarly, for $l = 2$, we have $D_2(Z) = Z^2 + 6Z + 12$ and $\gamma = \frac{\pm 3}{2} + \frac{3i}{2}$ are the roots of $\alpha_2(\gamma) + i\beta_2(\gamma) = 0$. However, for $l \geq 5$, these algebraic techniques are no longer possible and numerical techniques must be invoked to obtain solutions.

Two different numerical methods were employed in the determination of the zeros of $D_l(Z)$:

(1) A method based on contour integration:^{4,5}

For $f(Z)$ any function analytic inside and on the contour Γ except at a finite number of poles inside Γ , then

$$\frac{1}{2\pi i} \int_{\Gamma} dZ \frac{f'(Z)}{f(Z)} = Z - P, \quad (10)$$

where

Z = the number of zeros inside Γ ,

P = the number of poles inside Γ .

In our case for the function $D_{\ell}(Z)$, $P = 0$. The roots of $D_{\ell}(Z)$ are bounded^{1,2,3} in magnitude by 2ℓ . Thus, there is no problem in constructing a square enclosing the zeros. The method then consists quite simply in following $\arg(f(Z))$ around the contour in order to determine the number of times that the phase passes through 2π . The contour is then successively subdivided until only one zero and/or one pole is contained in a given subcontour of size ϵ , where ϵ is a small quantity. Then, a method of steepest descent is applied for an accurate determination of the position of the zero or pole.

⁴Whittaker, E.T. and Watson, G. N., A COURSE IN MODERN ANALYSIS, Cambridge Univ. Press, Cambridge, 1962.

⁵Adapted by Dr. Hans Ruppel of Los Alamos Scientific Laboratory from an existing IBM-7030 program.

(2) A method based on d'Alembert's Lemma^{6,7}

d'Alembert's Lemma states that given a polynomial $p(Z)$ over the complex numbers C , then $|p(Z)|$ for $Z \in C$ assumes a minimum if and only if $|p(Z)| = 0$: that is, Z is a root. The set of numbers M representable as floating point numbers on an electronic computer to a given precision is only a small subset of C . However, by the use of sufficient computing precision the machine graphs of $|p(Z)|$ over M may be made arbitrarily close to those of $|p(Z)|$ over C . From d'Alembert's Lemma we know that the surface $|p(Z)|$ above the Z plane has no relative minima except where it touches the plane and that with sufficiently high precision we can utilize this result to find roots on the computer.

The method then consists in stepping around the complex plane to locate the minima of $|p(Z)|$. All polynomial manipulations are carried out in double precision with care being taken to avoid underflow and overflow. The root searching procedure is as follows: At an origin Z_0 , $|p(Z)|$ is computed at Z_0 and at $re^{in\pi/2}$ ($n = 0, 1, 2, 3$) relative to Z_0 , where r is a scale factor. If $|p(Z)|$

⁶Mottler, Z. C., "A Simple Method for Finding Zeros of Polynomials Based on d'Alembert's Lemma," LASL Report LA-3355, unpublished; also Mottler, Z. C., SIAM Review, 7, 1965, p. 614.

⁷Delone, B. L., "Algebra: Theory of Algebraic Equations," in MATHEMATICS: ITS CONTENTS, METHODS, AND MEANING, A. D. Aleksandrov, ed., Cambridge MIT Press, Cambridge, 1964, p. 21.

is minimal at one of the points $re^{in\pi/2}$, then that point is taken as the new origin. If there is relatively little variation in $|p(Z)|$ over the points (i.e., the first few digits are the same), the scale r is expanded. If $|p(Z)|$ is minimal at the origin, the scale r is contracted. The acceptance of a root is composed of many conditions and safeguards such as neither too few nor too many trials, and that $|p(Z)|$ is sufficiently small or exactly zero, but the most important is that at a root Z

$$|p(Z)| < 2^{-N/2} |a_0|, \quad (11)$$

where N is the number of binary bits in the floating fraction and a_0 is the coefficient of unity of the polynomial $p(Z) = a_0 + a_1 Z^n + \dots + a_n Z^n$. Once a root is determined, it is divided out by synthetic division and the next root is sought.

Both methods were applied to the polynomial $D_\ell(Z)$ for $0 \leq \ell \leq 10$ and yielded the same results to over ten significant figures, the specified accuracy. The corresponding numerical roots of $\hat{\alpha}(\gamma) + i\hat{\beta}(\gamma) = 0$ are summarized in Table A-3.1 and graphically illustrated in Figure A-3.1.

Table A-3.1

Roots of $\hat{\alpha}_l(\gamma) + i\hat{\beta}_l(\gamma) = 0$

$$\gamma = \pm \bar{\gamma}_{lm} + i\bar{\bar{\gamma}}_{lm}$$

for $0 \leq l \leq 10$

l	m	$\bar{\gamma}_{lm}$	$\bar{\bar{\gamma}}_{lm}$
1	1	0	1
2	1	$\sqrt{3}/2$	$3/2$
3	1	0	2.3221 85855
	2	1.7543 80960	1.8389 07323
4	1	0.86723 41290	2.8962 10603
	2	2.6574 18042	2.1037 89398
5	1	0	3.6467 38595
	2	1.7426 61416	3.3519 56399
	3	3.5710 22920	2.3246 74303
6	1	0.86750 96732	4.2483 59459
	2	2.6262 72311	3.7357 08352
	3	4.4926 72954	2.5154 32248
7	1	0	4.9717 86859
	2	1.7392 36061	4.7582 90528
	3	3.5171 74048	4.0701 39163
	4	5.4206 44131	2.6856 76879
8	1	0.86761 44454	5.5878 86043
	2	2.6161 75153	5.2048 47041
	3	4.4144 42500	4.3682 89217
	4	6.3539 11299	2.8389 83949

Table A-3.1 continued

9	1	0	6.2970 19182
	2	1.7378 48383	6.1293 67904
	3	3.4981 56918	5.6044 21819
	4	5.3172 71671	4.6384 89887
	5	7.2914 63688	2.9792 60799
10	1	0.86766 51455	6.9220 44905
	2	2.6115 67921	6.6152 90965
	3	4.3849 47889	5.9675 28385
	4	6.2249 85483	4.8362 19567
	5	8.2326 99459	3.1089 16234

NOTE: There are no roots for the case $l = 0$ since

$$\hat{\alpha}_0(\gamma) + i\hat{\beta}_0(\gamma) = 1 \text{ for all } \gamma$$

Figure A-3.1

Roots of $\hat{\alpha}_\ell(\gamma) + i\beta_\ell(\gamma) = 0$, $\gamma = \bar{\gamma}_\ell + i\bar{\gamma}_{\ell m}$

A Residue Calculation

Finally, we examine the function $\tau^{2\ell} \Delta_\ell^2(\tau)$ as it might occur in a denominator after application of the residue theorem. We characterize the roots of $\gamma^{2\ell} \Delta_\ell^2(\gamma) = \alpha_\ell^2(\gamma) + \beta_\ell^2(\gamma) = 0$ by $\gamma_{\ell m}$ for $m = 1, 2, \dots, [\ell+1/2]$.⁸ Further, we let $\gamma_{\ell 1}$ denote the single imaginary root when ℓ is odd. We define

$$\gamma_{\ell m}^{(+)} \equiv \bar{\gamma}_{\ell m} + i\bar{\gamma}_{\ell m} \quad (12)$$

for $m = 1, 2, \dots, [\ell+1/2]$. Obviously, under complex conjugation

$$\overline{(\gamma_{\ell m}^{(+)})} = (\gamma_{\ell m}^{(-)}) \quad (13)$$

Thus, by direct observation (or if one prefers by 'rotating' the general results of references 1-3) the roots of $\alpha_\ell(\gamma) + i\beta_\ell(\gamma) = 0$, namely $\gamma_{\ell m}^{(+)}$ and $-\gamma_{\ell m}^{(-)}$, are all in the upper half plane, and the roots of $\alpha_\ell(\gamma) - i\beta_\ell(\gamma) = 0$, namely $\gamma_{\ell m}^{(-)}$ and $-\gamma_{\ell m}^{(+)}$, are all in the lower half plane.

Let us now define a function $R_{\ell m}(\gamma)$ which would result from application of the residue theorem to the function $\tau^{2\ell} \Delta_\ell^2(\tau)$ by

$$R_{\ell m}(\gamma) = \frac{\gamma^{2\ell} \Delta_\ell^2(\gamma)}{\gamma - \gamma_{\ell m}} \underset{\gamma \rightarrow \infty}{\rightarrow} \gamma^{2\ell-1} \quad (14)$$

8. $\left[\frac{\ell+1}{2} \right] \equiv$ greatest integer in $\ell/2$, namely $\ell/2$ if ℓ is even and $\frac{\ell+1}{2}$ if ℓ is odd.

Clearly from the above enumeration of the roots

$$R_{lm}(\gamma)$$

$$= \begin{cases} \frac{(\gamma - \gamma_{l1}^{(+)}) (\gamma - \gamma_{l1}^{(-)})}{\gamma - \gamma_{lm}} \prod_{m'=2}^{[l+1/2]} (\gamma - \gamma_{lm'}^{(+)}) (\gamma + \gamma_{lm'}^{(-)}) (\gamma - \gamma_{lm'}^{(-)}) (\gamma + \gamma_{lm'}^{(+)}) & \text{if } l \text{ odd,} \\ \frac{1}{\gamma - \gamma_{lm}} \prod_{m'=1}^{[l+1/2]} (\gamma - \gamma_{lm'}^{(+)}) (\gamma + \gamma_{lm'}^{(-)}) (\gamma - \gamma_{lm'}^{(-)}) (\gamma + \gamma_{lm'}^{(+)}) & \text{if } l \text{ even.} \end{cases} \quad (15)$$

For calculational purposes we would need explicit expressions for

$$R_{lm}(\gamma_{lm}) = \lim_{\gamma \rightarrow \gamma_{lm}} R_{lm}(\gamma) = \left(\begin{array}{c} \text{finite well-defined odd function} \\ \text{of the root } \gamma_{lm} \end{array} \right), \quad (16)$$

where

where $\gamma_{lm} = \pm \gamma_{lm}^{(+)}$ or $\pm \gamma_{lm}^{(-)}$, and it can be shown that

$$R_{lm}(-\gamma_{lm}^{(+)}) = -R_{lm}(\gamma_{lm}^{(+)}) ,$$

$$R_{lm}(\gamma_{lm}^{(+)}) = R_{lm}(\gamma_{lm}^{(-)}) . \quad (17)$$

The second of Eqs. (17) will be seen to be true from the explicit expressions below. The first of Eqs. (17) follows directly, Eq. (15) being an odd function of γ_{lm} when $\gamma \rightarrow \gamma_{lm}$.

It is useful to write out Eq. (15) more explicitly for four different cases:

$$(1) \quad \gamma_{lm} = \gamma_{lm}^{(+)}, \quad \gamma \rightarrow \gamma_{lm}^{(+)} \quad \text{for } l \text{ even and odd,}$$

$$(2) \quad \gamma_{lm} = \gamma_{lm}^{(-)}, \quad \gamma \rightarrow \gamma_{lm}^{(-)} \quad \text{for } l \text{ even and odd.}$$

For l even, $\gamma_{lm} = \gamma_{lm}^{(+)}$, and $\gamma \rightarrow \gamma_{lm}^{(+)}$, we have

$$\begin{aligned} R_{lm}(\gamma_{lm}^{(+)}) &= \prod_{\substack{m'=1 \\ m' \neq m}}^{l/2} (\gamma_{lm}^{(+)} - \gamma_{lm'}^{(+)}) \\ &\times \prod_{m'=1}^{l/2} (\gamma_{lm}^{(+)} + \gamma_{lm'}^{(-)}) (\gamma_{lm}^{(+)} + \gamma_{lm'}^{(+)}) (\gamma_{lm}^{(+)} - \gamma_{lm'}^{(-)}) \\ &= 2\gamma_{lm}^{(+)} \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm}^{(-)})^2 \right] \\ &\times \prod_{\substack{m'=1 \\ m' \neq m}}^{l/2} \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm'}^{(-)})^2 \right] \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm'}^{(+)})^2 \right]. \end{aligned}$$

(18)

For l even, $\gamma_{lm} = \gamma_{lm}^{(-)}$ and $\gamma \rightarrow \gamma_{lm}^{(-)}$, we have

$$\begin{aligned}
 R_{lm}(\gamma_{lm}^{(-)}) &= \prod_{\substack{m'=1 \\ m' \neq m}}^{l/2} (\gamma_{lm}^{(-)} - \gamma_{lm'}^{(-)}) \\
 &\times \prod_{m'=1}^{l/2} (\gamma_{lm}^{(-)} - \gamma_{lm'}^{(+)}) (\gamma_{lm}^{(-)} + \gamma_{lm'}^{(-)}) (\gamma_{lm}^{(-)} + \gamma_{lm'}^{(+)}) \\
 &= 2\gamma_{lm}^{(-)} \left[(\gamma_{lm}^{(-)})^2 - (\gamma_{lm}^{(+)})^2 \right] \\
 &\times \prod_{\substack{m'=1 \\ m' \neq m}}^{l/2} \left[(\gamma_{lm}^{(-)})^2 - (\gamma_{lm'}^{(+)})^2 \right] \left[(\gamma_{lm}^{(-)}) - (\gamma_{lm'}^{(-)})^2 \right]. \quad (19)
 \end{aligned}$$

For l odd, $\gamma_{lm} = \gamma_{lm}^{(+)}$, and $\gamma \rightarrow \gamma_{lm}^{(+)}$, we have two subcases to consider: $m = 1$ and $m \neq 1$. If $m = 1$

$$R_{l1}(\gamma_{l1}^{(+)}) = 2i\overline{\gamma}_{l1} \prod_{m'=2}^{\frac{l+1}{2}} \left[\overline{\gamma}_{l1}^2 + (\gamma_{lm'}^{(+)})^2 \right] \left[\overline{\gamma}_{l1}^2 + (\gamma_{lm'}^{(-)})^2 \right], \quad (20)$$

and if $m \neq 1$

$$\begin{aligned}
 R_{lm}(\gamma_{lm}^{(+)}) &= \left[(\gamma_{lm}^{(+)})^2 + \overline{\gamma}_{l1}^2 \right] \prod_{\substack{m'=2 \\ m' \neq m}}^{\frac{l+1}{2}} (\gamma_{lm}^{(+)} - \gamma_{lm'}^{(+)}) \\
 &\quad \prod_{m'=2}^{\frac{l+1}{2}} (\gamma_{lm}^{(+)} + \gamma_{lm'}^{(-)}) (\gamma_{lm}^{(+)} + \gamma_{lm'}^{(+)}) (\gamma_{lm}^{(+)} - \gamma_{lm'}^{(-)}) \\
 &= 2\gamma_{lm}^{(+)} \left[(\gamma_{lm}^{(+)})^2 + \overline{\gamma}_{l1}^2 \right] \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm}^{(-)})^2 \right] \\
 &\quad \prod_{\substack{m'=2 \\ m' \neq m}}^{\frac{l+1}{2}} \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm'}^{(-)})^2 \right] \left[(\gamma_{lm}^{(+)})^2 - (\gamma_{lm'}^{(+)})^2 \right]. \quad (21)
 \end{aligned}$$

For l odd, $\gamma_{lm} = \gamma_{lm}^{(-)}$, and $\gamma \rightarrow \gamma_{lm}^{(-)}$, we again have two subcases to consider: $m = 1$ and $m \neq 1$. If $m = 1$

$$R_{l1}(\gamma_{l1}^{(-)}) = -2i\overline{\gamma}_{l1} \prod_{m'=2}^{\frac{l+1}{2}} \left[\overline{\gamma}_{l1}^2 + (\gamma_{lm'}^{(+)})^2 \right] \left[\overline{\gamma}_{l1}^2 + (\gamma_{lm'}^{(-)})^2 \right], \quad (22)$$

and if $m \neq 1$

$$\begin{aligned}
 R_{lm}(\gamma_{lm}^{(-)}) &= \left[(\gamma_{lm}^{(-)})^2 + \gamma_{l1}^2 \right] \prod_{\substack{m'=2 \\ m' \neq m}}^{\frac{l+1}{2}} \left[(\gamma_{lm}^{(-)} - \gamma_{lm'}^{(-)}) \right] \\
 &\times \prod_{m'=2}^{\frac{l+1}{2}} (\gamma_{lm}^{(-)} - \gamma_{lm'}^{(+)}) (\gamma_{lm}^{(-)} + \gamma_{lm'}^{(-)}) (\gamma_{lm}^{(-)} + \gamma_{lm'}^{(+)}) \\
 &= 2\gamma_{lm}^{(-)} \left[(\gamma_{lm}^{(-)})^2 + \gamma_{l1}^2 \right] \left[(\gamma_{lm}^{(-)})^2 - (\gamma_{lm'}^{(+)})^2 \right] \\
 &\times \prod_{\substack{m'=2 \\ m' \neq m}}^{\frac{l+1}{2}} \left[(\gamma_{lm}^{(-)})^2 - (\gamma_{lm'}^{(+)})^2 \right] \left[(\gamma_{lm}^{(-)})^2 - (\gamma_{lm'}^{(-)})^2 \right]. \quad (23)
 \end{aligned}$$

From Eqs. (13), and (18) through (23), it is thus seen upon examination that the second of Eq. (17) is indeed valid.

APPENDIX 4

ON THE PRINCIPAL-VALUE INTEGRAL $K_{\ell}^{\alpha\beta}(k_0)$

The integral $K_{\ell}^{\alpha\beta}(k_0)$ is defined as

$$K_{\ell}^{\alpha\beta}(k_0) \equiv \int_0^{\infty} dk g_{\ell}^{\alpha}(k) g_{\ell}^{\beta}(k) P\left(\frac{1}{k^2 - k_0^2}\right), \quad (1)$$

where $g_{\ell}^{\alpha}(k)$, the momentum transformed potential, has the explicit form (see Eq. (16) in Appendix 1)

$$g_{\ell}^{\alpha}(\tau) \equiv \frac{\lambda_{\alpha}^{\mu} e^{\mu_{\alpha}}}{\tau^{\ell} \Delta_{\ell}(\tau)} \operatorname{Im} \left[H_{\ell}^{-}(\tau) Z_{\ell}(\mu_{\alpha}, \tau) \right], \quad (2)$$

and the functions involved in Eq. (2) are defined by

$$\left. \begin{aligned} H_{\ell}^{(\pm)}(\tau) &\equiv G_{\ell}(\tau) \pm iF_{\ell}(\tau) \\ Z_{\ell}(\mu_{\alpha}, \tau) &\equiv \sum_{s=0}^{\ell} \Gamma_{\ell s}(\tau) e^{-i(\ell-s)\pi/2} E_s(\mu_{\alpha} - i\tau) \\ \Gamma_{\ell s}(\tau) &\equiv \frac{(\ell+1/2, s)}{2^s} \tau^{\ell-s} \\ (\ell+1/2, s) &\equiv \text{Hankel symbol} \equiv \frac{(\ell+s)!}{(\ell-s)!s!} \\ E_n(z) &\equiv \int_1^{\infty} \frac{dt}{t^n} e^{-zt}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \\ \Delta_{\ell}^2(\tau) &= H_{\ell}^{(+)}(\tau) H_{\ell}^{(-)}(\tau) = G_{\ell}^2(\tau) + F_{\ell}^2(\tau) \end{aligned} \right\}, \quad (3)$$

and the dimensions τ and μ are defined by

$$\begin{aligned}\tau &\equiv ka \\ \mu &\equiv \lambda a .\end{aligned}\tag{4}$$

The kernel of Eq. (1) is symmetric under the transformation $k \rightarrow -k$. This may be deduced directly from $g_\ell^\alpha(k)$ being antisymmetric under $k \rightarrow -k$, which is easily proved as follows:

Since

$$\left. \begin{aligned}H_\ell^{(+)}(-\tau) &= (-)^{\ell} H_\ell^{(+)}(\tau) = (-)^{\ell} \overline{H_\ell^{(+)}(\tau)} \\ Z_\ell(\mu, -\tau) &= \overline{Z_\ell(\mu, \tau)} \\ \Delta_\ell(-\tau) &= \Delta_\ell(\tau)\end{aligned}\right\} ,\tag{5}$$

where the bar indicates complex conjugation, it follows, using Eq. (5) in (2) under the transformation $k \rightarrow -k$, that

$$g_\ell^\alpha(-\tau) = - g_\ell^\alpha(\tau) .\tag{6}$$

Thus

$$\begin{aligned}
g_{\ell}^{\alpha}(-\tau) &= \frac{\lambda_{\alpha}^{\mu_{\alpha}} e^{\mu_{\alpha}}}{(-\tau)^{\ell} \Delta_{\ell}(-\tau)} \operatorname{Im} \left[H_{\ell}^{(-)}(-\tau) Z_{\ell}(\mu_{\alpha}, -\tau) \right] \\
&= \frac{\lambda_{\alpha}^{\mu_{\alpha}} e^{\mu_{\alpha}} (-)^{\ell}}{\tau^{\ell} \Delta_{\ell}(\tau)} \operatorname{Im} \left[(-)^{\ell} H_{\ell}^{(+)}(\tau) \overline{Z_{\ell}(\mu_{\alpha}, \tau)} \right] \\
&= - \frac{\lambda_{\alpha}^{\mu_{\alpha}} e^{\mu_{\alpha}}}{\tau^{\ell} \Delta_{\ell}(\tau)} \operatorname{Im} \left[H_{\ell}^{(-)}(\tau) Z_{\ell}(\mu_{\alpha}, \tau) \right] \\
&= - g_{\ell}^{\alpha}(\tau),
\end{aligned}$$

inasmuch as for z complex $\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$.

The kernel of Eq. (1) being symmetric permits extension of the range of integration to $-\infty \leq k \leq \infty$. Transforming to the variable τ of Eq. (4), Eq. (1) may be rewritten as

$$a_{\ell}^{-1} K_{\ell}^{\alpha\beta}(\tau_0) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau g_{\ell}^{\alpha}(\tau) g_{\ell}^{\beta}(\tau) P\left(\frac{1}{\tau^2 - \tau_0^2}\right). \quad (7)$$

The evaluation of Eq. (7) is quite feasible via contour integration, using Cauchy's Residue Theorem. However, prior to specification of the contour, we examine the behavior of the kernel in two domains: $\tau \rightarrow 0$ and $\tau \rightarrow \infty$.

In Appendix 1 [see Eq. (31)], it is proved that as $\tau \rightarrow 0$

$$g_{\ell}^{\alpha}(\tau) \xrightarrow{\tau \rightarrow 0} \lambda_{\alpha} \frac{(\ell+1)!}{(2\ell+1)!!} C_{\ell}(\mu_{\alpha}) (\tau/\mu_{\alpha})^{\ell+1}, \quad (8)$$

where the coefficient $C_{\ell}(\mu_{\alpha})$ is given by

$$C_{\ell}(\mu_{\alpha}) \equiv \sum_{n=0}^{\ell+1} \frac{(\mu_{\alpha})^n}{n!} - \frac{\mu_{\alpha}^{\ell+2} e^{\mu_{\alpha}} E_{\ell}(\mu_{\alpha})}{(\ell+1)!}. \quad (9)$$

Thus, as $\tau \rightarrow 0$ the kernel of Eq. (7) behaves as

$$\frac{a^2 g_\ell^\alpha(\tau) g_\ell^\beta(\tau)}{\tau^2 - \tau_0^2} \rightarrow_{\tau \rightarrow 0} \frac{\mu_\alpha \mu_\beta e^{\mu_\alpha + \mu_\beta}}{\tau^2 - \tau_0^2} \left[\frac{(\ell+1)!}{(2\ell+1)!!} \right]^2 c_\ell(\mu_\alpha) c_\ell(\mu_\beta) (\tau^2 / \mu_\alpha \mu_\beta)^{\ell+1}. \quad (10)$$

Next we consider the domain $\tau \rightarrow \infty$. Since

$$H_\ell^{(-)}(\tau) \rightarrow_{\tau \rightarrow \infty} e^{i\ell\pi/2} e^{-i\tau}, \quad (11)$$

and

$$Z_\ell(\mu, \tau) \rightarrow_{\tau \rightarrow \infty} \frac{e^{-(\mu-i\tau)}}{\mu-i\tau} \tau^\ell e^{-i\ell\pi/2}, \quad (12)$$

the momentum transformed potential $g_\ell^\alpha(\tau)$ takes the asymptotic form

$$g_\ell^\alpha(\tau) \rightarrow_{\tau \rightarrow \infty} \frac{\lambda_\alpha \mu_\alpha \tau}{\mu_\alpha^2 + \tau^2}, \quad (13)$$

which also is the expression for all τ when $\ell = 0$.

The dominant behavior of the kernel of Eq. (7) as $\tau \rightarrow \infty$ is

$$\frac{a^2 g_\ell^\alpha(\tau) g_\ell^\beta(\tau)}{\tau^2 - \tau_0^2} \rightarrow_{\tau \rightarrow \infty} \frac{\mu_\alpha^2 \mu_\beta^2 \tau^2}{(\tau^2 - \tau_0^2)(\mu_\alpha^2 + \tau^2)(\mu_\beta^2 + \tau^2)} \rightarrow_{\tau \rightarrow \infty} O(\tau^{-4}). \quad (14)$$

Clearly the kernel is bounded and integrable.

Since the kernel vanishes as $\tau^{2\ell+2}$ for $\tau \rightarrow 0$ and there are no poles on the real axis, the integration path may be transformed into a contour from $-\infty$ to $+\infty$ with a semicircle deformation in the neighborhood of $\tau = 0$ into the upper half plane. The deformation gives zero contribution to the integral in the limit of its radius going to zero.

Figure 1. Deformed integration path in complex τ -plane.

The principal value term, $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$, will be treated

symmetrically in the contour integration by the definition

$$P\left(\frac{1}{\tau^2 - \tau_0^2}\right) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{2} \left[\frac{1}{(\tau + \tau_0 - i\epsilon)(\tau - \tau_0 + i\epsilon)} + \frac{1}{(\tau + \tau_0 + i\epsilon)(\tau - \tau_0 - i\epsilon)} \right] \right\}, \quad (15)$$

where the poles of Eq. (15) are situated as illustrated in Fig. 2.

Figure 2. Deformed integration path and poles of $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ in complex τ -plane.

With these conventions, Eq. (7) takes the form

$$\begin{aligned}
 & a k_{\ell}^{\alpha\beta}(\tau_0) \\
 &= \frac{1}{2} \mu_{\alpha}^2 \mu_{\beta}^2 e^{\mu_{\alpha} + \mu_{\beta}} \int_{\text{path}} \frac{d\tau}{\tau^{2\ell} \Delta_{\ell}^2(\tau)} \operatorname{Im} \left[H_{\ell}^{(-)}(\tau) Z_{\ell}(\mu_{\alpha}, \tau) \right] \\
 & \quad \operatorname{Im} \left[H_{\ell}^{(-)}(\tau) Z_{\ell}(\mu_{\beta}, \tau) \right] P\left(\frac{1}{\tau^2 - \tau_0^2}\right). \quad (16)
 \end{aligned}$$

The integrand in Eq. (16) may be written in more tractable form upon defining

$$Q_{\alpha}(\tau) \equiv H_{\ell}^{(-)}(\tau) Z_{\ell}(\mu_{\alpha}, \tau), \quad (17)$$

and noting that

$$\begin{aligned}
 & \operatorname{Im}(Q_{\alpha}(\tau)) \operatorname{Im}(Q_{\beta}(\tau)) \\
 &= \frac{1}{4} [Q_{\alpha}(\tau) \bar{Q}_{\beta}(\tau) + \bar{Q}_{\alpha}(\tau) Q_{\beta}(\tau) - Q_{\alpha}(\tau) Q_{\beta}(\tau) - \bar{Q}_{\alpha}(\tau) \bar{Q}_{\beta}(\tau)], \quad (18)
 \end{aligned}$$

where the bar indicates complex conjugation. Substitution of Eq. (18) into (16) yields

$$a_k^{\alpha\beta}(\tau_0) = \frac{\mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta}}{8} P \int_{\Gamma} \frac{d\tau}{\tau^{2\ell} (\tau^2 - \tau_0^2)^2} \left\{ \begin{aligned} & Z_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) + \bar{Z}_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau) \\ & - \frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} Z_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau) \\ & - \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \end{aligned} \right\}, \quad (19)$$

since $\Delta_\ell^2(\tau) \equiv H_\ell^{(+)}(\tau) H_\ell^{(-)}(\tau)$.

The portion of the complex plane, either upper or lower, in which the contour is closed is now determined strictly by the properties of the functions

Integration Details

The first two terms of the integrand in Eq. (19) may be treated in a similar fashion. From the definition of the exponential integral

$$E_n(z) \equiv \int_1^\infty \frac{dt e^{-zt}}{t^n}, \quad \text{Re}(z) > 0, \quad n = 0, 1, 2, \dots, \quad (20)$$

We may write the first term in Eq. (19), using Eq. (3), as

$$\begin{aligned}
& \underbrace{P \int}_{\rightarrow} \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} Z_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \\
&= \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} e^{i(s-t)\pi/2} \\
& \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t} \underbrace{\int}_{\rightarrow} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right), \quad (21)
\end{aligned}$$

where $\gamma_{\ell s} \equiv \Gamma_{\ell s}(1)$.

Consider the contour integral of Eq. (21) and define

$$I_{s,t}(x,y) \equiv P \underbrace{\int}_{\rightarrow} \frac{d\tau}{\tau^{s+t}(\tau^2 - \tau_0^2)} e^{i\tau(x-y)} \quad (22)$$

which can be written as

$$I_{s,t}(x,y) = I_{s,t}^{(+)}(x,y)\theta(x-y) + I_{s,t}^{(-)}(x,y)\theta(y-x), \quad (23)$$

where

$$\theta(w) = \begin{cases} 1 & \text{if } w > 0 \\ 0 & \text{if } w \leq 0. \end{cases} \quad (24)$$

If $x - y > 0$, we may close the contour of Eq. (22) in the upper half τ plane. Using the Cauchy Residue Theorem and Eq. (15) we obtain

$$\begin{aligned}
 I_{s,t}^{(+)}(x,y) &= P \int_{\mathcal{C}} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t}(\tau^2 - \tau_0^2)} \\
 &= \frac{1}{2} \int_{\mathcal{C}} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t}} \left[\frac{1}{(\tau + \tau_0 - i\varepsilon)(\tau - \tau_0 + i\varepsilon)} + \frac{1}{(\tau + \tau_0 + i\varepsilon)(\tau - \tau_0 - i\varepsilon)} \right] \\
 &= \frac{\pi i}{2\tau_0^{s+t+1}} \left[e^{i\tau_0(x-y)} + (-)^{s+t+1} e^{-i\tau_0(x-y)} \right], \tag{25}
 \end{aligned}$$

since the only poles within the contour are those due to the

$P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$ term located in the upper half plane at $\tau = \pm \tau_0 + i\varepsilon$.

If $x - y < 0$, we must close the contour in the lower half plane*.

*Whereas for $x - y > 0$ and $\tau = \tau_1 + i\tau_2$, we have $e^{i\tau(x-y)} = e^{i\tau_1(x-y)} e^{-\tau_2(x-y)}$ with $\tau_2 > 0$, if we have $x - y < 0$, the condition $\tau_2 < 0$ must be required for the contribution on the large semicircle to vanish as the radius goes to infinity. By this procedure, we obtain a representation for the integral for either sign of $(x-y)$.

Then, two contributions to the integral $I_{s,t}^{(-)}(x,y)$ arise, one from the term $P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$, similar to the previous case only with the poles located at $\tau = \pm \tau_0 - i\epsilon$, and one from an $(s+t)^{\text{th}}$ order pole at $\tau = 0$, interior to the contour for the integral. The result is readily obtained from

$$I_{s,t}^{(-)}(x,y) = - P \int \frac{d\tau e^{-i\tau(y-x)}}{\tau^{s+t}(\tau^2 - \tau_0^2)}, \quad (26)$$

where the minus sign arises from making the contour in the positive sense, as is necessary for the application of the Cauchy Residue theorem. Using Eq. (15) in (26) we obtain

$$\begin{aligned} I_{s,t}^{(-)}(x,y) &= - \frac{1}{2} \int \frac{d\tau e^{-i\tau(y-x)}}{\tau^{s+t}} \left[\frac{1}{(\tau + \tau_0 - i\epsilon)(\tau - \tau_0 + i\epsilon)} + \frac{1}{(\tau + \tau_0 + i\epsilon)(\tau - \tau_0 - i\epsilon)} \right] \\ &\quad - 2\pi i \operatorname{Res}_{\tau=0} \left[\frac{e^{-i\tau(y-x)}}{\tau^{s+t}(\tau^2 - \tau_0^2)} \right] \\ &= \frac{-\pi i}{2\tau_0^{s+t+1}} \left[e^{-i\tau_0(y-x)} + (-)^{s+t+1} e^{i\tau_0(y-x)} \right] - 2\pi i \operatorname{Res}_{\tau=0} \left[\frac{e^{-i\tau(y-x)}}{\tau^{s+t}(\tau^2 - \tau_0^2)} \right]. \end{aligned}$$

(27)

That $\tau = 0$ residue term in Eq. (27) is readily evaluated. However, we defer its explicit evaluation until later where we shall obtain significant cancellation with other terms.

Using the results obtained in Eqs. (25) and (27) in Eqs. (23) and (21), and realizing that

$$\int_1^\infty dx \int_1^\infty dy f(x,y) = \int_1^\infty dx \int_x^\infty dy f_1(x,y) + \int_1^\infty dy \int_y^\infty dx f_2(x,y), \quad (28)$$

where

$$f(x,y) \equiv f_1(x,y)\theta(y-x) + f_2(x,y)\theta(x-y), \quad (29)$$

we may write Eq. (21) as

$$\begin{aligned} & \underbrace{P \int}_{\rightarrow} \frac{d\tau Z_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \\ &= \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} e^{i(s-t)\pi/2} \int_1^\infty dx \int_1^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}(x,y) \\ &= \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} e^{i(s-t)\pi/2} \left\{ \begin{aligned} & \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}^{(-)}(x,y) \\ & + \int_1^\infty dy \int_y^\infty dx \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}^{(+)}(x,y). \end{aligned} \right. \quad (30) \end{aligned}$$

Substituting Eqs. (25) and (27) into Eq. (30) we obtain

$$\begin{aligned}
 & \underbrace{P \int}_{\rightarrow} \frac{d\tau Z_{\ell}(\mu_{\alpha}, \tau) \bar{Z}_{\ell}(\mu_{\beta}, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \\
 &= \frac{\pi i}{2\tau_0^{2\ell+1}} \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) e^{-i(\ell-s)\pi/2} e^{i(\ell-s)\pi/2} \\
 & \times \left\{ \begin{aligned} & e^{i(s+t)\pi/2} \left\{ \begin{aligned} & \int_1^{\infty} dx \int_x^{\infty} dy \frac{e^{-\mu_{\alpha}x}}{x^s} \frac{e^{-\mu_{\beta}y}}{y^t} e^{i\tau_0(y-x)} \\ & - e^{-(s+t)\pi/2} \int_1^{\infty} dx \int_x^{\infty} dy \frac{e^{-\mu_{\alpha}x}}{x^s} \frac{e^{-\mu_{\beta}y}}{y^t} e^{-i\tau_0(y-x)} \end{aligned} \right\} \\ & + e^{-i(s+t)\pi/2} \left\{ \begin{aligned} & \int_1^{\infty} dy \int_y^{\infty} dx \frac{e^{-\mu_{\alpha}x}}{x^s} \frac{e^{-\mu_{\beta}y}}{y^t} e^{i\tau_0(x-y)} \\ & - e^{-i(s+t)\pi/2} \int_1^{\infty} dy \int_y^{\infty} dx \frac{e^{-\mu_{\alpha}x}}{x^s} \frac{e^{-\mu_{\beta}y}}{y^t} e^{-i\tau_0(x-y)} \end{aligned} \right\} \end{aligned} \right\} \\
 & - 2\pi i \operatorname{Res}_{\tau=0} \left\{ \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \frac{e^{i(s-t)\pi/2}}{\tau^{s+t}} \int_1^{\infty} dx \int_x^{\infty} dy \frac{e^{-\mu_{\alpha}x}}{x^s} \frac{e^{-\mu_{\beta}y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} \right\}.
 \end{aligned}
 \tag{31}$$

Similarly the second term of Eq. (19) takes the form

$$\begin{aligned}
 & \underbrace{P \int \frac{d\tau \bar{Z}_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau)}{\tau^{2\ell} (\tau^2 - \tau_0^2)}}_{\rightarrow} \\
 &= \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} e^{-i(s-t)\pi/2} \int_1^\infty dx \int_1^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}(x,y) \\
 &= \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} e^{-i(s-t)\pi/2} \left\{ \int_1^\infty dy \int_y^\infty dx \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}^{(-)}(x,y) \right. \\
 & \quad \left. + \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} I_{s,t}^{(+)}(x,y) \right\}.
 \end{aligned}$$

(32)

Substitution of Eqs. (25) and (27) into Eq. (32) yields

$$P \int_{-\infty}^{\infty} \frac{d\tau \bar{Z}_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau)}{\tau^{2\ell} (\tau^2 - \tau_0^2)}$$

$$= \frac{\pi i}{2\tau_0^{2\ell+1}} \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) e^{i(\ell-s)\pi/2} e^{-i(\ell-t)\pi/2}$$

$$x \left\{ \begin{array}{l} e^{i(s+t)\pi/2} \left\{ \begin{array}{l} e^{i(s+t)\pi/2} \int_1^\infty dy \int_y^\infty dx \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} e^{i\tau_0(x-y)} \\ -e^{-i(s+t)\pi/2} \int_1^\infty dy \int_y^\infty dx \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} e^{-i\tau_0(x-y)} \end{array} \right\} \\ +e^{-i(s+t)\pi/2} \left\{ \begin{array}{l} e^{i(s+t)\pi/2} \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} e^{i\tau_0(y-x)} \\ -e^{-i(s+t)\pi/2} \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} e^{-i\tau_0(y-x)} \end{array} \right\} \end{array} \right\}$$

$$-2\pi i \operatorname{Res}_{\tau=0} \left\{ \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \frac{e^{-i(s-t)\pi/2}}{\tau^{s+t}} \int_1^\infty dy \int_y^\infty dx \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \frac{e^{-i(x-y)\tau}}{\tau^2 - \tau_0^2} \right\}.$$

(33)

If now we define the integral (see Appendix 6)

$$J_{s,t}(\alpha, \bar{\beta}) \equiv \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} e^{-i\tau_0(x-y)}$$

$$= \int_1^\infty dx \frac{e^{-\alpha x}}{x^{s+t-1}} E_t(\bar{\beta}x), \quad \text{Re}(\alpha, \beta) > 0, \quad (34)$$

where here,

$$\alpha \equiv \mu_\alpha + i\tau_0,$$

$$\bar{\beta} \equiv \mu_\beta - i\tau_0, \quad (35)$$

and

$$J_{s,t}(\alpha, \bar{\beta}) = \overline{J_{s,t}(\bar{\alpha}, \beta)}, \quad (36)$$

then upon combining Eqs. (30) and (33) we obtain

$$\begin{aligned}
 & \underset{\rightarrow}{P} \int \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)^2} [Z_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) + \bar{Z}_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau)] \\
 &= \frac{\pi i}{2\tau_0^{2\ell+1}} \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) \\
 & \cdot \left\{ \begin{aligned} & e^{i(s-t)\pi/2} [(-)^{s+t} J_{s,t}(\alpha, \bar{\beta}) - J_{s,t}(\bar{\alpha}, \beta) + J_{t,s}(\beta, \bar{\alpha}) \\ & \quad - (-)^{s+t} J_{t,s}(\bar{\beta}, \alpha)] \\ & + e^{-i(s-t)\pi/2} [(-)^{s+t} J_{t,s}(\beta, \bar{\alpha}) - J_{t,s}(\bar{\beta}, \alpha) + J_{s,t}(\alpha, \bar{\beta})] \end{aligned} \right\} \\
 & - 2\pi i \operatorname{Res}_{\tau=0} \left\{ \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \frac{e^{i(s-t)\pi/2}}{\tau^{s+t}} \right. \\
 & \cdot \left. \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} + \operatorname{sym}(\mu_\alpha \rightleftharpoons \mu_\beta) \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} \right\}, \\
 & \hspace{15em} (37)
 \end{aligned}$$

where $\operatorname{sym}(\mu_\alpha \rightleftharpoons \mu_\beta)$ means the same thing with $\mu_\alpha \rightleftharpoons \mu_\beta$ interchanged.

can be rewritten as

$$\begin{aligned}
 P \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} [Z_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) + \bar{Z}_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau)] \\
 = - \frac{2\pi}{\tau_0^{2\ell+1}} \operatorname{Im} \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) e^{-i(s-t)\pi/2} [J_{s,t}(\alpha, \bar{\beta}) + J_{s,t}(\beta, \bar{\alpha})] \\
 - 2\pi i \operatorname{Res}_{\tau=0} \left\{ \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \frac{e^{i(s-t)\pi/2}}{\tau^{s+t}} \int_1^{\infty} \right. \\
 \left. \cdot dx \int_x^{\infty} dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} + \operatorname{sym}(\mu_\alpha \leftrightarrow \mu_\beta) \right\}, \quad (38)
 \end{aligned}$$

which result is completely symmetric in μ_α and μ_β as it should be.

Equation (38) is obtained from Eq. (37) by collecting terms and relabeling the indices s and t , and then using the relation

$$\operatorname{Im} \left\{ e^{-i(s-t)\pi/2} J_{s,t}(\beta, \bar{\alpha}) \right\} = - \operatorname{Im} \left\{ e^{i(s-t)\pi/2} J_{s,t}(\bar{\beta}, \alpha) \right\}, \quad (39)$$

which follows from Eq. (36).

Now we turn our attention to the two remaining integrals of Eq. (19) and consider the determination of

$$\begin{aligned}
 -P \int_{\underbrace{\quad}_{\rightarrow}} \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} & \left[\frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} Z_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau) \right. \\
 & \left. + \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \right]. \quad (40)
 \end{aligned}$$

In spite of the fact that

$$H_\ell^{(+)}(\tau) \equiv G_\ell(\tau) \pm iF_\ell(\tau) \xrightarrow{\tau \rightarrow \infty} e^{\pm i\tau} e^{\mp i\ell\pi/2}, \quad (41)$$

the criteria for using the upper or lower half complex τ plane for closing the contours is determined strictly by the argument of $E_s(z)$ in the function $Z_\ell(\mu, \tau)$. The result is that if $z = \mu - i\tau$, we must close the contour in the upper half plane, and if $z = \mu + i\tau$, we must close the contour in the lower half plane. This conclusion can be verified by substituting the integral expressions [substitute Eq. (4) into Eq. (1) in Appendix 2] for $Z_\ell(\mu, \tau)$ into Eq. (40). See also the argument below Eq. (45).

The poles that can occur are those due to the term

$P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$, which are treated as before, those due to $1/\tau^{2\ell}$ when the contour is closed in the lower half plane, and those due

to the function $H_\ell^{(-)}(\tau)/H_\ell^{(+)}(\tau)$, or its reciprocal, which we examine now. Equivalently let us examine the zeros of $H_\ell^{(+)}(\tau)$. Now

$$H_\ell^{(+)}(\tau) \equiv \frac{e^{+i\tau}}{\tau^\ell} [\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)] , \quad (42)$$

where

$$\hat{\alpha}_\ell(\tau) \pm i\hat{\beta}_\ell(\tau) \equiv \sum_{s=0}^{\ell} \Gamma_{\ell s}(\tau) e^{\pm i(\ell-s)\pi/2} . \quad (43)$$

Taking the limit $\tau \rightarrow \infty$ in Eq. (42) we see that

$$H_\ell^{(+)}(\tau) \xrightarrow{\tau \rightarrow \infty} e^{+i\tau} e^{+i\ell\pi/2} .$$

Now using Eqs. (42) and (43), we see that

$$\frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} = e^{-2i\tau} \frac{[\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)]}{[\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)]} \quad (44)$$

will have ℓ zeros in the upper half plane, due to $\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)$, and ℓ poles in the lower half plane, due to the zeros of $\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)$, provided there are any. In Appendix 3 [see below Eqs. (6) and (13)], we showed that the zeros of $\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)$ and $\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)$ are located symmetrically about the $\text{Im}(\tau)$ axis at the points

$$\gamma = \pm \bar{\gamma}_{\ell,m} + i \bar{\gamma}_{\ell m} \quad \text{and} \quad \bar{\gamma} = \pm \bar{\gamma}_{\ell,m} - i \bar{\gamma}_{\ell m} ,$$

(45)

respectively, where $\bar{\gamma}_{\ell,m}$ and $\bar{\gamma}_{\ell m} > 0$, except when ℓ is odd in which case $\bar{\gamma}_{\ell,1} = 0$.

In order to always maintain the condition $\text{Re}(z) > 0$ on the exponential integrals, $E_s(z)$, involved in the $Z_\ell(\mu, \tau)$ functions, we must integrate the term

$$\frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} \frac{Z_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau)}{\tau^{2\ell}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$$

in the upper half plane, and the complex conjugate term in the lower half plane. But there are no pole contributions due to $H_\ell^{(-)}(\tau)/H_\ell^{(+)}(\tau)$ in the upper half plane, nor due to $H_\ell^{(+)}(\tau)/H_\ell^{(-)}(\tau)$ in the lower half plane. Therefore the only contributions to the two integrals in Eq. (40) arise from the $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ term in either half plane and from the $1/\tau^{2\ell}$ pole at $\tau = 0$ when the contour is closed in the lower half plane.

The first term of Eq. (40) then becomes (using Eq. (15) and (5))

$$\begin{aligned}
& -P \int_{\mathcal{L}} \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} Z_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau) \\
& = -\frac{\pi i}{2\tau_0^{2\ell+1}} \left[\frac{H_\ell^{(-)}(\tau_0)}{H_\ell^{(+)}(\tau_0)} Z_\ell(\mu_\alpha, \tau_0) Z_\ell(\mu_\beta, \tau_0) \right. \\
& \quad \left. - \frac{H_\ell^{(+)}(\tau_0)}{H_\ell^{(-)}(\tau_0)} \bar{Z}_\ell(\mu_\alpha, \tau_0) \bar{Z}_\ell(\mu_\beta, \tau_0) \right]. \quad (46)
\end{aligned}$$

Similarly, for the second integral we obtain

$$\begin{aligned}
& -P \int_{\mathcal{L}} \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \\
& = \frac{-\pi i}{2\tau_0^{2\ell+1}} \left[\frac{H_\ell^{(-)}(\tau_0)}{H_\ell^{(+)}(\tau_0)} Z_\ell(\mu_\alpha, \tau_0) Z_\ell(\mu_\beta, \tau_0) \right. \\
& \quad \left. - \frac{H_\ell^{(+)}(\tau_0)}{H_\ell^{(-)}(\tau_0)} \bar{Z}_\ell(\mu_\alpha, \tau_0) \bar{Z}_\ell(\mu_\beta, \tau_0) \right] \\
& \quad + 2\pi i \operatorname{Res}_{\tau=0} \left\{ \frac{H_\ell^{+}(\tau)}{H_\ell^{(-)}(\tau)} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \right\}. \quad (47)
\end{aligned}$$

Upon combining Eqs. (46) and (47) in Eq. (40), we have the result

$$\begin{aligned}
 & -P \int_{\mathcal{C}_\tau} \frac{d\tau}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \left[\frac{H_\ell^{(-)}(\tau)}{H_\ell^{(+)}(\tau)} Z_\ell(\mu_\alpha, \tau) Z_\ell(\mu_\beta, \tau) \right. \\
 & \quad \left. + \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \right] \\
 & = -\frac{2\pi}{\tau_0^{2\ell+1}} \operatorname{Im} \left[\frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau) \right] \\
 & \quad + 2\pi i \operatorname{Res}_{\tau=0} \left\{ \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \right\}. \quad (48)
 \end{aligned}$$

Now substituting Eqs. (48) and (38) into Eq. (19) we obtain

$$\begin{aligned}
 aK_\ell^{\alpha\beta}(\tau_0) & = -\frac{\pi}{4} \frac{\mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta}}{\tau_0^{2\ell+1}} \\
 & \quad \cdot \operatorname{Im} \left\{ \begin{aligned} & \frac{H_\ell^{(+)}(\tau_0)}{H_\ell^{(-)}(\tau_0)} \bar{Z}_\ell(\mu_\alpha, \tau_0) \bar{Z}_\ell(\mu_\beta, \tau_0) \\ & + \sum_{s,t=0}^{\ell} \Gamma_{\ell s}(\tau_0) \Gamma_{\ell t}(\tau_0) e^{-i(s-t)\pi/2} [J_{s,t}(\alpha, \bar{\beta}) + J_{s,t}(\beta, \bar{\alpha})] \end{aligned} \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\pi i}{4} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta} \operatorname{Res}_{\tau=0} \left\{ \begin{aligned} & \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell} (\tau^2 - \tau_0^2)} \\ & - \sum_{s, t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \frac{e^{i(s-t)\pi/2}}{\tau^{s+t}} \end{aligned} \right. \\
& \cdot \left(\int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} \right) \Bigg\} \\
& + \operatorname{sym}(\mu_\alpha \rightleftharpoons \mu_\beta). \tag{49}
\end{aligned}$$

Residue Terms at the Origin

The residue terms will now be considered. As $\tau \rightarrow 0$

$$\frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} = \frac{1 + iF_\ell(\tau)/G_\ell(\tau)}{1 - iF_\ell(\tau)/G_\ell(\tau)} \rightarrow 1 + \frac{2i\tau^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}, \tag{50}$$

since

$$\left. \begin{aligned} F_\ell(\tau) &\xrightarrow{\tau \rightarrow 0} \frac{\tau^{\ell+1}}{(2\ell+1)!!} \\ G_\ell(\tau) &\xrightarrow{\tau \rightarrow 0} \frac{(2\ell-1)!!}{\tau^\ell} \end{aligned} \right\} \tag{51}$$

From Eq. (42) of Appendix 2 we know that the low-momentum behavior of $\bar{Z}_\ell(\mu_\alpha, \tau)$ is given by

$$\bar{Z}_\ell(\mu_\alpha, \tau) \xrightarrow{\tau \rightarrow 0} X_\ell(\mu_\alpha, \tau) - iY_\ell(\mu_\alpha, \tau), \quad (52)$$

where $X_\ell(\mu_\alpha, \tau)$ is an even function of τ , finite at $\tau = 0$, and $Y_\ell(\mu_\alpha, \tau)$ is an odd function of τ vanishing as $O(\tau)^{2\ell+1}$. Thus, upon using Eq. (50) we see that the only contributing terms to the first residue term of Eq. (49) (for $\tau_0 \neq 0$) arise from

$$\frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} \xrightarrow{\tau \rightarrow 0} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)}. \quad (53)$$

Combining this result with the other residue term in Eq. (49) and using Eq. (52) and the arguments below, we obtain

$$\frac{\pi i}{4} \operatorname{Res}_{\tau=0} \left\{ \frac{H_\ell^{(+)}(\tau)}{H_\ell^{(-)}(\tau)} \frac{\bar{Z}_\ell(\mu_\alpha, \tau) \bar{Z}_\ell(\mu_\beta, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} - \sum_{s,t=0}^{\ell} \frac{\gamma_{\ell s} \gamma_{\ell t} e^{i(s-t)\pi/2}}{\tau^{s+t}} \right. \\ \left. \left(\int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} \right) \right. \\ \left. + \operatorname{sym} (\mu_\alpha \rightleftharpoons \mu_\beta) \right\}$$

$$\begin{aligned}
&= \frac{\pi i}{4} \operatorname{Res}_{\tau=0} \left\{ \sum_{s,t=0}^{\ell} \frac{\gamma_{\ell s} \gamma_{\ell t}}{\tau^{s+t}} \left\{ \int_1^{\infty} dx \int_x^{\infty} dy \frac{e^{-\mu_{\alpha} x}}{x^s} \frac{e^{-\mu_{\beta} y}}{y^t} \right. \right. \\
&\quad \left. \left. \frac{\left(e^{\frac{i(2\ell-s-t)\pi/2 - i(y+x)\tau}{\tau^2 - \tau_0^2}} e^{\frac{i(s-t)\pi/2 - i(y-x)\tau}{\tau^2 - \tau_0^2}} \right)}{\tau^2 - \tau_0^2} + \operatorname{sym} (\mu_{\alpha} \rightleftharpoons \mu_{\beta}) \right\} \right\} \\
&\hspace{15em} (54)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\pi}{2} \operatorname{Res}_{\tau=0} \left\{ \frac{1}{\tau^{\ell}(\tau^2 - \tau_0^2)} \sum_{s=0}^{\ell} \frac{\gamma_{\ell s}}{\tau^s} \right. \\
&\quad \cdot \left\{ \int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x} \bar{Z}_{\ell}(\mu_{\beta} x, \tau x)}{x^{\ell+s-1}} \operatorname{Im} \left(e^{i(\ell-s)\pi/2 - i\tau x} \right) \right. \\
&\quad \left. \left. + \operatorname{sym} (\mu_{\alpha} \rightleftharpoons \mu_{\beta}) \right\} \right\} \\
&= -\frac{\pi}{2} \operatorname{Res}_{\tau=0} \left\{ \frac{1}{\tau^{\ell}(\tau^2 - \tau_0^2)} \sum_{s=0}^{\ell} \frac{\gamma_{\ell s}}{\tau^s} \right. \\
&\quad \cdot \left\{ \int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x}}{x^{\ell+s-1}} X_{\ell}(\mu_{\beta} x, \tau x) \left\{ \begin{array}{l} \sin(\ell-s)\pi/2 \cos \tau x \\ -\cos(\ell-s)\pi/2 \sin \tau x \end{array} \right\} \right. \\
&\quad \left. \left. + \operatorname{sym} (\mu_{\alpha} \rightleftharpoons \mu_{\beta}) \right\} \right\}, \\
&\hspace{15em} (55)
\end{aligned}$$

where $\text{sym} (\mu_\alpha \rightleftharpoons \mu_\beta)$ means the same term with μ_α and μ_β interchanged.

If we consider Eq. (55) we see immediately that if $\ell \pm s$ is

$\begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix}$, we shall be taking an $\begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}$ number of derivatives with respect to τ of an $\begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}$ function of τ when evaluating the residue.

We can, therefore, obtain a non-vanishing result.

Explicit evaluation of the residue is readily achieved by returning to Eq. (54) and then performing the indicated integration. Defining $\alpha = y \pm x$ (for the moment), the residue term is of the form

$$\text{Res}_{\tau=0} \left\{ \frac{e^{-i\tau\alpha}}{\tau^{s+t}(\tau^2 - \tau_0^2)} \right\} = \frac{1}{(s+t-1)!} \lim_{\tau \rightarrow 0} \left\{ \frac{d^{s+t-1}}{d\tau^{s+t-1}} \left(\frac{e^{-i\tau\alpha}}{(\tau^2 - \tau_0^2)} \right) \right\} \quad (56)$$

Letting $n \equiv s + t - 1$ and using the chain rule for differentiation, we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{d^n}{d\tau^n} \left(\frac{e^{-i\tau\alpha}}{\tau^2 - \tau_0^2} \right) &= \lim_{\tau \rightarrow 0} \sum_{p=0}^n \binom{n}{p} \frac{d^p}{d\tau^p} (\tau^2 - \tau_0^2)^{-1} \frac{d^{n-p}}{d\tau^{n-p}} (e^{-i\tau\alpha}) \\ &= \sum_{p=0}^n \binom{n}{p} (-i\alpha)^{n-p} \left(\lim_{\tau \rightarrow 0} \frac{d^p}{d\tau^p} (\tau^2 - \tau_0^2)^{-1} \right). \quad (57) \end{aligned}$$

Now $\tau^2 - \tau_0^2 = (\tau + \tau_0)(\tau - \tau_0)$, so applying the chain rule again, we

obtain

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{d^p}{d\tau^p} (\tau^2 - \tau_o^2)^{-1} &= \lim_{\tau \rightarrow 0} \sum_{q=0}^p \binom{p}{q} \frac{d^q}{d\tau^q} (\tau + \tau_o)^{-1} \frac{d^{p-q}}{d\tau^{p-q}} (\tau - \tau_o)^{-1} \\ &= \lim_{\tau \rightarrow 0} \sum_{q=0}^p \binom{p}{q} \frac{(-)^q q!}{(\tau + \tau_o)^{q+1}} \frac{(-)^{p-q} (p-q)!}{(\tau - \tau_o)^{p-q+1}}, \end{aligned} \quad (58)$$

since

$$\frac{d^r}{dx^r} \left(\frac{1}{x+x_o} \right) = \frac{(-)^r r!}{(x+x_o)^{r+1}}. \quad (59)$$

Taking the limit $\tau \rightarrow 0$ in Eq. (58) we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{d^p}{d\tau^p} (\tau^2 - \tau_o^2)^{-1} &= \frac{-p!}{\tau_o^{p+2}} \sum_{q=0}^p (-)^q \\ &= \begin{cases} -p!/\tau_o^{p+2} & \text{if } p \text{ even} \\ 0 & \text{if } p \text{ odd} \end{cases}. \end{aligned} \quad (60)$$

Substituting Eq. (60) into (57) and transforming $p \rightarrow 2p$ we have
the result $(\alpha \equiv \underline{y+x})$

$$\text{Res}_{\tau=0} \left\{ \frac{e^{-i\tau\alpha}}{\tau^{s+t}(\tau^2 - \tau_0^2)} \right\} = - e^{-i(s+t-1)\pi/2} \sum_{p=0}^{\left(\frac{s+t-1}{2}\right)} \frac{(-)^p \alpha^{s+t-1-2p}}{(s+t-1-2p)! (\tau_0^2)^{p+1}}, \quad (61)$$

where

$$\left(\frac{n}{2}\right) \equiv \begin{cases} n/2 & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd.} \end{cases} \quad (62)$$

Upon substituting Eq. (61) into Eq. (54), the residue term becomes

$$\begin{aligned} \frac{\pi i}{4} \text{Res}_{\tau=0} & \left\{ \frac{H_{\ell}^{(+)}(\tau)}{H_{\ell}^{(-)}(\tau)} \frac{\bar{Z}_{\ell}(\mu_{\alpha}, \tau) \bar{Z}_{\ell}(\mu_{\beta}, \tau)}{\tau^{2\ell}(\tau^2 - \tau_0^2)} - \sum_{s,t=0}^{\ell} \frac{\gamma_{\ell s} \gamma_{\ell t} e^{i(s-t)\pi/2}}{\tau^{s+t}} \right. \\ & \left. \left(\int_1^{\infty} dx \int_x^{\infty} dy \frac{e^{-\mu_{\alpha} x}}{x^s} \frac{e^{-\mu_{\beta} y}}{y^t} \frac{e^{-i(y-x)\tau}}{\tau^2 - \tau_0^2} \right) \right. \\ & \left. + \text{sym} (\mu_{\alpha} \rightleftharpoons \mu_{\beta}) \right\} \\ & = \frac{\pi}{4} \sum_{s,t=0}^{\ell} \gamma_{\ell s} \gamma_{\ell t} \sum_{p=0}^{\left(\frac{s+t-1}{2}\right)} \frac{(-)^p \left(I_{\ell,s,t,p}^{\alpha\beta}\right)}{(s+t-1-2p)! (\tau_0^2)^{p+1}}, \quad (63) \end{aligned}$$

where

$$I_{l,s,t,p}^{\alpha\beta}$$

$$= \int_1^\infty dx \int_x^\infty dy \left\{ \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \begin{cases} (-)^{l+s+t} (y+x)^{s+t-1-2p} \\ + (-)^{t+1} (y-x)^{s+t-1-2p} \end{cases} \right. \\ \left. + \text{sym } (\alpha \rightleftharpoons \beta) \right\} \quad (64)$$

Equation (64) is evaluated explicitly in Appendix 7.

Summary of Results for $K_l^{\alpha\beta}(\tau_0)$

Collecting together Eqs. (49) and (63) and the results of Appendices 6 and 7, we have finally the result

$$aK_l^{\alpha\beta}(\tau_0) = -\frac{\pi}{4} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta} \text{Im} \left\{ \begin{aligned} & \frac{H_l^{(+)}(\tau_0)}{H_l^{(-)}(\tau_0)} \bar{Z}_l(\mu_\alpha, \tau_0) \bar{Z}_l(\mu_\beta, \tau_0) \\ & + \sum_{s,t=0}^l \Gamma_{ls}(\tau_0) \Gamma_{lt}(\tau_0) e^{-i(s-t)\pi/2} J_{s,t}(\alpha, \bar{\beta}) + J_{s,t}(\beta, \bar{\alpha}) \end{aligned} \right\} \\ + \frac{\pi}{4} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta} \sum_{s,t=0}^l \sum_{p=0}^{\frac{s+t-1}{2}} \frac{\gamma_{ls} \gamma_{lt} (-)^p I_{l,s,t,p}^{\alpha\beta}}{(s+t-1-2p)! (\tau_0^2)^{p+1}}, \quad (65)$$

where (see Eq. (14) in Appendix 6)

$$\begin{aligned}
 & J_{s,t}(\alpha, \bar{\beta}) \\
 &= \delta_{t,0} \frac{1}{\bar{\beta}} E_s(\alpha + \bar{\beta}) + \delta_{s,0} (1 - \delta_{t,0}) \left\{ \begin{aligned} & \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{t-k}(\alpha + \bar{\beta}) \\ & + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{1}{\alpha} [e^{-\alpha} E_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})] \end{aligned} \right\} \\
 &+ (1 - \delta_{s,0})(1 - \delta_{t,0}) \cdot \left\{ \begin{aligned} & \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{s+t-k}(\alpha + \bar{\beta}) \\ & + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{(-\alpha)^{s-1}}{(s-1)!} D_1(\alpha, \bar{\beta}) \\ & + \frac{\theta(s-1)}{(s-1)!} \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \sum_{k=1}^{s-1} (-\alpha)^{s-1-k} (k-1)! [e^{-\alpha} E_1(\bar{\beta}) - E_{k+1}(\alpha + \bar{\beta})] \end{aligned} \right\}, \quad (66)
 \end{aligned}$$

and with $\theta(0) = 0$, (see Eqs. (4) and (21) of Appendix 7)

$$I_{l,s,t,p}^{\alpha\beta} = (-)^{l+s+t} \left(L_{s,t,p}^{\alpha\beta(+)} + L_{s,t,p}^{\beta\alpha(+)} \right) + (-)^{t-1} \left(L_{s,t,p}^{\alpha\beta(-)} + L_{s,t,p}^{\beta\alpha(-)} \right), \quad (67)$$

with

$$L_{s,t,p}^{\alpha\beta(+)} = \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (+)^q$$

$$\left\{ \begin{aligned} & \theta(s-2p-q) \frac{(s-1-2p-q)!}{(\mu_\beta)^{s-2p-q}} \sum_{r=0}^{s-1-2p-q} \frac{(\mu_\beta)^r}{r!} E_{s-q-r}(\mu_\alpha + \mu_\beta) \\ & + \theta(q+2p+1-s) \left\{ \begin{aligned} & \frac{\theta(q+2p-s)}{(q+2p-s)!} \\ & \sum_{r=1}^{q+2p-s} (q+2p-s-r)! (-\mu_\beta)^{r-1} \left\{ \begin{aligned} & \theta(r-2p) \alpha_{r-1-2p}(\mu_\alpha + \mu_\beta) \\ & + \theta(2p+1-r) E_{2p+1-r}(\mu_\alpha + \mu_\beta) \end{aligned} \right\} \end{aligned} \right\} \\ & + \frac{(-\mu_\beta)^{q+2p-s}}{(q+2p-s)!} \left\{ \begin{aligned} & \theta(q-s+1) D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta) \\ & + \theta(s-q) D_{s-q,1}^{(2)}(\mu_\alpha, \mu_\beta) \end{aligned} \right\} \end{aligned} \right\}$$

(68)

The function $D_n(\alpha, \bar{\beta})$ is given by Eqs. (20) and (23) in Appendix 12, for the case $n = 1$, as

$$D_1(\alpha, \bar{\beta}) = \left\{ \begin{array}{l} \left\{ -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right\} - (\gamma + \ln \bar{\beta}) E_1(\alpha) \right. \\ \left. - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) \right\} \text{ if } |\bar{\beta}/\alpha| \leq 1 \\ \left\{ \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \bar{\beta})^2 \right\} + (\gamma + \ln \alpha + E_1(\alpha)) E_1(\bar{\beta}) \right. \\ \left. + \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}) \right\} \text{ if } |\alpha/\bar{\beta}| \leq 1 \end{array} \right\} . \quad (69)$$

When $n \geq 1$, Eq. (24) in Appendix 12 is

$$D_n(\alpha, \bar{\beta}) = \frac{(-\alpha)^{n-1}}{(n-1)!} D_1(\alpha, \bar{\beta}) + \theta(n-1) \sum_{p=1}^{n-1} \frac{(-\alpha)^{n-1-p} (p-1)!}{(n-1)!} \left[e^{-\alpha} E_1(\bar{\beta}) - E_{p+1}(\alpha + \bar{\beta}) \right] . \quad (70)$$

Also, according to Eq. (4) in Appendix 12, $D_0(\alpha, \bar{\beta}) = \frac{1}{\alpha} [e^{-\alpha} e_1(\bar{\beta})$

$- E_1(\alpha + \bar{\beta})]$ $[= D_{s-q}(\mu_\alpha, \mu_\beta)$ when $s \cong q]^*$. Finally, the function

$D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta)$ is determined by Eq. (27) in Appendix 12 to be

$$D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta) = \alpha_{q-s}(\mu_\alpha) E_1(\mu_\beta)$$

$$- \frac{(q-s)!}{(\mu_\alpha)^{q-s+1}} E_1(\mu_\alpha + \mu_\beta) - \sum_{r=1}^{q-s} \frac{(q-s)! e^{-(\mu_\alpha + \mu_\beta)} e_{r-1}(\mu_\alpha + \mu_\beta)}{r(\mu_\alpha)^{q-s+1-r} (\mu_\alpha + \mu_\beta)^r}, \quad (71)$$

where $\alpha_{-1}(z) = E_1(z)$ and

$$\alpha_n(z) \equiv \int_1^\infty dt e^{-zt} t^n, \quad \text{Re}(z) > 0, \quad n = 0, 1, 2, \dots \quad (72)$$

$$= \frac{n!}{z^{n+1}} e^{-z} e_n(z), \quad (73)$$

$$e_n(z) \equiv \sum_{p=0}^n \frac{z^p}{p!}. \quad (74)$$

Equations (65) - (74) constitute the complete analytical solution to the principal-value integral given by Eq. (1). The special case $\ell = 0$ is simple to derive directly from Eq. (7), with the aid of Eq. (16) in Appendix 1, and one finds the result

* See Eqs. (2) in Appendix 13 and (1) in Appendix 12.

$$a k_o^{\alpha\beta}(\tau_o) = \frac{\pi}{2} \frac{|\alpha^2| \beta^2 (\mu_\alpha \mu_\beta - \tau_o)}{(\tau_o^2 + \mu_\alpha^2)(\tau_o^2 + \mu_\beta^2)(\mu_\alpha + \mu_\beta)} . \quad (75)$$

ON THE EVALUATION OF $K_l^{\alpha\beta}(\tau_0)$ as $\tau_0 \rightarrow 0$

The limiting behavior of $K_l^{\alpha\beta}(\tau_0)$ as $\tau_0 \rightarrow 0$ is necessary for the low-energy scattering parameter analysis, where we require knowledge of terms up to cubic order in k_0^2 . This result could be obtained by taking the limit of the general expression for $K_l^{\alpha\beta}(\tau_0)$ (as given in Appendix 4). However, it is somewhat easier and more reasonable to determine the limiting form from the original integral and to evaluate the (few) needed integrals directly. In this analysis we use the notation of Appendix 4 unless otherwise stated.

The function $K_l^{\alpha\beta}(k_0)$ is an even function of k_0 , which may be expanded in an asymptotic series in k_0^2 about $k_0^2 = 0$; namely,

$$\begin{aligned}
 K_l^{\alpha\beta}(k_0^2) &= \int_0^\infty \frac{dk g_l^\alpha(k) g_l^\beta(k)}{k^2 - k_0^2} \\
 &\cong \sum_{n=0}^\infty \frac{(k_0^2)^n}{n!} \left. \frac{\partial^n K_l^{\alpha\beta}(k_0^2)}{\partial (k_0^2)^n} \right|_{k_0^2=0}, \quad (1)
 \end{aligned}$$

where

$$\left. \frac{\partial^n K_l^{\alpha\beta}(k_0^2)}{\partial (k_0^2)^n} \right|_{k_0^2=0} \equiv \int_0^\infty \frac{dk g_l^\alpha(k) g_l^\beta(k)}{k^{2(n+1)}}. \quad (2)$$

We are interested primarily in the case $n = 0$ for all ℓ values (since these give the values $K_{\ell}^{\alpha\beta}(0)$) and in the cases $n = (1,2)$ for $\ell = (0,2)$, which, it turns out, are the only values contributing to the derivatives involved in the low-energy scattering parameter analysis. For $n \leq \ell$, Eq. (2) is a convergent integral. (For $\ell = 0$ the small $\tau_0 (= k_0 a)$ expansion of $K_{\ell}^{\alpha\beta}(\tau_0)$ is readily performed from the explicit expression for $K_0^{\alpha\beta}(\tau_0)$ see Eqs. (43) - (46) at the end of this Appendix.)

We define the function

$$M_{\ell,n}^{\alpha\beta} \equiv \frac{1}{a^{2n-1}} \left. \frac{\partial^n K_{\ell}^{\alpha\beta}(k_0^2)}{\partial (k_0^2)^n} \right|_{k_0^2=0}, \quad (3)$$

so that

$$K_{\ell}^{\alpha\beta}(k_0^2) = \frac{1}{a} \sum_{n=0}^{\infty} \frac{(k_0 a)^{2n}}{n!} M_{\ell,n}^{\alpha\beta}, \quad (4)$$

and use Eq. (2) and the corresponding manipulations leading to Eq. (19) of Appendix 4. We may then rewrite Eq. (3) as

$$M_{\ell,n}^{\alpha\beta} = \frac{\mu_{\alpha}^2 \mu_{\beta}^2}{8} e^{\mu_{\alpha} \mu_{\beta}} (A_{\ell,n}^{\alpha\beta} + B_{\ell,n}^{\alpha\beta}) \quad (n \leq \ell), \quad (5)$$

where

$$A_{l,n}^{\alpha\beta} \equiv \int_{\Omega} \frac{d\tau}{\tau^{2l+2n+2}} [Z_l(\mu_\alpha, \tau) \bar{Z}_l(\mu_\beta, \tau) + \bar{Z}_l(\mu_\alpha, \tau) Z_l(\mu_\beta, \tau)] \quad (6)$$

$$B_{l,n}^{\alpha\beta} \equiv - \int_{\Omega} \frac{d\tau}{\tau^{2l+2n+2}} \left[\frac{H_l^{(-)}(\tau)}{H_l^{(+)}(\tau)} Z_l(\mu_\alpha, \tau) Z_l(\mu_\beta, \tau) + \frac{H_l^{(+)}(\tau)}{H_l^{(-)}(\tau)} \bar{Z}_l(\mu_\alpha, \tau) \bar{Z}_l(\mu_\beta, \tau) \right]. \quad (7)$$

Using the integral definition of $Z_l(\mu_\alpha, \tau)$ (see Eqs. (1) and (4) of Appendix 2), we may rewrite Eq. (6) as

$$A_{ln}^{\alpha\beta} = \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} e^{i(s-t)\pi/2} \left[\int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t} \int_{\Omega} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t+2n+2}} + \text{sym}(\alpha \rightleftharpoons \beta) \right], \quad (8)$$

where $\text{sym}(\alpha \rightleftharpoons \beta)$ means the same expression with the interchange $\alpha \rightleftharpoons \beta$,

and $\gamma_{ls} \equiv \Gamma_{ls}(1) = 2^{-s}(l + \frac{1}{2}, s)$.

Defining

$$I_{s,t}^n(x,y) \equiv \oint_{\gamma} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t+2n+2}} \quad (9)$$

$$= I_{s,t}^{(+n)}(x,y)\theta(x-y) + I_{s,t}^{(-n)}(x,y)\theta(y-x), \quad (10)$$

and noting that

$$I_{s,t}^{(+n)}(x,y) = \oint_{\gamma} \frac{d\tau e^{i\tau(x-y)}}{\tau^{s+t+2n+2}} = 0, \quad (11)$$

since there are no poles enclosed in the contour which must be closed in the upper half complex τ -plane, and

$$I_{s,t}^{(-n)}(x,y) = \oint_{\gamma} \frac{d\tau e^{-i\tau(y-x)}}{\tau^{s+t+2n+2}} - 2\pi i \operatorname{Res}_{\tau=0} \left[\frac{e^{-i\tau(y-x)}}{\tau^{s+t+2n+2}} \right], \quad (12)$$

we see that

$$I_{s,t}^n(x,y) = -2\pi i \operatorname{Res}_{\tau=0} \left[\frac{e^{-i\tau(y-x)}}{\tau^{s+t+2n+2}} \right] \theta(y-x) \quad (13)$$

since the contour integral of Eq. (12) vanishes also because there are no poles enclosed inside the specified contour.

The residue term of Eq. (13) is readily evaluated to yield

$$\begin{aligned}
 -2\pi i \operatorname{Res}_{\tau=0} \left[\frac{e^{-i\tau(y-x)}}{\tau^{s+t+2n+2}} \right] \\
 = -2\pi e^{-i(s+t+2n)\pi/2} (y-x)^{s+t+2n+1} .
 \end{aligned} \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (8), we obtain

$$\begin{aligned}
 A_{l,n}^{\alpha\beta} &= (-)^{n+1} 2\pi \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} (-)^t \\
 &\cdot \left[\int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_x^\infty \frac{dy e^{-\mu_\beta y}}{y^t} (y-x)^{s+t+2n+1} + \operatorname{sym}(\alpha \neq \beta) \right] .
 \end{aligned} \tag{15}$$

We now turn to the expression for $B_{l,n}^{\alpha\beta}$. From Eq. (7) it is readily seen that

$$B_{l,n}^{\alpha\beta} = 2\pi i \operatorname{Res}_{\tau=0} \left[\frac{H_l^{(+)}(\tau)}{H_l^{(-)}(\tau)} \frac{\bar{Z}_l(\mu_\alpha, \tau) \bar{Z}_l(\mu_\beta, \tau)}{\tau^{2l+2n+2}} \right] , \tag{16}$$

because

$$- \oint_{\mathcal{C}} \frac{d\tau}{\tau^{2l+2n+2}} \frac{H_l^{(-)}(\tau)}{H_l^{(+)}(\tau)} Z_l(\mu_\alpha, \tau) Z_l(\mu_\beta, \tau) = 0 , \tag{17}$$

and

$$- \oint_{\mathcal{C}} \frac{d\tau}{\tau^{2l+2n+2}} \frac{H_l^{(+)}(\tau)}{H_l^{(-)}(\tau)} \bar{Z}_l(\mu_\alpha, \tau) \bar{Z}_l(\mu_\beta, \tau) = 0, \quad (18)$$

where each of the contours in Eqs. (17) and (18) is closed according to the constituent exponential integrals (as is argued in detail in Appendix 4), inside of which contours there are no poles in $H_l^{(+)}(\tau)/H_l^{(-)}(\tau)$, respectively. Equation (16) then follows directly from Eqs. (7), (17), and (18), inasmuch as Eq. (18) excluded the pole term at $\tau = 0$.

Residue Terms at the Origin for $B_{l,n}^{\alpha,\beta}$

In order to evaluate the residue terms for $n \leq 2$ (the only cases of interest) it is convenient to expand the function $H_l^{(+)}(\tau)/H_l^{(-)}(\tau)$ in Eq. (16) to $O(\tau^{2l+5})$ as

$$\frac{H_l^{(+)}(\tau)}{H_l^{(-)}(\tau)} \xrightarrow{\tau \rightarrow 0} 1 + 2i\delta_0 \tau^{2l+1} + 2i\delta_1 \tau^{2l+3} + 2i\delta_2 \tau^{2l+5} + \dots, \quad (19)$$

since the resultant identification of δ_0 , δ_1 , and δ_2 will permit the determination of the $B_{l,n}^{\alpha,\beta}$ for the range of indices needed.

Now

$$\frac{H_l^{(+)}(\tau)}{H_l^{(-)}(\tau)} = \frac{1+iF_l(\tau)/G_l(\tau)}{1-iF_l(\tau)/G_l(\tau)} \quad (20)$$

$$\tau \rightarrow 0 \left\{ 1 + 2i \frac{F_l(\tau)}{G_l(\tau)} - 2 \left[\frac{F_l(\tau)}{G_l(\tau)} \right]^2 \right\} + \dots, \quad (21)$$

where $F_l(\tau)/G_l(\tau)_{\tau \rightarrow 0} \sim O(\tau^{2l+1})$. Using the low momentum expansions for $F_l(\tau)$ and $G_l(\tau)$ we have

$$\frac{F_l(\tau)}{G_l(\tau)}_{\tau \rightarrow 0} = \frac{\tau^{2l+1}}{(2l+1)[(2l-1)!!]^2} \left\{ \frac{1 - \frac{1}{2}\tau^2 + \frac{(\frac{1}{2}\tau^2)^2}{2(2l+3)(2l+5)}}{1 - \frac{1}{2}\tau^2 + \frac{(\frac{1}{2}\tau^2)^2}{2(1-2l)(3-2l)}} \right\} \quad (22)$$

$$\begin{aligned} \tau \rightarrow 0 \frac{1}{[(2l-1)!!]^2} & \left\{ \frac{\tau^{2l+1}}{2l+1} - \frac{\tau^{2l+3}}{(2l-1)(2l+3)} \right. \\ & \left. + \frac{(l-2)\tau^{2l+5}}{(2l-1)^2(2l-3)(2l-5)} + \dots \right\}, \end{aligned} \quad (23)$$

after expanding Eq. (22) for $\tau \rightarrow 0$.

Substituting Eq. (23) into Eq. (21) and using Eq. (19) leads to the identifications

$$\left. \begin{aligned} \delta_0 &= -\frac{1}{(2l+1)[(2l-1)!!]^2} \\ \delta_1 &= -\frac{1}{(2l-1)(2l+3)[(2l-1)!!]^2} \\ \delta_2 &= \frac{l-2}{(2l-1)^2(2l-3)(2l+5)[(2l-1)!!]^2} \end{aligned} \right\} \quad (24)$$

For $l = 2$ we have specifically that $\delta_1 = -1/189$ and $\delta_2 = 0$. Thus we may hence forth ignore the δ_2 contribution inasmuch as it vanishes for the only case for which we need its value. Substituting Eqs. (24) and (19) into Eq. (16) we obtain (remembering that the δ_1 contribution is only of interest for $l = 2$),

$$B_{l,n}^{\alpha\beta} = 2\pi i \operatorname{Res}_{\tau=0} \left\{ \frac{\bar{Z}_l(\mu_\alpha, \tau) \bar{Z}_l(\mu_\beta, \tau)}{\tau^{2l+2n+2}} \left[1 + 2i\delta_0 \tau^{2l+1} + 2i\delta_1 \tau^{2l+3} \delta_{l,2} \right] \right\} \quad (25)$$

Using the integral expressions for $Z_l(\mu_\alpha, \tau)$, Eq. (25) may be written as

$$\begin{aligned} B_{l,n}^{\alpha\beta} &= (-)^l 2\pi i \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} e^{-i(s+t)\pi/2} \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t} \\ &\times \operatorname{Res}_{\tau=0} \left\{ e^{-i\tau(y+x)} \left[\frac{1}{\tau^{s+t+2n+2}} + \frac{2i\delta_0}{\tau^{s+t+2n+1-2l}} + \frac{2i\delta_1 \delta_{l,2}}{\tau^{s+t+2n-1-2l}} \right] \right\} \quad (26) \end{aligned}$$

But, with $\theta(0) = 0$,

$$\text{Res}_{\tau=0} \left[\frac{e^{-i\tau(y+x)}}{\tau^q} \right] = \frac{\theta(q) e^{-i(q-1)\pi/2}}{(q-1)!!} (y+x)^{q-1}, \quad (27)$$

and substituting Eq. (27) into (26) gives the result that

$$B_{l,n}^{\alpha\beta} = 2\pi \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} (-)^{n+s+t} \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t}$$

$$\times \left\{ \frac{(-)^l (y+x)^{s+t+2n+1}}{(s+t+2n+1)!} - \frac{2\delta_0 (y+x)^{s+t+2n-2l} \theta(s+t+2n-2l+1)}{(s+t+2n-2l)!} \right.$$

$$\left. + \frac{2\delta_{l,2} \delta_1 (y+x)^{s+t+2n-2-2l} \theta(s+t+2n-1-2l)}{(s+t+2n-2-2l)!} \right\}. \quad (28)$$

Final Integrations of Residue Integrals

Now we turn to the explicit evaluation of the integrals involved in $A_{l,n}^{\alpha\beta}$ and $B_{l,n}^{\alpha\beta}$, as given in Eqs. (15) and (28). The integrals in question are of three forms, which we define as

$$g_{s,t,n}^{\alpha\beta} \equiv \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_x^\infty \frac{dy e^{-\mu_\beta y}}{y^t} (y-x)^{s+t+2n+1}, \quad (29)$$

$$\mathfrak{I}_{s,t,n}^{\alpha\beta} \equiv \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t} (y+x)^{s+t+2n+1}, \quad (30)$$

$$\mathfrak{I}_{l,s,t,n}^{\alpha\beta} \equiv \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_1^\infty \frac{dy e^{-\mu_\beta y}}{y^t} (y+x)^{s+t+2n-2l} \theta(s+t+2n-2l+1). \quad (31)$$

We examine Eq. (29) first. Transforming from the variable y to w by $y = wx$, and using the binomial theorem, we have

$$\begin{aligned} \mathfrak{I}_{s,t,n}^{\alpha\beta} &= \int_1^\infty dx e^{-\mu_\alpha x} x^{2n+2} \int_1^\infty \frac{dw e^{-\mu_\beta wx}}{w^t} (w-1)^{s+t+2n+1} \\ &= \sum_{r=0}^{s+t+2n+1} \binom{s+t+2n+1}{r} (-)^r \int_1^\infty dx e^{-\mu_\alpha x} x^{2(n+1)} \int_1^\infty dw e^{-\mu_\beta wx} w^{s+2n+1-r}, \end{aligned} \quad (32)$$

or

$$\begin{aligned} \mathfrak{I}_{s,t,n}^{\alpha\beta} &= \sum_{r=0}^{s+t+2n+1} \binom{s+t+2n+1}{r} (-)^r \int_1^\infty dx e^{-\mu_\alpha x} x^{2(n+1)} \\ &\quad \cdot \left[\theta(s+2n+2-r) \alpha_{s+2n+1-r}(\mu_\beta x) \right. \\ &\quad \left. + \theta(r-s-2n-1) E_{r-s-2n-1}(\mu_\beta x) \right]. \end{aligned} \quad (33)$$

The first integral of Eq. (33) has the solution (for $s+2n+1-r \geq 0$)

$$\int_1^{\infty} dx x^{2(n+1)} e^{-\mu_{\alpha} x} \alpha_{s+2n+1-r}(\mu_{\beta} x)$$

$$= \sum_{p=0}^{s+2n+1-r} \frac{(s+2n+1-r)!}{p!} \frac{1}{(\mu_{\beta})^{s+2n+2-r-p}} \left[\theta(p-s+r+1) \alpha_{p-s+r}(\mu_{\alpha} + \mu_{\beta}) + \theta(s-r-p) E_{s-r-p}(\mu_{\alpha} + \mu_{\beta}) \right], \quad (34)$$

which follows directly after using the definition

$$\alpha_n(z) \equiv \int_1^{\infty} dt t^n e^{-zt} = \frac{n!}{z^{n+1}} e^{-z} \sum_{p=0}^n \frac{z^p}{p!}, \quad \text{Re}(z) > 0. \quad (35)$$

Next we consider the second integral of Eq. (33) (for $r-s-2n-1 \geq 0$)

$$\int_1^{\infty} dx x^{2(n+1)} e^{-\mu_{\alpha} x} E_{r-s-2n-1}(\mu_{\beta} x) \equiv D_{2n+2, r-s-2n-1}^{(2)}(\mu_{\alpha}, \mu_{\beta}), \quad (36)$$

where from Eqs. (14) and (15) in Appendix 13 and (30) in Appendix 12, we have for $2n+2 \rightarrow n \geq 2$,

$$D_{n,k}^{(2)}(\lambda_1, \lambda_2) = \frac{\alpha_{n-1}(\lambda_1 + \lambda_2)}{\lambda_2} \delta_{0,k} \\ + \delta_{1,k} \left\{ \alpha_n(\lambda_1) E_1(\lambda_2) - \sum_{p=1}^n \frac{n!}{p!} \frac{\alpha_{p-1}(\lambda_1 + \lambda_2)}{\lambda_1^{n-p+1}} - \frac{n!}{\lambda_1^{n+1}} E_1(\lambda_1 + \lambda_2) \right\}$$

$$+ (1 - \delta_{0,k})(1 - \delta_{1,k})$$

$$\cdot \left\{ \begin{aligned} & \sum_{p=1}^{k-1} \frac{(k-1-p)!}{(k-1)!} (-\lambda_2)^{p-1} \alpha_{n+p-1}(\lambda_1 + \lambda_2) \\ & + \frac{(-\lambda_2)^{k-1}}{(k-1)!} \alpha_{n+k-1}(\lambda_1) E_1(\lambda_2) \\ & - \frac{(-\lambda_2)^{k-1}}{(k-1)!} \sum_{p=1}^{n+k-1} \frac{(n+k-1)!}{p!} \frac{\alpha_{p-1}(\lambda_1 + \lambda_2)}{\lambda_1^{n+k-p}} - \frac{(-\lambda_2)^{k-1} (n+k-1)! E_1(\lambda_1 + \lambda_2)}{\lambda_1^{n+k}} \end{aligned} \right\} \quad (37)$$

Combining Eqs. (34) and (36) we find the final expression for $\mathcal{D}_{s,t,n}^{(2)}$; namely,

$$y_{s,t,n}^{\alpha\beta} = \sum_{r=0}^{s+t+2n+1} \binom{s+t+2n+1}{r} (-)^r$$

$$\times \left\{ \begin{aligned} & \theta(s+2n+2-r) \sum_{p=0}^{s+2n+1-r} \frac{(s+2n+1-r)!}{p!} \frac{1}{\mu_{\beta}^{s+t+2-r-p}} \\ & \cdot \left[\begin{aligned} & \theta(p-s+r+1) \alpha_{p-s+r}(\mu_{\alpha} + \mu_{\beta}) \\ & + \theta(s-r-p) E_{s-r-p}(\mu_{\alpha} + \mu_{\beta}) \end{aligned} \right] \\ & + \theta(r-s-2n-1) D_{2n+2, r-s-2n-1}^{(2)}(\mu_{\alpha}, \mu_{\beta}) \end{aligned} \right\}$$

(38)

The expressions of Eqs. (30) and (31) integrate much more directly to give

$$y_{s,t,n}^{\alpha\beta} = \sum_{r=0}^{s+t+2n+1} \binom{s+t+2n+1}{r} \int_1^{\infty} dx e^{-\mu_{\alpha} x} x^{r-s} \int_1^{\infty} dy e^{-\mu_{\beta} y} y^{s+2n+1-r}$$

$$= \sum_{r=0}^{s+t+2n+1} \binom{s+t+2n+1}{r} \left\{ \begin{aligned} & \alpha_{s+2n+1-r}(\mu_{\beta}) \left[\begin{aligned} & \theta(r-s+1) \theta(s-r+2n+2) \alpha_{r-s}(\mu_{\alpha}) \\ & + \theta(s-r) E_{s-r}(\mu_{\alpha}) \end{aligned} \right] \\ & + \theta(r-s-2n-1) E_{r-s-2n-1}(\mu_{\beta}) \alpha_{r-s}(\mu_{\alpha}) \end{aligned} \right\},$$

(39)

and

$$x_{l,s,t,n}^{\alpha\beta}$$

$$= \sum_{r=0}^{s+t+2n-2l} \binom{s+t+2n-2l}{r} \theta(s+t+2n-2l+1)$$

$$\cdot \int_1^\infty dx e^{-\mu_\alpha x} x^{r-s} \int_1^\infty dy e^{-\mu_\beta y} y^{s+2n-2l-r}$$

$$= \theta(s+t+2n-2l+1) \sum_{r=0}^{s+t+2n-2l} \binom{s+t+2n-2l}{r}$$

$$\times \left\{ \begin{aligned} & [\theta(r-s+1) \alpha_{r-s}(\mu_\alpha) + \theta(s-r) E_{s-r}(\mu_\alpha)] \times \\ & \times [\theta(s+2n-2l-r+1) \alpha_{s+2n-2l-r}(\mu_\beta) + \theta(r+2l-2n-s) E_{r+2l-2n-s}(\mu_\beta)] \end{aligned} \right\} . \quad (40)$$

Clearly, since

$$A_{l,n}^{\alpha\beta} = (-)^{n+1} 2\pi \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} (-)^t (q_{s,t,n}^{\alpha\beta} + q_{s,t,n}^{\beta\alpha}) , \quad (41)$$

and

$$B_{l,n}^{\alpha\beta} = 2\pi \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} (-)^{n+s+t} \left\{ \frac{(-)^l \mathfrak{P}_{s,t,n}^{\alpha\beta}}{(s+t+2n+1)!} - \frac{2\delta_0 \mathfrak{P}_{l,s,t,n}^{\alpha\beta}}{(s+t+2n-2l)!} + \frac{2\delta_{l,2} \delta_1 \mathfrak{P}_{l,s,t,n-1}^{\alpha\beta}}{(s+t+2n-2-2l)!} \right\}, \quad (42)$$

where $\mathfrak{P}_{s,t,n}^{\alpha\beta}$, $\mathfrak{P}_{l,s,t,n}^{\alpha\beta}$, and $\mathfrak{P}_{l,s,t,n}^{\alpha\beta}$ are given by Eqs. (38) through (40), the functions $A_{l,n}^{\alpha\beta}$ and $B_{l,n}^{\alpha\beta}$ are completely determined. Knowing $A_{l,n}^{\alpha\beta}$ and $B_{l,n}^{\alpha\beta}$ we may therefore determine $M_{l,n}^{\alpha\beta}$ from Eq. (5), and hence the low-momentum expansion for $K_l^{\alpha\beta}(\tau_0^2)$ by using Eq. (4). The only values of interest are $n = 0$ for all l and $n = (1, 2)$, for $l = 2$ only.

For the special case $l = 0$, Eq. (75) in Appendix 4 can be expanded directly to give

$$K_0^{\alpha,\beta}(\tau_0) = \frac{\pi}{2a} \left(\frac{\mu_\alpha \mu_\beta}{\mu_\alpha + \mu_\beta} \right) \left\{ 1 - \left(\frac{1}{\mu_\alpha^2} + \frac{1}{\mu_\beta^2} + \frac{1}{\mu_\alpha \mu_\beta} \right) \tau_0^2 + \left(\frac{1}{\mu_\alpha^4} + \frac{1}{\mu_\beta^4} + \frac{1}{\mu_\alpha^2 \mu_\beta^2} + \frac{1}{\mu_\alpha^3 \mu_\beta} + \frac{1}{\mu_\alpha \mu_\beta^3} \right) \tau_0^4 + o(\tau_0^6) \right\}. \quad (43)$$

Therefore, since $K_0^{\alpha\beta}(\tau_0)$ is the integral in Eqs. (46) - (48) of Sec. I.2, we may conclude directly that the coefficients $a_n^\alpha(0)$ in Eq. (12) of Sec. II.1 are given by

$$a_0^c(0) = \frac{\sigma_c}{2}; \quad a_2^c(0) = -\frac{3\sigma_c}{2\mu_c^2}; \quad a_4^c(0) = \frac{5\sigma_c}{2\mu_c^4}; \quad (44)$$

$$a_0^T(0) = \frac{\sigma_T}{2} - \frac{1}{\sqrt{2}}; \quad a_2^T(0) = -\frac{3}{2} \frac{\sigma_T}{\mu_T^2}; \quad a_4^T(0) = \frac{5}{2} \frac{\sigma_T}{\mu_T^4}; \quad (45)$$

$$a_0^{cT}(0) = \left(\frac{\sigma_c \mu_T}{\mu_c + \mu_T} \right); \quad a_2^{cT}(0) = - \left(\frac{\sigma_c \mu_T}{\mu_c + \mu_T} \right) \left(\frac{1}{\mu_c^2} + \frac{1}{\mu_T^2} + \frac{1}{\mu_c \mu_T} \right) ,$$

$$a_4^{cT}(0) = \left(\frac{\sigma_c \mu_T}{\mu_c + \mu_T} \right) \left(\frac{1}{\mu_c^4} + \frac{1}{\mu_T^4} + \frac{1}{\mu_c^2 \mu_T^2} + \frac{1}{\mu_c^3 \mu_T} + \frac{1}{\mu_c \mu_T^3} \right) . \quad (46)$$

ON THE EVALUATION OF THE INTEGRAL $J_{s,t}(\alpha, \bar{\beta})$

The integral $J_{s,t}(\alpha, \bar{\beta})$ is defined by

$$J_{s,t}(\alpha, \bar{\beta}) \quad (1)$$

$$\equiv \int_1^{\infty} \frac{dx e^{-\alpha x} E_t(\bar{\beta} x)}{x^{s+t-1}}, \quad \operatorname{Re}(\alpha, \bar{\beta}) > 0 \quad s, t = 0, 1, 2, \dots,$$

where $\alpha \equiv \mu_{\alpha} + i\tau$ and $\bar{\beta} \equiv \mu_{\beta} - i\tau$. The exponential integral $E_n(z)$ is defined by

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \quad (2)$$

For $s \geq 0$ and $t = 0$, Eq. (1) yields

$$J_{s,0}(\alpha, \bar{\beta}) = \int_1^{\infty} \frac{dx e^{-\alpha x} E_0(\bar{\beta} x)}{x^{s-1}} = \frac{1}{\bar{\beta}} E_s(\alpha + \bar{\beta}). \quad (3)$$

For $s = 0$ and $t \geq 1$, we may use the expansion theorem for the exponential integral (integrate Eq. (2) by parts and use induction)

$$E_n(z) = \frac{\Theta(n-1)}{(n-1)!} e^{-z} \sum_{k=1}^{n-1} (n-1-k)! (-z)^{k-1} + \frac{(-z)^{n-1}}{(n-1)!} E_1(z), \quad (4)$$

where $\theta(0) \equiv 0$, to obtain

$$J_{0,t}(\alpha, \bar{\beta}) = \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)!(-\bar{\beta})^{k-1} \int_1^{\infty} \frac{dx e^{-(\alpha+\bar{\beta})x} x^{k-1}}{x^{t-1}} \\ + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \int_1^{\infty} dx e^{-\alpha x} E_1(\bar{\beta}x) \quad (5)$$

$$= \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)!(-\bar{\beta})^{k-1} E_{t-k}(\alpha + \bar{\beta}) + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} D_0(\alpha, \bar{\beta}) , \quad (6)$$

where

$$D_0(\alpha, \bar{\beta}) = \frac{1}{\alpha} [e^{-\alpha} E_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})] , \quad (7)$$

which can be determined by integration by parts of Eq. (10) below.

For $s \geq 1$ and $t \geq 1$ we use Eq. (4) in (1) and obtain

$$J_{s,t}(\alpha, \bar{\beta}) = \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)!(-\bar{\beta})^{k-1} \int_1^{\infty} \frac{dx e^{-(\alpha+\bar{\beta})x}}{x^{s+t-k}} \\ + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta}x)}{x^s} \quad (8)$$

$$= \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)!(-\bar{\beta})^{k-1} E_{s+t-k}(\alpha + \bar{\beta}) + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} D_s(\alpha, \bar{\beta}) , \quad (9)$$

where

$$D_s(\alpha, \bar{\beta}) \equiv J_{s,1}(\alpha, \bar{\beta}) = \int_1^\infty \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x^s} \quad (10)$$

is evaluated in Appendix 12. For $s = 1$ we have the results [see Eqs. (9), (20) and (23) of Appendix 12]

$$D_1(\alpha, \bar{\beta}) = E_1(\alpha)E_1(\bar{\beta}) - D_1(\bar{\beta}, \alpha), \quad (11)$$

$$D_1(\alpha, \bar{\beta})$$

$$= \left\{ \begin{array}{l} \left\{ -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right\} - (\gamma + \ln \bar{\beta}) E_1(\alpha) \right. \\ \left. - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) \right\} \quad \text{if } |\bar{\beta}/\alpha| \leq 1 \\ \left\{ \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \bar{\beta})^2 \right\} + (\gamma + \ln \alpha + E_1(\alpha)) E_1(\bar{\beta}) \right. \\ \left. + \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}) \right\} \quad \text{if } |\alpha/\bar{\beta}| \leq 1 \end{array} \right\} \quad (12)$$

($\gamma = \text{Euler's constant} = 0.577215664\dots$), and if $s \geq 1$ [see Eq. (24) of Appendix 12]

$$\begin{aligned}
D_s(\alpha, \bar{\beta}) &= \frac{(-\alpha)^{s-1}}{(s-1)!} D_1(\alpha, \bar{\beta}) \\
&+ \theta(s-1) \sum_{p=1}^{s-1} \frac{(-\alpha)^{s-1-p} (p-1)!}{(s-1)!} [e^{-\alpha} E_1(\bar{\beta}) - E_{p+1}(\alpha + \bar{\beta})] .
\end{aligned} \tag{13}$$

Collecting together Eqs. (3), (6), (7), (9), and (11) through (13), we obtain for $J_{s,t}(\alpha, \bar{\beta})$ the complete expression

$$\begin{aligned}
J_{s,t}(\alpha, \bar{\beta}) &= \delta_{t,0} \frac{1}{\bar{\beta}} E_s(\alpha + \bar{\beta}) \\
&+ \delta_{s,0} (1 - \delta_{t,0}) \left\{ \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{t-k}(\alpha + \bar{\beta}) \right. \\
&\quad \left. + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{1}{\alpha} [e^{-\alpha} E_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})] \right\} \\
&+ (1 - \delta_{s,0})(1 - \delta_{t,0}) \left\{ \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-1-k)! (-\bar{\beta})^{k-1} E_{s+t-k}(\alpha + \bar{\beta}) \right. \\
&\quad + \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \frac{(-\alpha)^{s-1}}{(s-1)!} D_1(\alpha, \bar{\beta}) \\
&\quad \left. + \frac{\theta(s-1)}{(s-1)!} \frac{(-\bar{\beta})^{t-1}}{(t-1)!} \sum_{k=1}^{s-1} (-\alpha)^{s-1-k} (k-1)! [e^{-\alpha} E_1(\bar{\beta}) - E_{k+1}(\alpha + \bar{\beta})] \right\}
\end{aligned} \tag{14}$$

See also the discussion of Eq. (13) in Appendix 13. A property similar to Eq. (11) holds also for $J_{s,t}(\alpha, \bar{\beta})$; namely,

$$J_{s,t}(\alpha, \bar{\beta}) + J_{t,s}(\bar{\beta}, \alpha) = E_s(\alpha) E_t(\bar{\beta}), \tag{15}$$

as can be proved in analogy with the proof of Eq. (9) in Appendix 12.

APPENDIX 7

ON THE EVALUATION OF THE INTEGRAL $I_{l,s,t,p}^{\alpha\beta}$

The integral $I_{l,s,t,p}^{\alpha\beta}$ is defined by

$$I_{l,s,t,p}^{\alpha\beta} \equiv \int_1^\infty dx \int_x^\infty dy \frac{e^{-\mu_\alpha x}}{x^s} \frac{e^{-\mu_\beta y}}{y^t} \left\{ \begin{aligned} &(-)^{l+s+t} (y+x)^{s+t-1-2p} \\ &+ (-)^{t+1} (y-x)^{s+t-1-2p} \end{aligned} \right\} \quad (1)$$

+ sym($\alpha \rightleftharpoons \beta$) ,

where sym ($\alpha \rightleftharpoons \beta$) means the same expression with the interchange of $\alpha \rightleftharpoons \beta$. The indices in Eq. (1) are constrained by the two conditions: $0 \leq 2p \leq s + t - 1$ and $0 \leq (s,t) \leq l$.

Upon making the substitution $y \equiv wx$ in Eq. (1), and defining

$$N_{s,t,p}^{\beta(\pm)}(x) \equiv \int_1^\infty \frac{dwe^{-\mu_\beta wx}}{w^t} (w \pm 1)^{s+t-1-2p} , \quad (2)$$

$$L_{s,t,p}^{\alpha\beta(\pm)} \equiv \int_1^\infty \frac{dxe^{-\mu_\alpha x}}{x^{2p}} N_{s,t,p}^{\beta(\pm)}(x) , \quad (3)$$

it is evident that Eq. (1) can be rewritten as

$$I_{l,s,t,p}^{\alpha\beta} = (-)^{l+s+t} (L_{s,t,p}^{\alpha\beta(+)} + L_{s,t,p}^{\beta\alpha(+)}) + (-)^{t-1} (L_{s,t,p}^{\alpha\beta(-)} + L_{s,t,p}^{\beta\alpha(-)}) . \quad (4)$$

In the determination of Eq. (4) our first consideration is the evaluation of Eq. (2) and then Eq. (3). Applying the binomial theorem to the $(w \pm 1)$ term in Eq. (2) and integrating yields

$$\begin{aligned}
N_{s,t,p}^{\beta(\pm)}(x) &= \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q \int_1^{\infty} dwe^{-\mu_{\beta} w x} w^{s-1-2p-q} \\
&= \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q \left\{ \begin{aligned} &\theta(s-2p-q) \alpha_{s-1-2p-q}(\mu_{\beta} x) \\ &+ \theta(q+2p+1-s) E_{q+2p+1-s}(\mu_{\beta} x) \end{aligned} \right\}
\end{aligned}
\tag{5}$$

with $\theta(0) \equiv 0$, since

$$\alpha_n(z) \equiv \int_1^{\infty} dt e^{-zt} t^n, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots
\tag{6}$$

$$= \frac{n!}{z^{n+1}} e^{-z} e_n(z),
\tag{7}$$

where

$$e_n(z) \equiv \sum_{q=0}^n \frac{z^q}{q!},
\tag{8}$$

and

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots
\tag{9}$$

For $n = 0$, $E_0(z) = \alpha_0(z) = e^{-z}/z$.

Upon substituting Eq. (5) into Eq. (3), we find

$$L_{s,t,p}^{\alpha\beta(\pm)}$$

$$= \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q \left\{ \begin{aligned} & \theta(s-2p-q) \int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x}}{x^{2p}} \alpha_{s-1-2p-q}(\mu_{\beta} x) \\ & + \theta(q+2p+1-s) \int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x}}{x^{2p}} E_{q+2p+1-s}(\mu_{\beta} x) \end{aligned} \right\} \quad (10)$$

Using Eqs. (7) and (8) in the first integral of Eq. (10) we readily find that

$$\begin{aligned} & \int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x}}{x^{2p}} \alpha_{s-1-2p-q}(\mu_{\beta} x) \\ &= \frac{(s-1-2p-q)!}{(\mu_{\beta})^{s-2p-q}} \sum_{r=0}^{s-1-2p-q} \frac{(\mu_{\beta})^r}{r!} E_{s-q-r}(\mu_{\alpha} + \mu_{\beta}) , \quad (11) \end{aligned}$$

since $r \leq s-q-1-2p < s-q$ for any $p \geq 0$.

The second integral of Eq. (10) is evaluated by utilizing the recursion relation for $E_n(z)$; namely, for $n > 0$,

$$E_n(z) = \theta(n-1) \frac{e^{-z}}{(n-1)!} \sum_{k=1}^{n-1} (n-1-k)! (-z)^{k-1} + \frac{(-z)^{n-1}}{(n-1)!} E_1(z) \quad (12)$$

with $\theta(0) \equiv 0$, to obtain

$$\int_1^{\infty} \frac{dx e^{-\mu_{\alpha} x}}{x^{2p}} E_{q+2p+1-s}(\mu_{\beta} x) = \left\{ \begin{aligned} & \frac{\theta(q+2p-s)}{(q+2p-s)!} \sum_{r=1}^{q+2p-s} (q+2p-s-r)! (-\mu_{\beta})^{r-1} \int_1^{\infty} \frac{dx e^{-(\mu_{\alpha}-\mu_{\beta})x}}{x^{2p}} x^{r-1} \\ & + \frac{(-\mu_{\beta})^{q+2p-s}}{(q+2p-s)!} \int_1^{\infty} dx x^{q-s} e^{-\mu_{\alpha} x} E_1(\mu_{\beta} x) \end{aligned} \right\} \quad (13)$$

The first integral of Eq. (13) is readily determined as

$$\begin{aligned} & \int_1^{\infty} \frac{dx e^{-(\mu_{\alpha}+\mu_{\beta})x}}{x^{2p}} x^{r-1} \\ & = \theta(r-2p) \alpha_{r-1-2p}(\mu_{\alpha} + \mu_{\beta}) + \theta(2p+1-r) E_{2p+1-r}(\mu_{\alpha} + \mu_{\beta}). \end{aligned} \quad (14)$$

The second integral in Eq. (13) is given by (see Eqs. (1) in Appendix 12 and (2) in Appendix (13))

$$\begin{aligned} & \int_1^{\infty} dx x^{q-s} e^{-\mu_{\alpha} x} E_1(\mu_{\beta} x) \\ & = \theta(q-s+1) D_{q-s+1}^{(2)}(\mu_{\alpha}, \mu_{\beta}) + \theta(s-q) D_{s-q}(\mu_{\alpha}, \mu_{\beta}), \end{aligned} \quad (15)$$

where according to Eq. (27) of Appendix 12,

$$D_{q-s}^{(2)}(\mu_\alpha, \mu_\beta) = \alpha_{q-s}(\mu_\alpha) E_1(\mu_\beta) - \frac{(q-s)!}{(\mu_\alpha)^{q-s+1}} E_1(\mu_\alpha + \mu_\beta) - \sum_{r=1}^{q-s} \frac{(q-s)! e^{-(\mu_\alpha + \mu_\beta)} e_{r-1}(\mu_\alpha + \mu_\beta)}{r(\mu_\alpha)^{q-s+1-r} (\mu_\alpha + \mu_\beta)^r}. \quad (16)$$

When $n+1 \equiv s-q > 0$, we have from Eq. (24) of Appendix 12

$$D_{n+1}(\mu_\alpha, \mu_\beta) = \frac{(-\mu_\alpha)^n}{n!} D_1(\mu_\alpha, \mu_\beta) + \sum_{r=1}^n \frac{(-\mu_\alpha)^{n-r}}{n!} (r-1)! \epsilon_{r+1}(\mu_\alpha, \mu_\beta), \quad (17)$$

where

$$\epsilon_{r+1}(\mu_\alpha, \mu_\beta) \equiv e^{-\mu_\alpha} E_1(\mu_\beta) - E_{r+1}(\mu_\alpha + \mu_\beta). \quad (18)$$

The function $D_1(\mu_\alpha, \mu_\beta)$ is given if $\mu_\alpha/\mu_\beta < 1$, by Eq. (23)

of Appendix 12 as

$$D_1(\mu_\alpha, \mu_\beta) = \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \mu_\beta)^2 \right\} + E_1(\mu_\beta) (\gamma + \ln \mu_\alpha + E_1(\mu_\alpha)) + \sum_{p=1}^{\infty} \frac{(-\mu_\beta)^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{(-\mu_\alpha/\mu_\beta)^p}{p^2} e^{-\mu_\beta} e_{p-1}(\mu_\beta), \quad (19)$$

and, if $\mu_\beta/\mu_\alpha < 1$, by Eq. (20) of Appendix 12 as

$$D_1(\mu_\alpha, \mu_\beta) = -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \mu_\alpha)^2 \right\} - E_1(\mu_\alpha)(\gamma + \ln \mu_\beta) \\ - \sum_{p=1}^{\infty} \frac{(-\mu_\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\mu_\beta/\mu_\alpha)^p}{p^2} e^{-\mu_\alpha} e_{p-1}(\mu_\alpha) \quad (20)$$

Substituting Eqs. (14) and (15) into Eq. (13), and Eqs. (11) and (13) into Eq. (10), we obtain for $L_{s,t,p}^{\alpha\beta(\pm)}$ the result that

$$L_{s,t,p}^{\alpha\beta(\pm)} = \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q \\ \theta(s-2p-q) \frac{(s-1-2p-q)!}{(\mu_\beta)^{s-2p-q}} \sum_{r=0}^{s-1-2p-q} \frac{(\mu_\beta)^r}{r!} E_{s-q-r}(\mu_\alpha + \mu_\beta)$$

$$+ \theta(q+2p+1-s)$$

$$\left\{ \frac{\theta(q+2p-s)}{(q+2p-s)!} \sum_{r=1}^{q+2p-s} (q+2p-s-r)! \mu_\beta^{r-1} \left\{ \begin{array}{l} \theta(r-2p) \alpha_{r-1-2p}(\mu_\alpha + \mu_\beta) \\ + \theta(2p+1-r) E_{2p+1-r}(\mu_\alpha + \mu_\beta) \end{array} \right\} \right. \\ \left. + \frac{(-\mu_\beta)^{q+2p-s}}{(q+2p-s)!} \left\{ \begin{array}{l} \theta(q-s+1) D_{q-s,1}^{(2)}(\mu_\alpha, \mu_\beta) \\ + \theta(s-q) D_{s-q,1}(\mu_\alpha, \mu_\beta) \end{array} \right\} \right\} \quad (21)$$

where $D_{q-s}^{(2)}(\mu_\alpha, \mu_\beta)$ and $D_{s-q}(\mu_\alpha, \mu_\beta)$ are defined by Eqs. (16) through (20).

Forming the combination $[L_{s,t,p}^{\alpha\beta\pm} + L_{s,t,p}^{\beta\alpha(\pm)}]$, we have

$$[L_{s,t,p}^{\alpha\beta(\pm)} + L_{s,t,p}^{\beta\alpha(\pm)}] = \sum_{q=0}^{s+t-1-2p} \binom{s+t-1-2p}{q} (\pm)^q$$

$$\theta(s-2p-q)(s-1-2p-q)! \sum_{r=0}^{s-1-2p-q} \frac{E_{s-q-r}(\mu_\alpha + \mu_\beta)}{r!}$$

$$\left(\frac{1}{\mu_\alpha^{s-2p-q-r}} + \frac{1}{\mu_\beta^{s-2p-q-r}} \right)$$

$$+ \theta(q+2p+1-s)$$

$$\left\{ \frac{\theta(q+2p-s)}{(q+2p-s)!} \sum_{r=1}^{q+2p-s} (q+2p-s-r)! (-\mu_\alpha)^{r-1} + (-\mu_\beta)^{r-1} \right. \\ \left. \left\{ \theta(r-2p)\alpha_{r-1-2p}(\mu_\alpha + \mu_\beta) \right. \right. \\ \left. \left. + \theta(2p+1-r)E_{2p+1-r}(\mu_\alpha + \mu_\beta) \right\} \right. \\ \left. + \theta(q-s+1)(-)^{q+2p-s} [\mu_\beta^{q+2p-s} D_{q-s}^{(2)}(\mu_\alpha, \mu_\beta) + \mu_\alpha^{q+2p-s} D_{q-s}^{(2)}(\mu_\beta, \mu_\alpha)] \right\}$$

$$+ \frac{\theta(s-q)}{(q+2p-s)!} \frac{1}{(s-q-1)!} \theta(q+2p+1-s)$$

$$\left\{ \begin{aligned} & (-\mu_\beta)^{q+2p-s} (-\mu_\alpha)^{s-q-1} D_1(\mu_\alpha, \mu_\beta) \\ & + (-\mu_\alpha)^{q+2p-s} (-\mu_\beta)^{s-q-1} D_1(\mu_\beta, \mu_\alpha) \\ & + \sum_{r=1}^{s-q-1} (r-1)! \left\{ \begin{aligned} & (-\mu_\beta)^{q+2p-s} (-\mu_\alpha)^{s-q-r-1} \epsilon_{r+1}(\mu_\alpha, \mu_\beta) \\ & + (-\mu_\alpha)^{q+2p-s} (-\mu_\beta)^{s-q-r-1} \epsilon_{r+1}(\mu_\beta, \mu_\alpha) \end{aligned} \right\} \end{aligned} \right\} \quad (22)$$

where $\theta(0) \equiv 0$, and the functions $\epsilon_{r+1}(\mu_\alpha, \mu_\beta)$ and $D_1(\mu_\alpha, \mu_\beta)$ are given

by Eqs. (18) through (20). The integral $I_{\ell, s, t, p}^{\alpha\beta}$ is then explicitly determined by substituting Eq. (22) into Eq. (4).

ON THE PRINCIPAL VALUE INTEGRAL $G_\ell(k_0 r, k_0 r')$

The integral $G_\ell(k_0 r, k_0 r')$ is defined by

$$G_\ell(k_0 r, k_0 r') \equiv \int_0^\infty dk F'_\ell(kr) F'_\ell(kr') P\left(\frac{1}{k^2 - k_0^2}\right) \quad (1)$$

Using the definition

$$F'_\ell(kr) \equiv \frac{\text{Im}[H_\ell^{(+)}(kr)H_\ell^{(-)}(ka)]}{\Delta_\ell(ka)} \quad (2)$$

in Eq. (1), we have

$$G_\ell(k_0 r, k_0 r') = aP \int_0^\infty \frac{d\tau}{\tau^2 - \tau_0^2} \frac{\text{Im}[H_\ell^{(+)}(\tau x)H_\ell^{(-)}(\tau)] \text{Im}[H_\ell^{(+)}(\tau x')H_\ell^{(-)}(\tau)]}{\Delta_\ell^2(\tau)} \quad (3)$$

where $\tau \equiv ka$ and $x = r/a$. Using the relationship that for z complex, $\text{Im}(z) = (z - \bar{z})/2i$, and noting that the kernel of Eq. (3) is an even bounded function of τ , we may extend the integration range to $-\infty \leq \tau \leq \infty$, and write Eq. (3) as an open indented (around the origin) contour integral; namely,

$$G_\ell(\tau_0 x, \tau_0 x') = -\frac{a}{8} P \int_{\gamma} \frac{d\tau}{\tau^2 - \tau_0^2} \left\{ \begin{aligned} & H_\ell^{(+)}(\tau x) H_\ell^{(+)}(\tau x') H_\ell^{(-)}(\tau) / H_\ell^{(+)}(\tau) \\ & + H_\ell^{(-)}(\tau x) H_\ell^{(-)}(\tau x') H_\ell^{(+)}(\tau) / H_\ell^{(-)}(\tau) \\ & - H_\ell^{(+)}(\tau x) H_\ell^{(-)}(\tau x') - H_\ell^{(+)}(\tau x') H_\ell^{(-)}(\tau x) \end{aligned} \right\} \quad (4)$$

where we have also used the definition,

$$\Delta_l^2(\tau) \equiv H_l^{(+)}(\tau) H_l^{(-)}(\tau) \quad . \quad (5)$$

We write the principle value, $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$, as

$$P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \equiv \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{2} \left[\frac{1}{(\tau + \tau_0 - i\epsilon)(\tau - \tau_0 + i\epsilon)} + \frac{1}{(\tau + \tau_0 + i\epsilon)(\tau - \tau_0 - i\epsilon)} \right] \right\} \quad (6)$$

for application of the Cauchy Residue theorem.

Now, using the definition

$$H_l^{\pm}(z) \equiv \frac{e^{\pm iz}}{z^l} [\hat{\alpha}_l(z) \mp i\hat{\beta}_l(z)] \quad , \quad (7)$$

where

$$\hat{\alpha}_l(z) \pm i\hat{\beta}_l(z) \equiv \sum_{s=0}^l \Gamma_{ls}(z) e^{\pm i(l-s)\pi/2} \quad , \quad (8)$$

and

$$\hat{\alpha}_l(-z) \pm i\hat{\beta}_l(-z) = \hat{\alpha}_l(z) \mp i\hat{\beta}_l(z) \quad , \quad (9)$$

we may write Eq. (4) as

$$G_{\ell}(\tau_0 x, \tau_0 x') = - \frac{a}{8(xx')^{\ell}} P \int_{\Gamma} \frac{d\tau}{(\tau^2 - \tau_0^2)} \frac{1}{\tau^{2\ell}}$$

$$\times \left\{ \begin{aligned} & e^{i\tau(x+x'-2)} \frac{[\hat{\alpha}_{\ell}(\tau x) - i\hat{\beta}_{\ell}(\tau x)][\hat{\alpha}_{\ell}(\tau x') - i\hat{\beta}_{\ell}(\tau x')][\hat{\alpha}_{\ell}(\tau) + i\hat{\beta}_{\ell}(\tau)]}{\hat{\alpha}_{\ell}(\tau) - i\hat{\beta}_{\ell}(\tau)} \\ & + e^{-i\tau(x+x'-2)} \frac{[\hat{\alpha}_{\ell}(\tau x) + i\hat{\beta}_{\ell}(\tau x)][\hat{\alpha}_{\ell}(\tau x') + i\hat{\beta}_{\ell}(\tau x')][\hat{\alpha}_{\ell}(\tau) - i\hat{\beta}_{\ell}(\tau)]}{\hat{\alpha}_{\ell}(\tau) + i\hat{\beta}_{\ell}(\tau)} \\ & - e^{i\tau(x-x')} [\hat{\alpha}_{\ell}(\tau x) - i\hat{\beta}_{\ell}(\tau x)][\hat{\alpha}_{\ell}(\tau x') + i\hat{\beta}_{\ell}(\tau x')] \\ & - e^{-i\tau(x-x')} [\hat{\alpha}_{\ell}(\tau x) + i\hat{\beta}_{\ell}(\tau x)][\hat{\alpha}_{\ell}(\tau x') - i\hat{\beta}_{\ell}(\tau x')] \end{aligned} \right\} \quad (10)$$

The roots of $\hat{\alpha}_{\ell}(z) \pm i\hat{\beta}_{\ell}(z)$ were determined below Eqs. (6) and (13) in Appendix 3 to be ℓ in number and to lie in the (+: upper, -: lower)-half complex τ plane, located symmetrically about the imaginary τ axis. If ℓ is odd, there is one purely imaginary root.

Since $x + x' - 2 \geq 0$ the first integral of Eq. (10) may be closed in the upper half plane, and only the $P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$ term contributes a residue. Similarly, the second integral may be closed in the lower half plane, with residue contributions coming from both the $P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$ term and the $1/\tau^{2\ell}$ pole at the origin.

Thus for the integral we obtain

$$P \int_{\Gamma} \frac{d\tau}{(\tau^2 - \tau_0^2)}$$

$$\begin{aligned}
& \cdot \frac{e^{i\tau(x+x'-2)} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)] [\hat{\alpha}_l(\tau x') - i\hat{\beta}_l(\tau x')] [\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)]}{\tau^{2l} [\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)]} \\
& = \frac{\pi}{\tau_0} (xx')^l \operatorname{Im}\{H_l^{(-)}(\tau_0 x) H_l^{(-)}(\tau_0 x') H_l^{(+)}(\tau_0)/H_l^{(-)}(\tau_0)\}, \quad (11)
\end{aligned}$$

and for the second integral,

$$P \int_{\Gamma} \frac{d\tau}{(\tau^2 - \tau_0^2)}$$

$$\begin{aligned}
& \cdot \frac{e^{-i\tau(x+x'-2)} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] [\hat{\alpha}_l(\tau x') + i\hat{\beta}_l(\tau x')] [\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)]}{\tau^{2l} [\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)]} \\
& = \frac{\pi}{\tau_0} (xx')^l \operatorname{Im}\{H_l^{(-)}(\tau_0 x) H_l^{(-)}(\tau_0 x') H_l^{(+)}(\tau_0)/H_l^{(-)}(\tau_0)\} \\
& - 2\pi i (xx') \operatorname{Res}_{\tau=0} \left\{ \frac{H_l^{(-)}(\tau x) H_l^{(-)}(\tau x') H_l^{(+)}(\tau)/H_l^{(-)}(\tau)}{\tau^2 - \tau_0^2} \right\} \quad (12)
\end{aligned}$$

The third and fourth integrals of Eq. (10) must be closed in the upper (lower) half τ plane depending on whether $x > x'$ (or $x < x'$). For the third integral, we obtain

$$\begin{aligned}
& -P \int_{-\infty}^{\infty} \frac{d\tau e^{i\tau(x-x')}}{(\tau^2 - \tau_0^2)\tau^{2l}} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau x') + i\hat{\beta}_l(\tau x')] \\
& = -\theta(x - x') \int_{\infty}^0 \frac{d\tau e^{i\tau(x-x')}}{(\tau^2 - \tau_0^2)\tau^{2l}} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau x') + i\hat{\beta}_l(\tau x')] \\
& \quad - \theta(x' - x) \int_0^{\infty} \frac{d\tau e^{-i\tau(x'-x)}}{(\tau^2 - \tau_0^2)\tau^{2l}} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau x') + i\hat{\beta}_l(\tau x')] \\
& = -\frac{\pi}{\tau_0} (xx')^l \operatorname{Im}\{H_l^{(-)}(\tau_0 x) H_l^{(+)}(\tau_0 x') [\theta(x - x') - \theta(x' - x)]\} \\
& \quad + 2\pi i (xx')^l \theta(x' - x) \operatorname{Res}_{\tau=0} \left\{ \frac{H_l^{(+)}(\tau x) H_l^{(-)}(\tau x')}{\tau^2 - \tau_0^2} \right\}, \tag{13}
\end{aligned}$$

and similarly for the fourth integral of Eq. (10) we have

$$\begin{aligned}
& -P \int_{-\infty}^{\infty} \frac{d\tau e^{-i\tau(x-x')}}{(\tau^2 - \tau_0^2)\tau^{2l}} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau x') - i\hat{\beta}_l(\tau x')] \\
& = -\frac{\pi}{\tau_0} (xx')^l \operatorname{Im}\{H_l^{(-)}(\tau_0 x) H_l^{(+)}(\tau_0 x') [\theta(x - x') - \theta(x' - x)]\} \\
& \quad + 2\pi i (xx')^l \theta(x - x') \operatorname{Res}_{\tau=0} \left\{ \frac{H_l^{(-)}(\tau x) H_l^{(+)}(\tau x')}{\tau^2 - \tau_0^2} \right\}. \tag{14}
\end{aligned}$$

Substituting Eqs. (11) through (14) into Eq. (10), we obtain the result

$$G_{\ell}(\tau_0 x, \tau_0 x')$$

$$= \frac{\pi a}{2\tau_0} [G'_{\ell}(\tau_0 x) F'_{\ell}(\tau_0 x') \Theta(x - x') + G'_{\ell}(\tau_0 x') F'_{\ell}(\tau_0 x) \Theta(x' - x)], \quad (15)$$

or

$$G_{\ell}(k_0 r, k_0 r')$$

$$= \frac{\pi}{2k_0} [G'_{\ell}(k_0 r) F'_{\ell}(k_0 r') \Theta(r - r') + G'_{\ell}(k_0 r') F'_{\ell}(k_0 r) \Theta(r' - r)], \quad (15')$$

where we have used the definition

$$\frac{H_{\ell}^{(-)}(\tau x) H_{\ell}^{(+)}(\tau)}{\Delta_{\ell}(\tau)} \equiv G'_{\ell}(\tau x) - i F'_{\ell}(\tau x) \quad (16)$$

and some simple algebraic manipulation to obtain Eq. (15). In particular, Eq. (16) is useful in demonstrating below that the residue terms of Eqs. (12) through (14), when combined, vanish because there is no pole at $\tau = 0$.

The residue terms at $\tau = 0$ take the form

$$\begin{aligned}
 2\pi i \operatorname{Res}_{\tau=0} & \left\{ - \frac{H_{\ell}^{(-)}(\tau x) H_{\ell}^{(-)}(\tau x') H_{\ell}^{(+)}(\tau) H_{\ell}^{(+)}(\tau)}{\Delta_{\ell}^2(\tau^2 - \tau_0^2)} \right. \\
 & + \Theta(x' - x) \frac{H_{\ell}^{(+)}(\tau x) H_{\ell}^{(-)}(\tau) H_{\ell}^{(-)}(\tau x') H_{\ell}^{(+)}(\tau)}{\Delta_{\ell}^2(\tau)(\tau^2 - \tau_0^2)} \\
 & \left. + \Theta(x - x') \frac{H_{\ell}^{(-)}(\tau x) H_{\ell}^{(+)}(\tau) H_{\ell}^{(+)}(\tau x') H_{\ell}^{(-)}(\tau)}{\Delta_{\ell}^2(\tau)(\tau^2 - \tau_0^2)} \right\} (xx')^{\ell} \\
 & = -4\pi (xx')^{\ell} \operatorname{Res}_{\tau=0} \left\{ \frac{F'_{\ell}(\tau x) H_{\ell}^{(-)}(\tau x') \Theta(x' - x) + F'_{\ell}(\tau x') H_{\ell}^{(-)}(\tau x) \Theta(x - x')}{\tau^2 - \tau_0^2} \right\}.
 \end{aligned} \tag{17}$$

But since as $\tau \rightarrow 0$,

$$H_{\ell}^{(-)}(\tau x) = G_{\ell}(\tau x) - i F_{\ell}(\tau x) \xrightarrow{\tau \rightarrow 0} \frac{(2\ell-1)!!}{(\tau x)^{\ell}} - \frac{i(\tau x)^{\ell+1}}{(2\ell+1)!!} \tag{18}$$

and

$$F'_{\ell}(\tau x) \xrightarrow{\tau \rightarrow 0} \frac{\tau^{\ell+1}}{(2\ell+1)!!} \frac{x^{2\ell+1}-1}{x^{\ell}} \tag{19}$$

there is no pole contribution from Eq. (17) as $\tau \rightarrow 0$ and the residue term vanishes identically.

In the limit $k_0 \rightarrow 0$, Eq. (15) becomes

$$G_\ell(\tau_0 x, \tau_0 x') \xrightarrow{\tau_0 \rightarrow 0} \frac{\pi a}{2(2\ell+1)} \left\{ \frac{((x')^{2\ell+1}-1)\theta(x-x') + ((x)^{2\ell+1}-1)\theta(x'-x)}{(xx')^\ell} \right\}, \quad (20)$$

where we have used Eq. (19) and

$$G'_\ell(\tau x) \xrightarrow{\tau \rightarrow 0} \frac{(2\ell-1)!!}{(\tau x)^\ell}. \quad (21)$$

ON THE PRINCIPAL-VALUE INTEGRAL $B_\ell^\alpha(k_0)$

The integral $B_\ell^\alpha(k_0)$ is defined as

$$B_\ell^\alpha(k_0) \equiv \int_0^\infty dk \frac{kg_\ell^\alpha(k)}{\Delta_\ell(ka)} P\left(\frac{1}{k^2 - k_0^2}\right), \quad (1)$$

where the momentum-transform potential, $g_\ell^\alpha(\tau)$, has the explicit form (see Eq. (16) in Appendix 1)

$$g_\ell^\alpha(\tau) = \frac{\lambda_\alpha^{\mu_\alpha} e^{\mu_\alpha}}{\tau^\ell \Delta_\ell(\tau)} \operatorname{Im}[H_\ell^{(-)}(\tau) Z_\ell(\mu_\alpha, \tau)], \quad (2)$$

where $\tau = ka$ and the various functions involved in Eq. (2) are defined in Appendix 1.

Since $g_\ell^\alpha(k)$ is antisymmetric and $\Delta_\ell(ka)$ is symmetric under the transformation $k \rightarrow -k$ [see Eqs. (5) and (6) in Appendix 4], we may extend the range of integration to be $-\infty \leq k \leq \infty$, and, as in Appendix 4 on $K_\ell^{\alpha\beta}$, we write the resultant integral as a contour integral

$$aB_\ell^\alpha(k_0) = \frac{a}{2} \int_{-\infty}^{\infty} d\tau \frac{\tau g_\ell^\alpha(\tau)}{\Delta_\ell(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) = \frac{a}{2} \int_{\text{C}} d\tau \frac{\tau g_\ell^\alpha(\tau)}{\Delta_\ell(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right), \quad (3)$$

where the deformation around the origin is permissible since there is no pole in the kernel at $\tau = 0$ (even for $\tau_0 = 0$). Using Eq. (2) and the definition

$$\Delta_l^2(\tau) = H_l^{(+)}(\tau)H_l^{(-)}(\tau) \quad (4)$$

in Eq. (3), we have

$$aB_l^\alpha(k_0) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{4i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau^{l-1}} \left[\frac{Z_l(\mu_\alpha, \tau)}{H_l^{(+)}(\tau)} - \frac{\bar{Z}_l(\mu_\alpha, \tau)}{H_l^{(-)}(\tau)} \right] P\left(\frac{1}{\tau^2 - \tau_0^2}\right), \quad (5)$$

where

$$Z_l(\mu_\alpha, \tau) \equiv \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} E_s(\mu_\alpha - i\tau), \quad (6)$$

$$\begin{aligned} H_l^{(+)}(\tau) &= G_l(\tau) + iF_l(\tau) \\ &= \frac{e^{\pm i\tau}}{\tau^l} [\hat{\alpha}_l(\tau) \mp i\hat{\beta}_l(\tau)], \end{aligned} \quad (7)$$

and

$$\hat{\alpha}_l(\tau) \pm i\hat{\beta}_l(\tau) = \sum_{s=0}^l \Gamma_{ls}(\tau) e^{\pm i(l-s)\pi/2}, \quad (8)$$

$$\hat{\alpha}_l(-\tau) \pm i\hat{\beta}_l(-\tau) = \hat{\alpha}_l(\tau) \mp i\hat{\beta}_l(\tau). \quad (9)$$

Substituting Eqs. (6) through (8) into Eq. (5) yields

$$aB_l^\alpha(\tau_0) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{4i}$$

$$\left\{ \begin{aligned} & \sum_{s=0}^l \frac{(l+\frac{1}{2}, s)}{2^s} e^{-i(l-s)\pi/2} \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_{\curvearrowright} \frac{d\tau \tau^{l+1-s} e^{i\tau(x-1)}}{\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\ & - \sum_{s=0}^l \frac{(l+\frac{1}{2}, s)}{2^s} e^{i(l-s)\pi/2} \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^s} \int_{\curvearrowright} \frac{d\tau \tau^{l+1-s} e^{-i\tau(x-1)}}{\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \end{aligned} \right\}, \quad (10)$$

where we have used the definitions

$$\begin{aligned} \Gamma_{ls}(\tau) &\equiv \frac{(l+\frac{1}{2}, s)}{2^s} \tau^{l-s}, \\ E_s(z) &\equiv \int_1^\infty \frac{dx e^{-zx}}{x^s}, \quad \text{Re}(z) > 0. \end{aligned} \quad (11)$$

Since $x - 1 > 0$ in Eq. (10), the first integral may be closed in the upper half plane, and the second integral may be closed in the lower half plane. In both instances, we use

$$P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left[\frac{1}{(\tau + \tau_0 - i\epsilon)(\tau - \tau_0 + i\epsilon)} + \frac{1}{(\tau + \tau_0 + i\epsilon)(\tau - \tau_0 - i\epsilon)} \right] \quad (12)$$

when applying the Cauchy Residue Theorem. The only further point requiring comment is the location of the zeros of the function

$$\hat{\alpha}_l(\tau) \pm i\hat{\beta}_l(\tau).$$

In Appendix 3, below, Eqs. (6) and (13), we determine that

$\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)$ has ℓ zeros located in the upper half plane, and that

$\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)$ has ℓ zeros located in the lower half plane.

Therefore, these functions do not contribute residues to the contour integrals in Eq. (10). Hence, the only residue contributions

arise from the terms of $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ as given by Eq. (12).

Substituting Eq. (12) into the integrals of Eq. (10) and evaluating them using the residue theorem, we obtain

$$\begin{aligned}
 \oint \frac{d\tau \tau^{\ell+1-s} e^{i\tau(x-1)}}{\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\
 = \frac{\pi i}{2} \tau_0^{\ell-s} \left[\frac{e^{i\tau_0(x-1)}}{\hat{\alpha}_\ell(\tau_0) - i\hat{\beta}_\ell(\tau_0)} + (-)^{\ell-s} \frac{e^{-i\tau_0(x-1)}}{\hat{\alpha}_\ell(\tau_0) + i\hat{\beta}_\ell(\tau_0)} \right] \\
 = \frac{\pi i}{2} \tau_0^{-s} \left[\frac{e^{i\tau_0 x}}{H_\ell^{(+)}(\tau_0)} + (-)^{\ell-s} \frac{e^{-i\tau_0 x}}{H_\ell^{(-)}(\tau_0)} \right], \quad (13)
 \end{aligned}$$

and similarly

$$\begin{aligned}
& \int \frac{d\tau \tau^{l+1-s} e^{-i\tau(x-1)}}{\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\
& = -\frac{\pi i}{2} \tau_0^{-s} \left[\frac{e^{-i\tau_0 x}}{H_l^{(-)}(\tau_0)} + (-)^{l-s} \frac{e^{i\tau_0 x}}{H_l^{(+)}(\tau_0)} \right] . \quad (14)
\end{aligned}$$

Substituting Eqs. (13) and (14) into Eq. (10) yields the result

$$\begin{aligned}
aB_l^\alpha(\tau_0) &= \frac{\pi}{8} \mu_\alpha^2 e^{\mu_\alpha} \sum_{s=0}^l \frac{(l+1/2, s)}{(2\tau_0)^s} \\
&\cdot \left\{ e^{-i(l-s)\pi/2} \left[\frac{1}{H_l^{(+)}(\tau_0)} \int_1^\infty \frac{dx e^{-(\mu_\alpha - i\tau_0)x}}{x^s} + \frac{(-)^{l-s}}{H_l^{(-)}(\tau_0)} \int_1^\infty \frac{dx e^{-(\mu_\alpha + i\tau_0)x}}{x^s} \right] \right. \\
&\quad \left. + e^{i(l-s)\pi/2} \left[\frac{1}{H_l^{(-)}(\tau_0)} \int_1^\infty \frac{dx e^{-(\mu_\alpha + i\tau_0)x}}{x^s} + \frac{(-)^{l-s}}{H_l^{(+)}(\tau_0)} \int_1^\infty \frac{dx e^{-(\mu_\alpha - i\tau_0)x}}{x^s} \right] \right\} .
\end{aligned}$$

Using Eqs. (4), (6), and (11) we have the final result that

$$aB_l^\alpha(\tau_0) = \frac{\pi}{2} \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau_0^l \Delta_l^2(\tau_0)} \operatorname{Re}[H_l^{(-)}(\tau_0) Z_l(\mu_\alpha, \tau_0)] \quad (15)$$

Low-Momentum Behavior

The low momentum behavior of $B_l^\alpha(\tau_0)$ is readily determined by referring to Appendices 1 [Eqs. (25)], and 2 [Eqs. (42)], where we obtained the results

$$H_l^{(-)}(\tau) = G_l(\tau) - iF_l(\tau) \xrightarrow{\tau \rightarrow 0} \frac{(2l-1)!!}{\tau^l} - \frac{i\tau^{l+1}}{(2l+1)!!}, \quad (16)$$

$$\tau^l \Delta_l(\tau) \xrightarrow{\tau \rightarrow 0} (2l-1)!! , \quad (17)$$

$$Z_l(\mu_\alpha, \tau) \equiv X_l(\mu_\alpha, \tau) + iY_l(\mu_\alpha, \tau) , \quad (18)$$

$$\begin{aligned} X_l(\mu_\alpha, \tau) \xrightarrow{\tau \rightarrow 0} & \sum_{p=0}^{\leq (l/2)} \frac{(2l-2p-1)!!}{2^p p!} E_{l-2p}(\mu_\alpha) \tau^{2p} \\ & + \sum_{p=0}^{\leq \frac{l-1}{2}} \frac{(2p-1)!! \alpha_{l-2p}(\mu_\alpha) \tau^{2l-2p}}{2^{l-p} (l-p)!} \\ & + \sum_{p=1}^{\infty} \frac{(-)^p \alpha_{l+2p}(\mu_\alpha) \tau^{2l+2p}}{(2p-1)!! 2^{p+l} (p+l)!} , \end{aligned} \quad (19)$$

$$Y_l(\mu_\alpha, \tau) \xrightarrow{\tau \rightarrow 0} \sum_{p=0}^{\infty} \frac{(-)^p \alpha_{l+2p+1}(\mu_\alpha) \tau^{2l+2p+1}}{2^p p! (2p+2l+1)!!} \quad (20)$$

Upon substituting Eqs. (16) through (20) into Eq. (15), it is clear that

$$\begin{aligned} aB_l^\alpha(\tau_0) \xrightarrow{\tau_0 \rightarrow 0} \frac{\pi}{2} \frac{\mu_\alpha^2 e^{\mu_\alpha^2}}{(2l-1)!!} \sum_{p=0}^{< l/2} \frac{(2l-2p-1)!!}{2^p p!} E_{l-2p}(\mu_\alpha) \tau_0^{2p} \\ \xrightarrow{\tau_0 \rightarrow 0} \frac{\pi}{2} \mu_\alpha^2 e^{\mu_\alpha^2} E_l(\mu_\alpha), \end{aligned} \quad (21)$$

which is a well-defined result, as it must be according to Eq. (1).

The result of Eq. (21) is identical with that which we would obtain by setting $\tau_0 = 0$ in Eq. (10), in which case the only possible pole term arises from setting $s = l$ and we have

$$\begin{aligned} aB_l^\alpha(0) = \frac{\mu_\alpha^2 e^{\mu_\alpha^2}}{4i} \frac{(l+\frac{1}{2}, l)}{2^l} \int_1^\infty \frac{dx e^{-\mu_\alpha x}}{x^l} \left\{ \int_{\rightarrow} \frac{d\tau}{\tau} \frac{e^{i\tau(x-1)}}{\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)} \right. \\ \left. + \int_{\leftarrow} \frac{d\tau}{\tau} \frac{e^{+i\tau(x-1)}}{\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)} \right\} \end{aligned}$$

where we have transformed $\tau \rightarrow -\tau$ in the second integral. We then obtain the immediate result

$$aB_f^\alpha(0) = \frac{\pi}{2} \mu_\alpha^2 e^{\mu_\alpha} E_f(\mu_\alpha) . \quad (22)$$

Since the first contour vanishes identically (no poles in the upper half plane), and the second contour (a simple pole at $\tau = 0$ in the upper half plane) yields $2\pi i(\hat{\alpha}_f(0) - i\hat{\beta}_f(0))^{-1}$, where $\hat{\alpha}_\ell(0) - i\hat{\beta}_\ell(0) = (\ell + \frac{1}{2}, \ell)/2^\ell$.

ON THE BOUND STATE INTEGRALS $B_l^{\alpha\beta}(\gamma)$ AND $C_l^{\alpha\beta}(\gamma)$

The deuteron bound state wave functions involve the coefficients $B_l^{\alpha\beta}(\gamma)$ and $C_l^{\alpha\beta}(\gamma)$ for $l = 0$ and 2 , where $(\alpha, \beta) = C$ or T (for Central or Tensor interaction). These coefficients are defined

$$B_l^{\alpha\beta}(\gamma) \equiv \frac{2\sigma_{\alpha}^r}{\pi} K_l^{\alpha\beta}(i\gamma) - \frac{1}{\sqrt{2}} \delta_{l,0} \delta_{\alpha\beta} \delta_{\alpha,T} \quad (1)$$

and

$$C_l^{\alpha\beta}(\gamma) \equiv - \frac{\pi}{4\gamma\sigma_{\alpha}^r} \frac{\partial}{\partial \gamma} (B_l^{\alpha\beta}(\gamma)), \quad (2)$$

in the terms of the integrals

$$K_l^{\alpha\beta}(i\gamma) \equiv a \int_0^{\infty} \frac{d\tau g_l^{\alpha}(\tau) g_l^{\beta}(\tau)}{\tau^2 + \gamma^2} \quad (3)$$

and

$$C_l^{\alpha\beta}(\gamma) \equiv a^3 \int_0^{\infty} \frac{d\tau g_l^{\alpha}(\tau) g_l^{\beta}(\tau)}{(\tau^2 + \gamma^2)^2}, \quad (4)$$

where $g_l^{\alpha}(\tau)$ is the momentum-transformed potential (Appendix 1),

$\tau^2 = (ka)^2$, and in the notation of the bound state problem

$B_l^{\alpha\alpha}(\gamma) \equiv B_l^{\alpha}(\gamma)$ and $C_l^{\alpha\alpha}(\gamma) \equiv C_l^{\alpha}(\gamma)$ for $\alpha = C$ or T .

Thus we may address ourselves to the specific determination of Eq. (3) and then determine $B_\ell^{\alpha\beta}(\gamma)$ and $C_\ell^{\alpha\beta}(\gamma)$ by using Eqs. (1) and (2). The solution to Eq. (3) can be obtained in a straightforward manner from the results for $K_\ell^{\alpha\beta}(\tau_0)$ (Appendix 4) upon making the transformation $\tau_0^2 = -\gamma^2$.

On the other hand, one may, if he wishes, repeat the contour integrations involved in $K_\ell^{\alpha\beta}(\tau_0)$ with the principle value part, $P \left(\frac{1}{\tau^2 - \tau_0^2} \right)$ replaced by $\frac{1}{\tau^2 + \gamma^2}$. The closing of the contours and the vanishing of the residue terms (which arise from the contour distortions about the origin), being independent of the parametric value $\tau_0 = i\gamma$, remains unchanged. The only crucial point will be in the correct application, whenever necessary, of the analytic continuation of

$$D_1(\alpha, \bar{\beta}) \equiv \int_1^\infty \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x} \quad (5)$$

$$= -D_1(\bar{\beta}, \alpha) + E_1(\alpha) E_1(\bar{\beta}) , \quad (6)$$

where $\alpha \equiv \mu_\alpha + i\tau$, $\bar{\beta} \equiv \mu_\beta - i\tau$ in the limit $\tau \rightarrow i\gamma$. If we proceed in this manner we arrive at the identical results as would be obtained directly upon setting $\tau_0 = i\gamma$ in the final expression for $K_\ell^{\alpha\beta}(\tau_0)$.

Making the substitution $\tau_0 = i\gamma$ in $K_l^{\alpha\beta}(\tau_0)$, Eq. (65) of Appendix 4, we obtain (using Eq. (1))

$$B_l^{\alpha\beta}(\gamma) = \frac{\sigma_\alpha}{\mu_\alpha}$$

$$\left\{ \begin{aligned} & \frac{\mu_\alpha^2 \mu_\beta^2 \mu_\alpha + \mu_\beta}{2\gamma^{2l+1}} \left\{ \begin{aligned} & \frac{(-)^{l+1} H_l^{(-)}(i\gamma) U_l^\alpha(\gamma) U_l^\beta(\gamma)}{H_l^{(+)}(i\gamma)} \\ & + \sum_{s,t=0}^l (-)^s \Gamma_{ls}(\gamma) \Gamma_{lt}(\gamma) [J_{s,t}(\alpha, \bar{\beta}) + J_{s,t}(\beta, \bar{\alpha})]_{\tau_0=i\gamma} \end{aligned} \right\} \\ & - \frac{1}{2} \mu_\alpha^2 \mu_\beta^2 \mu_\alpha + \mu_\beta \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} \sum_{p=0}^{\left(\frac{s+t-1}{2}\right)} \frac{I_{l,s,t,p}^{\alpha\beta}}{(s+t-1-2p)! \gamma^{2p+2}} \\ & - \frac{1}{\sqrt{2}} \delta_{l,0} \delta_{\alpha,\beta} \delta_{\alpha,c} \end{aligned} \right\}, \quad (7)$$

where

$$U_l^\alpha(\gamma) \equiv Z_l(\mu_\alpha, i\gamma) = \sum_{s=0}^l \Gamma_{ls}(\gamma) E_s(\mu_\alpha + \gamma) \quad (8)$$

$$H_l^{(\pm)}(i\gamma) \equiv \frac{e^{\mp\gamma}}{i^l \gamma^l} [\hat{\alpha}_l(i\gamma) \mp i \hat{\beta}_l(i\gamma)] \quad (9)$$

$$\hat{\alpha}_l(i\gamma) \pm i \hat{\beta}_l(i\gamma) \equiv \sum_{s=0}^l \Gamma_{ls}(\mp\gamma) \quad (10)$$

$$[J_{s,t}(\alpha, \bar{\beta}) + J_{s,t}(\beta, \bar{\alpha})]_{r_0=i\gamma}$$

$$\equiv J_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma) \quad (11)$$

and $I_{s,t,p}$ is determined in Appendix 7. Upon taking the indicated limit in Eq. (11) of the corresponding expression of Appendix 6 and using Eq. (6), we obtain

$$J_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma) =$$

$$\left(\frac{1}{\mu_\alpha + \gamma} + \frac{1}{\mu_\beta + \gamma} \right) \{ E_0(\mu_\alpha + \mu_\beta) \delta_{s,0} \delta_{t,0} + E_1(\mu_\alpha + \mu_\beta) \delta_{s,1} \delta_{t,0} \} \\ + \left\{ \frac{e^{-(\mu_\alpha - \gamma)} E_1(\mu_\beta + \gamma) - E_1(\mu_\alpha + \mu_\beta)}{\mu_\alpha - \gamma} + \frac{e^{-(\mu_\beta - \gamma)} E_1(\mu_\alpha + \gamma) - E_1(\mu_\alpha + \mu_\beta)}{\mu_\beta - \gamma} \right\} \delta_{s,0} \delta_{t,1}$$

$$\begin{aligned}
& + \theta(s)\theta(t) \\
& \left\{ \frac{\theta(t-1)}{(t-1)!} \sum_{k=1}^{t-1} (t-k-1)! (-)^{k-1} [(\mu_\alpha + \gamma)^{k-1} + (\mu_\beta + \gamma)^{k-1}] E_{s+t-k}(\mu_\alpha + \mu_\beta) \right. \\
& + \frac{\theta(s-1)}{(s-1)!(t-1)!} \sum_{k=1}^{s-1} (s-k-1)! (-)^{k+t} \\
& \quad \left\{ \begin{aligned} & (\mu_\beta + \gamma)^{t-1} (\mu_\alpha - \gamma)^{k-1} [e^{-(\mu_\alpha - \gamma)} E_1(\mu_\beta + \gamma) - E_{s-k+1}(\mu_\alpha + \mu_\beta)] \\ & + (\mu_\alpha + \gamma)^{t-1} (\mu_\beta - \gamma)^{k-1} [e^{-(\mu_\beta - \gamma)} E_1(\mu_\alpha + \gamma) - E_{s-k+1}(\mu_\alpha + \mu_\beta)] \end{aligned} \right\} \\
& + \frac{(-)^{s+t}}{(s-1)!(t-1)!} [(\mu_\beta + \gamma)^{t-1} (\mu_\alpha - \gamma)^{s-1} D_1(\mu_\alpha - \gamma, \mu_\beta + \gamma) \\
& \quad + (\mu_\alpha + \gamma)^{t-1} (\mu_\beta - \gamma)^{s-1} D_1(\mu_\beta - \gamma, \mu_\alpha + \gamma)] \left. \right\}
\end{aligned}
\tag{12}$$

Using Eq. (6), and Eqs. (20) and (23) of Appendix 12; namely,

$$D_1(\alpha, \bar{\beta})$$

$$= \left\{ \begin{array}{l} \left\{ -\frac{1}{2} \left(\frac{\pi^2}{6} + (C + \ln \alpha)^2 \right) - (C + \ln \bar{\beta}) E_1(\alpha) \right. \\ \left. - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) \right\} \quad \text{if } |\bar{\beta}/\alpha| \leq 1 \\ \\ \left\{ \frac{1}{2} \left(\frac{\pi^2}{6} + (C + \ln \bar{\beta})^2 \right) + (C + \ln \alpha + E_1(\alpha) E_1(\bar{\beta})) \right. \\ \left. + \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2} + \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}) \right\} \quad \text{if } |\alpha/\bar{\beta}| \leq 1 \end{array} \right\}, \quad (13)$$

where $C = \text{Euler's constant} = 0.577215\dots$, we take the limit $\tau \rightarrow i\gamma$ and readily obtain the results

$$D_1(\mu_\alpha - \gamma, \mu_\beta + \gamma) = \begin{cases} D_1^{\alpha\beta} & \text{if } \frac{\mu_\beta}{\gamma} \geq \frac{\mu_\alpha}{\gamma} + 2 \\ D_2^{\alpha\beta} & \text{if } \frac{\mu_\alpha}{\gamma} \geq \frac{\mu_\beta}{\gamma} - 2 \end{cases}, \quad (14)$$

$$D_1(\mu_\beta - \gamma, \mu_\alpha + \gamma) = \begin{cases} D_1^{\beta\alpha} & \text{if } \frac{\mu_\alpha}{\gamma} \geq \frac{\mu_\beta}{\gamma} + 2 \\ D_2^{\beta\alpha} & \text{if } \frac{\mu_\beta}{\gamma} \geq \frac{\mu_\alpha}{\gamma} - 2 \end{cases}, \quad (15)$$

where

$$D_1^{\alpha\beta} \equiv \frac{1}{2} \left[\frac{\pi^2}{6} + (C + \ln(\mu_\beta + \gamma))^2 \right] - E_1(\mu_\beta + \gamma) \sum_{p=1}^{\infty} \frac{(\gamma - \mu_\alpha)^p}{p p!} \\ + \sum_{p=1}^{\infty} \frac{(-)^p (\gamma + \mu_\beta)^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{1}{p^2} \left(\frac{1 - \frac{\mu_\alpha}{\gamma}}{1 + \frac{\mu_\beta}{\gamma}} \right)^p e^{-(\mu_\beta + \gamma)} e_{p-1}(\mu_\beta + \gamma) , \quad (16)$$

$$D_2^{\alpha\beta} \equiv -\frac{1}{2} \left[\frac{\pi^2}{6} + (C + \ln(\mu_\alpha - \gamma))^2 \right] - E_1(\mu_\alpha - \gamma) (C + \ln(\mu_\beta + \gamma)) \\ - \sum_{p=1}^{\infty} \frac{(\gamma - \mu_\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{1}{p^2} \left(\frac{1 + \frac{\mu_\beta}{\gamma}}{1 - \frac{\mu_\alpha}{\gamma}} \right)^p e^{-(\mu_\alpha - \gamma)} e_{p-1}(\mu_\alpha - \gamma) . \quad (17)$$

The domain of these functions is illustrated in Figure A-10.1.

Figure A-10.1

The domains of the functions $D_1^{\alpha\beta}$, $D_2^{\alpha\beta}$ and $D_1^{\beta\alpha}$, $D_2^{\beta\alpha}$.

Thus, the determination of $B_l^{\alpha\beta}(\gamma)$ is given by Eqs. (7), (12), (14) through (17), with $I_{lstp}^{\alpha\beta}$ given in Appendix 7.

The Function $C_l^{\alpha\beta}(\gamma)$

The determination of $C_l^{\alpha\beta}(\gamma)$ involves the differentiation of Eq. (7) with respect to γ , a straightforward process, with the result that

$$C_l^{\alpha\beta}(\gamma) = \frac{\pi}{\sigma_\alpha r_\alpha^4 \gamma^2} (2l+1) B_l^{\alpha\beta}(\gamma) + \alpha_l^{\alpha\beta}(\gamma) + \beta_l^{\alpha\beta}(\gamma) + \xi_l^{\alpha\beta}(\gamma), \quad (18)$$

where

$$\alpha_l^{\alpha\beta}(\gamma) = - \frac{\pi a}{8\gamma^{2l+2}} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \gamma} e^{\mu_\beta + \gamma}$$

$$\left\{ \begin{aligned} & \frac{U_l^\alpha(\gamma) \dot{U}_l^\beta(\gamma)}{\sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} (2l+s-t) (-)^{l-t} \gamma^{2l-s-t-1}} \\ & \frac{\sum_{t=0}^l \Gamma_{lt}(-\gamma)}{\sum_{s=0}^l \Gamma_{ls}(\gamma)} \frac{\sum_{r,p=0}^l \Gamma_{lr}(\gamma) \Gamma_{lp}(\gamma)}{\left\{ \begin{aligned} & \frac{2l-r-p}{\gamma} E_r(\mu_\alpha + \gamma) E_p(\mu_\beta + \gamma) \\ & - E_{r-1}(\mu_\alpha + \gamma) E_p(\mu_\beta + \gamma) - E_r(\mu_\alpha + \gamma) E_{p-1}(\mu_\beta + \gamma) \end{aligned} \right\}} \end{aligned} \right\} \quad (19)$$

and where $E_{-1}(\lambda) \equiv \alpha_1(\lambda)$. For the quantity $\gamma_l^{\alpha\beta}(\gamma)$ we find

$$\beta_l^{\alpha\beta}(\gamma) = - \frac{\pi a}{8\gamma^{2l+2}} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta}$$

$$\sum_{s,t=0}^l (-)^s \Gamma_{ls}(\gamma) \Gamma_{lt}(\gamma) \left\{ \begin{aligned} & \frac{2l-s-t}{\gamma} [J_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma)] \\ & + [J'_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J'_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma)] \end{aligned} \right\} \quad (20)$$

where the term $[J_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma)]$ is given by Eq.

(12) and

$$[J'_{s,t}(\mu_\alpha - \gamma, \mu_\beta + \gamma) + J'_{s,t}(\mu_\beta - \gamma, \mu_\alpha + \gamma)]$$

$$= - \left(\frac{1}{(\mu_\alpha + \gamma)^2} + \frac{1}{(\mu_\beta + \gamma)^2} \right) \{ E_0(\mu_\alpha + \mu_\beta) \delta_{s,0} \delta_{t,0} + E_1(\mu_\alpha + \mu_\beta) \delta_{s,1} \delta_{t,0} \}$$

$$+ \left\{ \begin{aligned} & \alpha_1(\mu_\alpha - \gamma) E_1(\mu_\beta + \gamma) - \alpha_0(\mu_\alpha - \gamma) \alpha_0(\mu_\beta + \gamma) - \frac{E_1(\mu_\alpha + \mu_\beta)}{(\mu_\alpha - \gamma)^2} \\ & + \alpha_1(\mu_\beta - \gamma) E_1(\mu_\alpha + \gamma) - \alpha_0(\mu_\beta - \gamma) \alpha_0(\mu_\alpha + \gamma) - \frac{E_1(\mu_\alpha + \mu_\beta)}{(\mu_\beta - \gamma)^2} \end{aligned} \right\} \delta_{s,0} \delta_{t,1}$$

$$\theta(s)\theta(t)$$

$$\left\{ \begin{aligned} & \frac{\theta(t-2)}{(t-1)!} \sum_{k=2}^{t-1} (t-k-1)!(k-1)(-)^{k-1} [(\mu_{\alpha}+\gamma)^{k-2} + (\mu_{\beta}+\gamma)^{k-2}] E_{s+t-k}(\mu_{\alpha}+\mu_{\beta}) \\ & + \frac{\theta(s-1)}{(s-1)!(t-1)!} \sum_{k=1}^{s-1} (s-k-1)!(-)^{k+t} \\ & \left\{ \begin{aligned} & (\mu_{\beta}+\gamma)^{t-1}(\mu_{\alpha}-\gamma)^{k-1} \\ & \left\{ \left(\frac{(t-1)}{\mu_{\beta}+\gamma} - \frac{(k-1)}{\mu_{\alpha}-\gamma} + 1 \right) (e^{-(\mu_{\alpha}-\gamma)} E_1(\mu_{\beta}+\gamma) - E_{s-k+1}(\mu_{\alpha}+\mu_{\beta})) \right\} \\ & + E_{s-k+1}(\mu_{\alpha}+\mu_{\beta}) - \frac{e^{-(\mu_{\alpha}+\mu_{\beta})}}{\mu_{\beta}+\gamma} \end{aligned} \right\} \\ & + (\mu_{\alpha}+\gamma)^{t-1}(\mu_{\beta}-\gamma)^{k-1} \\ & \left\{ \left(\frac{(t-1)}{\mu_{\alpha}+\gamma} - \frac{(k-1)}{\mu_{\beta}-\gamma} + 1 \right) (e^{-(\mu_{\beta}-\gamma)} E_1(\mu_{\alpha}+\gamma) - E_{s-k+1}(\mu_{\alpha}+\mu_{\beta})) \right\} \\ & + E_{s-k+1}(\mu_{\alpha}+\mu_{\beta}) - \frac{e^{-(\mu_{\alpha}+\mu_{\beta})}}{\mu_{\alpha}+\gamma} \end{aligned} \right\} \\ & + \frac{(-)^{s+t}}{(s-1)!(t-1)!} \\ & \left\{ \begin{aligned} & \left(\frac{(t-1)}{\mu_{\beta}+\gamma} - \frac{(s-1)}{\mu_{\alpha}-\gamma} \right) (\mu_{\beta}+\gamma)^{t-1}(\mu_{\alpha}-\gamma)^{s-1} D_1(\mu_{\alpha}-\gamma, \mu_{\beta}+\gamma) \\ & + \left(\frac{(t-1)}{\mu_{\alpha}+\gamma} - \frac{(s-1)}{\mu_{\beta}-\gamma} \right) (\mu_{\alpha}+\gamma)^{t-1}(\mu_{\beta}-\gamma)^{s-1} D_1(\mu_{\beta}-\gamma, \mu_{\alpha}+\gamma) \\ & + (\mu_{\beta}+\gamma)^{t-1}(\mu_{\alpha}-\gamma)^{s-1} \left(\frac{e^{-(\mu_{\alpha}-\gamma)}}{\mu_{\alpha}-\gamma} E_1(\mu_{\beta}+\gamma) - \left(\frac{1}{\mu_{\alpha}-\gamma} + \frac{1}{\mu_{\beta}+\gamma} \right) E_1(\mu_{\alpha}+\mu_{\beta}) \right) \\ & + (\mu_{\alpha}+\gamma)^{t-1}(\mu_{\beta}-\gamma)^{s-1} \left(\frac{e^{-(\mu_{\beta}-\gamma)}}{\mu_{\beta}-\gamma} E_1(\mu_{\alpha}+\gamma) - \left(\frac{1}{\mu_{\beta}-\gamma} + \frac{1}{\mu_{\alpha}+\gamma} \right) E_1(\mu_{\alpha}+\mu_{\beta}) \right) \end{aligned} \right\} \end{aligned} \right\} \quad (21)$$

In the determination of Eq. (21), we have used the result

$$\begin{aligned} \frac{\partial}{\partial \gamma} [D_1(\mu_\alpha - \gamma, \mu_\beta + \gamma)] \\ = \frac{e^{-(\mu_\alpha - \gamma)}}{\mu_\alpha - \gamma} E_1(\mu_\beta + \gamma) - \left(\frac{1}{\mu_\alpha - \gamma} + \frac{1}{\mu_\beta + \gamma} \right) E_1(\mu_\alpha + \mu_\beta), \end{aligned} \quad (22)$$

which follows directly from differentiating Eq. (5) after taking the limit $\tau \rightarrow i\gamma$.

Finally we have

$$\begin{aligned} C_l^{\alpha\beta}(\gamma) \\ = \frac{\pi a}{8\gamma^{2l+2}} \mu_\alpha^2 \mu_\beta^2 e^{\mu_\alpha + \mu_\beta} \sum_{s,t=0}^l \gamma_{ls} \gamma_{lt} \sum_{p=0}^{\left(\frac{s+t-1}{2}\right)} \frac{(2l-2p-1) I_{l,s,t,p}^{\alpha\beta} \gamma^{2(l-1-p)}}{(s+t-1-2p)!} \end{aligned} \quad (23)$$

where $I_{l,s,t,p}^{\alpha\beta}$ is given in Appendix 7.

Thus, $C_l^{\alpha\beta}(\gamma)$ is given by Eqs. (18), (19), (20), and (23).

ON THE PRINCIPAL-VALUE INTEGRAL $F_\ell^\alpha(r)$

The integral $F_\ell^\alpha(r)$ is defined by

$$F_\ell^\alpha(r) = \int_0^\infty dk \, g_\ell^\alpha(k) F'_\ell(kr) P\left(\frac{1}{k^2 - k_0^2}\right), \quad (1)$$

where $g_\ell^\alpha(k)$, the momentum transformed potential, is determined by Eq. (16) in Appendix 1 as, with $\tau \equiv ka$,

$$g_\ell^\alpha(\tau) = \frac{\lambda_\alpha \mu_\alpha e^{\mu_\alpha}}{\tau^\ell \Delta_\ell(\tau)} \operatorname{Im}[H_\ell^{(-)}(\tau) Z_\ell(\mu_\alpha, \tau)]. \quad (2)$$

In Eq. (2)

$$H_\ell^{(+)}(\tau) \equiv G_\ell(\tau) \pm iF_\ell(\tau), \quad (3)$$

and $F'_\ell(kr)$ is defined by

$$F'_\ell(kr) \equiv \frac{F_\ell(kr)G_\ell(ka) - F_\ell(ka)G_\ell(kr)}{\Delta_\ell(ka)}, \quad (4)$$

where

$$\Delta_\ell^2(\tau) \equiv H_\ell^{(+)}(\tau)H_\ell^{(-)}(\tau). \quad (5)$$

We may rewrite Eq. (4) as

$$F_l'(kr) = \frac{\text{Im}[H_l^{(+)}(kr)H_l^{(-)}(ka)]}{\Delta_l(ka)} \quad (6)$$

Substituting Eqs. (2) and (6) into Eq. (1), we obtain

$$F_l^\alpha(r) = \mu_\alpha^2 e^{\mu_\alpha} \int_0^\infty \frac{d\tau}{\tau^l} \frac{\text{Im}[H_l^{(-)}(\tau)Z_l(\mu_\alpha, \tau)]\text{Im}[H_l^{(+)}(\tau x)H_l^{(-)}(\tau)]}{H_l^{(+)}(\tau)H_l^{(-)}(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right),$$

where $\tau \equiv ka$, $\mu \equiv \lambda a$, and $x \equiv r/a \geq 1$.

Since the kernel of Eq. (7) is symmetric under the transformation $\tau \rightarrow -\tau$, we may extend the integration range to $-\infty \leq \tau \leq \infty$. There being no poles at the origin (even for $\tau_0 = 0$), we may write the result as an open contour above the real axis, indented at the origin, i.e.,

$$F_l^\alpha(r) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{2} \int_{-\infty}^\infty \frac{d\tau}{\tau^l} \frac{\text{Im}[H_l^{(-)}(\tau)Z_l(\mu_\alpha, \tau)]\text{Im}[H_l^{(+)}(\tau x)H_l^{(-)}(\tau)]}{H_l^{(+)}(\tau)H_l^{(-)}(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$$

$$= \frac{\mu_\alpha^2 e^{\mu_\alpha}}{2} \int_{\Omega} \frac{d\tau}{\tau^l} \frac{\text{Im}[H_l^{(-)}(\tau)Z_l(\mu_\alpha, \tau)]\text{Im}[H_l^{(+)}(\tau x)H_l^{(-)}(\tau)]}{H_l^{(+)}(\tau)H_l^{(-)}(\tau)} P\left(\frac{1}{\tau^2 - \tau_0^2}\right), \quad (8)$$

where we write the principle value, $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$, in Eq. (8) as

$$P\left(\frac{1}{\tau^2 - \tau_0^2}\right) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{2} \left[\frac{1}{(\tau + \tau_0 - i\epsilon)(\tau - \tau_0 + i\epsilon)} + \frac{1}{(\tau + \tau_0 + i\epsilon)(\tau - \tau_0 - i\epsilon)} \right] \right\}. \quad (9)$$

Now the spherical Hankel functions, $H_l^{(+)}(z)$, are defined by

$$H_l^{(+)}(z) \equiv \frac{e^{+iz}}{z^l} [\hat{\alpha}_l(z) + i\hat{\beta}_l(z)], \quad (10)$$

where

$$\hat{\alpha}_l(z) + i\hat{\beta}_l(z) \equiv \sum_{s=0}^l \Gamma_{ls}(z) e^{+i(l-s)\pi/2}. \quad (11)$$

Substituting Eqs. (10) and (11) into Eq. (8), we obtain

$$F_l^\alpha(r) = \frac{-\mu_\alpha^2 e^{\mu_\alpha}}{8} \int_{\Omega_r} \frac{d\tau}{\tau^l} \cdot \left[\frac{H_l^{(-)}(\tau) Z_l(\mu_\alpha, \tau) H_l^{(+)}(\tau x) + H_l^{(+)}(\tau) \bar{Z}_l(\mu_\alpha, \tau) H_l^{(-)}(\tau x)}{H_l^{(+)}(\tau)} \right. \\ \left. - \frac{H_l^{(-)}(\tau)}{-Z_l(\mu_\alpha, \tau) H_l^{(-)}(\tau x) - \bar{Z}_l(\mu_\alpha, \tau) H_l^{(+)}(\tau x)} \right] P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$$

or

$$F_l^\alpha(r) = \frac{-\mu_\alpha^2 e^{\mu_\alpha}}{8x^l} \int_{-\infty}^{\infty} \frac{d\tau}{\tau^l} P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$$

$$x \left\{ \begin{aligned} & Z_l(\mu_\alpha, \tau) e^{i\tau(x-2)} \frac{[\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)][\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)]}{\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)} \\ & + \bar{Z}_l(\mu_\alpha, \tau) e^{-i\tau(x-2)} \frac{[\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)][\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)]}{\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)} \\ & - Z_l(\mu_\alpha, \tau) e^{-i\tau x} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] \\ & - \bar{Z}_l(\mu_\alpha, \tau) e^{i\tau x} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)] \end{aligned} \right\}, \quad (12)$$

where the superscript bar indicates complex conjugation. The function $Z_l(\mu_\alpha, \tau)$ (see Eqs. (13) in Appendix 1) is defined by

$$Z_l(\mu_\alpha, \tau) \equiv \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} E_s(\mu_\alpha - i\tau), \quad (13)$$

where the exponential integral, $E_n(z)$, is defined by

$$E_n(z) \equiv \int_1^\infty \frac{d\tau}{\tau^n} e^{-z\tau}, \quad \text{Re}(z) > 0, \quad \text{for } n = 0, 1, 2, \dots \quad (14)$$

Upon substituting Eqs. (13) and (14) into Eq. (12), we obtain

$$F_{\ell}^{\alpha}(r) = \frac{-\mu_{\alpha}^2 e^{\mu_{\alpha}}}{8x^{\ell}} \sum_{s=0}^{\ell} \frac{(\ell+\frac{1}{2}, s)}{2^s} \int_1^{\infty} \frac{dy e^{-\mu_{\alpha} y}}{y^s}$$

$$\left\{ \begin{aligned} & e^{-i(\ell-s)\pi/2} \int_{\Gamma_{\uparrow}} \frac{d\tau}{\tau^{\ell+s}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) e^{i\tau(x+y-2)} \\ & \quad \frac{[\hat{\alpha}_{\ell}(\tau) - i\hat{\beta}_{\ell}(\tau)][\hat{\alpha}_{\ell}(\tau x) + i\hat{\beta}_{\ell}(\tau x)]}{[\hat{\alpha}_{\ell}(\tau) + i\hat{\beta}_{\ell}(\tau)]} \\ & + e^{i(\ell-s)\pi/2} \int_{\Gamma_{\downarrow}} \frac{d\tau}{\tau^{\ell+s}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) e^{-i\tau(x+y-2)} \\ & \quad \frac{[\hat{\alpha}_{\ell}(\tau) + i\hat{\beta}_{\ell}(\tau)][\hat{\alpha}_{\ell}(\tau x) - i\hat{\beta}_{\ell}(\tau x)]}{[\hat{\alpha}_{\ell}(\tau) - i\hat{\beta}_{\ell}(\tau)]} \\ & - e^{-i(\ell-s)\pi/2} \int_{\Gamma_{\uparrow}} \frac{d\tau}{\tau^{\ell+s}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) e^{i\tau(y-x)} [\hat{\alpha}_{\ell}(\tau) + i\hat{\beta}_{\ell}(\tau)] \\ & + e^{i(\ell-s)\pi/2} \int_{\Gamma_{\downarrow}} \frac{d\tau}{\tau^{\ell+s}} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) e^{-i\tau(y-x)} [\hat{\alpha}_{\ell}(\tau) - i\hat{\beta}_{\ell}(\tau)] \end{aligned} \right\} \quad (15)$$

Integration Details

The roots of $\hat{\alpha}_{\ell}(\tau) \pm i\hat{\beta}_{\ell}(\tau)$ were determined below Eqs. (6) and (13) in Appendix 3 to be ℓ in number and to lie in the (+: upper, -: lower)-half complex τ -plane, located symmetrically about the imaginary τ axis. If ℓ is odd, there is one purely imaginary root.

Since $x \geq 1$ and $y \geq 1$, the first integral of Eq. (15) must be closed in the upper-half plane. Then the only pole contribution to Cauchy's Residue Theorem will come from the $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ term. Similarly, the second integral must be closed in the lower-half

plane and will have residue contributions from the $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ term and the $(\ell + s)^{\text{th}}$ order pole at $\tau = 0$.

The third and fourth integrals must be examined in more detail, depending on whether $y - x > 0$ or $y - x < 0$. If $y - x > 0$, the third integral is closed in the upper-half plane, and only a residue contribution from the $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ term, and the fourth integral is closed in the lower-half plane with residue contributions from $P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$ and the $(\ell + s)^{\text{th}}$ order pole at $\tau = 0$. If $y - x < 0$, the rôles of the third and fourth integrals are interchanged.

Using Eq. (9) and, for the moment, ignoring the contributions arising from the $1/\tau^{\ell+s}$ poles in the lower half plane, we have (upon applying the Cauchy Residue Theorem) the results for the first integral in Eq. (15), that

$$\oint \frac{d\tau}{\tau^{\ell+s}} e^{i\tau(x+y-2)} \frac{[\hat{\alpha}_\ell(\tau x) - i\hat{\beta}_\ell(\tau x)][\hat{\alpha}_\ell(\tau) + i\hat{\beta}_\ell(\tau)]}{[\hat{\alpha}_\ell(\tau) - i\hat{\beta}_\ell(\tau)]} P\left(\frac{1}{\tau^2 - \tau_0^2}\right)$$

$$= \frac{\pi i}{2} \frac{x^\ell}{\tau_0^{\ell+1}} \left[e^{i\tau_0 y} \frac{H_\ell^{(+)}(\tau_0 x) H_\ell^{(-)}(\tau_0)}{H_\ell^{(+)}(\tau_0)} - (-)^{\ell+s} e^{-i\tau_0 y} \frac{H_\ell^{(-)}(\tau_0 x) H_\ell^{(+)}(\tau_0)}{H_\ell^{(-)}(\tau_0)} \right].$$

(16)

For the second integral we obtain

$$\begin{aligned}
 & \oint \frac{d\tau}{\tau^{l+s}} e^{-i\tau(x+y-2)} \frac{[\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)]}{[\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)]} P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\
 &= -\frac{\pi i}{2} \frac{x^l}{\tau_0^{s+1}} \left[e^{-i\tau_0 y} \frac{H_l^{(-)}(\tau_0 x) H_l^{(+)}(\tau_0)}{H_l^{(-)}(\tau_0)} - (-)^{l+s} \frac{e^{i\tau_0 y} H_l^{(+)}(\tau_0 x) H_l^{(-)}(\tau_0)}{H_l^{(+)}(\tau_0)} \right] \\
 &\quad + (\text{pole term at the origin}), \\
 &\hspace{15em} (17)
 \end{aligned}$$

and for the third we get

$$\begin{aligned}
 & \int_{\Omega_+} \frac{d\tau}{\tau^{l+s}} e^{i\tau(y-x)} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\
 &= \theta(y-x) \oint \frac{d\tau}{\tau^{l+s}} e^{i\tau(y-x)} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] P\left(\frac{1}{\tau^2 - \tau_0^2}\right) \\
 &\quad + \theta(y-x) \oint \frac{d\tau}{\tau^{l+s}} e^{-i\tau(x-y)} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] P\left(\frac{1}{\tau^2 - \tau_0^2}\right)
 \end{aligned}$$

$$= \frac{\pi i x^l}{2\tau_o^{s+1}} \left\{ \begin{aligned} & \left[e^{i\tau_o y} H_l^{(-)}(\tau_o x) - (-)^{l+s} e^{-i\tau_o y} H_l^{(+)}(\tau_o x) \right] \Theta(y-x) \\ & - \left[e^{i\tau_o y} H_l^{(-)}(\tau_o x) - (-)^{l+s} e^{-i\tau_o y} H_l^{(+)}(\tau_o x) \right. \\ & \quad \left. + \left(\text{pole term} \right) \right] \Theta(x-y) \end{aligned} \right\} \quad (18)$$

Finally, for the fourth integral of Eq. (15) we obtain

$$\oint \frac{d\tau}{\tau^{l+s}} e^{-i\tau(y-x)} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)] P\left(\frac{1}{\tau^2 - \tau_o^2}\right)$$

$$= - \frac{\pi i x^l}{2\tau_o^{s+1}} \left\{ \begin{aligned} & \left[e^{-i\tau_o y} H_l^{(+)}(\tau_o x) - (-)^{l+s} e^{i\tau_o y} H_l^{(-)}(\tau_o x) \right. \\ & \quad \left. + \left(\text{pole term} \right) \right] \Theta(y-x) \\ & - \left[e^{-i\tau_o y} H_l^{(+)}(\tau_o x) - (-)^{l+s} e^{i\tau_o y} H_l^{(-)}(\tau_o x) \right] \Theta(x-y) \end{aligned} \right\} \quad (19)$$

Thus, upon substituting Eqs. (16) - (19) into Eq. (15) and collecting terms we find

$$F_l^\alpha(r) = \pi \frac{\mu_\alpha^2 e^{\mu_\alpha}}{4 \tau_o^{l+1}} \operatorname{Im} \left[Z_l(\mu_\alpha, \tau_o) \frac{H_l^{(+)}(\tau_o x) H_l^{(-)}(\tau_o)}{H_l^{(+)}(\tau_o)} \right]$$

$$- \frac{\pi}{4} \frac{\mu_{\alpha}^2 e^{\mu_{\alpha}}}{\tau_0^{l+1}}$$

$$\text{Im} \left[H_l^{(-)}(\tau_0 x) \sum_{s=0}^l \Gamma_{ls}(\tau_0) e^{-i(l-s)\pi/2} \int_1^{\infty} \frac{dy e^{-(\mu_{\alpha} - i\tau_0)y}}{y^s} (\theta(y-x) - \theta(x-y)) \right]$$

$$+ (\text{pole terms at the origin}) . \quad (20)$$

Now, upon examining the remaining integral in Eq. (20) we note that

$$\int_1^{\infty} dy f(y) \theta(y - x) = \int_x^{\infty} dy f(y) \quad x \geq 1 , \quad (21)$$

$$\int_1^{\infty} dy f(y) \theta(x - y) = \int_1^x dy f(y) , \quad (22)$$

and

$$\int_1^{\infty} dy f(y) = \int_1^x dy f(y) + \int_x^{\infty} dy f(y) , \quad (23)$$

so that we may write

$$\int_1^{\infty} dy f(y) [\theta(y - x) - \theta(x - y)] = 2 \int_x^{\infty} dy f(y) - \int_1^{\infty} dy f(y). \quad (24)$$

Using Eq. (24) in Eq. (20), and noting that

$$\int_x^\infty \frac{dy}{y^s} e^{-(\mu_\alpha - i\tau_0)y} = \frac{1}{x^{s-1}} E_s [(\mu_\alpha - i\tau_0)x] , \quad (25)$$

we derive

$$\begin{aligned} F_l^\alpha(r) = & \frac{\pi}{4} \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau_0^{l+1}} \operatorname{Im} \left(Z_l(\mu_\alpha, \tau_0) \left[\frac{H_l^{(+)}(\tau_0 x) H_l^{(-)}(\tau_0)}{H_l^{(+)}(\tau_0)} + H_l^{(-)}(\tau_0 x) \right] \right) \\ & - \frac{\pi}{2} \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau_0^{l+1}} \operatorname{Im} \left(\frac{H_l^{(-)}(\tau_0 x) Z_l(\mu_\alpha x, \tau_0 x)}{x^{l-1}} \right) \\ & + (\text{pole terms at the origin}) . \end{aligned} \quad (26)$$

Ignoring the pole terms (which vanish identically as we prove below), using Eq. (2) and the definition of $G_l'(\tau x)$, i.e.,

$$G_l'(\tau x) = \frac{G_l(\tau x) G_l(\tau) + F_l(\tau x) F_l(\tau)}{\Delta_l(\tau)} \xrightarrow{x \rightarrow 1} \Delta_l(\tau) , \quad (27)$$

We find that Eq. (26) may be rewritten as

$$F_l^\alpha(r) = \frac{\pi}{2} \frac{a}{\tau_0} [g_l^\alpha(\tau_0) G_l'(\tau_0 x) - x \Delta_l(\tau_0 x) g_l^\alpha(\tau_0 x)] . \quad (28)$$

In the limit as $x \rightarrow 1$, we obtain

$$F_l^\alpha(r) \xrightarrow{x \rightarrow 1} 0 \quad (29)$$

as we should.

Cancellation of Residue Terms at Origin

Now let us examine the pole terms arising from the residue due to $1/\tau^{l+s}$ type terms in the neighborhood $\tau \rightarrow 0$, which we neglected previously. It will be shown that these terms vanish identically since, in fact, when they are collected together, there is no pole as $\tau \rightarrow 0$. Let us denote the contribution to $F_l^\alpha(r)$ due to these pole terms at the origin by $W_l^\alpha(r)$, where

$$W_l^\alpha(r) = - \frac{\mu_\alpha^2 e^{\mu_\alpha}}{8x^l} \sum_{s=0}^l \frac{(l+\frac{1}{2}, s)}{2^s} \int_1^\infty \frac{dy e^{-\mu_\alpha y}}{y^s} \left\{ \begin{aligned} & - 2\pi i e^{i(l-s)\pi/2} \operatorname{Res}_{\tau=0} \left\{ \frac{e^{-i\tau(x+y-2)}}{\tau^{l+s}(\tau^2 - \tau_0^2)} \frac{[\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)][\hat{\alpha}_l(\tau) - i\hat{\beta}_l(\tau)]}{[\hat{\alpha}_l(\tau) + i\hat{\beta}_l(\tau)]} \right\} \\ & + 2\pi i e^{-i(l-s)\pi/2} \theta(x-y) \operatorname{Res}_{\tau=0} \left\{ \frac{e^{-i\tau(x-y)}}{\tau^{l+s}(\tau^2 - \tau_0^2)} [\hat{\alpha}_l(\tau x) + i\hat{\beta}_l(\tau x)] \right\} \\ & + 2\pi i e^{i(l-s)\pi/2} \theta(y-x) \operatorname{Res}_{\tau=0} \left\{ \frac{e^{-i\tau(y-x)} [\hat{\alpha}_l(\tau x) - i\hat{\beta}_l(\tau x)]}{\tau^{l+s}(\tau^2 - \tau_0^2)} \right\} \end{aligned} \right\} \quad (30)$$

or, upon collecting terms

$$W_l^\alpha(r) = -\frac{\pi}{2} \mu_\alpha^2 e^{\mu_\alpha} \operatorname{Res}_{\tau=0} \left(\frac{1}{\tau^l (\tau^2 - \tau_0^2)} \left\{ \begin{aligned} & \operatorname{Im} \left[\frac{Z_l(\mu_\alpha x, \tau x) H_l^{(-)}(\tau x)}{x^{l-1}} \right] \\ & - \frac{H_l^{(-)}(\tau x)}{H_l^{(-)}(\tau)} \operatorname{Im} \left[Z_l(\mu_\alpha, \tau) H_l^{(-)}(\tau) \right] \end{aligned} \right\} \right). \quad (31)$$

Equation (31) vanishes identically for $x \rightarrow 1$. For $x > 1$ we write

$$H_l^\pm(\tau x) H_l^{(\mp)}(\tau) = \Delta_l(\tau) [G_l'(\tau x) \pm i F_l'(\tau x)] , \quad (32)$$

where (see Eqs. (4) and (27))

$$G_l'(\tau x) = \frac{G_l(\tau x) G_l(\tau) + F_l(\tau x) F_l(\tau)}{\Delta_l(\tau)} \quad (33)$$

$$F_l'(\tau x) = \frac{F_l(\tau x) G_l(\tau) - G_l(\tau x) F_l(\tau)}{\Delta_l(\tau)} . \quad (34)$$

Substituting Eqs. (2) and (32) into Eq. (31) we find

$$W_l^\alpha(r) = -\frac{\pi a}{2} \operatorname{Res}_{\tau=0} \left[\frac{x \Delta_l(\tau x) g_l^\alpha(\tau x) - (G_l'(\tau x) - i F_l'(\tau x)) g_l^\alpha(\tau)}{\tau^2 - \tau_0^2} \right], \quad (35)$$

where (see Eq. (2))

$$ag_l^\alpha(\tau x) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{(\tau x)^l \Delta_l(\tau x)} \operatorname{Im} [H_l^{(-)}(\tau x) Z_l(\mu_\alpha x, \tau x)] . \quad (36)$$

Examination of Eq. (35) in the limit $\tau = 0$ indicates a zero residue since there are, in fact, no poles. Thus, as $\tau \rightarrow 0$ it is readily seen by using Eqs. (25) and (26) in Appendix 1, Eqs. (33) and (34) that

$$\left. \begin{aligned} \Delta_l(\tau x) &\xrightarrow{\tau \rightarrow 0} \frac{(2l-1)!!}{(\tau x)^l} \\ xag_l^\alpha(\tau x) &\xrightarrow{\tau \rightarrow 0} \frac{\mu_\alpha^{l+1}}{(2l+1)!!} (\tau/\mu_\alpha)^{l+1} c_l(\mu_\alpha x) e^{-\mu_\alpha(x-1)} \\ G_l'(\tau x) &\xrightarrow{\tau \rightarrow 0} \Delta_l(\tau x) \rightarrow \frac{(2l-1)!!}{(\tau x)^l} \\ F_l'(\tau x) &\xrightarrow{\tau \rightarrow 0} \frac{\tau^{l+1}}{(2l+1)!!} \frac{x^{2l+1}-1}{x^l} \end{aligned} \right\} , \quad (37)$$

where

$$c_l(\mu_\alpha) \equiv \sum_{n=0}^{l+1} \frac{(\mu_\alpha)^n}{n!} - \frac{(\mu_\alpha)^{l+2} e^{\mu_\alpha} E_l(\mu_\alpha)}{(l+1)!} . \quad (38)$$

Substituting the low-momentum limits of Eqs. (37) into Eq. (35), we obtain

$$W_l^\alpha(r) = -\frac{\pi}{2} \mu_\alpha^2 e^{\mu_\alpha}$$

$$\text{Res}_{\tau=0} \left(\frac{\tau(l+1)!}{(\tau^2 - \tau_0^2)(2l+1)} \left[\begin{array}{l} C_l(\mu_\alpha x) e^{-\mu_\alpha(x-1)} \\ -C_l(\mu_\alpha) \left(1 - i \frac{[\tau x]^{2l+1} - \tau^{2l+1}}{(2l+1)!!(2l-1)!!} \right) \end{array} \right] \frac{1}{(\mu_\alpha x)^l} \right) \quad (39)$$

which residue vanishes at $\tau = 0$, so that

$$W_l^\alpha(r) = 0 \quad (40)$$

Returning to Eq. (28) and using Eqs. (37) and (40), we may examine the limiting cases of $F_l^\alpha(r)$ as $\tau_0 \rightarrow 0$ and $x \rightarrow 0$ to obtain

$$F_l^\alpha(r) = \frac{\pi a}{2\tau_0} \left\{ G_l^i(\tau_0 x) g_l^\alpha(\tau_0) - x \Delta_l(\tau_0 x) g_l^\alpha(\tau_0 x) \right\}$$

$$\tau_0 \rightarrow 0 \quad \frac{\pi}{2} \frac{(l+1)!}{(2l+1)(\mu_\alpha x)^l} \left\{ C_l(\mu_\alpha) - C_l(\mu_\alpha x) e^{-\mu_\alpha(x-1)} \right\} \quad (41)$$

$$\begin{array}{l} \rightarrow 0 \\ x \rightarrow 1 \\ \tau_0 \rightarrow 0 \end{array} ,$$

where $g_l^\alpha(\tau_0 x)$, $G_l^i(\tau_0 x)$, and $C_l(\mu_\alpha x)$ are defined by Eqs. (36), (33), and (38), respectively.

ON THE EVALUATION OF THE INTEGRAL $D_n(\alpha, \bar{\beta})$

The integral $D_n(\alpha, \bar{\beta})$ is defined by

$$D_n(\alpha, \bar{\beta}) \equiv \int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x^n}, \quad \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\bar{\beta}) > 0 \quad (1)$$

where the bar over β indicates complex conjugation.

For $n > 0$, Eq. (1) satisfies the recursion relationship

$$D_{n+1}(\alpha, \bar{\beta}) + \frac{\alpha}{n} D_n(\alpha, \bar{\beta}) = \frac{1}{n} [e^{-\alpha} E_1(\bar{\beta}) - E_{n+1}(\alpha + \bar{\beta})], \quad (2)$$

which follows directly from Eq. (1) upon integrating by parts:

$$\begin{aligned} D_{n+1}(\alpha, \bar{\beta}) &= \int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x^{n+1}} \\ &= - \frac{e^{-\alpha x} E_1(\bar{\beta} x)}{n x^n} \Big|_1^{\infty} - \frac{1}{n} \int_1^{\infty} \frac{dx e^{-(\alpha + \bar{\beta})x}}{x^{n+1}} - \frac{\alpha}{n} \int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x^n} \quad (3) \\ &= - \frac{\alpha}{n} D_n(\alpha, \bar{\beta}) + \frac{1}{n} [e^{-\alpha} E_1(\bar{\beta} x) - E_{n+1}(\alpha + \bar{\beta})] . \end{aligned}$$

For $n = 0$, we find

$$D_0(\alpha, \bar{\beta}) = \frac{1}{\alpha} [e^{-\alpha} E_1(\bar{\beta}) - E_1(\alpha + \bar{\beta})], \quad (4)$$

either by integrating Eq. (1) by parts or by multiplying Eq. (2) by n and then taking the limit $n \rightarrow 0$ (assuming $D_1(\alpha, \bar{\beta})$ is well-defined which it is).

Defining

$$\epsilon_{k+1}(\alpha, \bar{\beta}) \equiv e^{-\alpha} E_1(\bar{\beta}) - E_{k+1}(\alpha + \bar{\beta}) , \quad (5)$$

We may express Eq. (2) for $n > 0$ as

$$D_{n+1}(\alpha, \bar{\beta}) = \frac{(-\alpha)^n}{n!} D_1(\alpha, \bar{\beta}) + \sum_{k=1}^n \frac{(-\alpha)^{n-k} (k-1)!}{n!} \epsilon_{k+1}(\alpha, \bar{\beta}) . \quad (6)$$

Clearly Eqs. (2) and (6) are identical for $n = 1$. Let us assume that Eq. (6) holds for $n = p - 1$, apply transfinite induction, and deduce that it holds for all n . Using Eq. (2) (which holds for all $n > 0$), we have

$$D_{p+1}(\alpha, \bar{\beta}) = -\frac{\alpha}{p} D_p(\alpha, \bar{\beta}) + \frac{1}{p} \epsilon_{p+1}(\alpha, \bar{\beta}) . \quad (7)$$

Assuming Eq. (6) holds for $n = p - 1$, its substitution into Eq.

(7) yields

$$\begin{aligned} D_{p+1}(\alpha, \bar{\beta}) &= \frac{(-\alpha)^p}{p!} D_1(\alpha, \bar{\beta}) + \sum_{k=1}^{p-1} \frac{(-\alpha)^{p-k} (k-1)!}{p!} \epsilon_{k+1}(\alpha, \bar{\beta}) + \frac{1}{p} \epsilon_{p+1}(\alpha, \bar{\beta}) \\ &= \frac{(-\alpha)^p}{p!} D_1(\alpha, \bar{\beta}) + \sum_{k=1}^p \frac{(-\alpha)^{p-k} (k-1)!}{p!} \epsilon_{k+1}(\alpha, \bar{\beta}) . \end{aligned} \quad (8)$$

Then using transfinite induction, since Eq. (6) is true for $n = 1$ and $n = p - 1$ implies that it is true for $n = p$, it is true for all $n \geq 1$.

Hence, any $D_n(\alpha, \bar{\beta})$ for $n > 1$ is determined from Eq. (6) if we know $D_1(\alpha, \bar{\beta})$, insasmuch as the remaining terms in Eq. (6) are expressing in terms of known functions.

The Function $D_1(\alpha, \bar{\beta})$

Before performing the explicit evaluation for $D_1(\alpha, \bar{\beta})$, we first note that

$$D_1(\alpha, \bar{\beta}) + D_1(\bar{\beta}, \alpha) = E_1(\alpha)E_1(\bar{\beta}) , \quad (9)$$

see end of Appendix 6, which is derived most readily by integrating the identity

$$\frac{e^{-\alpha x} E_1(\bar{\beta} x)}{x} + \frac{e^{-\bar{\beta} x} E_1(\alpha x)}{x} \equiv - \frac{d}{dx} [E_1(\alpha x) E_1(\bar{\beta} x)] ; \quad (10)$$

namely,

$$\int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x} + \int_1^{\infty} \frac{dx e^{-\bar{\beta} x} E_1(\alpha x)}{x} = - E_1(\alpha) E_1(\bar{\beta}) \Big|_1^{\infty} ,$$

or

$$D_1(\alpha, \bar{\beta}) + D_1(\bar{\beta}, \alpha) = E_1(\alpha)E_1(\bar{\beta}) .$$

Equation (9) provides, in fact, a method for analytic continuing the explicit expressions for $D_1(\alpha, \bar{\beta})$ which we derive now.

The exponential integral of order 1, $E_1(z)$, has the representation

$$E_1(z) = -\gamma - \ln z - \sum_{p=1}^{\infty} \frac{(-z)^p}{pp!} , \quad \text{Re}(z) > 0 \quad (11)$$

where

$$\gamma = \text{Euler's constant} = \lim_{s \rightarrow \infty} \left(\sum_{m=1}^s \frac{1}{m} - \ln s \right)$$

$$= - \left[E_1(1) + \sum_{p=1}^{\infty} \frac{(-)^p}{pp!} \right]$$

$$= 0.577215 \dots$$

Upon substituting Eq. (11) into Eq. (1) for $n = 1$, we find

$$\begin{aligned}
D_1(\alpha, \bar{\beta}) &= \int_1^{\infty} \frac{dx e^{-\alpha x} E_1(\bar{\beta} x)}{x} \\
&= - \int_1^{\infty} \frac{dx e^{-\alpha x}}{x} [\gamma + \ln \bar{\beta} + \ln x + \sum_{p=1}^{\infty} \frac{(-\bar{\beta} x)^p}{p p!}] \\
&= -(\gamma + \ln \bar{\beta}) E_1(\alpha) - \int_1^{\infty} \frac{dx e^{-\alpha x} \ln x}{x} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p p!} \int_1^{\infty} dx e^{-\alpha x} x^{p-1}.
\end{aligned} \tag{12}$$

First we evaluate the last integral of Eq. (12)

$$\alpha_{p-1}(\alpha) \equiv \int_1^{\infty} dx e^{-\alpha x} x^{p-1} = \frac{(p-1)!}{\alpha^p} e^{-\alpha} e_{p-1}(\alpha), \quad \text{Re}(\alpha) > 0, \tag{13}$$

where

$$e_{p-1}(\alpha) \equiv \sum_{n=0}^{p-1} \frac{\alpha^n}{n!}. \tag{14}$$

Next we consider the integral

$$\int_1^{\infty} \frac{dx e^{-\alpha x} \ln x}{x} = \frac{\alpha}{2} \int_1^{\infty} dx e^{-\alpha x} (\ln x)^2 + \frac{1}{2} (\ln x)^2 e^{-\alpha x} \Big|_1^{\infty}. \tag{15}$$

The last term of Eq. (15) vanishes at both limits. Now

$$\begin{aligned}
 \alpha \int_0^{\infty} dx (\ln x)^2 e^{-\alpha x} &= \frac{1}{2} \int_0^{\infty} dy e^{-y} (\ln y - \ln \alpha)^2 \\
 &= \frac{1}{\alpha} [(\ln \alpha)^2 - 2 \ln \alpha \int_0^{\infty} dy \ln y e^{-y} + \int_0^{\infty} dy (\ln y)^2 e^{-y}] \\
 &= \frac{1}{2} [(\ln \alpha)^2 + 2(\ln \alpha)\gamma + (\frac{\pi^2}{6} + \gamma^2)] \\
 &= \frac{1}{2} [\frac{\pi^2}{6} + (\ln \alpha + \gamma)^2],
 \end{aligned} \tag{16}$$

$$\alpha \int_0^1 dx e^{-\alpha x} (\ln \frac{1}{x})^2 = \sum_{p=0}^{\infty} \frac{(-\alpha)^{p+1}}{p!} \int_0^1 dx x^p (\ln \frac{1}{x})^2, \tag{17}$$

$$\int_0^1 dx x^{n-1} (\ln \frac{1}{x})^m = \frac{\Gamma(m+1)}{n^{m+1}}, \tag{18}$$

where $\Gamma(n)$ is the gamma function. Substituting Eqs. (16) through (18) into Eq. (15) we have the result that

$$\begin{aligned}
 \int_1^{\infty} \frac{dx e^{-\alpha x} \ln x}{x} &= \frac{\alpha}{2} \left\{ \int_0^{\infty} dx e^{-\alpha x} (\ln x)^2 - \int_0^1 dx e^{-\alpha x} (\ln \frac{1}{x})^2 \right\} \\
 &= \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right\} + \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!},
 \end{aligned} \tag{19}$$

where we have used relabeling to rewrite the series

$$\sum_{p=0}^{\infty} \frac{(-\alpha)^{p+1}}{(p+1)^3 p!} = \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} .$$

The answer obtained in Eq. (19) is valid for $\text{Re}(\alpha) > 0$.

Substituting Eqs. (13) and (19) into Eq. (12) for $D_1(\alpha, \bar{\beta})$ yields the result

$$\begin{aligned} D_1(\alpha, \bar{\beta}) = & -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right\} - (\gamma + \ln \bar{\beta}) E_1(\alpha) \\ & - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) . \end{aligned} \quad (20)$$

The result obtained in Eq. (20) for $D_1(\alpha, \bar{\beta})$ is valid for $|\bar{\beta}/\alpha| \leq 1$ only. This may be seen as follows: As $p \rightarrow \infty$ the term $e^{-\alpha} e_{p-1}(\alpha)$ in the last term of Eq. (20) approaches unity. Thus,

$$\sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) < \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} . \quad (21)$$

But the right hand side of Eq. (21) is just Abel's series, which converges only for $|\bar{\beta}/\alpha| \leq 1$.

In order to evaluate $D_1(\alpha, \bar{\beta})$ whenever $|\bar{\beta}/\alpha| > 1$, we utilize the analytic continuation of $D_1(\alpha, \bar{\beta})$, as expressed in Eq. (9).

Thus

$$D_1(\bar{\beta}, \alpha) = -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \bar{\beta})^2 \right\} - (\gamma + \ln \alpha) E_1(\bar{\beta}) \\ - \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}), \quad (22)$$

which is valid for $|\alpha/\bar{\beta}| < 1$, i.e., $|\bar{\beta}/\alpha| > 1$, so that

$$D_1(\alpha, \bar{\beta}) = \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \bar{\beta})^2 \right\} + E_1(\bar{\beta})(\gamma + \ln \alpha + E_1(\alpha)) \\ + \sum_{p=1}^{\infty} \frac{(-\bar{\beta})^p}{p^2 p!} + \sum_{p=1}^{\infty} \frac{(-\alpha/\bar{\beta})^p}{p^2} e^{-\bar{\beta}} e_{p-1}(\bar{\beta}) \quad (23)$$

for $|\alpha/\bar{\beta}| < 1$, and

$$D_1(\alpha, \bar{\beta}) = -\frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \alpha)^2 \right\} - E_1(\alpha)(\gamma + \ln \bar{\beta}) \\ - \sum_{p=1}^{\infty} \frac{(-\alpha)^p}{p^2 p!} - \sum_{p=1}^{\infty} \frac{(-\bar{\beta}/\alpha)^p}{p^2} e^{-\alpha} e_{p-1}(\alpha) \quad (24)$$

for $|\bar{\beta}/\alpha| < 1$.

Then, in summary for positive n , the integral of Eq. (1) has the solution Eq. (24) for $n = 0$ and for $n > 0$,

$$D_n(\alpha, \bar{\beta}) = \frac{(-\alpha)^{n-1}}{(n-1)!} D_1(\alpha, \bar{\beta}) \\ + \theta(n-1) \sum_{k=1}^{n-1} \frac{(-\alpha)^{n-k-1} (k-1)!}{(n-1)!} [e^{-\alpha} E_1(\bar{\beta}) - E_{k+1}(\alpha + \bar{\alpha})] \quad (25)$$

where $D_1(\alpha, \bar{\beta})$ is given by Eqs. (20) and (23) for $|\alpha/\bar{\beta}| \geq 1$ and $|\alpha/\bar{\beta}| \leq 1$, respectively.

When n is negative we consider the integral ($|n| \rightarrow n$)

$$D_{-n}(\alpha, \bar{\beta}) \equiv \int_1^{\infty} dx x^n e^{-\alpha x} E_1(\bar{\beta} x), \quad (26)$$

and make the observation that

$$D_{-n}(\alpha, \bar{\beta}) = \left(-\frac{\partial}{\partial \alpha} \right)^n D_0(\alpha, \bar{\beta}). \quad (27)$$

Using the chain rule for differentiation we have

$$\begin{aligned}
D_{-n}(\alpha, \bar{\beta}) &= \sum_{k=0}^n \binom{n}{k} \left(-\frac{\partial}{\partial \alpha}\right)^{n-k} \left(\frac{1}{\alpha}\right) \left(-\frac{\partial}{\partial \alpha}\right)^k [e^{-\alpha E_1(\bar{\beta})} - E_1(\alpha + \bar{\beta})] \\
&= \sum_{k=1}^n \binom{n}{k} \frac{(n-k)!}{\alpha^{n-k+1}} \left(-\frac{\partial}{\partial \alpha}\right)^{k-1} [e^{-\alpha E_1(\bar{\beta})} - \frac{e^{-(\alpha+\bar{\beta})}}{\alpha+\bar{\beta}}] \\
&\quad + \left[-\frac{\partial}{\partial \alpha}\right]^n \left(\frac{1}{\alpha}\right) [e^{-\alpha E_1(\bar{\beta})} - E_1(\alpha+\bar{\beta})] \\
&= \sum_{k=1}^n \frac{n!}{k!} \frac{1}{\alpha^{n-k+1}} [e^{-\alpha E_1(\bar{\beta})} - \sum_{r=1}^{k-1} \binom{k-1}{r} \\
&\quad + \left(\frac{\partial}{\partial \alpha}\right)^r (e^{-(\alpha+\bar{\beta})} \left(\frac{-1}{\partial \alpha}\right)^{k-1-r} \left(\frac{1}{\alpha+\bar{\beta}}\right)] \\
&\quad + \frac{n!}{\alpha^{n+1}} [e^{-\alpha E_1(\bar{\beta})} - E_1(\alpha+\bar{\beta})] , \tag{28}
\end{aligned}$$

or

$$\begin{aligned}
D_{-n}(\alpha, \bar{\beta}) &= n! e^{-\alpha} \sum_{k=1}^n \frac{1}{k! \alpha^{n-k+1}} [E_1(\bar{\beta}) - e^{-\bar{\beta}} (k-1)! \sum_{r=1}^{k-1} \frac{1}{r!} \frac{1}{(\alpha+\bar{\beta})^{k-r}}] \\
&\quad + \frac{n!}{\alpha^{n+1}} [e^{-\alpha E_1(\bar{\beta})} - E_1(\alpha+\bar{\beta})] \tag{29}
\end{aligned}$$

or, finally

$$D_{-n}(\alpha, \bar{\beta}) = \alpha_n(\alpha) E_1(\bar{\beta}) - \sum_{k=0}^n \frac{n!}{k!} \frac{\alpha_{k-1}(\alpha+\bar{\beta})}{\alpha^{n-k+1}} \tag{30}$$

which completes the analysis of $D_n(\alpha, \overline{3})$. In Eq. (28) $\alpha_p(\alpha)$ is given by Eq. (13) and $\alpha_{-1}(z) = E_1(z)$.

ON THE EVALUATION OF THE INTEGRALS

$$\underline{D_{n,s}^{(1)}(\Lambda), D_{n,s}^{(2)}(\Lambda_1, \Lambda_2), D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3), \text{ and } D_n^{(4)}(\Lambda_1, \Lambda_2)}$$

The integrals $D_{n,s}^{(1)}(\Lambda)$, $D_{n,s}^{(2)}(\Lambda_1, \Lambda_2)$, $D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3)$ and $D_n^{(4)}(\Lambda_1, \Lambda_2)$ are defined by

$$D_{n,s}^{(1)}(\Lambda) \equiv \int_1^\infty dx x^n E_s(\Lambda x), \operatorname{Re}(\Lambda) > 0 \quad (1)$$

$$D_{n,s}^{(2)}(\Lambda_1, \Lambda_2) \equiv \int_1^\infty dx x^n e^{-\Lambda_1 x} E_s(\Lambda_2 x), \operatorname{Re}(\Lambda_1 + \Lambda_2) > 0 \quad (2)$$

$$D_{n,s}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \equiv \int_1^\infty dx x^n e^{\Lambda_1 x} E_1(\Lambda_2 x) E_1(\Lambda_3 x), \operatorname{Re}(-\Lambda_1 + \Lambda_2 + \Lambda_3) > 0 \quad (3)$$

$$D_n^{(4)}(\Lambda_1, \Lambda_2) \equiv \int_1^\infty dx x^n E_1(\Lambda_1 x) E_1(\Lambda_2 x), \operatorname{Re}(\Lambda_1 + \Lambda_2) > 0 \quad (4)$$

For Eqs. (1) and (2) we are interested in the values $s \geq 0$ and $n \geq 0$, $n < 0$ (since for the deuteron quadrupole moment $n=3-s-t$ with $0 \leq (s,t) \leq 2$). Equation (1) may be reduced to somewhat simpler form by using the expansion theorem for $E_s(z)$; namely,

$$E_s(z) \equiv \delta_{s,0} E_0(z)$$

$$+ (1 - \delta_{s,0}) \left\{ \theta(s-1) e^{-z} \sum_{k=1}^{s-1} \frac{(s-1-k)!}{(s-1)!} (-z)^{k-1} + \frac{(-z)^{s-1}}{(s-1)!} E_1(z) \right\}, \quad (5)$$

where $\theta(0) = 0$. Upon substituting Eq. (5) into Eq. (1), and using the definitions

$$\alpha_n(z) \equiv \int_1^\infty dt e^{-zt} t^n, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2$$

$$E_n(z) \equiv \int_1^\infty dt \frac{e^{-zt}}{t^n}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2. \quad (6)$$

we obtain

$$D_{n,s}^{(1)}(\Lambda) = \frac{\delta_{s,0}}{\Lambda} \{ \theta(n) \alpha_{n-1}(\Lambda) + \theta(1-n) E_{|n|+1}(\Lambda) \}$$

$$+ (1 - \delta_{s,0}) \left\{ \theta(s-1) \sum_{k=1}^{s-1} \frac{(s-1-k)!}{(s-1)!} (-\Lambda)^{k-1} \left\{ \theta(n+k) \alpha_{n+k-1}(\Lambda) + \theta(1-n+k) E_{|n|-k+1}(\Lambda) \right\} + \frac{(-\Lambda)^{s-1}}{(s-1)!} D_{n+s-1,1}^{(1)}(\Lambda) \right\} \quad (7)$$

Thus, the determination of $D_{n,s}^{(1)}(\Lambda)$ has been reduced to the evaluation of the integral

$$D_{p,1}^{(1)}(\lambda) \equiv \int_1^{\infty} dx x^p E_1(\lambda x), \operatorname{Re}(\lambda) > 0 \quad (8)$$

for $p \geq 0$ and $p < 0$.

For $p \geq 0$ integration of Eq. (18) by parts yields

$$\begin{aligned} D_{p,1}^{(1)}(\lambda) &= \left. \frac{x^{p+1}}{p+1} E_1(\lambda x) \right|_1^{\infty} + \frac{1}{p+1} \int_1^{\infty} dx x^p e^{-\lambda x} \\ &= \frac{1}{p+1} \left\{ \alpha_{p+1}(\lambda) - E_1(\lambda) \right\} \end{aligned} \quad (9)$$

For $p < 0$ we have two cases: $p = -1$ and $p < -1$.

For $p = -1$, integration of Eq. (8) by parts and using Eq. (19) of Appendix 12 yields

$$\begin{aligned} D_{1,1}^{(1)}(\lambda) &= \int_1^{\infty} \frac{dx E_1(\lambda x)}{x} \\ &= \left. \ln(x) E_1(\lambda x) \right|_1^{\infty} + \int_1^{\infty} dx e^{-\lambda x} \frac{\ln(x)}{x} \\ &= \int_1^{\infty} dx e^{-\lambda x} \frac{\ln(x)}{x} \\ &= 1/2 \left\{ \frac{\pi^2}{6} + (\gamma + \ln \lambda)^2 \right\} + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p^2 p!} \end{aligned} \quad (10)$$

where γ = Euler's constant = 0.577215 ...

For $p < -1$ integration of Eq. (8) by parts yields-

$$D_{p,1}^{(1)}(\Lambda) = - \frac{(|p|-1)}{x^{|p|-1}} E_1(\Lambda x) \Big|_1^\infty - (|p|-1) \int_1^\infty dx \frac{e^{-\Lambda x}}{x^{|p|}}$$

$$= (|p|-1) (E_1(\Lambda) - E_{|p|}(\Lambda)) \quad (11)$$

Collecting Eqs. (9) - (11), Eq. (8) becomes

$$D_{p,1}^{(1)}(\Lambda) = \left\{ \begin{aligned} & \frac{\theta(p+1)}{p+1} \{ \alpha_{p+1}(\Lambda) - E_1(\Lambda) \} \\ & + \delta_{p,-1} \left\{ \frac{1}{2} \left\{ \frac{\pi^2}{6} + (\gamma + \ln \Lambda)^2 \right\} \sum_{p=1}^{\infty} \frac{(-\Lambda)^p}{p^2 p!} \right\} \\ & + \theta(-1-p) (|p|-1) \{ E_1(\Lambda) - E_{|p|}(\Lambda) \} \end{aligned} \right\} \quad (12)$$

Substituting Eq. (12) into Eq. (7) (where $p = n+s-1$) we obtain the final expression for Eq. (1):

$$\delta_{s,0} \frac{1}{\Lambda} \{ \theta(n) \alpha_{n-1}(\Lambda) + \theta(1-n) E_{|n|+1}(\Lambda) \}$$

$$+ (1-\delta_{s,0}) \frac{\theta(s-1)}{(s-1)!} \sum_{k=1}^{s-1} (s-1-k)! (-\Lambda)^{k-1}$$

$$D_{n,s}^{(1)}(\Lambda) = \cdot \{ \theta(n+k) \alpha_{n+k-1}(\Lambda) + (1+k-n) E_{|n|+1-k}(\Lambda) \}$$

$$+ (1-\delta_{s,0}) \frac{(-\Lambda)^{s-1}}{(s-1)!} \left\{ \frac{\theta(n+s)}{n+s} \{ \alpha_{n+s}(\Lambda) - E_1(\Lambda) \} \right.$$

$$+ \delta_{n+s-1,-1} \left\{ \frac{1}{2} \left\{ \frac{\pi^2}{6} + \gamma + \ln \Lambda \right\}^2 + \sum_{p=1}^{\infty} \frac{(-\Lambda)^p}{p^2 p!} \right\}$$

$$+ \theta(-n-s) \{ |n+s-1| - 1 \} \{ E_1(\Lambda) - E_{|n+s-1|}(\Lambda) \} \quad (13)$$

The integral for $D_{n,s}^{(2)}(\Lambda_1, \Lambda_2)$ is treated in an analogous manner.

Upon substituting Eq. (5) into Eq. (2) and using Eq. (3) we obtain

$$D_{n,s}^{(2)}(\Lambda_1, \Lambda_2) = \frac{\delta_{s,0}}{\Lambda_2} \left\{ \theta(n) \alpha_{n-1} (\Lambda_1 + \Lambda_2) + \theta(1-n) E_{|n|+1}(\Lambda_1 + \Lambda_2) \right. \\ \left. + (1 - \delta_{s,c}) \left\{ \frac{\theta(s-1)}{(s-1)!} \sum_{k=1}^{s-1} (s-1-k)! (-\Lambda_2)^{k-1} \right. \right. \\ \left. \left\{ \theta(n+k) \alpha_{n+k-1} (\Lambda_1 + \Lambda_2) \right. \right. \\ \left. \left. + \theta(1-n+k) E_{|n|-k+1}(\Lambda_1 + \Lambda_2) \right\} \right. \\ \left. \left. \frac{(-\Lambda_2)^{s-1}}{(s-1)!} D_{n+s-1,1}^{(1)}(\Lambda_1, \Lambda_2) \right\} \right\} \quad (14)$$

where from Appendix 12

$$D_{p,1}^{(2)}(\Lambda_1, \Lambda_2) = \int_1^\infty dx x^p e^{-\Lambda_1 x} E_1(\Lambda_2 x) = D_{-p}(\Lambda_1, \Lambda_2) \quad (15)$$

Now, from Appendix 12, Eq. (22), (24), and (30), Eq. (15) is completely determined.

The integral $D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3)$ given by Eq. (3) may be evaluated for $n \geq 0$ from the relationship

$$D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = \frac{\partial^n}{\partial \Lambda_1^n} D_0^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \quad (16)$$

where the integral of interest, $D_0^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3)$, is given by

$$D_0^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = \int_1^\infty dx e^{-\Lambda_1 x} E_1(\Lambda_2 x) E_1(\Lambda_3 x), \quad \text{Re}(\Lambda_2 + \Lambda_3 - \Lambda_1) > 0 \quad (17)$$

Using the identity

$$\frac{d}{dx} \{ e^{\Lambda_1 x} E_1(\Lambda_2 x) - E_1(\Lambda_2 - \Lambda_1)x \} = \Lambda_1 e^{\Lambda_1 x} E_1(\Lambda_2 x) \quad (18)$$

and integrating Eq. (17) symmetrically in $E_1(\Lambda_2 x)$ and $E_1(\Lambda_3 x)$

by parts, we find

$$\begin{aligned} D_0^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = & -\frac{1}{\Lambda_1} e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) \\ & + \frac{1}{2\Lambda_1} \{ E_1(\Lambda_2 - \Lambda_1) E_1(\Lambda_3) + E_1(\Lambda_3 - \Lambda_1) E_1(\Lambda_2) \} \\ & + \frac{1}{2\Lambda_1} \{ D_1(\Lambda_3 - \Lambda_1, \Lambda_2) - D_1(\Lambda_3, \Lambda_2 - \Lambda_1) + D_1(\Lambda_2 - \Lambda_1, \Lambda_3) \\ & - D_1(\Lambda_2, \Lambda_3 - \Lambda_1) \}, \end{aligned} \quad (19)$$

where

$$D_1(\Lambda_1, \Lambda_2) \equiv \int_1^\infty dx \frac{e^{-\Lambda_1 x} E_1(\Lambda_2 x)}{x} \quad (20)$$

is examined in detail in Appendix 12. Using the identity (which can be derived from Eq. (2) by integrating by parts)

$$D_1(\Lambda_1, \Lambda_2) + D_1(\Lambda_2, \Lambda_1) = E_1(\Lambda_1) E_1(\Lambda_2) \quad (21)$$

in Eq. (19), we obtain

$$\begin{aligned} D_0^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = & -\frac{1}{\Lambda_1} \{ e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) + D_1(\Lambda_2, \Lambda_3 - \Lambda_1) \\ & + D_1(\Lambda_3, \Lambda_2 - \Lambda_1) \} \end{aligned} \quad (22)$$

Substituting Eq. (22) into (16) and using the chain rule for differentiation we find ($n > 0$)

$$\begin{aligned}
 D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) &= - \sum_{k=0}^n \binom{n}{k} \frac{\partial^k}{\partial \Lambda_1^k} \frac{1}{\Lambda_1} \frac{\partial^{n-k}}{\partial \Lambda^{n-k}} \{ e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) \\
 &\quad + D_1(\Lambda_2, \Lambda_3 - \Lambda_1) + D_1(\Lambda_3, \Lambda_2 - \Lambda_1) \} \\
 &= - \sum_{k=0}^n \binom{n}{k} \frac{(-)^k k!}{\Lambda_1^{k+1}} e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) + \frac{\partial^{n-k}}{\partial \Lambda^{n-k}} \\
 &\quad \{ D_1(\Lambda_2, \Lambda_3 - \Lambda_1) + D_1(\Lambda_3, \Lambda_2 - \Lambda_1) \} \quad (23)
 \end{aligned}$$

Applying the chain rule for differentiation to Eq. (20) for $r > 0$ we obtain

$$\frac{\partial^r}{\partial \Lambda_1^r} D_1(\Lambda_2, \Lambda_3 - \Lambda_1) = \frac{(r-1)!}{(\Lambda_3 - \Lambda_1)^r} \sum_{p=0}^{r-1} \frac{(\Lambda_3 - \Lambda_1)^p}{p!} \alpha_{p-1}(\Lambda_2 + \Lambda_3 - \Lambda_1) \quad (24)$$

where we have used

$$\frac{\partial^r}{\partial \Lambda_1^r} E_1((\Lambda_3 - \Lambda_1) x) = x^r \alpha_{r-1}((\Lambda_3 - \Lambda_1) x) \quad (25)$$

and ($n > 0$)

$$\alpha_n(z) = \int_1^\infty dt e^{-zt} t^n = \frac{n!}{z^{n+1}} e^{-z} e_n(z) \quad (26)$$

where

$$e_n(z) = \sum_{p=0}^n \frac{z^p}{p!} \quad (27)$$

(and $\alpha_{-1}(z) \equiv E_1(z)$). Substituting Eq. (24) into (23), the final expression for Eq. (3) becomes ($n \geq 0$)

$$D_n^{(3)}(\lambda_1, \lambda_2, \lambda_3) = \delta_{n,0} D_0^{(3)}(\lambda_1, \lambda_2, \lambda_3) + (1 - \delta_{n,0}) \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k k!}{\lambda_1^{k+1}} \left\{ e^{\lambda_1} E_1(\lambda_2) E_1(\lambda_3) + \delta_{n,k} \{ D_1(\lambda_2, \lambda_3 - \lambda_1) + D_1(\lambda_3, \lambda_2 - \lambda_1) \} \right. \\ \left. + \theta(n-k)(n-k-1)! \sum_{p=0}^{n-k-1} \frac{\alpha_{p-1}(\lambda_2 + \lambda_3 - \lambda_1)}{p!} \right\} \\ \frac{1}{(\lambda_3 - \lambda_1)^{n-k-p}} + \frac{1}{(\lambda_2 - \lambda_1)^{n-k-p}} \quad (28)$$

Next we consider the cases of Eq. (3) for $n < 0$. It turns out that the integral

$$D_{-1}^{(3)}(\lambda_1, \lambda_2, \lambda_3) \equiv \int_1^\infty dx \frac{e^{\lambda_1 x} E_1(\lambda_2 x) E_1(\lambda_3 x)}{x}, \quad \text{Re}(\lambda_2 + \lambda_3 - \lambda_1) > 0 \quad (29)$$

as it occurs in the deuteron quadrupole moment expressions cancels exactly. We consider, therefore, only those integrals for which $n < -1$,

$$D_n^{(3)}(\lambda_1, \lambda_2, \lambda_3) \equiv \int_1^\infty dx e^{\lambda_1 x} E_1(\lambda_2 x) E_1(\lambda_3 x), \quad |n| > 1 \quad (30)$$

Integrating Eq. (30) by parts ($n \neq -1$) yields

$$D_{-|n|}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = (|n| - 1) \{ e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) + D_{-(n+1)}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \\ - D_{|n|}(\Lambda_2 - \Lambda_1, \Lambda_3) - D_{-|n|}(\Lambda_3 - \Lambda_1, \Lambda_2) \} \quad (31)$$

Where $D_n(\Lambda_1, \Lambda_2)$ is defined by

$$D_n(\Lambda_1, \Lambda_2) \equiv \int_1^\infty dx \frac{e^{-\Lambda_1 x} E_1(\Lambda_2 x)}{x}, \quad \text{Re}(\Lambda_1 + \Lambda_2) > 0 \quad (32) \\ = D_{-|n|,1}^{(2)}(\Lambda_1, \Lambda_2)$$

In particular (for the deuteron quadrupole moment) we are interested in $n = -2$ in which case we obtain

$$D_{-2}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) + \Lambda_1 D_{-1}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \\ - D_2(\Lambda_2 - \Lambda_1, \Lambda_3) - D_2(\Lambda_3 - \Lambda_1, \Lambda_2) \quad (33)$$

But from Appendix 12 we have

$$D_2(\Lambda_1, \Lambda_2) = -\Lambda_1 D_1(\Lambda_1, \Lambda_2) + e^{-\Lambda_1} E_1(\Lambda_2) - E_2(\Lambda_1 + \Lambda_2) \quad (34)$$

which allows us to write Eq. (33) as

$$D_{-2}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) = e^{\Lambda_1} E_1(\Lambda_2) E_1(\Lambda_3) + \Lambda_1 D_{-1}^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \\ + (\Lambda_2 - \Lambda_1) D_1(\Lambda_2 - \Lambda_1, \Lambda_3) + (\Lambda_3 - \Lambda_1) D_1(\Lambda_3 - \Lambda_1, \Lambda_2) \\ - e^{-(\Lambda_2 - \Lambda_1)} E_1(\Lambda_3) - e^{-(\Lambda_3 - \Lambda_1)} E_1(\Lambda_2) \\ + 2 E_2(-\Lambda_1 + \Lambda_2 + \Lambda_3) \quad (35)$$

Finally we consider the integral for $D_n^{(4)}(\Lambda_1, \Lambda_2)$ defined by Eq. (4). For $n \geq 0$ if we integrate by parts we obtain

$$D_n^{(4)}(\Lambda_1, \Lambda_2) = \frac{1}{n+1} \left\{ -E_1(\Lambda_1) E_1(\Lambda_2) + D_{n,1}^{(2)}(\Lambda_1, \Lambda_2) + D_{n,1}^{(2)}(\Lambda_2, \Lambda_1) \right\} \quad (36)$$

Similarly for $n < -1$, integration by parts yields

$$D_{-n}^{(4)}(\Lambda_1, \Lambda_2) = (n-1) \left\{ E_1(\Lambda_1) E_1(\Lambda_2) - D_{|n|}(\Lambda_1, \Lambda_2) - D_{|n|}(\Lambda_2, \Lambda_1) \right\} \quad (37)$$

The case $n = -1$ is not considered because the terms involving

$D_{-1}^{(4)}(\Lambda_1, \Lambda_2, \Lambda_3)$ in the deuteron quadrupole moment cancel exactly.

We note that $D_n^{(4)}(\Lambda_2, \Lambda_3) = \lim_{\Lambda_1 \rightarrow 0} \{ D_n^{(3)}(\Lambda_1, \Lambda_2, \Lambda_3) \}$ and that Eq. (31)

and (37) satisfy this condition.

APPENDIX 14

ON THE EVALUATION OF THE INTEGRALS $h_l^\alpha(k)$ and $\hat{h}_l^\alpha(k)$

The integrals $h_l^\alpha(k)$ and $\hat{h}_l^\alpha(k)$ are defined as

$$\left. \begin{aligned} h_l^\alpha(k) &\equiv \frac{1}{k} \int_a^\infty dr F_l(kr) f_\alpha(r) \\ \hat{h}_l^\alpha(k) &\equiv \frac{1}{k} \int_a^\infty dr G_l(kr) f_\alpha(r) \end{aligned} \right\}, \quad (1)$$

where

$$f_\alpha(r) \equiv \lambda_\alpha^2 e^{-\lambda_\alpha(r-a)}, \quad (2)$$

and the spherical bessel functions are defined by

$$\rho^l \begin{Bmatrix} G_l(\rho) \\ F_l(\rho) \end{Bmatrix} \equiv \begin{Bmatrix} \text{Re} \\ \text{Im} \end{Bmatrix} (e^{i\rho} [\hat{\alpha}_l(\rho) - i\hat{\beta}_l(\rho)]), \quad (3)$$

with

$$\hat{\alpha}_l(\rho) - i\hat{\beta}_l(\rho) \equiv \sum_{s=0}^l \Gamma_{ls}(\rho) e^{-i(l-s)\pi/2}, \quad (4)$$

$$\Gamma_{ls}(\rho) \equiv \frac{(l+\frac{1}{2}, s)}{2^s} \rho^{l-s}, \quad (5)$$

$$(l+\frac{1}{2}, s) \equiv \frac{(l+s)!}{(l-s)!s!}. \quad (6)$$

Upon inserting Eqs. (2) through (4) into Eq. (1), and changing variables to $\tau \equiv ka$, $\mu \equiv \lambda a$, and $\rho = kr$, we obtain

$$h_l^\alpha(\tau) = \frac{\mu_\alpha^2}{\tau^2} \int_\tau^\infty \frac{d\rho}{\rho^l} \operatorname{Im} \left\{ e^{i\rho} \sum_{s=0}^l \Gamma_{ls}(\rho) e^{-i(l-s)\pi/2} e^{-(\frac{\mu_\alpha}{\tau}\rho - \mu_\alpha)} \right\} . \quad (7)$$

Upon setting $x = \rho/\tau = r/a$, we find

$$h_l^\alpha(\tau) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau^{l+1}} \operatorname{Im} \left\{ \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} \int_1^\infty \frac{dx e^{-(\mu_\alpha - i\tau)x}}{x^s} \right\} , \quad (8)$$

or

$$h_l^\alpha(\tau) = \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau^{l+1}} \operatorname{Im} \left\{ \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} E_s(\mu_\alpha - i\tau) \right\} \quad (9)$$

where the exponential integral $E_n(z)$ is defined by

$$E_n(z) \equiv \int_1^\infty \frac{dt}{t^n} e^{-zt}, \quad \operatorname{Re}(z) > 0, \quad n = 0, 1, 2, \dots \quad (10)$$

But from Eq. (1) of Appendix 2

$$Z_l(\mu_\alpha, \tau) \equiv \sum_{s=0}^l \Gamma_{ls}(\tau) e^{-i(l-s)\pi/2} E_s(\mu_\alpha - i\tau) , \quad (11)$$

so that Eq. (9) becomes

$$h_l^\alpha(\tau) = \frac{\mu_\alpha^{2l+1} e^{\mu_\alpha}}{\tau^{l+1}} \operatorname{Im} \left| Z_l(\mu_\alpha, \tau) \right| . \quad (12)$$

In a quite similar fashion, we find

$$\hat{h}_l^\alpha(\tau) = \frac{\mu_\alpha^{2l+1} e^{\mu_\alpha}}{\tau^{l+1}} \operatorname{Re} \left| Z_l(\mu_\alpha, \tau) \right| . \quad (13)$$

In the limit $\tau \rightarrow 0$, we have from Eqs. (42) of Appendix 2,

$$\operatorname{Im} \left| Z_l(\mu_\alpha, \tau) \right| \xrightarrow{\tau \rightarrow 0} \tau^{2l+1} \sum_{k=0}^{\infty} \frac{(-)^k \alpha_{l+2k+1}(\mu_\alpha) \tau^{2k}}{2^k k! (2k+2l+1)!!} , \quad (14)$$

and

$$\operatorname{Re} \left| Z_l(\mu_\alpha, \tau) \right| \xrightarrow{\tau \rightarrow 0} \left. \begin{aligned} & \sum_{p=0}^{\leq(l/2)} \frac{(2l-2p-1)!!}{2^p p!} E_{l-2p}(\mu_\alpha) \tau^{2p} \\ & + \sum_{p=0}^{\leq(\frac{l-1}{2})} \frac{(2p-1)!! \alpha_{l-2p}(\mu_\alpha) \tau^{2l-2p}}{2^{l-p} (l-p)!} \\ & + \sum_{k=1}^{\infty} \frac{(-)^k \alpha_{l+2k}(\mu_\alpha) \tau^{2l+2k}}{(2k-1)!! 2^{k+l} (k+l)!} \end{aligned} \right\} , \quad (15)$$

which, when substituted into Eqs. (12) and (13) gives

$$h_l^\alpha(\tau) \xrightarrow{\tau \rightarrow 0} \frac{(l+1)!!}{(2l+1)!!} (\tau/\mu_\alpha)^l \sum_{k=0}^{l+1} \frac{(\mu_\alpha)^k}{k!}, \quad (16)$$

$$\begin{aligned} \tilde{h}_l^\alpha(\tau) \xrightarrow{\tau \rightarrow 0} \frac{\mu_\alpha^2 e^{\mu_\alpha}}{\tau^{l+1}} \sum_{p=0}^{\leq(l/2)} \frac{(2l-2p-1)!!}{2^p p!} E_{l-2p}(\mu_\alpha) \tau^{2p} \\ \sum_{p=0}^{\leq(l-1)/2} \frac{(l-2p)!(2p-1)!! e_{l-2p}(\mu_\alpha) (\tau/\mu_\alpha)^{l-2p-1}}{2^{l-p} (l-p)!} + O(\tau^{l+1}) \\ = O(1/\tau^{l+1}). \end{aligned} \quad (17)$$

These limiting forms have the correct τ -dependence, as can be checked directly from Eqs. (1). In Eq. (17) we have used the definition

$$e_n(x) \equiv \sum_{p=0}^n \frac{(x)^p}{p!}, \quad (18)$$

and the convention

$$\left(\frac{l-1}{2}\right) \equiv \begin{cases} \frac{l-2}{2} & \text{if } l \text{ is even} \\ \frac{l-1}{2} & \text{if } l \text{ is odd.} \end{cases} \quad (19)$$

There is a constant term for $l = \text{odd}$ from the second term in Eq. (17).

Acknowledgments

The author wishes to express his gratitude and appreciation to Professor Franz Mohling for suggesting this investigation and for his continued encouragement; to Professor James Young for his many useful discussions; and, to all his professors and colleagues over the years who have taught and inspired him.

A debt of gratitude is also due the National Science Foundation and the University of Colorado for research and teaching assistantships in graduate school and to the Atomic Energy Commission under whose auspices this research was performed.