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Linear Theory of Parametric Instabilities in Plasma*

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ABSTRACT

In these lectures, the linear theory of parametric instabilities will be reviewed. The general dispersion relation will be derived from the Vlasov equation. The dispersion relation will be analyzed in terms of the linear dielectric function and its derivatives. The thresholds and growth rates for the decay instability, the purely growing mode, and the kinetic instability will be obtained.

The analysis will be applied to discuss some specific modes, such as the interaction of electron plasma waves and the ion acoustic waves, the upper hybrid and the lower hybrid modes, Bernstein waves and ion acoustic waves, and the decay of the whistler wave and the lower hybrid wave.

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INTRODUCTION

In this survey, our interest will be focused upon studying the coupling of a low frequency mode (ion mode) with frequency $\omega < \omega_{pi}$, to a relatively high frequency electron mode ($\omega \pm \omega_o$), due to the presence of an rf electric field, ω_o , in the plasma.

We shall see that electrostatic plasma waves may become unstable in such a situation, the energy being provided by the external electromagnetic field. Thus, this sort of process may provide a mechanism for absorbing the external field energy by the plasma in a rather efficient way, since the excited short wavelength oscillations would then rapidly heat the plasma particles by ordinary collisional absorption or Landau damping. On the other hand, the long wavelength electromagnetic waves would deposit their energy due to linear dissipative mechanisms rather slowly. Thus, we have here the possibility of enhanced absorption and heating of the plasma due to "anomalous" high frequency resistivity.

1. Derivation of the Dispersion Relation

Let us derive the general dispersion relation of electrostatic plasma waves in the presence of a strong, externally applied electric field. We shall consider only the simplest case, namely that of a uniform, homogeneous plasma immersed in a dc magnetic field, $\vec{B}_o = B_o \hat{z}$. We shall see later that the presence of a magnetic field can introduce a large variety of interesting modes. We shall also assume that the spatial variation of the external

electric field can be neglected (dipole approximation). Thus, the analysis is valid only for short wavelengths compared with that of the external rf "pump" field. We assume that the external electric field is of the form

$$\vec{E}_0(x, t) = (E_{0\perp} \hat{x} + E_{0\parallel} \hat{z}) \cos \omega_0 t \quad (1)$$

where E_0 is at an angle to the dc magnetic field, with $E_{0\perp}$ in the $\hat{x}$ direction of a right handed Cartesian coordinate system. The ion temperature is taken to be zero in this analysis.*

Let us begin with the Vlasov equation, which can be written in the following form:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_x f + \frac{q}{m} \left(\frac{\vec{v} \times \vec{B}}{c} + \vec{E} + \vec{E}_0 \cos \omega_0 t \right) \cdot \nabla_v f = 0 \quad (2)$$

where we used standard notation, with $q = \pm e$ for ions (electrons), e being the magnitude of electronic charge.

Let us first look at the equilibrium:

$$\frac{\partial f_0}{\partial t} + \vec{v} \cdot \nabla_x f_0 + \frac{q}{m} \left(\frac{\vec{v} \times \vec{B}}{c} + \vec{E}_0 \cos \omega_0 t \right) \cdot \nabla_v f_0 = 0 \quad (3)$$

The characteristics of this equation are given by:

$$\frac{d\vec{x}}{dt} = \vec{v} \quad ; \quad \frac{d\vec{v}}{dt} = \vec{v} \times \vec{\Omega} + \frac{q}{m} \vec{E}_0 \cos \omega_0 t \quad (4)$$

where $\Omega = qB/mc$ is the cyclotron frequency. The solutions of these orbit equations are the following:

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* An exception to this assumption is the inclusion of possible ion Landau damping in the damping of ion acoustic and ion cyclotron waves (see Appendix A).

$$v_x = u_x + \frac{q\omega_o E_{o\perp} \sin\omega_o t}{m(\omega_o^2 - \Omega^2)}$$

$$v_y = u_y + \frac{q}{m} \Omega \frac{E_{o\perp} \cos\omega_o t}{\omega_o^2 - \Omega^2} \quad (5)$$

$$v_z = u_z + \frac{qE_{o\parallel}}{m\omega_o} \sin\omega_o t$$

where u^2 is a constant of the motion, and the particular solutions giving the drift terms have been written out explicitly. We note that due to the presence of the external electric field the velocity components oscillate in time with frequency ω_o . This oscillation causes displacements of the particles which are given by the integrals of Eq. (5), namely

$$x_d = - \frac{q}{m} \frac{E_{o\perp} \cos\omega_o t}{\omega_o^2 - \Omega^2}$$

$$y_d = \frac{\Omega}{\omega_o} \frac{q}{m} \frac{E_{o\perp} \sin\omega_o t}{\omega_o^2 - \Omega^2} \quad (6)$$

$$z_d = - \frac{q}{m\omega_o} E_{o\parallel} \cos\omega_o t$$

where we gave only the particular solution, and the sub d in the foregoing expressions refer to "drift." We see that the rf drift velocities and displacements of electrons and ions will be

different due to their different masses. Thus, even in equilibrium charge neutrality will not be satisfied at all times, and under suitable conditions instability can result from such free energy sources.

Let us now consider perturbations about such an equilibrium. The linearized Vlasov equation can be written as follows:

$$\frac{\partial f^{(1)}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^{(1)} + \frac{q}{m} \left(\frac{\vec{v} \times \vec{B}}{c} + E_0 \cos \omega_0 t \right) \cdot \nabla_{\mathbf{v}} f^{(1)} = - \frac{q}{m} \vec{E}^{(1)} \cdot \nabla_{\mathbf{v}} f_0 \quad (7)$$

where $f_0 = f_0(\mathbf{v} - \mathbf{v}_d)$, $\mathbf{v}_d$ being the drift parts of Eq. (5) and where the superscript (1) refers to first order quantities. To expedite solution, we introduce the Ansatz

$$f_k^{(1)}(\vec{r}, \vec{v}, t) = F(\vec{r}, \vec{u}, t) \exp(-i\vec{k} \cdot \vec{x}_d) \quad (8)$$

where we considered only the $\vec{k}$ -th Fourier component, since we are using the dipole approximation ($\vec{k}_0 = 0$). Note that this approximation is essential in this whole treatment, and solutions with arbitrary finite $\vec{k}_0$ are still being worked on by theorists.

Note that the Ansatz (8) is essentially a transformation to the oscillating frame of reference in which there is no induced drift.

Thus, transforming to the $\vec{u}$ variables, Eq. (7) becomes

$$\frac{\partial F}{\partial t} + \vec{u} \cdot \nabla_{\mathbf{x}} F + (\vec{u} \times \vec{\Omega}) \cdot \nabla_{\mathbf{u}} F = - \frac{q}{m} E_k^{(1)} \cdot (\nabla_{\mathbf{u}} f_0) e^{i\vec{k} \cdot \vec{x}_d} \quad (9)$$

where we can write $\vec{k} \cdot \vec{x}_d$ in the following form:

$$\vec{k} \cdot \vec{x}_d = \mu \sin(\omega_0 t - \beta) \quad (10)$$

where

$$\mu = \frac{q}{m} \left[\left(\frac{E_{0z} k_z}{\omega_0^2} + \frac{E_{0x} k_x}{\omega_0^2 - \Omega^2} \right)^2 + \frac{(E_{0x} k_y)^2 \Omega^2}{(\omega_0^2 - \Omega^2)^2 \omega_0^2} \right]^{1/2} \quad (11)$$

and β is an angle which depends on the various parameters contained in (8). We shall, however, not need the exact form of β here since it represents a phase angle of some sort which cancels out at the end. The important point to note is that Eq. (10) is a harmonic function, and that Eq. (9) is very similar to the homogeneous Vlasov equation, with the exception of the term $\exp(i\vec{k} \cdot \vec{x}_d)$ on the right-hand side.

The solution of Eq. (9) is obtained immediately by the method of characteristics, i.e.,

$$F(r, u, t) = \frac{q}{m} \int_{-\infty}^t dt' \nabla_{\mathbf{x}'} \phi(r', t') \cdot \nabla_{\mathbf{u}'} f_0 e^{i\mu \sin(\omega_0 t - \beta)} \quad (12)$$

where we used $\vec{E} = -\vec{\nabla}\phi$ and where the characteristics are the well known orbits in a magnetic field:

$$\begin{aligned} x' &= x - (u_{\perp}/\Omega) [\sin(\Omega\tau + \alpha) - \sin\alpha] \\ y' &= y + (u_{\perp}/\Omega) [\cos(\Omega\tau + \alpha) - \cos\alpha] \\ z' &= z + u_{\parallel}\tau \quad ; \quad \tau = t - t' \end{aligned} \quad (13)$$

By expanding in a Bessel series

$$\exp[i\mu \sin(\omega_0 t - \beta)] = \sum_{n=-\infty}^{\infty} J_n(\mu) \exp[in(\omega_0 t - \beta)] \quad (14)$$

the integration of Eq. (12) is straightforward, the main effect being the appearance of the beat frequency $n\omega_0$ in the susceptibility (we shall leave the integration as an exercise).

Now note that the charge density is given by the integral

$$\rho = q \int du F(\mathbf{r}, u, t)$$

We see that because of the oscillating exponential in Eq. (12), the charge density also exhibits oscillations with beat frequency $n\omega_0$ [see Eq. (12)]. Furthermore, since the electrons and ions oscillate by different amounts, the charge density of electrons and ions oscillates by different amounts, i.e., we obtain the earlier mentioned charge separation.

Introducing now the Fourier transforms

$$\phi(\vec{r}, t) = \int d\omega \phi(\vec{k}, \vec{\omega}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$F(\vec{r}, \vec{u}, t) = \int d\omega F(\vec{k}, \vec{u}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

we have

$$\rho(\vec{k}, \omega) = q \int d^3u F(\vec{k}, \vec{u}, \omega)$$

The charge density as obtained from the foregoing integrations can be written as follows:

$$\rho_0(\mathbf{k}, \omega) = - (k^2 / 4\pi) \sum_n J_n(\mu) e^{-in\beta} \chi_0(\omega, k) \phi(\omega + n\omega_0, k) \quad (15)$$

where $\chi_0(\omega, k)$ is just the linear susceptibility. For Poisson's equation

$$\nabla^2 \phi = - 4\pi \sum_{\sigma} q \int f^{(1)} d^3v$$

we obtain

$$\phi(k, \omega) = (4\pi/k^2) \sum_{\sigma} \sum_n J_n(-\mu) e^{-in\beta} \rho_{\sigma}(\omega + n\omega_0, k) \quad (16)$$

The problem now is essentially solved. The dispersion relation is obtained by substituting Eq. (15) into Eq. (16). Unfortunately, we obtained an infinite set of equations of order n . In order to make progress, we shall consider sufficiently weak electric fields so that $\mu < 1$, and then we can concentrate only on the modes ω , $(\omega \pm \omega_0)$, as shown in Fig. 1. Furthermore, if we assume that $(\omega, \vec{k})$ is an "ion" mode, and that $|\omega \pm \omega_0| \gg \omega$ and that $[(\omega \pm \omega_0), \vec{k}]$ is an electron mode, we can easily solve the set of three coupled equations to yield the following dispersion relation:

$$\frac{\epsilon_i(\omega)}{\chi_i(\omega)} = J_0^2(\mu_e) \frac{\chi_e(\omega)}{\epsilon_e(\omega)} + J_1^2(\mu_e) \left(\frac{\chi_e(\omega + \omega_0)}{\epsilon_e(\omega + \omega_0)} + \frac{\chi_e(\omega - \omega_0)}{\epsilon_e(\omega - \omega_0)} \right) \quad (17)$$

where we defined $\epsilon_j(\omega) = 1 + \chi_j(\omega)$. (The argument $\vec{k}$ is understood and has been suppressed.) We remark that Eq. (17) is valid only if $\epsilon(\omega \pm \omega_0)$ is dependent only on the electron contribution to the "side band" charge density. If the pump frequency is near some mode which also depends on ions then additional terms due to ions must also be retained. We shall say a few words about this later.

In order to make further progress, we shall now consider the case of weak fields, i.e., $\mu < 1$ so that the Bessel functions can be expanded in Eq. (17). Thus, using the expansions $J_0(\mu) \approx 1 - \mu^2/4$, $J_1(\mu) \approx \mu/2$, we can rewrite Eq. (17) in the following form:

$$\epsilon(\omega) + \frac{\mu^2}{4} \chi_i(\omega) \left(\frac{\epsilon_e(\omega)}{\epsilon_e(\omega - \omega_0)} + \frac{\epsilon_e(\omega)}{\epsilon_e(\omega + \omega_0)} - 2 \right) = 0 \quad (18)$$

which is our final dispersion relation. Note that here we used $\epsilon(\omega) = 1 + \chi_e(\omega) + \chi_i(\omega)$, $\epsilon_e(\omega \pm \omega_0) = 1 + \chi_e(\omega \pm \omega_0)$.

2. Solutions for Thresholds and Growth Rates

a. Decay Instability

We shall now show that Eq. (18) predicts three different types of instabilities, and the growth rates and threshold field strengths will be obtained for a number of modes of current interest. We note that the growth rate and the threshold is the same for both side bands and the low frequency mode. Depending on the relative values of the damping rates, different thresholds are obtained. In particular, if the damping rates γ_2 of the high frequency modes is larger than the acoustic frequency, both upper and lower side bands may be on resonance simultaneously, and both terms must be retained. As we can guess, the upper side band reduces the available energy for driving unstable the lower side band and the acoustic wave. (This can be seen by noting that the energy $U \propto \omega$, $|\omega \pm \omega_0|$.) In what follows we shall assume that $\gamma_2 < \omega_A$, and so it is sufficient to consider the interaction of only the lower side band and the acoustic wave, i.e., we shall be interested in the case when

$$\epsilon(\omega) \approx \epsilon_e(\omega - \omega_0) \approx 0$$

Thus, Eq. (18) reduces to

$$\epsilon(\omega) + \frac{\mu^2}{4} \chi_i(\omega) \frac{\epsilon_e(\omega)}{\epsilon_e(\omega - \omega_0)} \approx 0. \quad (19)$$

Let us now expand Eq. (19) about the resonant mode ω_1 ,

$$\epsilon(\omega) \approx \epsilon(\omega_1) + \delta\omega \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} \quad (20)$$

where $\epsilon(\omega_1) = 0$.

$$\epsilon_e(\omega - \omega_0) \approx \epsilon_e(\omega_2) - \delta\omega \frac{\partial \epsilon_e}{\partial \omega_2}(\omega_2) \quad (21)$$

where $\epsilon_e(\omega_1 - \omega_0) \approx \epsilon_e(\omega_0) - \omega_1 (\partial \epsilon_e / \partial \omega_0) \approx 0$ and $\omega_2 = \omega_0 - \omega_1$, i.e., the side band is resonant simultaneously with the acoustic mode.

Then (19) becomes

$$\delta\omega \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} = \frac{\mu^2}{4} \frac{\chi_i(\omega_1) \epsilon_e(\omega_1)}{\delta\omega \frac{\partial \epsilon_e(\omega_2)}{\partial \omega_2}}$$

or

$$(\delta\omega)^2 = \frac{\mu^2 \chi_i(\omega_1) \epsilon_e(\omega_1)}{4 \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} \frac{\partial \epsilon_e(\omega_2)}{\partial \omega_2}} \quad (22)$$

Now since usually $\chi_i(\omega)$ is negative, and $\partial \epsilon / \partial \omega_1$ is positive (assuming positive energy waves) we see that Eq. (22) predicts instability since $\delta\omega$ is imaginary. Note that the coupling depends on finite $\chi_i(\omega)$, i.e., the ion susceptibility.

Let us now take the linear damping of the modes into account.

Assume that the low frequency mode in the absence of interaction is damped by damping rate γ_1 , and that the high frequency mode is damped by γ_2 . Then for Eq. (22) we obtain

$$(\delta\omega + i\gamma_1)(\delta\omega + i\gamma_2) = \frac{\mu^2 \chi_i(\omega_1) \epsilon_e(\omega_1)}{4 \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} \frac{\partial \epsilon_e(\omega_2)}{\partial \omega_2}} \quad (23)$$

Thus, for minimum threshold we have $\delta\omega = 0$ and

$$- \frac{\mu^2 \chi_i(\omega_1) \epsilon_e(\omega_1)}{4 \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} \frac{\partial \epsilon_e(\omega_2)}{\partial \omega_2}} > \gamma_1 \gamma_2 \quad (24)$$

Note that in all the foregoing treatment we assumed that the condition of exact matching of frequencies and resonances can be satisfied. In the case of mismatch of frequencies, the threshold is generally increased, and exists even in the absence of linear damping mechanisms. The expressions become more complicated, and the reader is referred to literature. (See the list of references provided at the end.)

b. The Purely Growing Instability

Here we shall be interested in the case where $\omega = 0$. Before discussing this instability, we shall introduce the susceptibilities for the low frequency mode. In particular, for acoustic modes (i.e., $\omega_1/k v_{te} \ll 1$) at frequencies above the ion cyclotron frequency (or for acoustic waves in the absence of magnetic field) we have:

$$\cos\theta = k_{||}/k ; \chi_e \approx 1/k^2 \lambda_D^2 ; \chi_i = -\omega_{pi}^2/\omega^2 \quad (25)$$

where $\lambda_D = v_t/\sqrt{2}\omega_{pe}$ is the electron Debye length, and ω_{pi} is the ion plasma frequency. Thus, from Eq. (18), we write:

$$1 - \frac{\omega_{pi}^2}{(\delta\omega)^2} + \frac{1}{k^2\lambda_D^2} = \frac{\mu^2}{4} \frac{\omega_{pi}^2}{(\delta\omega)^2} \frac{1}{k^2\lambda_D^2} \left(\frac{1}{\epsilon_R(\omega_0) + i\epsilon_I(\omega_0)} + \frac{1}{\epsilon_R(\omega_0) - i\epsilon_I(\omega_0)} \right) \quad (26)$$

where $\eta = \cos^2\theta$ for $\delta\omega \approx 0$, $\eta = 1$ for $|\delta\omega| > \Omega_i$ and where we used the approximation $k^2\lambda_D^2 \ll 1$, and that for most modes of interest $\epsilon_R(+\omega_0) = \epsilon_R(-\omega_0)$ and $\epsilon_I(-\omega_0) = -\epsilon_I(\omega_0)$ (where R, I refer to the real and imaginary parts, respectively). Here we shall retain small but finite values of $\epsilon_R(\omega_0)$. In particular, we shall find that $\epsilon_R(\omega_0) < 0$ will be necessary for instability. Rewriting Eq. (26), at threshold $\delta\omega = 0$ and we obtain:

$$\frac{\mu^2}{2} \frac{\epsilon_R}{\epsilon_R^2 + \epsilon_I^2} + k^2\lambda_D^2 \quad (27)$$

or

$$\mu^2 = -2k^2\lambda_D^2 \frac{(\epsilon_R^2 + \epsilon_I^2)}{\epsilon_R}$$

We see from Eq. (27) that we need $\epsilon_R(\omega_0) < 0$ at threshold. The minimum threshold is given for $\epsilon_R(\omega_0) = -\epsilon_I(\omega_0)$. [This can be found directly by differentiating Eq. (27) with respect to $\epsilon_R(\omega_0)$.] Thus, the threshold is given by

$$\mu_t^2 = 4k^2\lambda_D^2\epsilon_I(\omega_0) \quad (28)$$

We note that this threshold would be independent of ion collisions (i.e., for ion collisions $\chi_i(\omega) = -\omega_{pi}^2/\omega(\omega + i\nu_i)$, and for $\omega \rightarrow 0$ ion collisions cancel out).

To obtain the maximum growth rate, we neglect damping and use the expansion $\epsilon_R(\omega \pm \omega_0) = \epsilon_R(\omega_0) \pm \delta\omega \partial\epsilon(\omega_0)/\partial\omega_0$, hence, instead of Eq. (26) we have for $|\delta\omega| > \Omega_i$

$$-\frac{\omega_A^2}{(\delta\omega)^2} + 1 = \frac{\mu^2}{4} \frac{\omega_{pi}^2}{(\delta\omega)^2} \left(\frac{2\epsilon_R}{\epsilon_R^2 - (\delta\omega)^2 (\partial\epsilon_R/\partial\omega_0)^2} \right) \quad (29)$$

where $\epsilon_R = \epsilon_R(\omega_0)$, and $\omega_A^2 = k^2 v_A^2 = k^2 \lambda_D^2 \omega_{pi}^2 = k^2 kT_e/M_i$.

Rearranging Eq. (29), we have

$$\left[(\delta\omega)^2 - \omega_A^2 \right] \left[\epsilon_R^2 - (\delta\omega)^2 \left(\frac{\partial\epsilon_R}{\partial\omega_0} \right)^2 \right] = \frac{\mu^2}{2} \epsilon_R \omega_{pi}^2 \quad (30)$$

We can now consider a number of different limits, whether $(\delta\omega)^2$ is larger or smaller than ω_A^2 . For $(\delta\omega)^2 < \omega_A^2$,

$$(\delta\omega)^2 = \frac{\mu^2}{2} \frac{\epsilon_R}{\left(\frac{\partial\epsilon_R}{\partial\omega_0} \right)^2 k^2 \lambda_D^2} + \epsilon_R^2 / \left(\frac{\partial\epsilon_R}{\partial\omega_0} \right)^2 \quad (31)$$

and after differentiating with respect to ϵ_R , the maximum growth rate is found to be at $\epsilon_R = -\mu^2/4k^2\lambda_D^2$ and is given by

$$\gamma = \frac{\mu^2}{4k^2\lambda_D^2 (\partial\epsilon_R/\partial\omega_0)} \quad (32)$$

On the other hand, for $(\delta\omega)^2 \gg \omega_A^2$, we obtain the maximum growth rate for $\epsilon_R = \mu^2 \omega_{pi}^2 / 4(\delta\omega)^2$ (differentiate Eq. (30) with respect to ϵ_R) and is given by

$$\gamma = \left(\frac{\mu^2 \omega_{pi}^2}{4(\partial\epsilon/\partial\omega_0)} \right)^{-1/3} \quad (33)$$

which is very similar to the usual beam-plasma instability growth rate. In particular, for beam plasma instabilities the dispersion is given by

$$\epsilon_R(\omega) = \frac{\omega_{pb}^2}{(\omega - kv_b)^2}$$

(where ω_{pb}^2 is the beam plasma frequency) and for $\omega \approx kv_b$ and $\epsilon_R(kv_b) \approx 0$ we have

$$\delta\omega \left. \frac{\partial \epsilon_R}{\partial \omega} \right|_{\omega=kv_b} = \frac{\omega_{pb}^2}{(\delta\omega)^2}$$

or

$$(\delta\omega) = \left(\frac{\omega_{pb}^2}{(\partial \epsilon_R / \partial \omega)} \right)^{1/3} \Big|_{\omega=kv_b} \quad (34)$$

Thus, by identifying $\omega_{pb}^2 \rightarrow (\mu^2 \omega_{pi}^2 / 4)$, the growth rates are the same. Hence, we see that in this limit the purely growing mode takes on the appearance of a beam like instability, and hence this instability is also called the "oscillating two-stream" instability. In particular, the oscillating ions move through the electrons as if they formed a beam.

c. The Kinetic Instability

This instability depends on the second derivative of the distribution function, and hence it is called "kinetic" instability. In order to show the essential points of this instability, which is mainly of interest in the case of intermediate and high

strength fields (i.e., $\mu \approx 1$) and underdense plasmas, we shall look only at the simplest case. In particular, we shall assume that $(\omega \pm \omega_0)$ is considerably above the resonance frequency of ϵ_R , and thus $\epsilon_R(\omega_0) \approx 1$. We may expand

$$\epsilon_e(\omega \pm \omega_0) \approx \epsilon_R(\omega_0) \pm \omega \frac{\partial \epsilon_R}{\partial \omega_0} \pm i \epsilon_I(\omega_0) + i \omega \frac{\partial \epsilon_I}{\partial \omega_0}$$

where we used the symmetry relationships of the dielectric function of an even distribution function. Now assume that, $\omega(\partial \epsilon_R / \partial \omega_0)$ is negligibly small. Then if the acoustic wave is near resonance at $\omega = \omega_1 = k v_A$, then Eq. (18) becomes

$$\delta \omega \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} = - \frac{\mu^2 \chi_i(\omega)}{4} \left(\frac{\epsilon_e(\omega_1)}{1 + i \epsilon_I + i \omega \frac{\partial \epsilon_I}{\partial \omega_0}} + \frac{\epsilon_e(\omega_1)}{1 - i \epsilon_I + i \omega \frac{\partial \epsilon_I}{\partial \omega_0}} - 2 \right)$$

or for $\epsilon_I \ll 1$, $\omega_1(\partial \epsilon_I / \partial \omega_0) \ll 1$, $\chi_i(\omega_1) = -\omega_{pi}^2 / \omega_1^2$, $\epsilon_e = 1 / k^2 \lambda_D^2$, we obtain

$$\delta \omega \frac{\partial \epsilon(\omega_1)}{\partial \omega_1} = - \frac{\mu^2 \omega_{pi}^2}{4 \omega_1^2} \left(\frac{2 i \omega (\partial \epsilon_I / \partial \omega_0)}{k^2 \lambda_D^2} - 2 \right) \quad (35)$$

Thus, we see that $\delta \omega$ picks up an imaginary part due to $\partial \epsilon_I / \partial \omega$. Now since $\partial \epsilon_I / \partial \omega_0 \propto (\partial^2 f_0 / \partial v^2)$ is negative for a number of distribution functions (in particular for a Maxwellian) instability results with growth rate

$$\frac{\gamma}{\omega_1} \approx - \frac{\mu^2}{2 k^2 \lambda_D^2} \left(\frac{\omega_{pi}^2}{\omega_1^2} \right) \frac{(\partial \epsilon_I / \partial \omega_0)}{(\partial \epsilon / \partial \omega_1)} \quad (36)$$

To finish this section, we give $\partial \epsilon_I / \partial \omega_0$ for a Maxwellian distribution:

$$\frac{\partial \epsilon_I}{\partial \omega_0} \approx - \frac{\sqrt{\pi}}{k^2 \lambda_D^2} \sum_{p=-\infty}^{\infty} \left(\frac{2\omega_0 (\omega_0 - p\Omega)}{k_{\parallel}^2 v_t^2} - 1 \right) \frac{I_p(b) e^{-b}}{k_{\parallel} v_t} \exp\left(-\frac{(\omega_0 - p\Omega)^2}{k_{\parallel}^2 v_t^2}\right)$$

where $b = k_{\perp}^2 v_t^2 / 2\Omega^2$, with $v_t^2 = 2kT/m$. Thus, instability results for $2\omega_0 (\omega_0 - p\Omega) / k_{\parallel}^2 v_t^2 > 1$.

d. Explicit Results for Modes of Interest

In order to illustrate the previously obtained results, we shall consider here some cases of interest in recent experimental research. We shall study only the decay instabilities and the purely growing mode. The thresholds will be obtained in explicit forms. The growth rates well above threshold for the decay instability are obtained from the threshold formulas by replacing $\gamma_1 \gamma_2$ with γ^2 , (where γ is now the growth rate). For the purely growing mode the growth rates are obtained by simply substituting the formulas for μ and $\partial \epsilon / \partial \omega_2$ given below into Eqs. (32) and (33).

i) Electron plasma waves, $B = 0$ (or $E_0 \parallel B$).

In this case, for $k^2 \lambda_D^2 \ll 1$,

$$\epsilon_R(\omega_0) = 1 - \frac{\omega_{pe}^2}{\omega_0^2} ; \quad \epsilon(\omega) = 1 + \frac{1}{k^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2} ; \quad (37)$$

where

$$\chi_e(\omega) = 1/k^2 \lambda_D^2, \quad \chi_i(\omega) = -\omega_{pi}^2/\omega^2, \quad \text{and}$$

$$\epsilon(\omega_1) = 0 \quad \text{gives}$$

$$\omega_1^2 = \frac{k^2 v_A^2}{1+k^2 \lambda_D^2} = \frac{\omega_A^2}{1+k^2 \lambda_D^2} \quad (38)$$

where $v_A^2 = \kappa T_e/m_i$ is the acoustic speed. We also have

$$\partial \epsilon / \partial \omega_1 = 2/(k^2 \lambda_D^2 \omega_1), \quad \partial \epsilon_R / \partial \omega_0 = 2\omega_{pe}^2/\omega_0^3, \quad \mu = eE_0 k/m\omega_0^2,$$

and the threshold for the decay instability is

$$\gamma_1 \gamma_2 \leq \frac{\frac{e^2 E_0^2 k^2}{4m^2 \omega_0^4} \left(\frac{1}{k^2 \lambda_D^2} \right)^2}{\frac{2\omega_{pe}^2}{\omega_0^3} \frac{2\omega_{pi}^2}{(k\lambda_D \omega_{pi})^3}}$$

or after reducing, and using $\omega_0 \approx \omega_{pe}$, we get

$$\sqrt{\frac{\gamma_1 \gamma_2}{\omega_A \omega_0}} \leq \frac{E_0}{8\sqrt{\pi n_0 \kappa T}} \quad (39)$$

The threshold for the purely growing mode is given in terms of the damping rate γ_2

$$\epsilon_I = \text{Im} \left(1 - \frac{\omega_{pe}^2}{(\omega_0 + i\gamma_2)^2} \right) = \frac{2i\gamma_2}{\omega_0} \frac{\omega_{pe}^2}{\omega_0^2}$$

and for $\omega_0 \approx \omega_{pe}$, the threshold given by Eq. (28) is

$$\frac{eE_0 k}{m\omega_{pe}^2} \gg 2k\lambda_D \sqrt{\frac{2\gamma_2}{\omega_{pe}}}$$

or

$$\frac{eE_0}{m\omega_0 v_t} \gg \sqrt{\frac{4\gamma_2}{\omega_{pe}}} \quad (40)$$

where $v_t = \sqrt{2kT_e/m_e}$. We see that the left hand side is just the ratio of the rf drift velocity to the electron thermal velocity. Since in typical laboratory experiments $\gamma_2/\omega_{pe} \sim 10^{-4}$ to 10^{-3} , we see that the threshold drift velocity is only a few percent of the electron thermal velocity.

ii) Electron plasma waves in a magnetic field.

The dispersion relation is

$$\epsilon(\omega_0) = \frac{k_n^2}{k^2} \left(1 - \frac{\omega_{pe}^2}{\omega_0^2}\right) + \frac{k_1^2}{k^2} \left(1 - \frac{\omega_{pe}^2}{\omega_0^2 - \Omega^2}\right) \quad (41)$$

where $k_1^2 = k_x^2 + k_y^2$, $k^2 = k_n^2 + k_1^2$. Thus, $\epsilon(\omega_2) = 0$ gives

$$\omega_2^2 = \frac{\omega_p^2 + \Omega^2 \pm \sqrt{(\omega_p^2 + \Omega^2)^2 - 4\omega_p^2 \Omega^2 \cos^2 \theta}}{2} \quad (42)$$

for which we obtain

$$\frac{\partial \epsilon}{\partial \omega_2} = \frac{2}{\omega_2} \left(1 + \frac{\sin^2 \theta}{(\omega_2^2 - \Omega^2)^2} \frac{\omega_p^2 \Omega^2}{1}\right) \quad (43)$$

where we used $k_{||}/k = \cos \theta$, $k_{\perp}/k = \sin \theta$. Equation (42) predicts waves propagating for $\omega_0 \gtrsim \omega_{LH}$. The ion acoustic waves retain the same form as in Eq. (38) in the previous section. The imaginary part of ϵ in terms of the damping rates is given by

$$\epsilon_I = \gamma \frac{\partial \epsilon}{\partial \omega_2}$$

Thus, the threshold for the decay instability becomes

$$\sqrt{\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2}} \leq \frac{\mu}{4k\lambda_D \sqrt{1 + \sin^2 \theta \omega_p^2 \Omega^2 / (\omega_2^2 - \Omega^2)^2}} \quad (44)$$

The threshold for the purely growing mode is

$$\mu \geq 2 \cdot k\lambda_D \sqrt{\frac{2\gamma}{\omega_2} \left(1 + \frac{\sin^2 \theta \omega_p^2 \Omega^2}{(\omega_0^2 - \Omega^2)^2} \right)} \quad (45)$$

In general these thresholds depend on the type of pump field, frequencies and angles in a rather complex way. However, the thresholds obtained are often lower than that of the magnetic field free case. A particularly simple form can be obtained for $\omega_0 \ll \Omega$, $E_0 \approx E_{0\perp}$. In this case (which may be of interest for heating fusion devices because of the low frequencies) μ reduces to

$$\mu = \frac{qE_{0x} k_y}{m\Omega \omega_0} \quad (46)$$

i.e., $E_{01} \perp k_y$ (i.e., in a cylindrical geometry for $E_{ox} \rightarrow E_{or}$, $k_y \rightarrow k_\theta$, or vice versa). We see that in this approximation $\mu \approx (cE/B_0)/(\omega_0/k_y)$, i.e., μ is the ratio of the $\vec{E} \times \vec{B}$ drift velocity of electrons to the perpendicular phase velocity. In this approximation for the threshold of the decay instability we obtain from Eq. (44) (using $V_0 = cE_0/B$ and $v_t = \sqrt{2kT_e/m_e}$),

$$\frac{\partial \epsilon}{\partial \omega_2} = \frac{2}{\omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \quad (47a)$$

and

$$\frac{V_0}{v_t} \geq \frac{4\omega_0}{\sqrt{2}\omega_{pe}} \left[\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \quad (47b)$$

For the purely growing mode, from Eq. (45) we obtain

$$\frac{V_0}{v_t} \geq \frac{2\omega_0}{\omega_{pe}} \left[\frac{\gamma_2}{\omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \quad (48)$$

where $\omega_0 \approx \omega_2$. These formulas appear to give a reduced threshold by the factor of $\omega_0/\omega_{pe} \ll 1$ if $\omega_{pe} > \omega_{ce}$. However, one must be careful in concluding much lower thresholds since the damping rate γ_2/ω_2 is increased as ω_2 is decreased, and thus to some extent such gains are canceled. Nevertheless, in general Eqs. (47) and (48) predict fairly low thresholds.

iii) Lower hybrid waves.

One may consider lowering ω_o all the way down to the lower hybrid frequency, for which dielectric function must be modified in Eq. (18). In particular, it can be shown that in the limit $\mu^2 < 1$, $k^2 \lambda_D^2 \ll 1$, the only modification to Eq. (18) is that the ion susceptibility must also be added to $\epsilon_e(\omega \pm \omega_o)$. This is to be expected, since we are considering the case when $(\omega \pm \omega_o) \rightarrow \omega_{LH}$ and ions also contribute to the wave motion. Thus, instead of Eq. (41), for the side band dispersion relation we use

$$\epsilon(\omega_o) = \frac{k_n^2}{k^2} \left(1 - \frac{\omega_{pe}^2}{\omega_o^2} \right) + \frac{k_i^2}{k^2} \left(1 - \frac{\omega_{pi}^2}{\omega_o^2 - \Omega_i^2} \right) - \frac{\omega_{pi}^2}{\omega_o^2} \quad (49)$$

Thus, $\epsilon(\omega_2) = 0$ gives

$$\omega^2 = \omega_{LH}^2 \left(1 + \frac{k_{||}^2}{k^2} \frac{m_i}{m_e} \right) \quad (50)$$

where ω_{LH} is the lower hybrid frequency, and is given by

$$\omega_L^2 = \frac{\omega_{pi}^2}{1 + (\omega_{pi}^2 / \Omega_e \Omega_i)} \quad (51)$$

Now it is easy to show from (49) that $(\partial \epsilon / \partial \omega_2)$ is given by exactly the same expression as before, i.e., Eqs. (43) and (47). Thus, the threshold for frequencies approaching the lower hybrid frequency [i.e., $1 < (k_{||}^2 / k^2) (m_i / m_e)$] still retain the same as the electron plasma wave case (i.e., we just have a modified electron plasma wave). However, this theory breaks down for

$(k_{\parallel}^2/k^2) (m_i/m_e) \lesssim 1$, the criterion for true lower hybrid waves.

In particular, we may write for the lower hybrid wave condition

$$\xi = \frac{k_{\parallel}^2}{k^2} \frac{m_i}{m_e} = \frac{k_{\parallel}^2}{k^2} \frac{\kappa T_e}{m_e} \frac{m_i}{\kappa T_e} = \frac{k_{\parallel}^2}{k^2} \frac{v_{te}^2}{v_A^2} = \frac{k_{\parallel}^2 v_{te}^2}{\omega_1^2} < 1 \quad (52)$$

But this means that the condition for ion acoustic waves, i.e., $\omega_1/k_{\parallel} v_{te} < 1$, is violated. Thus, we cannot satisfy simultaneously the condition for coupling lower hybrid waves and ion acoustic waves in the dipole approximation. In fact, Eq. (52) says that the low frequency wave becomes a lower hybrid wave. Thus, we have the case of decay into two lower hybrid waves. In this case it is doubtful that we can ignore the pump wave number, and thus the present theory is not valid (i.e., we need nonlinear mode-mode coupling theory).

iv. Bernstein waves, $\omega \approx n\Omega$ (upper hybrid waves).

These waves may decay into ion acoustic waves for

$\omega_1 < k_{\parallel} v_{te}$ and $k_{\parallel}^2 \ll k_1^2$, or lower hybrid waves for $\omega_1 > k_{\parallel} v_{te}$ and $k_{\parallel}^2 \ll k_1^2$. We take $E_0 \approx E_{01}$, and thus

$$\mu \approx \frac{e}{m} \frac{E_1 k_1}{\omega_0^2 - \Omega^2}$$

where we took $k_1 \approx k_x$, and k_y will be ignored. In this case the dielectric is obtained from Eq. (41), where we may drop the $k_{\parallel}$ terms, and thus

$$\epsilon(\omega_0) \approx 1 - \frac{\omega_{pe}^2}{\omega_0^2 - \Omega^2} \quad (53)$$

and $\epsilon(\omega_2) = 0$ gives $\omega_u^2 = \omega_{pe}^2 + \Omega^2$ (the upper hybrid frequency) in the cold plasma approximation. It can be shown that for $b = k_1^2 v_t^2 / 2\Omega_e^2 < 1$, with the hot plasma correction the dielectric is

$$\epsilon(\omega_o) = 1 - \frac{\omega_p^2}{\Omega^2} \left(\frac{\Omega^2(1-b)}{\omega_o^2 - \Omega^2} + \frac{b(1-b)\Omega^2}{\omega_o^2 - 4\Omega^2} \right) \quad (54)$$

where we have both the first and second harmonic terms. Let us assume $\omega_p < \Omega$, and then the upper hybrid frequency is below the second harmonic. Thus, for the upper hybrid wave we have

$$\omega_2^2 \approx \Omega^2 + \omega_p^2(1-b) \quad (55)$$

and for the second harmonic we have

$$\omega_2^2 \approx 4\Omega^2 + \omega_p^2 b \quad (56)$$

Furthermore, for the upper hybrid wave we have

$$\epsilon_I(\omega_o) = + \frac{\omega_p^2}{\Omega^2} \frac{\Omega^2(1-b)}{(\omega_o^2 - \Omega^2)^2} \quad (2\gamma\omega_o) = \gamma \frac{\partial \epsilon}{\partial \omega_o} ,$$

and

$$\epsilon_I(\omega_2) = \frac{2\gamma_2 \omega_2}{\omega_2^2 - \Omega^2} ; \quad \frac{\partial \epsilon(\omega_2)}{\partial \omega_2} = \frac{2\omega_2}{\omega_2^2 - \Omega^2} .$$

On the other hand, for the lower hybrid wave $(\partial \epsilon / \partial \omega_1)$ is given by (47a),

$$\chi_i(\omega_1) = -\omega_{pi}^2/\omega_1^2, \quad \epsilon_e(\omega_1) = \omega_{pe}^2/\Omega_e^2.$$

Thus, for the decay instability threshold we have

$$\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2} < \frac{1}{16} \left(\frac{e}{m}\right)^2 \frac{E_i^2 k_i^2 \omega_{pi}^2}{\omega_1^2 \omega_2^2 (\omega_0^2 - \Omega_e^2)} \quad (57)$$

and using $\omega_0^2 = \omega_2^2 = \Omega^2 + \omega_p^2(1-b)$, $\omega_u^2 = \omega_p^2 + \Omega_e^2$, we have

$$\frac{k_i^2 v_t^2}{2} = \frac{(\omega_{UH}^2 - \omega_2^2) \Omega^2}{\omega_{pe}^2}$$

and so at threshold

$$\frac{eE}{m v_t \omega_2} \geq \frac{4\omega_1}{\sqrt{(\omega_2^2 - \omega_u^2)}} \frac{\omega_{pe}^2}{\Omega \omega_{pi}} \sqrt{\frac{\gamma_1 \gamma_2}{2\omega_1 \omega_2}} \quad (58)$$

where ω_1 is the lower hybrid wave frequency. On the other hand, for decay into ion acoustic waves we get for $k^2 \lambda_D^2 \ll 1$,

$$\omega_0^2 - \Omega^2 \approx \omega_{pe}^2$$

$$\frac{eE}{m \omega_2 v_t} \approx 4 \sqrt{\frac{\gamma_1 \gamma_2}{2\omega_1 \omega_2}} \quad (59)$$

where ω_1 is the ion acoustic wave and $v_t = \sqrt{2kT_e/m_e}$. Finally, for the purely growing mode in the limit of small but finite k , we have

$$\frac{eE}{m\omega_2 v_z} \approx 2 \sqrt{\frac{\gamma_2}{\omega_2}} \quad (60)$$

We note that in Appendix A we indicate the kind of damping mechanisms which can be used in evaluating these thresholds. With this, we finish this introductory discussion of parametric instabilities. We see that a large number of modes exists even in the simplest situation. For more accurate discussion the reader is referred to the number of references provided at the end.

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APPENDIX A. DAMPING RATES OF PLASMA WAVES

In the analysis presented in this lecture we assumed that the damping of the waves was relatively weak. In this approximation the damping rates γ_1, γ_2 can be obtained relatively easily in terms of Landau damping and collision damping. Here we shall briefly summarize the relevant damping mechanisms of the waves discussed. For collision damping we shall use only the simplest type of term, namely that obtained from a Krook-type collision model. Thus, strictly speaking this collision term is valid only for electron-neutral (ν_{eo}) and ion-neutral (ν_{io}) collisions. However, for relatively long wavelength waves (i.e., $k_1 \gamma_{ce} \ll 1$) electron-ion collisions (ν_{ei}) can also be included by adding it to the electron-neutral collision term, i.e., $\nu_e = \nu_{eo} + \nu_{ei}$. We note that viscous-damping terms due to like-particle collisions will not be included here.

a) Electron Plasma Waves

The total dielectric is given by the expression:

$$\begin{aligned} \epsilon(\omega) = & \cos^2 \theta \left[1 - \frac{\omega_{pe}^2}{\omega(\omega + i\nu_e)} \right] + \sin^2 \theta \left[1 - \frac{\omega_{pe}^2 (1 + \frac{i\nu_e}{\omega})}{(\omega + i\nu_e)^2 - \Omega^2} \right] \\ & + i\sqrt{\pi} \frac{\omega}{k_{\parallel} v_t} \frac{e^{-\omega^2/(k_{\parallel} v_t)^2}}{k^2 \lambda_D^2} \end{aligned} \quad (A1)$$

where all quantities refer to electrons, and where the last term represents electron Landau damping. Note that for short wavelengths and for $\omega > \omega_{pe}$, the Bohm-Gross waves are obtained by replacing in the first brace

$$\omega_{pe}^2 \rightarrow \omega_{pe}^2 \left(1 + \frac{3}{2} \frac{k_{\parallel}^2 v_t^2}{\omega^2} \right),$$

and take $\cos\theta \approx 1$, $\sin\theta \approx 0$. Now by taking $\omega \rightarrow (\omega + i\gamma)$ in the first two braces of Eq. (A1), we obtain in the weak damping limit (i.e., $\gamma \ll \omega$, $v_e \ll \omega$)

$$\epsilon_R(\omega_2) + i\gamma \left(\frac{\partial \epsilon_R(\omega_2)}{\partial \omega_2} \right) + i\epsilon_I(\omega_2) = 0 \quad (A2)$$

Using the resonance approximation for the real part, i.e., $\epsilon_R(\omega_2) \approx 0$, we obtain for the damping rate:

$$\gamma_2 = - \frac{v_e \left(1 + \frac{2\Omega^2 \omega_{pe}^2 \sin^2 \theta}{(\omega_2^2 - \Omega^2)^2} \right) + \frac{\sqrt{\pi} \omega_2^2 \exp(-\omega_2^2/k_{\parallel}^2 v_t^2)}{k_{\parallel} v_{te} k^2 \lambda_D^2}}{2 \left(1 + \frac{\Omega^2 \omega_{pe}^2 \sin^2 \theta}{(\omega_2^2 - \Omega^2)^2} \right)^2} \quad (A3)$$

b) Lower Hybrid Waves

In this case $\Omega_i \ll \omega \ll \Omega_e$, and the ion contribution should also be added to Eq. (A1). Thus, in Eq. (A1), we add to $\epsilon(\omega)$ the term $-\omega_{pi}^2/\omega(\omega + iv_i)$. It is easy to show that in the resonance approximation $(\partial \epsilon/\partial \omega_2)$ remains the same as before, and ϵ_I picks up the additional contribution $\omega_{pi}^2 v_i/\omega_2^3$.

The result is (taking $\sin\theta \approx 1$)

$$\gamma_2 \approx - \frac{v_e [\omega_{pe}^2 / \Omega_e^2 + \cos^2 \theta \omega_{pe}^2 / \omega_2^2] + v_i \omega_{pi}^2 / \omega_2^2 + L_e}{2(1 + \omega_{pe}^2 / \Omega_e^2)} \quad (A4)$$

where ω_2 designates the lower hybrid frequency with finite $k_{||}$ correction (Eq. 50), and L_e is the electron Landau damping term (the same as the last term in Eq. (A3)).

c) Cyclotron-Harmonic Waves (Upper Hybrid Waves, Bernstein Waves)

In this case the imaginary part of the dielectric is given by

$$\begin{aligned} \epsilon_I = & \frac{v_e}{\omega} \frac{\omega_{pe}^2}{\Omega_e^2} \left(\frac{\Omega_e^2 (\omega^2 + \Omega_e^2)}{(\omega^2 - \Omega_e^2)^2} + \frac{b \Omega_e^2 (\omega^2 + 4\Omega_e^2)}{(\omega^2 - 4\Omega_e^2)^2} \right) \\ & + \frac{\omega}{k_{||} v_t} \frac{\sqrt{\pi}}{k_{||}^2 \lambda_D^2} \sum_p I_p(b) e^{-b} e^{-(\omega - p\Omega)^2 / k_{||}^2 v_t^2} \end{aligned} \quad (A5)$$

where we considered only the first and the second harmonics, and assumed $b = k_{||}^2 v_t^2 / 2 \ll 1$. From Eq. 54 we have

$$\frac{\partial \epsilon}{\partial \omega} = \frac{2\omega \omega_p^2}{\Omega_e^2} \left(\frac{\Omega_e^2 (1-b)}{(\omega^2 - \Omega_e^2)^2} + \frac{\Omega_e b (1-b)}{(\omega^2 - 4\Omega_e^2)^2} \right) \quad (A6)$$

Thus, the damping rate is given by

$$\gamma_2 \approx - \epsilon_I(\omega_2) / \left(\frac{\partial \epsilon(\omega_2)}{\partial \omega_2} \right) \quad (A7)$$

where the roots are given approximately (for $\omega_p^2 \ll 3\Omega^2$)

$$\omega_2^2 = \begin{cases} \Omega^2 + \omega_p^2(1-b) \\ 4\Omega^2 + \omega_p^2 b \end{cases} \quad (A8)$$

Note that for accurate work and for $b \approx 0(1)$ numerical work is indispensable.

d) Ion Acoustic Waves

Here the main damping mechanisms are electron Landau damping, ion collision damping, and ion Landau damping. Thus, for $b \ll 1$, $\omega \ll k v_{te}$, we have

$$\epsilon(\omega) = 1 + \frac{1}{k^2 \lambda_{De}^2} \left(1 + \frac{i\sqrt{\pi} \omega I_0(b) e^{-b}}{k v_{te}} \right) - \frac{\omega_{pi}^2}{\omega(\omega + i\nu_i)} + \frac{i\omega \sqrt{\pi} \exp[-\omega_1^2 / (k v_{ti})^2]}{k v_{ti} k^2 \lambda_{Di}^2} \quad (A9)$$

where $\hat{\omega} = \omega - \vec{k} \cdot \vec{V}_D$, V_D being some possible DC electron drift relative to ions and $\lambda_{Di} = v_{ti} / \omega_{pi} \sqrt{2}$. Thus, for $v_i \ll \omega_1$, $\gamma \ll \omega_1$, we obtain

$$\gamma_1 = - \frac{\nu_i \frac{\omega_{pi}^2}{\omega_1^3} + \frac{\sqrt{\pi} \omega_1 I_0(b) e^{-b}}{k^2 \lambda_{De}^2 (k v_{te})} + \frac{\sqrt{\pi} \omega_1 \exp(-\omega_1^2 / k^2 v_{ti}^2)}{k^2 \lambda_{Di}^2 k v_{ti}}}{(2\omega_{pi}^2 / \omega_1^3)} \quad (A10)$$

where

$$\omega_1^2 = \frac{k^2 v_A^2}{1 + k^2 \lambda_{De}^2} = \frac{k^2 \lambda_{De}^2 \omega_{pi}^2}{1 + k^2 \lambda_{De}^2} \quad (A11)$$

and where for simplicity the hot ion plasma effects have been ignored in the real part of the dispersion relation (these can be added easily if needed).

e) Ion Cyclotron Waves

A special form of ion acoustic wave is the electrostatic ion cyclotron wave (or its harmonics) which may be excited either with lower hybrid waves or with cyclotron harmonic waves as sidebands. The dielectric function is given by (for $b_i \ll 1$)

$$\epsilon(\omega) = 1 + \frac{1}{\lambda_{De}^2 k^2} \left(1 + \frac{i\sqrt{\pi}\omega I_0(b_e)e^{-b_e}}{k_{\parallel}v_{te}} \right) - \frac{\omega_{pi}^2 \sin^2 \theta}{\omega^2 - \Omega_i^2} - \frac{\omega_{pi}^2}{\omega^2} \cos^2 \theta + iL_i$$

where

$$L_i = \frac{i\sqrt{\pi}\omega}{k_{\parallel}v_{ti}k^2\lambda_{Di}^2} \int_0^{\infty} I_p(b_i)e^{-b_i} \exp[-(\omega - p\Omega_i)^2/k_{\parallel}^2v_{ti}^2] p \quad (A12)$$

where L_i designates ion cyclotron damping. Note that for $\omega^2 \gg \Omega_i^2$, Eq. (A12) reduces to Eq. (A9). Here we shall be interested in $\sin\theta \approx 0(1)$, $\omega^2 \sim \Omega_i^2$, and thus Eq. (A12) reduces to

$$\epsilon(\omega) = 1 + \frac{1}{\lambda_{De}^2 k^2} \left(1 + \frac{i\sqrt{\pi}\omega I_0(b)e^{-b}}{k_{\parallel}v_{te}} \right) - \frac{\omega_{pi}^2 \sin^2 \theta}{\omega^2 - \Omega_i^2} \left(1 - \frac{iv_i}{\omega} \frac{(\omega^2 + \Omega_i^2)}{(\omega^2 - \Omega_i^2)} \right) + iL_i \quad (A13)$$

Here we introduced ion collision damping by the substitution

$m_i \rightarrow m_i(1 + iv_i/\omega)$. Now letting $\omega \rightarrow \omega + i\gamma$, we obtain for $k^2\lambda_{De}^2 \ll 1$,

$$\omega_1^2 = \Omega_i^2 (1 + k_{\parallel}^2 v_A^2 / \Omega_i^2) \quad (A14)$$

$$\left(\frac{\partial \epsilon}{\partial \omega_1} \right) = \frac{2\omega_1 \omega_{pi}^2 \sin^2 \theta}{(\omega_1^2 - \Omega_i^2)^2} \quad (A15)$$

$$\epsilon_I = - \frac{\sqrt{\pi}\omega_1 I_0(b_e)e^{-b_e}}{\lambda_{De}^2 k^2 k_{\parallel}v_{te}} + \frac{v_i \omega_{pi}^2 \sin^2 \theta (\omega^2 + \Omega_i^2)}{\omega (\omega^2 - \Omega_i^2)^2} + L_i \quad (A16)$$

and again $\gamma_1 = -\epsilon_I(\omega_1)/(\partial\epsilon_R(\omega_1)/\partial\omega_1)$. Note that the thresholds and growth rates are given by Eqs. (22) and (24),

$$\epsilon_e(\omega) = 1 + 1/k^2 \lambda_{De}^2 \quad (A17)$$

as for the ion acoustic waves, and from Eq. (A13).

$$\chi_i(\omega) = - \frac{\omega_{pi}^2 \sin^2 \theta}{\omega^2 - \Omega_i^2} \quad (A18)$$

The sideband contributions $\partial\epsilon/\partial\omega_2$ and γ_2 are obtained from this Appendix depending on whether the pump is near the lower hybrid frequency or the electron cyclotron harmonics.

Note that in a hot ion plasma coupling with ion Bernstein waves may also be possible. In that case $\omega_1/k v_{te} \gg 1$, and hot plasma theory is necessary to treat ions. We shall not deal with such a case here.

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PARAMETRIC INSTABILITY

Fig. 1

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