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STRESS ANALYSIS OF A REACTOR CORE SUPPORT STRUCTURE CONSISTING OF TWO INTERCONNECTED MULTIREGION PLATES

by

A. H. Marchertas

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by

A. H. Marchertas

Reactor Engineering Division

June 1967

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FOREWORD

The analysis presented in this report owes its beginning to work done for the FARET* project. Recurrent changes in configuration of the reactor core support structure, brought about by considerations of the overall vessel design, often made the necessity for an improved method of stress analysis obvious. This need activated the formulation of an analysis for a rather general core support structure, where the design parameters could be varied arbitrarily.

Formulation of the analysis, and the effort devoted to the development of the associated computer program, took place after the FARET project had been terminated. Although the analysis formulated is general and encompasses all the previously considered variations of the support structure, termination of the FARET project and the unexpectedly long effort needed for developing the computer program have prevented completion of this task at present. However, the computer program will be published as a supplement to this report in the near future.

*The abbreviation FARET stands for Fast Reactor Test Facility, a former sodium-cooled fast reactor project at the Argonne National Laboratory; see ANL-7168.¹

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NOMENCLATURE

Symbol	Definition	Symbol	Definition
a^{-1}	Coefficient of the inverse matrix	T	Temperature distribution, °F
a	Average radius of the shell, in.	u, v, w	Displacements in r, θ , z directions, respectively
A	Cross-sectional area of tube, in. ²	x, y	Rectangular coordinates
A	Temperature coefficient for the plates (see Eq. 83)	X, Y	Temperature coefficients for the shells (see Eq. 87)
b	Element of the vector	α	Coefficient of thermal expansion, °F ⁻¹
b	Plate ring width, in. (see Fig. 14)	β	Characteristic shell constant, in. ⁻¹ (see Eq. 36)
B	Temperature coefficient for the plates (see Eq. 83)	γ	Shear strain, in./in.
C	Matrix coefficient	Δ	Variation from the average temperature
C_1, C_2, C_3, C_4	Constants of integration	ϵ	Normal strain, in./in.
D	Flexural rigidity of plate or shell, lb-in.	κ	Percentage of radial restraint at the edge
e	Axial tube deformation, in. (see Eq. 23)	λ	Number of tubes in a given circular distribution
E	Modulus of elasticity, lb/in. ²	ν	Poisson's ratio
F	Pressure coefficient of the plate (see Eq. 82)	ξ	Variable in the radial coordinate direction
g	Average plate thickness at a junction, in.	σ	Bending or normal stress, lb/in. ²
G	Temperature coefficient for the shells (see Eq. 86)	τ	Shear stress, lb/in. ²
h	Plate thickness, in.	ϕ	Angular displacement (see Fig. 14)
i, j	Summation indicies	Subscripts:	
I	Moment of inertia, in. ⁴	A	Top plate
k	Percentage of bending restraint at the edge supports	B	Bottom plate
l	Length of reinforcing tubes, in.	C	Central shell
m	Total number of regions in a plate	i, j	Individual terms of series
M	Bending moment in tubes or ring moment on plates, lb-in. or lb	L	Lower surface
n	Index to any representative region of the plate or its circumscribing radius, i.e., 1, 2, 3, ..., n, ..., m - 2, m - 1, m (see Fig. 5)	m	Plate region at the outer edge
N	Membrane or normal load, lb/in.	n	Any individual plate region, or its outer boundary
p	Transverse pressure, lb/in. ²	p	Particular solution of a given differential equation
P	Axial load in tubes or ring loads on plates, lb or lb/in.	P	Corresponding load
q	Effective pressure and temperature component in bending (see Eqs. 2 and 36)	r, θ , z	Cylindrical coordinates
Q	Shear load in tubes, plates, or shells, lb or lb/in.	T	Temperature component
r, θ , z	Radial, angular, and axial coordinates of the cylindrical coordinate system, respectively	U	Upper surface
S	Slope or first derivative of deflection	x	Rectangular coordinate
t	Shell thickness, in.	η	Location of shell ($r_n = r\eta$)
		o	Origin of a given coordinate system

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ABSTRACT

A general method is presented for the stress-deflection solution of a typical nuclear-reactor core support structure. The structure under consideration consists of two axisymmetric multi-region plates, located one above the other and interconnected by concentric rings of tubes. This main body of the structure is supported at the outer circumference, or at some intermediate radius, by circular cylindrical shells.

Loading of the structure may be twofold: static transverse pressure on the plane of the plates or the shell walls, and steady-state temperature gradients, both along the radius and across the thickness of the plates and the shells. This loading could be different for the two plates, their individual regions, and the separate shells. Plates with perforations of triangular configuration may also be included in the list of region properties to be considered.

Solution to this composite structure problem is obtained by mathematical superposition of elementary thin-plate, circular-cylindrical-shell, and simple beam problems. This is accomplished by the matrix-inversion procedure to solve the resulting simultaneous equations. Results are presented in the form of a matrix, giving the expressions for the coefficients in terms of the design variables. Special effort is taken to make the results particularly suitable for computer programming.

I. INTRODUCTION

Several analyses on axisymmetric coupled plates, primarily intended for service as nuclear-reactor support structures, have been presented in the literature. All these analyses assume that the interconnecting members between the plates produce an ideal continuous effect on the plane of the plates. In turn, this implies that only closely spaced tubes would realize the assumed condition. Although the approximation involved is good for many applications, its validity may at least become doubtful in other cases.

The treatment in this report assumes that the collective tube reactions can be represented by axisymmetric line moments and loads on the plates. Thus, the analysis assumes the structure is composed of finite structural elements, where the properties or loads may vary from one element to the next. The connection, or junction, between the individual elements is taken to be infinitely small.

The analyses given thus far and the present approach give two extreme cases of one relevant assumption: One approach takes the effect of the tubes to be a uniformly varying action on the plates; the other assumes a concentrated load effect. Both methods should provide the limits to any practical problem. It is believed, however, that the latter method, presented in this report, would be more versatile for design purposes. Here the planar uniformity of tube distributions would be irrelevant. In addition, the treatment of annular plate regions with different properties, such as heavy rims at the edges, would be most convenient, as opposed to the inner often perforated portion of the plate.

II. STATEMENT OF THE PROBLEM

This report describes one method of structural analysis for a reactor core support structure, such as that shown in Fig. 1. The structure consists of two multiregion plates, one above the other, interconnected by concentric rings of tubes. Furthermore, the regions of the plates from one ring of the tubes to the next have uniform properties throughout. These properties may vary from region to region and from plate to plate. The same applies to the tubes: dimensions are identical for the tubes in the same ring, but may change from one ring to the next. The position of the tubes is normal to the plane of the plates. This necessarily sets the planar dimensions of the regions to be identical for both plates.

A semi-infinite circular cylindrical shell may be rigidly connected at some radius of each plate. This would take place at the top surface of the top plate and the bottom surface of the bottom plate, as depicted in Fig. 1.

Edge support of the plates may be twofold: Both plates may be embedded into the walls of a right circular cylindrical shell of "infinite length," where the properties of the top, bottom, and intermediate sections may not necessarily be the same. The fixity at the edges may also be arbitrarily prescribed, as for example, 100% fixity, 50% fixity, etc.

Loading of the structure is also axisymmetrical, consisting of transverse pressure and thermal effects. This loading must be continuous throughout the individual regions, but may be different for the individual regions and also for the top and bottom plates. Transverse pressure and differential temperature gradients may be the components of load in the shells. Representation of the loading is shown in Fig. 2.

Fig. 1. Pictorial of a Reactor Core Support Structure

Fig. 2. Representation of Load Variation in the Structure

III. GENERAL ANALYSIS

The type of analysis proposed for the solution of a given problem must be weighed against the accuracy of the results desired. The ability to incorporate common design variables should also play a basic role in the selection. This section of the report therefore considers all possibilities and, consequently, chooses the method to be used in the analysis. The corresponding set of assumptions is also enumerated.

A. Method of Analysis

A uniform tube distribution throughout the planar area of the support structure can seldom be assumed in practical cases. The usual plate-tube combination core support structure consists of a central perforated plate area and a nonperforated solid rim at the edges. In most designs, additional variations are likely to take place which, in many cases, will have the near characteristic of being axisymmetrical. Treatment of such structures is only natural if the support plates can be assumed to be composed of annular plate regions with rings of tubes or shells at their intersections. If the effect of the tubes can be represented by axisymmetric line reactions at the intersections, then the method of superposition of elementary structural components is ideal for this solution. The accuracy of elementary solutions such as beam problems and thin plates is sufficiently accurate for most engineering purposes. The method of superposition is thus used in the analysis of the described core support structure.

B. Associated Assumptions

Two types of assumptions are incorporated in the method of superposition: First, since the analysis involves ready sample solutions of simple plate, tube, and shell problems, the same assumptions also hold in the combined structure. The second set of assumptions pertains to the conditions at the junctions of the regions. The assumptions involved in the analysis are as follows:

1. Thin-plate theory, beam analysis, and circular cylindrical shell solutions are applicable.
2. The action of the central reinforcing tube member is taken as that of a concentrated load.
3. The effect of the ring of tubes is assumed to be equivalent to that of a uniformly distributed ring force and moment.
4. The interaction of the tube's end remains perpendicular to the plane of the plate.

5. The effect of tubes at the intersection with the plates is set equal to that on the geometrical neutral surface of the plate.

6. Plane stress conditions exist in the plates.*

7. In case of perforated plates, the modified properties of modulus of elasticity and Poisson's ratio (due to O'Donnell and Langer²) are taken to be valid.

8. The possibility of buckling in the structure is assumed to be nonexistent.

* This assumption is used only in the approximation of membrane stresses.

IV. PRELIMINARY CONSIDERATIONS

The solution given in this report involves the superposition of circular-plate problems with those of slender beams and circular cylindrical shells. Before superposition is attempted, however, it is first necessary to have the "building block." Literature sources could be referred to for individual solutions, but because of nonuniform notation, some confusion may arise. To alleviate possible inconvenience, we chose to recapitulate the requirements to be satisfied for the individual plate-beam-shell solutions and then present the "building blocks" from which the final problem would be molded. In this way the material contained here would be self-sufficient.

A. Thin Circular Plate under Thermoelastic Loading

The thermoelastic plate analysis covered in this report is divided into two portions: One part has to do with deformation in the transverse direction; the other part deals with effects in the plane of the plate. Transverse bending and membrane effects are treated in the subsequent sections.

1. Conditions for Bending

For a thin plate, where only bending is taken into consideration, the differential equation of the deflection surface is given as³

$$\nabla^2 \nabla^2 w = \frac{1}{D} q(r, z), \quad (1)$$

where w is the plate deflection, D is the flexural rigidity of the plate, ∇^2 is the Laplacian operator, $q(r, z)$ stands for the transverse load component associated with bending, and r is the radial coordinate of the cylindrical coordinate system. The bending component $q(r, z)$ in thermoelasticity may be regarded as a linear combination of transverse bending and the effect of thermal gradient. This can be shown as follows:

$$q(r, z) = p(r) - \nabla^2 M_T, \quad (2)$$

where $p(r)$ is the transverse pressure, and M_T represents the temperature effect by means of the integral

$$M_T = \frac{E\alpha}{1-\nu} \int_{-h/2}^{h/2} T(r, z) z \, dz. \quad (3)$$

Here, α is the coefficient of thermal expansion, ν is Poisson's ratio, h is the thickness of the plate, and $T(r, z)$ is the temperature distribution in the plate where z is the axial coordinate.

In the axisymmetric case, the Laplacian operator, ∇^2 , is only dependent on the radial coordinate; i.e., $\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}$. Thus, if the left-hand terms of Eq. 1 are rearranged, they can be represented in the following way:

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] \right\} = \frac{1}{D} q(r, z). \quad (4)$$

This is a readily integrable expression from which w can be evaluated explicitly. Hence, when the terms of Eq. 4 are successively rearranged and integration performed, the result is

$$w = C_1 + C_2 r^2 + C_3 \log_e \left(\frac{r}{r_n} \right) + C_4 r^2 \log_e \left(\frac{r}{r_n} \right) + w_p, \quad (5)$$

where C_1 , C_2 , C_3 , and C_4 are the constants of integration to be determined from the boundary conditions at hand, r_n is the outside radius of the plate, and w_p is the particular solution (particular deflection) of the differential equation. The expression for w_p , obtained by the simultaneous evaluation of Eq. 4, is

$$w_p = \frac{1}{D} \int \frac{1}{\xi} \int \xi \int \frac{1}{\xi} \int q(\xi, z) \xi d\xi d\xi d\xi d\xi. \quad (6)$$

Expressed in terms of transverse pressure and temperature distribution, Eq. 6 becomes

$$w_p = \frac{1}{D} \int \frac{1}{\xi} \int \xi \int \frac{1}{\xi} \int p(\xi, z) \xi d\xi d\xi d\xi d\xi - \frac{12(1+\nu)\alpha}{h^3} \int \frac{1}{\xi} \int \xi \int_{-h/2}^{h/2} T(\xi, z) z dz d\xi d\xi. \quad (7)$$

Thus, Eq. 5, together with Eq. 6 or 7, represents the deflection expression of an annular thin plate under arbitrary axisymmetric boundary conditions. Equation 5 pertains also to a plate with no hole in the center, provided the constants, C_3 and C_4 , and the radius, r_0 , are set equal to zero.

Thermoelasticity, in turn, provides the relations of deflection with slope, radial and tangential moments, and shear load in the plate. These relationships are as follows:

$$\left. \begin{aligned}
 S_n &= \frac{dw_n}{dr}, \\
 M_{rn} &= -D_n \left[\frac{d^2 w_n}{dr^2} + \frac{\nu_n}{r} \frac{dw_n}{dr} \right] - M_{Tn}, \\
 M_{\theta n} &= -D_n \left[\nu_n \frac{d^2 w_n}{dr^2} + \frac{1}{r} \frac{dw_n}{dr} \right] - M_{Tn}, \\
 Q_{rn} &= -\frac{d}{dr} \left\{ D_n \left[\frac{d^2 w_n}{dr^2} + \frac{1}{r} \frac{dw_n}{dr} \right] + M_{Tn} \right\},
 \end{aligned} \right\} \quad (8)$$

where the subscript n pertains to the n th region in a multiregion plate. When Eq. 5 is substituted into Eqs. 8, the same expressions, in terms of the constants of integration and the load parameters, yield

$$\left. \begin{aligned}
 S_n &= 2C_{2n}r + C_{3n}\left(\frac{1}{r}\right) + C_{4n}r \left[1 + 2 \log_e \left(\frac{r}{r_n} \right) \right] + S_{pn}, \\
 M_{rn} &= -D_n \left\{ 2C_{2n}(1 + \nu_n) - C_{3n}\left(\frac{1 - \nu_n}{r^2}\right) \right. \\
 &\quad \left. + C_{4n} \left[3 + \nu_n + 2(1 + \nu_n) \log_e \left(\frac{r}{r_n} \right) \right] \right\} + M_{rpn}, \\
 M_{\theta n} &= -D_n \left\{ 2C_{2n}(1 + \nu_n) + C_{3n}\left(\frac{1 - \nu_n}{r^2}\right) \right. \\
 &\quad \left. + C_{4n} \left[1 + 3\nu_n + 2(1 + \nu_n) \log_e \left(\frac{r}{r_n} \right) \right] \right\} + M_{\theta pn}, \\
 Q_{rn} &= 4D_n C_{4n} \left(\frac{1}{r} \right) + Q_{rpn},
 \end{aligned} \right\} \quad (9)$$

where the subscript p indicates relation to the particular deflection, w_p . Here, in particular, S_{pn} , M_{rpn} , $M_{\theta pn}$, and Q_{rpn} pertain to Eqs. 8, when instead of w , the particular deflection w_p is used. Hence,

$$\left. \begin{aligned}
 S_{pn} &= \frac{dw_{pn}}{dr}, \\
 M_{rpn} &= -D_n \left[\frac{d^2 w_{pn}}{dr^2} + \frac{\nu_n}{r} \frac{dw_{pn}}{dr} \right] - M_{Tn},
 \end{aligned} \right\} \quad (10)$$

Contd.

$$\left. \begin{aligned} M_{\theta pn} &= -D_n \left[\nu_n \frac{d^2 w_{pn}}{dr^2} + \frac{1}{r} \frac{dw_{pn}}{dr} \right] - M_{Tn}, \\ Q_{rpn} &= -\frac{d}{dr} \left\{ D_n \left[\frac{d^2 w_{pn}}{dr^2} + \frac{1}{r} \frac{dw_{pn}}{dr} \right] + M_{Tn} \right\}. \end{aligned} \right\} \quad \begin{array}{l} (10) \\ \text{Contd.} \end{array}$$

Fig. 3. Superposition of the Central-plate Region. (a) Plate under Concentrated Load. (b) Plate under Distributed Load.

The equations of deflection, slope, moments, and shear load have thus been derived in terms of the effective boundary conditions and loading parameters. This makes the solution of any thermoelastic, axisymmetrical, thin-plate problem readily available whenever the boundary conditions are specified.

Before the superposition can be performed, it is necessary to have the individual component solutions of which the composite structure is made. The plates may be regarded as being composed of two types of simple plate solutions (central plate solution and annular plate solution), which are discussed in the following sections.

a. Central-plate Solution.

The first plate element is the centermost region of the composite plate and may have a free-body diagram, as shown in Fig. 3. This may also be regarded as a combination of two separate solutions shown by Figs. 3a and 3b. The case represented in Fig. 3a, a simply supported, centrally loaded circular plate, can be found in the literature.⁴ Thus, the deflection, slope, radial moment, and radial shear, respectively, pertaining to the case of Fig. 3a, are*

*Deflection, slope, radial moment, and radial shear load are the quantities that will be used later in the process of composite superposition.

$$\left. \begin{aligned} w &= - \frac{P_0 L - P_0 U}{16\pi D(1+\nu)} \left[(3+\nu)(r_1^2 - r^2) + 2(1+\nu) r^2 \log_e \left(\frac{r}{r_1} \right) \right], \\ S &= \frac{P_0 L - P_0 U}{4\pi D(1+\nu)} \left[r - (1+\nu) r \log_e \left(\frac{r}{r_1} \right) \right], \\ M_r &= \frac{P_0 L - P_0 U}{4\pi} (1+\nu) \log_e \left(\frac{r}{r_1} \right), \\ Q_r &= - \frac{P_0 L - P_0 U}{2\pi r}. \end{aligned} \right\} \quad (11)$$

Consideration of the plate problem given in Fig. 3b yields a simplified Eq. 5, as mentioned previously. Effectively, Eqs. 9 then reduce to

$$\left. \begin{aligned} w &= C_1 + C_2 r^2 + w_p, \\ S &= 2C_2 r + S_p, \\ M_r &= 2C_2 D(1+\nu) + M_{rp}, \\ Q_r &= Q_{rp}. \end{aligned} \right\} \quad (12)$$

The two boundary conditions needed to evaluate the constants of integration may be taken as follows:

$$\left. \begin{aligned} w|_{r=r_1} &= w_1|_{r_1}, \\ M_r|_{r=r_1} &= M_{r1}|_{r_1}. \end{aligned} \right\} \quad (13)$$

When Eqs. 13 are substituted into Eqs. 12, the constants of integration, C_1 and C_2 , become known. Substitution then can be made back into Eqs. 12. Combining these with the corresponding parts of Eqs. 11 produces the resultant expressions (pertaining to Fig. 3). Consequently, the expressions for the innermost region of the combined plate yield

$$\left. \begin{aligned} w_1 &= - \frac{P_0 L - P_0 U}{16\pi D_1(1+\nu_1)} \left[(3+\nu_1)(r_1^2 - r^2) + 2(1+\nu_1) r^2 \log_e \left(\frac{r}{r_1} \right) \right] \\ &+ \frac{r_1^2 - r^2}{2D_1(1+\nu_1)} \left[M_{r1}|_{r_1} - M_{rp1}|_{r_1} \right] - w_{p1}|_{r_1} + w_{p1} + w_1|_{r_1}, \end{aligned} \right\} \quad \begin{matrix} (14) \\ \text{Contd.} \end{matrix}$$

annular plate represents the regions from $n = 2$ through $n = m$, where m is the total number of regions. Equation 5, together with Eqs. 9, is representative for this case. Four boundary conditions must be specified before explicit solutions can be obtained. These can be chosen as follows:

$$\left. \begin{aligned} w|_{r=r_n} &= w_n|_{r_n}, & M_r|_{r=r_{n-1}} &= M_{rn}|_{r_{n-1}}, \\ M_r|_{r=r_n} &= M_{rn}|_{r_n}, & Q_r|_{r=r_{n-1}} &= Q_{rn}|_{r_{n-1}} \end{aligned} \right\} \quad (15)$$

Thus, when Eqs. 5 and 9 are subjected to the boundary conditions of Eqs. 15, the following expressions result:

$$\left. \begin{aligned} C_{4n} &= \frac{r_{n-1}}{4D_n} \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right], \\ C_{3n} &= \frac{r_n^2 r_{n-1}^2}{D_n(1-\nu_n)(r_n^2 - r_{n-1}^2)} \left\{ M_{rn}|_{r_{n-1}} - M_{rpn}|_{r_{n-1}} - M_{rn}|_{r_n} \right. \\ &\quad \left. + M_{rpn}|_{r_n} + \frac{1}{2}(1+\nu_n) r_{n-1} \log_e \left(\frac{r_{n-1}}{r_n} \right) \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right] \right\}, \\ C_{2n} &= \frac{1}{2D_n(1+\nu_n)(r_n^2 - r_{n-1}^2)} \left\{ r_{n-1}^2 \left[M_{rn}|_{r_{n-1}} - M_{rpn}|_{r_{n-1}} \right] \right. \\ &\quad \left. - r_n^2 \left[M_{rn}|_{r_n} - M_{rpn}|_{r_n} \right] - \frac{1}{4} r_{n-1} \left[(3+\nu_n)(r_n^2 - r_{n-1}^2) \right. \right. \\ &\quad \left. \left. - 2(1+\nu_n) r_{n-1}^2 \log_e \left(\frac{r_{n-1}}{r_n} \right) \right] \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right] \right\}, \\ C_{1n} &= -C_{2n} r_n^2 - w_{pn}|_{r_n} + w_{n+1}|_{r_n} \end{aligned} \right\} \quad (16)$$

Equations 16, together with Eqs. 5 and 9, give the steady-state, axisymmetric thermoelastic solution for the plate shown in Fig. 4.

2. Conditions under Membrane Loading

In thermoelasticity, the membrane stress and radial deformation equations for the plane stress condition are given as³

$$\left. \begin{aligned} \sigma_{rn} &= \frac{E_n}{1-\nu_n} C_{1n} - \frac{1}{r^2} \frac{E_n}{1+\nu_n} C_{2n} - \frac{E_n \alpha_n}{r^2} \int_{r_{n-1}}^r T_n(r) r dr, \\ \sigma_{\theta n} &= \frac{E_n}{1-\nu_n} C_{1n} + \frac{1}{r^2} \frac{E_n}{1+\nu_n} C_{2n} + E_n \alpha_n \left[\frac{1}{r^2} \int_{r_{n-1}}^r T_n(r) r dr - T_n(r) \right], \\ u_n &= r C_{1n} + \frac{1}{r} C_{2n} + \frac{(1+\nu_n) \alpha_n}{r} \int_{r_{n-1}}^r T_n(r) r dr, \end{aligned} \right\} \quad (17)$$

where C_{1n} and C_{2n} are constants of integration in the n th region dependent on the radial boundary conditions. Equations 17 pertain to an annular plate where r_{n-1} and r_n are the inside and outside radii, respectively, as shown in Fig. 5. The same expressions also hold for a plate without a central hole, provided C_2 and r_0 are assigned values of zero.

Fig. 5. Membrane Loads in the Plates

B. Beam Subjected to End Loads

The only type of tube (or beam) appearing in the support structure is one loaded at its ends. A free-body diagram is represented in Fig. 6, where the deformation of the beam is considered very small and the axial-load contribution to bending is taken to be negligible. Because of this assumption, the effects of axial load and bending can be decoupled, and the two contributions treated separately.

1. Bending Solution

The elastic-curve equation of bending for a beam is

$$EI \frac{d^2 y}{dx^2} = M_x, \quad (18)$$

Fig. 6. Deformed Tube

where the coordinates x and y are identified in Fig. 6, M_x is the internal bending moment of the beam along its length, and E and I are the modulus of elasticity and moment of inertia of the beam, respectively, both taken to be constant along the length of the beam. The moment, M_x , can be expressed in terms of the end loads and then written as $M_x = xQ_A - M_A$. When this is substituted into Eq. 8, integrated twice in succession, the slope and deflection of the elastic curve are obtained. The resulting equations are as follows:

$$\left. \begin{aligned} EI \frac{d^2 y}{dx^2} &= xQ_A - M_A, \\ EI \frac{dy}{dx} &= \frac{1}{2} x^2 Q_A - xM_A + C_1, \\ EI y &= \frac{1}{6} x^3 Q_A - \frac{1}{2} x^2 M_A + xC_1 + C_2. \end{aligned} \right\} \quad (19)$$

Two unknown constants of integration, C_1 and C_2 , are to be evaluated from Eqs. 19. In addition, the end forces, Q_A and M_A , must be expressed in terms of the slope and displacement at the end of the tube. Hence, four boundary conditions must be specified to secure the equations in the desired form. Thus, the conditions pertaining to deflections and slopes at the end of the tube are

$$\left. \begin{aligned} y|_{x=0} &= y_A, & y|_{x=l} &= -y_B, \\ \frac{dy}{dx}|_{x=0} &= S_A, & \frac{dy}{dx}|_{x=l} &= S_B. \end{aligned} \right\} \quad (20)$$

These conditions refer to Fig. 6. When Eqs. 19 are subjected to these conditions and rearranged, the end loads at $x = 0$ become

$$\left. \begin{aligned} Q_A &= \frac{6EI}{l^3} \left[l(S_A + S_B) + 2(y_A + y_B) \right], \\ M_A &= \frac{2EI}{l^2} \left[l(2S_A + S_B) + 3(y_A + y_B) \right]. \end{aligned} \right\} \quad (21)$$

From the equilibrium considerations of Fig. 6, two other relationships can be obtained. Since the transverse deflection of the tube is assumed to be small, then the expressions for loads at $x = l$ are

$$\left. \begin{aligned} Q_B &= Q_A, \\ M_B &= lQ_A - M_A. \end{aligned} \right\} \quad (22)$$

Consequently, Eqs. 21 and 22 provide the expressions of the end loadings of the tube in terms of its displacement.

2. Axial-load Solution

It was initially assumed that the transverse deformation of the tubes is very small. This assumption, in turn, implies that the axial load is invariant throughout the length of each tube. Hence, the stress intensity at each cross section is uniform, and the uniaxial strain yields

$$\epsilon = \frac{\sigma}{E} + \alpha T(x),$$

where σ is the axial stress and $T(x)$ is the temperature variation along the length of the tube. The corresponding deformation of the tube in the axial direction becomes

$$e = \frac{P\ell}{EA} + \int_0^\ell \alpha \Delta T(x) dx, \quad (23)$$

where P is the axial load, A is the cross-sectional area of the tube, ℓ is its length, and $\Delta T(x)$ represents the difference between the tube's actual temperature at a point and the average across the entire length.

C. Circular Cylindrical Shell under Thermoelastic Loading

Analysis of circular cylindrical shell under axisymmetric loading is standard and can be found in practically any reference book on shells. The inclusion of temperature gradients in the walls of the shell is less common. Thus, for ready reference, the temperature effect on the shell is given in this presentation.

The material given here is confined to a thin circular cylindrical shell of Fig. 7, where the cylindrical coordinates r , θ , and z are shown. Relation of strain to the displacements in the direction of the coordinates is provided by the theory of elasticity as follows:

$$\left. \begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, & \gamma_{r\theta} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \\ \epsilon_\theta &= \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}, & \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \\ \epsilon_z &= \frac{\partial w}{\partial z}, & \gamma_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}, \end{aligned} \right\} \quad (24)$$

where u , v , and w are the conventional displacements in the respective r , θ , and z coordinate directions. Hooke's law, on the other hand, gives relationships of stress and strain. Hence, for the same system of coordinates,

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{E} [\sigma_r - \nu(\sigma_\theta + \sigma_z)] + \alpha T(r, z), & \gamma_{r\theta} &= \frac{2(1+\nu)}{E} \tau_{r\theta}, \\ \epsilon_\theta &= \frac{1}{E} [\sigma_\theta - \nu(\sigma_r + \sigma_z)] + \alpha T(r, z), & \gamma_{rz} &= \frac{2(1+\nu)}{E} \tau_{rz}, \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \alpha T(r, z), & \gamma_{\theta z} &= \frac{2(1+\nu)}{E} \tau_{\theta z}, \end{aligned} \right\} \quad (25)$$

where $T = T(r, z)$ is the temperature distribution in the wall of the shell. The internal components of stress are shown on an element of the shell's wall in Fig. 7a.

Fig. 7. Circular Cylindrical Shell

Because of the special symmetry of the shell and because loading and temperature variation is taken to be axisymmetric, the expressions of strain become ordinary differential equations and can be simplified considerably. Thus, the displacements and stresses are independent of the coordinate θ ; some even are identically zero, namely:

$$v = 0, \quad \tau_{r\theta} = 0, \quad \tau_{\theta z} = 0.$$

Upon further examination, and considering that the wall thickness is much smaller than the radius of the shell, we can make other simplifying assumptions. The normal stresses, σ_θ and σ_z , are the predominating stresses, while σ_r and τ_{rz} have secondary importance and may be neglected altogether in further discussions.

It will be convenient to assume that axial external forces do not take part in this presentation. This is done solely for simplicity. Later, if this load does happen to be different from zero, the deformation and stress corresponding to such constant force can be easily calculated and superimposed on stresses and deformations produced by lateral loads. It must be certain, however, that the magnitude of the added load will not be so great as to make it possible for buckling to impend.

In other calculations, it will also be advantageous and sufficiently accurate to use the average radius, a , of the shell instead of the variable, r . This can be taken advantage of in the first term of the circumferential strain of Eqs. 24; namely, $u/r \approx u/a$. Other such approximations will be introduced later.

When all the mentioned simplifications are considered, two coordinates (θ and z) appear to have primary importance. The predominance of stresses and strains in these directions overshadow the others, which will be neglected. Thus, the strain of the shell from Eqs. 24 and 25, when simplifications are introduced, becomes

$$\epsilon_{\theta} = \frac{u}{a} = \frac{1}{E} (\sigma_{\theta} - \nu \sigma_z) + \alpha T(r, z),$$

$$\epsilon_z = \frac{dw}{dz} = \frac{1}{E} (\sigma_z - \nu \sigma_{\theta}) + \alpha T(r, z).$$

Solving the above expressions for stresses, in terms of displacements, yields

$$\left. \begin{aligned} \sigma_{\theta} &= \frac{E}{1 - \nu^2} \left[\frac{u}{a} + \nu \frac{dw}{dz} - (1 + \nu) \alpha T(r, z) \right], \\ \sigma_z &= \frac{E}{1 - \nu^2} \left[\nu \frac{u}{a} + \frac{dw}{dz} - (1 + \nu) \alpha T(r, z) \right]. \end{aligned} \right\} \quad (26)$$

Fig. 8

Deformed Section of the Shell

For analysis, the radius r to any point in the wall of the shell is expressed in terms of the average radius, a , as follows:

$$r = a + \xi,$$

where ξ is the length of the normal from that point to the neutral surface, as shown in Fig. 8. In the deformed state, the radial and axial displacements of the given point may be related to those of the neutral surface at the intersection of the normal, as shown in the same figure, and yield

$$\left. \begin{aligned} u &= u_0, \\ w &= w_0 - \xi \frac{du_0}{dz}, \end{aligned} \right\} \quad (27)$$

where the approximation of small deflection and rotation has been used and the subscript 0 refers to the neutral surface. Hence, differentiating and substituting Eqs. 27 into Eqs. 26 results in

$$\left. \begin{aligned} \sigma_{\theta} &= \frac{E}{1-\nu^2} \left[\frac{u}{a} + \nu \frac{dw_0}{dz} - \nu \xi \frac{d^2u}{dz^2} - (1+\nu) \alpha T(\xi, z) \right], \\ \sigma_z &= \frac{E}{1-\nu^2} \left[\nu \frac{u}{a} + \frac{dw_0}{dz} - \xi \frac{d^2u}{dz^2} - (1+\nu) \alpha T(\xi, z) \right]. \end{aligned} \right\} \quad (28)$$

Since the radius of the shell is large in comparison to its thickness, the area of a circumferential element, $2\pi r dr$, may be taken to be nearly equal to $2\pi a d\xi$. This assumption can be usefully applied in the condition that the resultant axial load is zero; i.e.,

$$\int_{-t/2}^{t/2} 2\pi a \sigma_z d\xi = \frac{2\pi a E}{1-\nu^2} \int_{-t/2}^{t/2} \left[\nu \frac{u}{a} + \frac{dw_0}{dz} - \xi \frac{d^2u}{dz^2} - (1+\nu) \alpha T(\xi, z) \right] d\xi = 0. \quad (29)$$

When Eq. 29 is integrated, the following explicit relation for dw_0/dz becomes available:

$$\frac{dw_0}{dz} = -\nu \frac{u}{a} + \frac{(1+\nu) \alpha}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi. \quad (30)$$

Upon substitution of Eq. 30 into Eqs. 28, the stress expressions yield

$$\left. \begin{aligned} \sigma_{\theta} &= \frac{E}{1-\nu^2} \left[(1-\nu^2) \frac{u}{a} - \nu \xi \frac{d^2u}{dz^2} - (1+\nu) \alpha T(\xi, z) + \frac{\nu(1+\nu) \alpha}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi \right], \\ \sigma_z &= \frac{E}{1-\nu^2} \left[-\xi \frac{d^2u}{dz^2} - (1+\nu) \alpha T(\xi, z) + \frac{(1+\nu) \alpha}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi \right]. \end{aligned} \right\} \quad (31)$$

Figure 9 is a free-body diagram of an element of the shell wall. Performing the summation of moments and loads in the radial direction and neglecting secondary terms, we obtain two equations of equilibrium. These become

$$\left. \begin{aligned} Q_z - \frac{dM_z}{dz} &= 0, \\ \frac{dQ_z}{dz} - \frac{N_{\theta}}{a} + p(z) &= 0. \end{aligned} \right\} \quad (32)$$

Eliminating Q_z from Eqs. 32 provides a single equation of equilibrium; i.e.,

Fig. 9. Representative Stressed Element of the Shell Wall

$$\frac{d^2 M_z}{dz^2} - \frac{N_\theta}{a} + p(z) = 0. \quad (33)$$

The loads M_θ , M_z , and N_θ on the element of Fig. 9 may be expressed in integral form in terms of the stress. Thus,

$$\left. \begin{aligned} M_\theta &= \int_{-t/2}^{t/2} \sigma_\theta \xi \, d\xi, \\ M_z &= \int_{-t/2}^{t/2} \sigma_z \xi \, d\xi, \\ N_\theta &= \int_{-t/2}^{t/2} \sigma_\theta \, d\xi. \end{aligned} \right\} \quad (34)$$

When Eqs. 31 are substituted into Eqs. 34 and the integration is partly performed, the following results:

$$\left. \begin{aligned} M_\theta &= -\nu D \frac{d^2 u}{dz^2} - \frac{E\alpha}{1-\nu} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{\nu}{t} \int_{-t/2}^{t/2} T(\xi, z) \, d\xi \right] \xi \, d\xi, \\ M_z &= D \frac{d^2 u}{dz^2} - \frac{E\alpha}{1-\nu} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{1}{t} \int_{-t/2}^{t/2} T(\xi, z) \, d\xi \right] \xi \, d\xi, \\ N_\theta &= \frac{Et u}{a} - \frac{E\alpha}{1-\nu} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{\nu}{t} \int_{-t/2}^{t/2} T(\xi, z) \, d\xi \right] d\xi, \end{aligned} \right\} \quad (35)$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}.$$

Substituting in turn the needed expressions of Eqs. 35 into Eq. 33 results in the differential equation of the thin circular cylindrical shell. This becomes

$$\frac{d^4 u}{dz^4} + 4\beta^4 u = \frac{1}{D} q(r, z), \quad (36)$$

where

$$\beta = \sqrt[4]{\frac{3(1-\nu^2)}{a^2 t^2}},$$

and

$$q(r, z) = p(z) + \frac{E\alpha}{1-\nu} \left\{ \frac{1}{a} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{\nu}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi \right] d\xi \right. \\ \left. - \frac{d^2}{dz^2} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{1}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi \right] \xi d\xi \right\}.$$

The general solution of Eq. 36 is

$$u = C_1 \sin \beta z \sinh \beta z + C_2 \sin \beta z \cosh \beta z + C_3 \cos \beta z \sinh \beta z \\ + C_4 \cos \beta z \cosh \beta z + u_p, \quad (37)$$

where C_1 , C_2 , C_3 , and C_4 are the constants of integration to be determined in each case from the conditions at the ends of the shell, and u_p is a particular solution of the differential equation. In general, the particular integral of Eq. 36 may be expressed as

$$u_p = \frac{e^{(1-i)\beta z}}{D} \int e^{2i\beta z} \int e^{-2\beta z} \int e^{-2i\beta z} \int e^{(1+i)\beta z} q(r, z) dz dz dz dz. \quad (38)$$

In its full content, Eq. 37 pertains to a finite cylinder, where two boundary conditions can be specified for each end. The same expression can also be adapted to a solution of a semi-infinitely long shell, provided the following adjustments are made to the constants:*

$$C_1 = -C_2, C_3 = -C_4.$$

From the analysis point of view, the term "infinite" refers to a shell of length such that the effects of any steady-state loading at one end can be felt only to a very small degree at the other end. Timoshenko and Woinowski-Krieger⁴ tabulated and plotted the quantities that determine the bending of the shell and found that the effect of a load at the edge dies out at about a distance of π/β from that edge. Thus, if loading takes place at both ends, the shell length must be more than $2\pi/\beta$ to be considered infinite from the point of view of the analysis.

*The reason for these relations can best be realized if the hyperbolic terms of Eq. 37 are written in exponential form.

The slope, moment, and shear of the shells can be expressed in terms of the deflection as follows:

$$\left. \begin{aligned} S_z &= \frac{du}{dz}, \\ M_z &= -D \frac{d^2u}{dz^2} - M_z T, \\ Q_z &= \frac{d}{dz} [M_z], \end{aligned} \right\} \quad (39)$$

where

$$M_{zT} = \frac{E\alpha}{1-\nu} \int_{-t/2}^{t/2} \left[T(\xi, z) - \frac{1}{t} \int_{-t/2}^{t/2} T(\xi, z) d\xi \right] \xi d\xi.$$

Since these quantities will be used later to satisfy the boundary conditions, they may as well be given also in terms of the constants of integration. Substituting Eq. 37 into Eqs. 39 and performing the indicated operations results in

$$\left. \begin{aligned} S_z &= \beta [C_1(\cos \beta z \sinh \beta z + \sin \beta z \cosh \beta z) + C_2(\cos \beta z \cosh \beta z + \sin \beta z \sinh \beta z) \\ &\quad + C_3(\cos \beta z \cosh \beta z - \sin \beta z \sinh \beta z) + C_4(\cos \beta z \sinh \beta z - \sin \beta z \cosh \beta z)] \\ &\quad + S_p, \\ M_z &= -2D\beta^2 [C_1 \cos \beta z \cosh \beta z + C_2 \cos \beta z \sinh \beta z - C_3 \sin \beta z \cosh \beta z \\ &\quad - C_4 \sin \beta z \sinh \beta z] + M_{zp}, \\ Q_z &= -2D\beta^3 [C_1(\cos \beta z \sinh \beta z - \sin \beta z \cosh \beta z) + C_2(\cos \beta z \cosh \beta z - \sin \beta z \sinh \beta z) \\ &\quad - C_3(\cos \beta z \cosh \beta z + \sin \beta z \sinh \beta z) - C_4(\cos \beta z \sinh \beta z + \sin \beta z \cosh \beta z)] \\ &\quad + Q_{zp}, \end{aligned} \right\} \quad (40)$$

where S_p , M_{zp} , and Q_{zp} are the quantities associated with the particular solution of Eq. 36 and are defined with the help of Eqs. 39 as follows:

$$\left. \begin{aligned} S_{zp} &= \frac{du_p}{dz}, \\ M_{zp} &= -D \frac{d^2u_p}{dz^2} - M_{zT}, \\ Q_{zp} &= \frac{d}{dz} [M_{zp}]. \end{aligned} \right\} \quad (41)$$

1. Semi-infinite Shell

As mentioned previously, loading at only one edge should be specified for an infinitely long, circular, cylindrical shell. From Fig. 10a, the conditions may be taken as follows:

$$\left. \begin{aligned} u|_{z=0} &= u_0, \\ S_z|_{z=0} &= S_0, \end{aligned} \right\} \quad (42)$$

where u_0 and S_0 are the radial and angular displacements at the edge, respectively. Setting $C_1 = -C_2$, $C_3 = -C_4$, and the conditions of Eqs. 42 into Eqs. 37 and 40 result in expressions for C_2 and C_4 . This provides relations between the applied loads and the corresponding displacements. Thus, the resultant moment and shear expressions of Eqs. 40, evaluated at $z = 0$, yield

$$\left. \begin{aligned} M_z|_{z=0} &= 2D\beta^2 \left[u_0 - u_p|_{z=0} + \frac{1}{\beta} (S_0 - S_p|_{z=0}) \right] + M_{zp}|_{z=0}, \\ Q_z|_{z=0} &= -2D\beta^3 \left[2u_0 - 2u_p|_{z=0} + \frac{1}{\beta} (S_0 - S_p|_{z=0}) \right] + Q_{zp}|_{z=0}. \end{aligned} \right\} \quad (43)$$

2. Finite Shell

The finite shell of length ℓ , shown in Fig. 10b, requires four boundary conditions to be specified for the evaluation of the unknown constants. However, for the sake of using consistent symbols, we add two more. These conditions may be taken to be as follows:

$$\left. \begin{aligned} u|_{z=0} &= u_A, & u|_{z=\ell} &= -u_B, \\ S_z|_{z=0} &= S_A, & S_z|_{z=\ell} &= S_B, \\ M_z|_{z=0} &= M_A, & M_z|_{z=\ell} &= M_B. \end{aligned} \right\} \quad (44)$$

When these conditions are substituted into Eqs. 37 and 40, the constants of integration yield

$$\left. \begin{aligned} C_1 &= \frac{1}{2D\beta^2} [M_A - M_{zp}|_{z=0}], \\ C_2 &= \frac{1}{\sin \beta \ell \cosh \beta \ell - \cos \beta \ell \sinh \beta \ell} \left\{ \frac{1}{2D\beta^2} [M_A - M_{zp}|_{z=0}] \sin \beta \ell \sinh \beta \ell \right. \\ &\quad - \frac{1}{\beta} [S_A - S_{zp}|_{z=0}] \cos \beta \ell \sinh \beta \ell - [u_A - u_p|_{z=0}] \cos \beta \ell \cosh \beta \ell \\ &\quad \left. - [u_B + u_p|_{z=\ell}] \right\}, \end{aligned} \right\} \quad (45) \quad \text{Contd.}$$

a. Semi-infinite Shell

b. Finite Shell

Fig. 10
Shell End Loading

$$\left. \begin{aligned}
C_3 &= -\frac{1}{\sin \beta l \cosh \beta l - \cos \beta l \sinh \beta l} \left\{ \frac{1}{2D\beta^2} [M_A - M_{zp}|_{z=0}] \sin \beta l \sinh \beta l \right. \\
&\quad - \frac{1}{\beta} [S_A - S_{zp}|_{z=0}] \sin \beta l \cosh \beta l - [u_A - u_p|_{z=0}] \cos \beta l \cosh \beta l \\
&\quad \left. - [u_B + u_p|_{z=l}] \right\}, \\
C_4 &= u_A - u_p|_{z=0}.
\end{aligned} \right\} \text{Contd. (45)}$$

The moment and shear expressions, evaluated at the edges and expressed in terms of the constants of integration, become

$$\left. \begin{aligned}
M_z|_{z=0} &= -2D\beta^2 C_1 + M_{zp}|_{z=0}, \\
Q_z|_{z=0} &= -2D\beta^3 (C_2 - C_3) + Q_{zp}|_{z=0}, \\
M_z|_{z=l} &= -2D\beta^2 [C_1 \cos \beta l \cosh \beta l + C_2 \cos \beta l \sinh \beta l \\
&\quad - C_3 \sin \beta l \cosh \beta l - C_4 \sin \beta l \sinh \beta l] + M_{zp}|_{z=l}, \\
Q_z|_{z=l} &= -2D\beta^3 [C_1 (\cos \beta l \sinh \beta l - \sin \beta l \cosh \beta l) \\
&\quad + C_2 (\cos \beta l \cosh \beta l - \sin \beta l \sinh \beta l) \\
&\quad - C_3 (\cos \beta l \cosh \beta l + \sin \beta l \sinh \beta l) \\
&\quad - C_4 (\cos \beta l \sinh \beta l + \sin \beta l \cosh \beta l)] + Q_{zp}|_{z=l}.
\end{aligned} \right\} (46)$$

V. SUPERPOSITION CONSIDERATIONS IN BENDING

The basic elements of the composite structure having been established, the structure itself must now be molded. This is done by systematically combining the individual components, matching the proper quantities at the junctions, and forming the required product depicted in Fig. 1.

To attain this end, we propose to proceed in several steps. First of all, the formation of a combined plate, as shown in Fig. 11, is suggested. Next, the effect of the tubes on the plate can be accounted for, resulting in a coupled two-plate structure. Finally, the edge effect can be added to complete the final structure. This general trend will be followed in the formulation of the analysis in the succeeding sections of this report.

Fig. 11. Plate Composed of Many Axisymmetric Regions

Before delving into the superposition head-on, we shall describe the convention used for designation of symbols.

Practical and efficient use of any complicated analysis depends on the availability of a corresponding computer program patterned from the analysis. Since the construction of a computer program was anticipated from the beginning as the final undertaking of this investigation, the symbolism chosen in this analysis was inclined toward the use of the FORTRAN language. Thus, the analysis contains hybrid symbolism; one type of symbols can be directly used in the logic of the computer program, the other is purely analytical in nature for which an equivalent FORTRAN designation must be formulated.

Symbols used solely for the analysis are due to expressions that could have several meanings. It is expedient in the analysis to refer, for example, to the radial moment at the junction in general, i.e., pertaining to both plates. Later the same expression is subscripted to particularize it to the top or the bottom plate. This could not be done too well with the computer symbols.

The use of the subscripts derives its order from the chronological appearance of the symbols in the analysis. For example, the radial moment of the plates in general is taken to be M_r . If the same thermoelastic relation of radial moment and deflection is to apply to the particular deflection w_p instead, as in Eq. 10, then the lower-case letter p is attached; i.e., M_{rp} . When the same moment later pertains to the same particular region n in the combined plate, then the letter n , or some other relevant index, is used as the subscript: M_{rpn} , M_{rpi} , or M_{rpij} . If the moment refers specifically to the top plate, then the capital letter A is also added to the subscripts; i.e., M_{rpnA} .

Evaluation of the quantity in question at any particular boundary is designated in a conventional manner: A vertical bar is set after the symbol, and the evaluation to be performed is given by the radius shown on the lower right side of the bar. If the previously described radial moment is evaluated at the outer boundary of region n , that is, at $r = r_n$, then the bar follows all the other subscripts in question and the designation becomes $M_{rpnA}|_{r=r_n}$ or $M_{rpnA}|_{r_n}$.

A. Multiregion Plate

The individual solutions of the plate regions having been determined, the object now is to combine them in a proper fashion and form the intended result. To this end, the junction of two representative adjacent regions is examined as to the relationship requirement of the common properties. Several conditions will be established to ensure the continuity of deflection and compatibility of internal forces from one region to the next.

The assumption that the same material extends from one region to the next implies that no abrupt discontinuities in deflection and slope can take place. Consequently, the deflection and slope at the junction of the two regions must be the same. The compatibility of internal and external forces at the junction can best be examined by referring to Fig. 12, where the radial moments and shear loads are shown. The equilibrium between the internal and the external loads must be maintained. In summary, then, all these conditions at every junction can be represented mathematically as follows:

$$\begin{aligned} M_{r(n+1)}|_{r_n} &= M_{rn}|_{r_n} + M_{PnU} + M_{PnL} \\ Q_{r(n+1)}|_{r_n} &= Q_{rn}|_{r_n} + P_{nU} - P_{nL} \end{aligned}$$

Fig. 12. Stress Condition at the Plate Junctions

$$\left. \begin{aligned} w_n|_{r_n} &= w_{n+1}|_{r_n}, \\ S_n|_{r_n} &= S_{n+1}|_{r_n}, \\ M_{rn}|_{r_n} &= M_{r(n+1)}|_{r_n} - M_{PnL} - M_{PnU}, \\ Q_{rn}|_{r_n} &= Q_{r(n+1)}|_{r_n} + P_{nL} - P_{nU}, \end{aligned} \right\} \quad (47)$$

where the junction of the n th and the $(n+1)$ th region has been taken as the representative for all; i.e., $n = 1, 2, 3, \dots, m-1$. Some comment to each of the individual continuity conditions appears to be in order.

The first condition of Eqs. 47 ensures the deflection continuity from one region to the next. This requirement can be satisfied later in the procedure, making certain that the deflection accumulated by one region is added to the deflection of the adjacent region.

This will have to be done at all junctions of the regions, so that the accumulative effect of all the regions will be accounted for.

Contrary to the previous case, the slope condition of Eq. 47 is not of the accumulative type. To satisfy this condition, the slope expressions of two adjacent regions must be set equal to each other. This results in an equation with several unknowns. Repeating this for each junction should result in as many such equations as there are unknowns. Before this is done here, it is advisable to refer to the remaining two conditions of radial moment and radial shear.

If the slope condition is used to form the equations mentioned in the previous section, the radial moment at the junction of the two regions will be the unknown. The moment condition of Eqs. 47 can then be used to express one moment at the junction in terms of the other internal moment. Thus, if $M_{r(n+1)}|_{r_n}$ is substituted into the mentioned expressions, $M_{rn}|_{r_n}$ remains the only unknown moment at that junction. As a result, there will be as many such unknown moments as there are common junctions.

The radial shear condition provides for the equilibrium of the external and internal shears from adjacent regions. As of this point, however, the shear is not known in terms of the external loads. Nevertheless, this can be determined without much difficulty.

The last expression of Eqs. 14, shows a direct relationship of the external loads with those of the shear at the edge. Using the last condition of Eqs. 9 in conjunction with the expression of C_{4n} in Eqs. 16, we can derive similar expressions for $n = 2, 3, 4$, etc., for shear. These successive relations contain two unknown shear loads, $Q_{rn}|_{r_n}$ and $Q_{r(n+1)}|_{r_n}$, at each junction. Here the shear condition of Eqs. 47 can be used to advantage. Thus by induction, the shear at any joint can be determined. Writing this expression in the form of its most frequent appearance in the mentioned expressions yields

$$Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} = - \frac{P_{0L} - P_{0U}}{2\pi r_{n-1}} - \sum_{j=1}^{n-1} \frac{r_j}{r_{n-1}} \left\{ [P_{jL} - P_{jU}] - [Q_{rpj}|_{r_j} - Q_{rp(j+1)}|_{r_j}] \right\}, \quad (48)$$

where the subscripts L and U designate the lower and upper surfaces of the plate, respectively.

Having discussed the role of the moment and shear conditions, we can pursue the matter of the slope further. Thus, the first derivative of deflection represents the slope expression. The moment condition of Eqs. 47 and 48 can now be substituted and set equal to each other, as specified by the slope requirement of Eqs. 47. The resultant equation contains the radial moment at the junctions as the unknowns.

Recall that the slope expression of the first region is different from that of the remaining regions. Upon careful consideration, however, the general slope expression can be applied to the first region, provided in the first region $r_0 = 0$ and $r_0 \log_e(r_0) = 0$. Consequently, the equation derived from the superposition at the first junction is made to be the same as those derived at the other junctions. Hence, the resultant equations, corresponding to the superposition of the junction at the outside of each specific region, can be presented in the following way:

$$\begin{aligned} & ZA(n-1) ZC(n+1) M_{r(n-1)}|_{r_{n-1}} - [ZC(n+1) ZE(n) + ZC(n) ZD(n+1)] M_{rn}|_{r_n} \\ & + ZA(n+1) ZC(n) M_{r(n+1)}|_{r_{n+1}} \\ & = - \frac{1}{2\pi r_n} [ZC(n+1) ZG(n) - ZC(n) ZF(n+1)] [P_{0L} - P_{0U}] \end{aligned} \quad (49)$$

Contd.

$$\begin{aligned}
& - \left[ZC(n+1) ZG(n) - ZC(n) ZF(n+1) \right] \sum_{j=1}^{n-1} \frac{r_j}{r_n} \left[P_{jL} - P_{jU} \right] \\
& + ZC(n) ZF(n+1) \left[P_{nL} - P_{nU} \right] - ZA(n-1) ZC(n+1) \left[M_{P(n-1)L} \right. \\
& \left. + M_{P(n-1)U} \right] + ZC(n) ZD(n+1) \left[M_{PnL} + M_{PnU} \right] \\
& + ZC(n+1) \left[ZA(n-1) M_{rpn} \Big|_{r_{n-1}} - ZE(n) M_{rpn} \Big|_{r_n} \right] \\
& - ZC(n) \left[ZD(n+1) M_{rP(n+1)} \Big|_{r_n} - ZA(n+1) M_{rP(n+1)} \Big|_{r_{n+1}} \right] \\
& + \left[ZC(n+1) ZG(n) - ZC(n) ZF(n+1) \right] \sum_{j=1}^{n-1} \frac{r_j}{r_n} \left[Q_{rpj} \Big|_{r_j} - Q_{rp(j+1)} \Big|_{r_j} \right] \\
& - ZC(n) ZF(n+1) \left[Q_{rpn} \Big|_{r_n} - Q_{rp(n+1)} \Big|_{r_n} \right] - ZB(n) \left[S_{pn} \Big|_{r_n} \right. \\
& \left. - S_{p(n+1)} \Big|_{r_n} \right] \quad (n = 1, 2, 3, \dots, m-1), \quad \text{Contd. (49)}
\end{aligned}$$

where

$$ZA(n) = 2r_n^2,$$

$$ZB(n) = \frac{1}{r_n} D_n D_{n+1} (1 - \nu_n^2)(1 - \nu_{n+1}^2)(r_n^2 - r_{n-1}^2)(r_{n+1}^2 - r_n^2),$$

$$ZC(n) = D_n (1 - \nu_n^2)(r_n^2 - r_{n-1}^2),$$

$$ZD(n) = (1 + \nu_n) r_n^2 + (1 - \nu_n) r_{n-1}^2,$$

$$ZE(n) = (1 - \nu_n) r_n^2 + (1 + \nu_n) r_{n-1}^2,$$

$$ZF(n) = \frac{1}{2} r_{n-1} \left[(1 - \nu_n)(r_n^2 - r_{n-1}^2) - 2(1 + \nu_n) r_n^2 \log_e \left(\frac{r_{n-1}}{r_n} \right) \right],$$

and

$$ZG(n) = \frac{1}{2} r_n \left[(1 - \nu_n)(r_n^2 - r_{n-1}^2) - 2(1 + \nu_n) r_{n-1}^2 \log_e \left(\frac{r_{n-1}}{r_n} \right) \right].$$

Thus, Eqs. 49 represent a set of linear, simultaneous algebraic equations, where $M_{rn}|_{r_n}$ (for $n = 1, 2, 3, \dots, m-1$) should be regarded as the unknowns. These are the internal junction moments of the plate. The external loads of Fig. 11 appear on the right-hand side of the equations. If Fig. 11 were to represent the final goal of an analysis, the external loads would be given (making the right-hand side constant), and the solution of the problem would be represented in the form of Eqs. 49.

The intended product of this analysis, however, is a composite structure of coupled plates, which has been represented in Fig. 1. Thus, some of the loads on the right-hand side of Eqs. 49 would be interdependent of other structural characteristics of the components and thus could not be known at this point. It remains, therefore, to solve for the known internal moments, $M_{rn}|_{r_n}$, in terms of the quantities appearing on the right-hand side of Eqs. 49.

The moments $M_{rn}|_{r_n}$ (for $n = 1, 2, 3, \dots, m-1$) are solved by the matrix-inversion procedure. Mathematically, when the determinant belonging to a system of $m-1$ linear equations with $m-1$ unknowns is nonzero, the system has a unique solution given by

$$M_{rn}|_{r_n} = \sum_{i=1}^{m-1} a_{ni}^{-1} b_i \quad (n = 1, 2, 3, \dots, m-1), \quad (50)$$

where the a_{ni}^{-1} are the coefficients of the inverse matrix of the determinant, and b_i are the elements of the vector as represented by the right-hand side of Eqs. 49. Hence, Eq. 50 represents the moments as a linear combination of the right-hand terms of Eqs. 49.

It is convenient to represent the set of simultaneous equations in the form particularly suited for the matrix-inversion procedure described above. This is especially true since computer programs of the matrix-inversion procedure are available. For this purpose, the expressions of the vector elements b_i are revised and formalized by grouping common terms together, yielding

$$\begin{aligned}
b_i = & -ZA(m) ZC(m-1) M_{rm} \Big|_{r_m} + ZPO(i) [P_{oL} - P_{oU}] + \sum_{j=1}^i ZP(i,j) [P_{jL} - P_{jU}] \\
& + ZMPO(i) [M_{P(i-1)L} + M_{P(i-1)U}] + ZMP1(i) [M_{PiL} + M_{PiU}] + ZMPS(i).
\end{aligned} \tag{51}$$

Thus, Eqs. 49 can be transformed into the coefficients of the determinant and the elements of the vector as follows:

Matrix Coefficients for $n = 1, 2, 3, \dots, m-1$

$$\begin{aligned}
C(n,s) &= 0 \quad (s = 1, 2, 3, \dots, n-2), \\
C(n,n-1) &= ZA(n-1) ZC(n+1), \\
C(n,n) &= -ZC(n+1) ZF(n) - ZC(n) ZD(n+1), \\
C(n,n+1) &= ZA(n+1) ZC(n), \\
C(n,s) &= 0 \quad (s = n+2, n+3, n+4, \dots, m-3).
\end{aligned}$$

Vector Elements

$$\begin{aligned}
ZPO(n) &= -\frac{1}{2\pi r_n} [ZC(n+1) ZG(n) - ZC(n) ZF(n+1)], \\
ZP(n,j) &= -\frac{r_j}{r_n} [ZC(n+1) ZG(n) - ZC(n) ZF(n+1)] \quad (j = 1, 2, 3, \dots, n-1), \\
ZP(n,n) &= ZC(n) ZF(n+1), \\
ZMPO(n) &= -ZC(n+1) ZA(n-1), \\
ZMP1(n) &= ZC(n) ZD(n+1), \\
ZMPS(n) &= ZC(n+1) ZA(n-1) M_{rpn} \Big|_{r_{n-1}} - ZC(n+1) ZE(n) M_{rpn} \Big|_{r_n} \\
&\quad - ZC(n) ZD(n+1) M_{rp(n+1)} \Big|_{r_n} + ZC(n) ZA(n+1) M_{rp(n+1)} \Big|_{r_{n+1}} \\
&\quad + [ZC(n+1) ZG(n) - ZC(n) ZF(n+1)] \sum_{j=1}^{n-1} \frac{r_j}{r_n} [Q_{rpj} \Big|_{r_j} - Q_{rp(j+1)} \Big|_{r_j}] \\
&\quad - ZC(n) ZF(n+1) [Q_{rpn} \Big|_{r_n} - Q_{rp(n+1)} \Big|_{r_n}] - ZB(n) [S_{pn} \Big|_{r_n} - S_{p(n+1)} \Big|_{r_n}].
\end{aligned} \tag{52}$$

Hence, Eq. 50, with the associated coefficients of Eqs. 51 and 52, provides the solution for the internal moments at the junctions of the regions in the combined plate. Since the equations of deflection and moments are all in terms of these internal moments (by way of the constants of integration), this may also be regarded as a solution of the multiregion plate problem depicted in Fig. 11.

In anticipation of building up to the composite structure by utilizing this multiregion plate solution, additional information might be derived here for later use. This may consist of deflection and slope equations at the junctions of the regions, expressed in terms of the external loads acting on the plate.

This purpose is attained by proceeding backwards from the here-to-fore-established process: using the latest expressions being substituted into its predecessor formulas, and substituting again, until the intended expression is formed. Thus, starting with Eq. 50, the new expression, b_i , of Eq. 51 is substituted and common terms grouped together yielding

$$\begin{aligned}
 M_{rn}|_{r_n} = & -ZA(m) ZC(m-1) a_{(m-1)(m-1)}^{-1} M_{rm}|_{r_m} \\
 & + \sum_{i=1}^{m-1} a_{ni}^{-1} ZPO(i) [P_{0L} - P_{0U}] + \sum_{i=1}^{m-1} \sum_{j=1}^i a_{ni}^{-1} ZP(i,j) [P_{jL} - P_{jU}] \\
 & + \sum_{i=1}^{m-2} \left[a_{n(i+1)}^{-1} ZMPO(i+1) + a_{ni}^{-1} ZMPl(i) \right] [M_{PiL} - M_{PiU}] \\
 & + a_{n(m-1)}^{-1} ZMPl(m-1) [M_{p(m-1)L} + M_{p(m-1)U}] \\
 & + \sum_{i=1}^{m-1} a_{ni}^{-1} ZMPS(i). \tag{53}
 \end{aligned}$$

The deflection expression, $w_n|_{r_{n-1}}$, using Eq. 5, is next expressed in terms of the constants of integration given in Eqs. 16. Consequently, the deflection at the junctions becomes

$$\begin{aligned}
w_n|_{r_{n-1}} = & - ZJ(n) \left[M_{r(n-1)}|_{r_{n-1}} + M_{P(n-1)} - M_{rpn}|_{r_{n-1}} \right] \\
& + ZK(n) \left[M_{rn}|_{r_n} - M_{rpn}|_{r_n} \right] \\
& + r_{n-1} ZL(n) \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right] \\
& + w_{pn}|_{r_{n-1}} - w_{pn}|_{r_n} + w_{n+1}|_{r_n} \quad (n = 2, 3, 4, \dots, m), \quad (54)
\end{aligned}$$

where

$$ZJ(n) = r_{n-1} \frac{ZF'(n)}{ZC(n)},$$

$$ZK(n) = r_n \frac{ZG(n)}{ZC(n)},$$

$$ZL(n) = \frac{1}{8} \frac{1}{ZC(n)} \left[(3 + \nu_n)(1 - \nu_n)(r_n^2 - r_{n-1}^2)^2 + 2(1 + \nu_n)^2 r_n^2 r_{n-1}^2 \log_e \left(\frac{r_{n-1}}{r_n} \right) \right].$$

The above deflection expression of $w_n|_{r_{n-1}}$ is given, among other variables, in terms of its succeeding deflection, $w_{n+1}|_{r_n}$. If a similar expression is obtained for $w_{n+1}|_{r_n}$ using Eq. 54 and this procedure is repeated again, we can project that a series of such expressions will exist ranging from n to $m - 1$. Thus, making this summation of the individual terms of Eq. 54, noticing that at the edge, $w_{m+1}|_{r_m} = 0$, and factoring out common terms, we obtain

$$\begin{aligned}
w_n|_{r_{n-1}} = & \sum_{k=n}^{m-1} \left\{ -ZJ(k) \left[M_{r(k-1)}|_{r_{k-1}} + M_{P(k-1)} - M_{rpk}|_{r_{k-1}} \right] \right. \\
& + ZK(k) \left[M_{rk}|_{r_k} - M_{rpk}|_{r_k} \right] \\
& \left. + r_{n-1} ZL(k) \left[Q_{rk}|_{r_{k-1}} - Q_{rpk}|_{r_{k-1}} \right] + w_{pk}|_{r_{k-1}} - w_{pk}|_{r_k} \right\} \\
& (n = 2, 3, 4, \dots, m). \quad (55)
\end{aligned}$$

Equation 55 is of the form into which Eqs. 48 and 53 can be readily substituted. When this is done and the corresponding loads are grouped together, the result becomes

$$\begin{aligned}
 w_n|_{r_{n-1}} &= \sum_{j=0}^{m-1} ZWP(n, j+1) [P_{jL} - P_{jU}] \\
 &+ \sum_{i=1}^{m-1} ZWM(n, i) [M_{PiL} + M_{PiU}] + ZWM(n, m) M_{rm}|_{r_m} \\
 &+ ZWB(n) \quad (n = 2, 3, 4, \dots, m),
 \end{aligned} \tag{56}$$

where

$$\begin{aligned}
 ZWP(n, 1) &= \sum_{k=n}^m \left[\sum_{i=1}^{m-1} ZKJ(i, k) ZPO(i) - \frac{1}{2\pi} ZL(k) \right], \\
 ZWP(n, j+1) &= \sum_{k=n}^m \left[\sum_{i=j}^{m-1} ZKJ(i, k) ZP(i, j) - r_j ZL(k) \right] \quad (j = 1, 2, 3, \dots, n-1), \\
 ZWP(n, j+1) &= \sum_{k=n}^m \sum_{i=j}^{m-1} ZKJ(i, k) ZP(i, j) \\
 &- \sum_{k=j+1}^m r_j ZL(k) \quad (j = n, n+1, n+2, \dots, m-1), \\
 ZWM(n, i) &= \sum_{k=n}^m \left[ZKJ(i+1, k) ZMPO(i+1) \right. \\
 &\left. + ZKJ(i, k) ZMP1(i) \right] \quad (i = 1, 2, 3, \dots, n-2), \\
 ZWM(n, i) &= \sum_{k=n}^m \left[ZKJ(i+1, k) ZMPO(i+1) + ZKJ(i, k) ZMP1(i) \right] \\
 &- ZJ(i+1) \quad (i = n-1, n, n+1, \dots, m-2),
 \end{aligned}$$

$$ZWM(n, m-1) = \sum_{k=n}^m ZKJ(m-1, k) ZMP1(m-1) - ZJ(m),$$

$$ZWM(n, m) = -ZA(m) ZC(m-1) \sum_{k=n}^m ZKJ(m-1, k) + AK(m),$$

$$\begin{aligned} ZWB(n) = \sum_{k=n}^m \left\{ \sum_{i=1}^{m-1} ZKJ(i, k) ZMPS(i) + ZJ(k) M_{rp k} \Big|_{r_{k-1}} \right. \\ \left. - ZK(k) M_{rp k} \Big|_{r_k} + w_{pk} \Big|_{r_{k-1}} - w_{pk} \Big|_{r_k} + \sum_{j=1}^{k-1} r_j ZL(k) \left[Q_{rp j} \Big|_{r_j} \right. \right. \\ \left. \left. - Q_{rp(j+1)} \Big|_{r_j} \right] \right\}, \end{aligned}$$

and

$$ZWM(m-1, m) = -ZA(m) ZC(m-1) \sum_{k=m-1}^m ZKJ(m-1, k) - ZK(m),$$

$$ZKJ(i, n) = \alpha_{ni}^{-1} ZK(n) - \alpha_{(n-1)i}^{-1} ZJ(n),$$

$$ZKJ(i, 1) = \alpha_{ii}^{-1} ZK(1),$$

$$ZKJ(i, m) = -\alpha_{(m-1)i}^{-1} ZJ(m).$$

Recall again that deflection expression of the first region ($n=1$) is different from that of the others; consequently, Eq. 56 does not apply there. Apparently the deflection at the center of the plate cannot be determined by Eq. 56 alone. It appears necessary to refer back to Eqs. 14 for the proper expression of w_1 . Since the deflection at the center is in terms of $w_2 \Big|_{r_1}$, Eq. 56 will be used here to provide the contribution of all the other regions. Thus, using the deflection expression of Eqs. 14 in conjunction with Eq. 56 and combining common terms, we obtain the following expression for the deflection in the center of the multiregion plate:

$$\begin{aligned}
w_1|_{r_0} = & \sum_{j=0}^{m-1} ZWP(i, j+1) [P_{jL} - P_{jU}] + \sum_{i=1}^{m-1} ZWM(1, i) [M_{PiL} + M_{PiU}] \\
& + ZWM(1, m) M_{rm}|_{r_m} + ZWB(1),
\end{aligned} \tag{57}$$

where

$$ZWP(1, 1) = ZWP(2, 1) - \frac{1}{2\pi} ZL(1) + ZK(1) \sum_{i=1}^{m-1} a_{1i}^{-1} ZPO(i),$$

$$ZWP(1, j+1) = ZWP(2, j+1) + ZK(1) \sum_{i=j}^{m-1} a_{1i}^{-1} ZP(i, j) \quad (j = 1, 2, 3, \dots, m-1),$$

$$\begin{aligned}
ZWM(1, i) = & ZWM(2, i) + ZK(1) \left[a_{1(i+1)}^{-1} ZMPO(i+1) \right. \\
& \left. + a_{1i}^{-1} ZMP1(i) \right] \quad (i = 1, 2, 3, \dots, m-2),
\end{aligned}$$

$$ZWM(1, m-1) = ZWM(2, m-1) + ZK(1) a_{1(m-1)}^{-1} ZMP1(m-1),$$

$$ZWM(1, m) = ZWM(2, m) - ZA(m) ZC(m-1) a_{1(m-1)}^{-1} ZK(1),$$

$$ZWB(1) = ZWB(2) + ZK(1) \sum_{i=1}^{m-1} a_{1i}^{-1} ZMPS(i) - ZK(1) M_{1p}|_{r_1} - w_{1p}|_{r_1} + w_{1p}|_{r_0},$$

and

$$ZWM(m-1, m) = ZWM(2, m) - ZA(m) ZC(m-1) a_{1(m-1)}^{-1} ZK(1) - ZK(m).$$

Hence, Eq. 56 represents the deflections at all the junctions of the annular regions of the combined multiregion plate, provided the edge of the plate is taken to be the reference. Equation 57 expresses the deflection in the center of the combined plate. Both expressions are given in terms of the external loads, which would have to be specified in any independent analysis of the multiregion plate.

The slope at the junctions of the regions is similarly determined as was the deflection in the previous paragraphs. This time, however, the first derivative of Eq. 5, given in Eqs. 14, is used as the basis. When the constants of integration are substituted therein, and common terms of the external loads combined, the slope at the inner boundary of a region becomes

$$\begin{aligned}
S_n|_{r_{n-1}} &= ZM(n) \left[M_{r(n-1)}|_{r_{n-1}} + M_{P(n-1)} - M_{rpn}|_{r_{n-1}} \right] \\
&\quad - ZN(n) \left[M_{rn}|_{r_n} - M_{rpn}|_{r_n} \right] \\
&\quad - r_{n-1} ZO(n) \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right] \\
&\quad + S_{pn}|_{r_{n-1}} \quad (n = 2, 3, 4, \dots, m), \quad (58)
\end{aligned}$$

where

$$ZM(n) = r_{n-1} ZD(n)/ZC(n)$$

$$ZN(n) = 2r_{n-1}^2 r_n / ZC(n)$$

$$ZO(n) = ZF(n)/ZC(n).$$

The same expression is obtained in terms of the external loads after Eqs. 48 and 53 are substituted into Eq. 58, yielding

$$\begin{aligned}
S_n|_{r_{n-1}} &= \sum_{j=0}^{m-1} ZSP(n, j+1) \left[P_{jL} - P_{jU} \right] + \sum_{i=1}^{m-1} ZSM(n, i) \left[M_{PiL} + M_{PiU} \right] \\
&\quad + ZSM(n, m) M_{rm}|_{r_m} + ZSB(n) \quad (n = 2, 3, 4, \dots, m), \quad (59)
\end{aligned}$$

where

$$ZSP(n, 1) = \sum_{i=1}^{m-1} ZMN(i, n) ZPO(i) + \frac{1}{2\pi} ZO(n),$$

$$ZSP(n, j+1) = \sum_{i=j}^{m-1} ZMN(i, n) ZP(i, j) + r_j ZO(n) \quad (j = 1, 2, 3, \dots, n-1)$$

$$ZSP(n, j+1) = \sum_{i=j}^{m-1} ZMN(i, n) ZP(i, j) \quad (j = n, n+1, n+2, \dots, m-1),$$

$$\begin{aligned}
ZSM(n, i) &= ZMN(i+1, n) ZMPO(i+1) \\
&\quad + ZMN(i, n) ZMPl(i) \quad (i = 1, 2, 3, \dots, m-2),
\end{aligned}$$

$$ZSM(n, m-1) = ZMN(m-1, n) ZMP1(m-1),$$

$$ZSM(n, m) = -ZA(m) ZC(m-1) ZMN(m-1, n),$$

$$ZSB(n) = \sum_{i=1}^{m-1} ZMN(i, n) ZMPS(i) - ZM(n) M_{rpn}|_{r_{n-1}} + ZN(n) M_{rpn}|_{r_n} \\ + S_{rpn}|_{r_{n-1}} - ZQ(n) \sum_{j=1}^{n-1} r_j [Q_{rpj}|_{r_j} - Q_{rp(j+1)}|_{r_j}],$$

and

$$ZSM(m-1, m) = -ZA(m) ZC(m-1) ZMN(m-1, m-1) - ZN(m),$$

$$ZMN(i, n) = \alpha_{(n-1)i}^{-1} ZM(n) - \alpha_{ni}^{-1} ZN(n),$$

$$ZMN(i, 1) = -\alpha_{1i}^{-1} ZN(1),$$

$$ZMN(i, m) = \alpha_{(m-1)i}^{-1} ZM(m).$$

The expressions given by Eqs. 56, 57, and 59 represent deflections and slopes where the evaluation was performed at the inside boundary of each region. This was done for the express purpose of using the least number of equations applying to all the regions of the combined plate. Thus, if the formulas had been evaluated at the outside boundary of each region, then they could not apply to the first function because of the uniqueness of the first region.

It will be necessary later in the analysis portion of this report to refer to the slope at the edge of the combined plate. Equation 59 could not be used for this purpose. Thus, for reasons of ready reference, another slope expression, evaluated at the outer boundary of the respective region, is given here. This slope equation is

$$S_n|_{r_n} = \frac{r_n}{ZC(n)} \left\{ ZA(n-1) \left[M_{r(n-1)}|_{r_{n-1}} + M_{P(n-1)} - M_{rpn}|_{r_{n-1}} \right] \right. \\ \left. - ZE(n) \left[M_{rn}|_{r_n} - M_{rpn}|_{r_n} \right] \right. \\ \left. - \frac{r_{n-1}}{r_n} ZG(n) \left[Q_{rn}|_{r_{n-1}} - Q_{rpn}|_{r_{n-1}} \right] \right\} + S_{pn}|_{r_n}. \quad (60)$$

In terms of the external loads, Eq. 60 becomes

$$\begin{aligned}
 S_n|_{r_n} = & \frac{r_n}{ZC(n)} \left\{ - \left[ZA(n) ZC(n-1) ZA(n-1) a_{(n-1)(m-1)}^{-1} + ZE(n) \right] M_{rm}|_{r_m} \right. \\
 & + \left[ZA(n-1) \sum_{i=1}^{m-1} a_{(n-1)i}^{-1} ZPO(i) + \frac{ZG(n)}{2\pi r_n} \right] [P_{0L} - P_{0U}] \\
 & + \sum_{j=1}^{m-1} \left[ZA(n-1) \sum_{i=j}^{m-1} a_{(n-1)i}^{-1} ZP(i, j) + \frac{r_j}{r_n} ZG(n) \right] [P_{jL} - P_{jU}] \\
 & + ZA(n-1) \sum_{i=1}^{m-2} \left[a_{(n-1)(i+1)}^{-1} ZMPO(i+1) \right. \\
 & \left. + a_{(n-1)i}^{-1} ZMPL(i) \right] [M_{PiL} + M_{PiU}] \\
 & + ZA(n-1) \left[a_{(n-1)(m-1)}^{-1} ZMPL(m-1) + 1 \right] [M_{P(m-1)L} + M_{P(m-1)U}] \\
 & - ZA(n-1) M_{rpn}|_{r_{n-1}} + ZE(n) M_{rpn}|_{r_n} \\
 & - ZG(n) \sum_{j=1}^{m-1} \frac{r_j}{r_n} [Q_{rpj}|_{r_j} - Q_{rp(j+1)}|_{r_j}] \\
 & \left. + ZA(n-1) \sum_{i=1}^{m-1} a_{(n-1)i}^{-1} ZMPS(i) \right\} + S_{pn}|_{r_n} \quad (61)
 \end{aligned}$$

Having the deflection at the joints, we can readily determine the properties in the interior of any region. This is done by evaluating the constants of integration for the region in question by Eqs. 16, and then going to Eqs. 14 for the evaluation of the quantity desired.

B. Coupled Multiregion Plate Structure

Having obtained a solution for the multiregion plate in terms of its external loads, we now combine two such plates with tubes to form the coupled plate shown in Fig. 13. This is done by matching the deformations at the junctions of plates and tubes, and then evaluating the corresponding loads.

Fig. 13. Coupled Multiregion Plate Structure

Before this is done, however, some minor revisions might be introduced for smoother operation later. The information in this report about the connecting members pertains only to a single tube. But most of the tubes in the structure may be regarded as merely integral elements of a ring of tubes. For consistency, and in line with the other structure component's solutions, it is easier to regard the ring of tubes as a unit. In other words, the previously developed equations pertaining to a single tube should be modified for axisymmetric analysis, thus making them applicable to a ring of tubes. To do this, we can give relationships between the singular loads and loads per unit length as used in the axisymmetric analysis. Consequently, to adapt the previously developed expressions of load we must multiply them by a ratio, $2\pi r_n / \lambda_n$, where λ_n is the number of tubes in the r_n th row.

The matching of boundary conditions of the plates and the tubes needs some elaboration. We can predict, for instance, that during the deformation of the whole structure, the tube will deform the plate locally. To assume that the slope at the outside surface of the plate is the same as that of the neutral surface of the plate would not be quite correct. For more realistic calculations, therefore, the effective length of the tubes may be modified to partly counteract the possible discrepancy. To this end, the dimension ℓ may be arbitrarily changed. No experimental or analytical results are available to this date to be used as a guide for establishing the modified distance. It is left up to the judgment of the stress analyst, by observing the relative stiffness of the plate and the tubes, to specify the modifications at his discretion.

Having made the needed comments about the effective location of the end points of the tubes, we can resume the superposition of the plates-tubes. Two types of compatibility conditions must be satisfied at the effective joints. The first one is the relationship between the axial deformation of the tubes and the relative deflections of the plates. Figure 13 indicates that the difference in deflection of the top and bottom plates at the junctions must be equal to the axial deformation of the tubes represented by Eq. 23. Thus, if we make the adjustment for the axisymmetric loading, the same condition can be given mathematically, as follows:

$$\left. \begin{aligned} w_{1A}|_{r_0} - w_{1B}|_{r_0} &= \frac{P_0 \ell_0}{E_0 A_0} + N_0, \\ w_{(n+1)A}|_{r_n} - w_{(n+1)B}|_{r_n} &= \frac{2\pi r_n}{\lambda_n} \left[\frac{P_n \ell_n}{E_n A_n} + N_n \right] \quad (n = 1, 2, 3, \dots, m-1), \end{aligned} \right\} \quad (62)$$

where

$$N_n = \alpha_n \int_0^{\ell_n} \Delta T_n(x) dx$$

and the subscripts A and B designate reference to the top and bottom plates, respectively.

The second condition to be satisfied is the matching of angular deformations at the junctions. The slope of the tubes is taken to be the same as that of the neutral surface of the plate, and the transverse end displacements, y_A and y_B , are determined from geometrical considerations. Symbolically, the relationship between y_A and y_B and their corresponding angular displacements yields

$$\left. \begin{aligned} S_{An} = -S_{nA}|_{r_n} &= 2 \frac{y_{nA}}{g_{nA}} \\ S_{Bn} = -S_{nB}|_{r_n} &= 2 \frac{y_{nB}}{g_{nB}} \end{aligned} \quad (n = 1, 2, 3, \dots, m-1), \right\} \quad (63)$$

The relationship of tube moments and slopes at the ends was expressed by Eqs. 21 and 22. Conditions 63 can be incorporated there to yield the compatibility conditions. When the adjustment for the axisymmetric loading is made, the transfer of load from the effective end condition to the neutral surface is affected, and the expressions become

$$\left. \begin{aligned} M_{PnA} &= -\frac{\lambda_n E_n I_n}{2\pi r_n \ell_n^3} \left[AH(n) S_{(n+1)} A \Big|_{r_n} + AI(n) S_{(n+1)} B \Big|_{r_n} \right] \quad (n = 1, 2, 3, \dots, m-1), \\ M_{PnB} &= -\frac{\lambda_n E_n I_n}{2\pi r_n \ell_n^3} \left[BI(n) S_{(n+1)} A \Big|_{r_n} + BH(n) S_{(n+1)} B \Big|_{r_n} \right] \quad (n = 1, 2, 3, \dots, m-1), \end{aligned} \right\} \quad (64)$$

where

$$AH(n) = 4\ell_n^2 + 6\ell_n g_{nA} + 3g_{nA}^2,$$

$$BH(n) = 4\ell_n^2 + 6\ell_n g_{nB} + 3g_{nB}^2,$$

$$AI(n) = 2\ell_n^2 + 3\ell_n g_{nA} + 3\ell_n g_{nB} + 3g_{nA}g_{nB},$$

and

$$BI(n) = AI(n).$$

Thus, Eqs. 62 and 64 provide the needed compatibility conditions at the plate-tube junctions.

If the compatibility conditions are satisfied at every junction, then the formation of a coupled plate-tube structure is accomplished. In these conditions, we must separate the known and the unknown parameters. To this end, still the same definition is used as before: The so-called known parameters will be the loads external to the structure, while the unknowns are the internal loads. Equations 56, 57, and 59 provide the expressions for the deflections and slopes at the junctions in terms of the external loads to the combined plate of Fig. 11. These equations hold true for both the top and bottom plates of the coupled structure, provided the relevant quantities are appropriately designated by subscripts A and B, respectively. With reference to Fig. 13, however, the loads on the bottom surface of the top plate and the top surface of the bottom plate would be internal to the coupled structure. Therefore, these loads are considered to be the unknown parameters. Because of practical considerations, it may be convenient to consider the end moments as unknowns also. Only rarely would the magnitude of the end moments be known numerically. More likely the edge condition could be specified as either free or fixed, or more generally, as part of composite shell structure. In either case, relations of the moment could be derived in terms of other variables which affect the edge condition at hand. The unknowns having been thus defined, it remains to group the known and the unknown parameters together and present them in the form convenient to solve; as was done previously with Eqs. 49.

As a result of the compatibility conditions of the coupled structure, there are four separate sets of equations. The first set pertains to the equality of the axial deformation of the central tube with the central deflections of the two plates. This equation is obtained by substituting Eq. 57 into the first of Eqs. 62 yielding:

Matrix Coefficients

$$\begin{aligned} C(1, 1) &= AWP(1, 1) + BWP(1, 1), \\ C(1, j+1) &= AWP(1, j+1) + BWP(1, j+1) \quad (j = 1, 2, 3, \dots, m-1), \\ C(1, m+i) &= AWM(1, i) \quad (i = 1, 2, 3, \dots, m), \\ C(1, 2m+i) &= -BWM(1, i) \quad (i = 1, 2, 3, \dots, m); \end{aligned}$$

Revised Special Coefficient

$$C(1, 1) = C(1, 1) - \left[\frac{\ell_0}{E_0 A_0} + e_0 \right];$$

Vector Element

$$\begin{aligned} b_1 &= AWP(1, 1) P_{0AU} + BWP(1, 1) P_{0BL} \\ &+ \sum_{j=1}^{m-1} [AWP(1, j+1) P_{jAU} + BWP(1, j+1) P_{jBL}] \\ &- \sum_{i=1}^{m-1} [AWM(1, i) M_{PiAU} - BWM(1, i) M_{PiBL}] \\ &- AWB(1) + BWB(1), \end{aligned}$$

(65)

where the subscripts A and B refer to the top and bottom plates, respectively. Similarly, A and B have been substituted for the first letter Z of the previously used symbols to make the same designation.

The unknown parameters in the matrix of Eqs. 65 were first taken to be $P_0, P_1, P_2, \dots, P_{m-1}$;* next, the moments of the top plate, $MP_{1AL}, MP_{2AL}, MP_{3AL}, \dots, MP_{(m-1)AL}, M_{rmA}|_{r_m}$; and last, the moments of the bottom plate, $MP_{1BU}, MP_{2BU}, MP_{3BU}, \dots, MP_{(m-1)BU}, M_{rmB}|_{r_m}$. The second

*Note that the law of transmissibility has been applied here; namely, $P_{nAL} = P_{nBU} = P_n$.

compatibility condition represents the axial deformation and transverse deflection equivalence for all the other tubes. This set of conditions is obtained by substituting Eq. 56 into the remaining Eqs. 62, yielding:

Matrix Coefficients for $n = 1, 2, 3, \dots, m - 1$

$$\begin{aligned} C(n+1, 1) &= AWP(n+1, 1) + BWP(n+1, 1), \\ C(n+1, j+1) &= AWP(n+1, j+1) \\ &\quad + BWP(n+1, j+1) \quad (j = 1, 2, 3, \dots, m-1), \\ C(n+1, m+i) &= AWM(n+1, i) \quad (i = 1, 2, 3, \dots, m), \\ C(n+1, 2m+i) &= -BWM(n+1, i) \quad (i = 1, 2, 3, \dots, m); \end{aligned}$$

Revised Special Coefficients

$$C(n+1, n+1) = C(n+1, n+1) - \frac{2\pi r_n}{\lambda_n} \left[\frac{\ell_n}{E_n A_n} + e_n \right]; \quad (66)$$

Vector Elements

$$\begin{aligned} b_{n+1} &= AWP(n+1, 1) P_{0AU} + BWP(n+1, 1) P_{0BL} \\ &\quad + \sum_{j=1}^{m-1} [AWP(n+1, j+1) P_{jAU} + BWP(n+1, j+1) P_{jBL}] \\ &\quad - \sum_{i=1}^{m-1} [AWM(n+1, i) M_{PiAU} - BWM(n+1, i) M_{PiBL}] \\ &\quad - AWB(n+1) + BWB(n+1). \end{aligned}$$

The third set of constants pertains to the equivalence of the tube rotation to the slope of the top plate at the respective junction. The set of constants is obtained by substituting Eq. 59 into the first set of Eqs. 64, yielding:

Matrix Coefficients for $n = 1, 2, 3, \dots, m - 1$

$$\begin{aligned} C(m+n, 1) &= AH(n) ASP(n+1, 1) - AI(n) BSP(n+1, 1), \\ C(m+n, j+1) &= AH(n) ASP(n+1, j+1) \\ &\quad - AI(n) BSP(n+1, j+1) \quad (j = 1, 2, 3, \dots, m-1), \end{aligned} \quad \left. \begin{array}{l} (67) \\ \text{Contd.} \end{array} \right\}$$

$$\begin{aligned} C(m+n, m+i) &= AH(n) ASM(n+1, i) & (i = 1, 2, 3, \dots, m), \\ C(m+n, 2m+i) &= AI(n) BSM(n+1, i) & (i = 1, 2, 3, \dots, m); \end{aligned}$$

Revised Special Coefficients

$$C(m+n, m+n) = AH(n) ASM(n+1, n) + \frac{2\pi r_n \ell_n^3}{\lambda_n E_n A_n};$$

Vector Elements

$$b_{m+n} = AH(n) ASP(n+1, 1) P_{0AU} - AI(n) BSP(n+1, 1) P_{0BL}$$

$$+ \sum_{j=1}^{m-1} \left[AH(n) ASP(n+1, j+1) P_{jAU} \right. \\ \left. - AI(n) BSP(n+1, j+1) P_{jBL} \right]$$

$$- \sum_{i=1}^{m-1} \left[AH(n) ASM(n+1, i) M_{PiAU} \right. \\ \left. + AI(n) BSM(n+1, i) M_{PiBL} \right]$$

$$- AH(n) ASB(n+1) - AI(n) BSB(n+1).$$

Contd.
(67)

The fourth and last condition satisfies the compatibility condition of angular deformation at the junctions of the lower plate. The set of constants representing this condition is obtained by substituting Eq. 59 into the second set of Eqs. 64, yielding:

Matrix Coefficients for $n = 1, 2, 3, \dots, m-1$

$$\begin{aligned} C(2m+n-1, 1) &= BI(n) ASP(n+1, 1) - BH(n) BSP(n+1, 1), \\ C(2m+n-1, j+1) &= BI(n) ASP(n+1, j+1) \\ &\quad - BH(n) BSP(n+1, j+1) \quad (j = 1, 2, 3, \dots, m-1), \\ C(2m+n-1, m+i) &= BI(n) ASM(n+1, i) \quad (i = 1, 2, 3, \dots, m), \\ C(2m+n-1, 2m+i) &= BH(n) BSM(n+1, i) \quad (i = 1, 2, 3, \dots, m); \end{aligned}$$

(68)
Contd.

Revised Special Coefficients

$$C(2m+n-1, 2m+n) = BH(n) BSM(n+1, n) + \frac{2\pi r_n \ell_n^3}{\lambda_n E_n A_n};$$

Vector Elements

$$b_{2m+n-1} = BI(n) ASP(n+1, 1) P_{0AU} - BH(n) BSP(n+1, 1) P_{0BL}$$

$$+ \sum_{j=1}^{m-1} [BI(n) ASP(n+1, j+1) P_{jAU}$$

$$- BH(n) BSP(n+1, j+1) P_{jBL}]$$

$$- \sum_{i=1}^{m-1} [BI(n) ASM(n+1, i) M_{PiAU}$$

$$+ BH(n) BSM(n+1, i) M_{PiBL}]$$

$$- BI(n) ASB(n+1) - BH(n) BSB(n+1).$$

Contd.
(68)

Equations 65-68 provide coefficients of a $3m$ by $3m - 2$ array for the coefficient matrix, and the respective number $(3m - 2)$ of vectors. In the above equations, there are $3m$ unknowns with $3m - 2$ conditions, and the unknowns cannot be explicitly evaluated. The additional information needed can be derived from the conditions at the edge of the plates. Only when these two conditions are specified and solved together with the above equations, will the explicit solution be possible.

C. Additional Accessories

The accessories to the two-plate coupled structure are all the remaining components to form the final structure shown in Fig. 1. Additional components are the end supports and the intermediate shells. Conditions associated with them are treated separately in the following paragraphs.

1. Edge Effects

The edge condition, described at the beginning of this report, may be twofold: The first type is the arbitrary specification of the degree of fixity at the edges; the second type is a support made up of circular cylindrical shells shown in Fig. 1.

a. Arbitrary Edge Conditions. Usually the numerical values of the edge moments of the plates are unknown and are difficult to predict. However, the edge moments may be expressed in terms of other parameters, the magnitude of which is easier to approximate. Thus, two extreme conditions can be specified: The edge moments have a value of zero if the edges are free of bending restraint, and the edges are considered fixed when a moment exists that makes the slope zero. To make the consideration of intermediate restraint conditions also feasible, the edge-slope expression of Eq. 61 is set equal to zero and solved for the edge moment $M_{rm}|_{r_m}$. This expression is multiplied by a factor k , which may be regarded as the percentage indicator of edge restraint. As an example, a value of unity for k signifies 100% fixity or ideal rigidity, and, on the other extreme, a value of zero pertains to no bending restraint at all, or a free-edge condition.

In line with the final presentation of Section B above, the equation of the top plate is given here in terms of the coefficients of the coefficient matrix and the vector. This is achieved by rearranging the modified slope expression, attaching the appropriate subscript A, and separating the known-unknown variables. This yields:

Matrix Coefficients

$$\begin{aligned}
 C(3m-1, 1) &= k_A \left[AA(m-1) \sum_{i=1}^{m-1} a_{(m-1)iA}^{-1} APO(i) + \frac{AG(m)}{2\pi r_m} \right], \\
 C(3m-1, j+1) &= k_A \sum_{j=1}^{m-1} \left[AA(m-1) \sum_{i=j}^{m-1} a_{(m-1)iA}^{-1} AP(i, j) \right. \\
 &\quad \left. + \frac{r_j}{r_m} AG(m) \right] \quad (j = 1, 2, 3, \dots, m-1), \\
 C(3m-1, m+i) &= k_A AA(m-1) \sum_{i=1}^{m-2} \left[a_{(m-1)(i+1)A}^{-1} AMPO(i+1) \right. \\
 &\quad \left. + a_{(m-1)iA}^{-1} AMP1(i) \right] \quad (i = 1, 2, 3, \dots, m-2), \\
 C(3m-1, 2m-1) &= k_A AA(m-1) \left[1 + a_{(m-1)(m-1)A}^{-1} AMP1(m-1) \right],
 \end{aligned} \tag{69}$$

Contd.

$$C(3m-1, 2m) = -k_A AA(m-1) AA(m) AC(m-1) a_{(m-1)(m-1)}^{-1} A - AE(m),$$

$$C(3m-1, 2m+s) = 0 \quad (s = 1, 2, 3, \dots, m);$$

Vector Element

$$\begin{aligned} b_{3m-1} = k_A \left\{ \right. & AA(m-1) M_{rpmA} \Big|_{r_{m-1}} - AE(m) M_{rpmA} \Big|_{r_m} \\ & + AG(m) \sum_{j=1}^{m-1} \frac{r_j}{r_m} [Q_{rpjA} \Big|_{r_j} - Q_{rp(j+1)A} \Big|_{r_j}] \\ & - \frac{1}{r_m} AC(m) S_{pmA} \Big|_{r_m} - AA(m) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} A \text{ AMPS}(i) \Big\} \\ & + C(3m-1, 1) P_{0AU} + \sum_{j=1}^{m-1} C(3m-1, j+1) P_{jAU} \\ & - \sum_{i=1}^{m-1} C(3m-1, m+i) M_{PiAU}. \end{aligned} \quad \left. \begin{array}{l} \text{Contd.} \\ (69) \end{array} \right\}$$

Similarly, the edge condition of the bottom plate becomes

Matrix Coefficients

$$\begin{aligned} C(3m, 1) &= k_B \left[BA(m-1) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} BPO(i) + \frac{BG(m)}{2\pi r_m} \right], \\ C(3m, j+1) &= k_A \sum_{j=1}^{m-1} \left[BA(m-1) \sum_{i=j}^{m-1} a_{(m-1)i}^{-1} BPO(i, j) \right. \\ &\quad \left. + \frac{r_j}{r_m} BG(m) \right] \quad (j = 1, 2, 3, \dots, m-1), \\ C(3m, s) &= 0. \quad (s = 1, 2, 3, \dots, m), \end{aligned} \quad \left. \begin{array}{l} (70) \\ \text{Contd.} \end{array} \right\}$$

$$C(3m, 2m+i) = k_B BA(m-1) \sum_{i=1}^{m-2} \left[a_{(m-1)(i+1)B}^{-1} BMPO(i+1) + a_{(m-1)iB}^{-1} BMP1(i) \right], \quad (i = 1, 2, 3, \dots, m-2)$$

$$C(3m, 3m-1) = k_B BA(m-1) \left[1 + a_{(m-1)(m-1)B}^{-1} BMP1(m-1) \right],$$

$$C(3m, 3m) = -k_B BA(m-1) BA(m) BC(m-1) a_{(m-1)(m-1)B}^{-1} - BE(m).$$

Vector Element

$$\begin{aligned} b_{3m} = k_D \left\{ BA(m-1) M_{rpmB} \Big|_{r_{m-1}} - BE(m) M_{rpmB} \Big|_{r_m} \right. \\ + BG(m) \sum_{j=1}^{m-1} \frac{r_j}{r_m} \left[Q_{rpjB} \Big|_{r_j} - Q_{rp(j+1)B} \Big|_{r_j} \right] \\ - \frac{1}{r_m} BC(m) S_{pmB} \Big|_{r_m} - BA(m-1) \sum_{i=1}^{m-1} a_{(m-1)iB}^{-1} BMPS(i) \Big\} \\ - C(3m, 1) P_{0BL} - \sum_{j=0}^{m-1} C(3m, j+1) P_{jBL} \\ - \sum_{i=1}^{m-1} C(3m, 2m+i) M_{PiBL} \end{aligned} \quad \left. \vphantom{\sum_{j=1}^{m-1}} \right\} \text{Contd. (70)}$$

Equations 69 and 70 supply two additional conditions to be used in conjunction with Eqs. 65-68, if the general edge condition is specified. Another, more realistic edge condition may be useful in design. This is discussed in Section b below.

b. Edges Made up of Shells. A sample structure of a pressure vessel edge support is shown in Fig. 14. This is a simplified version of a structure usually found in an actual pressure vessel. Nevertheless, it is a good guide for design purposes. Other analytical models may also be constructed to serve the design in question and could be incorporated in the analysis instead. Without any particular construction in mind, the one shown in Fig. 14 should serve well for the illustrative purposes of this analysis.

Fig. 14. Sample Edge Supports

The end support structure in question consists of two rings with finite and semi-infinite shells attached. For the ring shown in the free-body diagrams of Fig. 14, Timoshenko⁵ gives a relationship between the angle of twist and the resultant circumferential moment. In terms of the present symbols, this relation becomes

$$M_c = \left(1 - \frac{b}{2r_m}\right) \phi_m ZT, \quad (71)$$

where

$$ZT = \frac{Ebh_m^3}{12r_m \left(r_m - \frac{b}{2}\right)},$$

ϕ_m is the angular displacement, and M_c is the resultant moment. When Eq. 71 is applied to each ring separately, and M_c is expressed in terms of the loads shown in the free-body diagrams of Fig. 14, the resultant expressions can be shown to be

$$\left. \begin{aligned} M_{rmA}|_{r_m} &= M_{mA U} - M_{mA L} + \frac{h_{mA}}{2} (Q_{mA L} - Q_{mA U}) - AT \cdot \phi_{mA}, \\ M_{rmB}|_{r_m} &= M_{mB U} - M_{mB L} + \frac{h_{mB}}{2} (Q_{mB L} - Q_{mB U}) - BT \cdot \phi_{mB}, \end{aligned} \right\} \quad (72)$$

where the first letter of the symbol ZT has been replaced by A or B , designating reference to the top or bottom plate, respectively.

Before Eqs. 72 can be used in conjunction with the basic equations of the coupled plate structure of Fig. 13, they must be expressed in terms of the same unknowns as before. This is accomplished by forming the composite edge structure using Eqs. 43 and 46, and then superimposing these resultant expressions with those at the edge of the plates.

The moments shown in the free-body diagrams of Fig. 14 and presented in Eqs. 72 are the loads due to the finite and semi-infinite shells, the expressions of which have been derived in Eqs. 43 and 46. The correspondence of the different designations used depends on the choice of coordinate axes and may be defined as follows:

$$\left. \begin{aligned} M_{mA U} &= M_{zA}|_{z=0}, & Q_{mA U} &= Q_{zA}|_{z=0}, \\ M_{mA L} &= M_{zC}|_{z=0}, & Q_{mA L} &= Q_{zC}|_{z=0}, \\ M_{mB U} &= M_{zC}|_{z=l}, & Q_{mB U} &= Q_{zC}|_{z=l}, \\ M_{mB L} &= M_{zB}|_{z=0}, & Q_{mB L} &= Q_{zB}|_{z=0}, \end{aligned} \right\} \quad (73)$$

where the subscripts A , B , and C refer to the top, bottom, and central shells, respectively.

As indicated by Eqs. 43 and 46 and the associated Eqs. 45, the load expressions of Eqs. 73 are all given in terms of the edge moments, $M_{rmA}|_{r_m}$ and $M_{rmB}|_{r_m}$; the angular rotations, ϕ_{mA} and ϕ_{mB} ; the radial translations, u_{mA} and u_{mB} ; and other variables. The radial translation of the shell can be related to the rotation of the rings. For small displacements, the relationships become

$$\left. \begin{aligned} u_{mA} &= \frac{1}{2}h_{mA} \phi_{mA}, \\ u_{mB} &= \frac{1}{2}h_{mB} \phi_{mB}. \end{aligned} \right\} \quad (74)$$

The slope expression at the edge of the plates is in terms of internal loads of the coupled structure, i.e., the unknowns. Thus, matching the angular rotation of the rings with the slope of the plates at the edge results in the superposition of the two portions of the final structure. When this is done and Eqs. 72 are expressed in terms of these unknowns, the equations are then compatible with the final equations of the coupled-plate structure. The result of the many substitutions can be presented by giving the component expressions of Eqs. 72 as follows:

$$\left. \begin{aligned} M_{mAL} &= MALP + MALA \cdot \phi_{mA}|_{r_m} + MALB \cdot \phi_{mB}|_{r_m}, \\ M_{mBU} &= MBUP + MBUA \cdot \phi_{mA}|_{r_m} + MBUB \cdot \phi_{mB}|_{r_m}, \\ Q_{mAL} &= QALP + QALA \cdot \phi_{mA}|_{r_m} + QALB \cdot \phi_{mB}|_{r_m}, \\ Q_{mBU} &= QBUP + QBUA \cdot \phi_{mA}|_{r_m} + QBUB \cdot \phi_{mB}|_{r_m}, \\ M_{mAU} &= MAUP + MAUA \cdot \phi_{mA}|_{r_m}, \\ M_{mBL} &= MBLP + MBLB \cdot \phi_{mB}|_{r_m}, \\ Q_{mAU} &= QAUP + QAUA \cdot \phi_{mA}|_{r_m}, \\ Q_{mBL} &= QBLP + QBLB \cdot \phi_{mB}|_{r_m}, \end{aligned} \right\} \quad (75)$$

where

$$\text{MAUA} = -2D_A \beta_A^2 \left(\frac{h_{mA}}{2} + \frac{1}{\beta_A} \right),$$

$$\text{MAUP} = -2D_A \beta_A^2 \left[u_{pA}|_{z=0} + \frac{1}{\beta_A} S_{pA}|_{z=0} \right] + M_{zpA}|_{z=0},$$

$$\text{QAUA} = 2D_A \beta_A^2 (\beta_A h_{mA} + 1),$$

$$\text{QAUP} = 2D_A \beta_A^2 \left[2\beta_A u_{pA}|_{z=0} + S_{pA}|_{z=0} \right] + Q_{zpA}|_{z=0},$$

$$\text{MBLB} = 2D_B \beta_B^2 \left(\frac{h_{mB}}{2} + \frac{1}{\beta_B} \right),$$

$$\text{MBLP} = -2D_B \beta_B^2 \left[u_{pB}|_{z=0} + \frac{1}{\beta_B} S_{pB}|_{z=0} \right] + M_{zpB}|_{z=0},$$

$$\text{QBLB} = -2D_B \beta_B^2 (\beta_B h_{mB} + 1),$$

$$\text{QBLP} = 2D_B \beta_B^2 \left[2\beta_B u_{pB}|_{z=0} + S_{pB}|_{z=0} \right] + Q_{zpB}|_{z=0},$$

$$\text{MALA} = \frac{2D_c \beta_c^2}{\sin^2 - \sinh^2} \left[\frac{1}{\beta_c} (\sin \cdot \cos - \sinh \cdot \cosh) - \frac{h_{mA}}{2} (\sin^2 + \sinh^2) \right],$$

$$\text{MALB} = - \frac{2D_c \beta_c^2}{\sin^2 - \sinh^2} \left[\frac{1}{\beta_c} (\sin \cdot \cosh - \cos \cdot \sinh) + h_{mb} \cdot \sin \cdot \sinh \right],$$

$$\begin{aligned} \text{MALP} = & \frac{2D_c \beta_c^2}{\sin^2 - \sinh^2} \left[(\sin^2 + \sinh^2) u_{pc}|_{z=0} - 2 \cdot \sin \cdot \sinh \cdot u_{pc}|_{z=\ell_m} \right. \\ & - \frac{1}{\beta_c} (\sin \cdot \cos - \sinh \cdot \cosh) S_{pc}|_{z=0} \\ & \left. + \frac{1}{\beta_c} (\sin \cdot \cosh - \cos \cdot \sinh) S_{pc}|_{z=\ell_m} \right] + M_{zpc}|_{z=0}, \end{aligned}$$

$$\text{MBUA} = \frac{2D_c \beta_c^2}{\sin^2 - \sinh^2} \left[\frac{1}{\beta_c} (\sin \cdot \cosh - \cos \cdot \sinh) + h_{mA} \cdot \sin \cdot \sinh \right],$$

$$\text{MBUB} = - \frac{2D_c \beta_c^2}{\sin^2 - \sinh^2} \left[\frac{1}{\beta_c} (\sin \cdot \cos - \sinh \cdot \cosh) - \frac{h_{mB}}{2} (\sin^2 + \sinh^2) \right],$$

$$\begin{aligned} \text{MBUP} = & -\frac{2D_c\beta_c^2}{\text{SIN}^2 - \text{SINH}^2} \left[2 \cdot \text{SIN} \cdot \text{SINH} \cdot u_{pc} \Big|_{z=0} - (\text{SIN}^2 + \text{SINH}^2) u_{pc} \Big|_{z=\ell_m} \right. \\ & + \frac{1}{\beta_c} (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) S_{pc} \Big|_{z=0} \\ & \left. - \frac{1}{\beta_c} (\text{SIN} \cdot \text{COS} - \text{SINH} \cdot \text{COSH}) S_{pc} \Big|_{z=\ell_m} \right] + M_{zpc} \Big|_{z=\ell_m}, \end{aligned}$$

$$\begin{aligned} \text{QALA} = & \frac{2D_c\beta_c^3}{\text{DENOM}} \left\{ \frac{1}{\beta_c} (\text{SIN}^2 + \text{SINH}^2) (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) \right. \\ & \left. + h_{mA} \left[\text{SIN} \cdot \text{SINH} (\text{SIN}^2 + \text{SINH}^2) + \text{COS} \cdot \text{COSH} (\text{SIN}^2 - \text{SINH}^2) \right] \right\}, \end{aligned}$$

$$\begin{aligned} \text{QALB} = & \frac{2D_c\beta_c^3}{\text{DENOM}} \left[\frac{2}{\beta_c} \text{SIN} \cdot \text{SINH} (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) \right. \\ & \left. + h_{mB} (\text{SIN}^2 \cdot \text{COSH}^2 - \text{COS}^2 \cdot \text{SINH}^2) \right], \end{aligned}$$

$$\begin{aligned} \text{QALP} = & -\frac{2D_c\beta_c^3}{\text{DENOM}} \left\{ 2 \left[\text{SIN} \cdot \text{SINH} (\text{SIN}^2 + \text{SINH}^2) \right. \right. \\ & \left. \left. + \text{COS} \cdot \text{COSH} (\text{SIN}^2 - \text{SINH}^2) \right] u_{pc} \Big|_{z=0} \right. \\ & - 2 (\text{SIN}^2 \cdot \text{COSH}^2 - \text{COS}^2 \cdot \text{SINH}^2) u_{pc} \Big|_{z=\ell_m} \\ & + \frac{1}{\beta_c} (\text{SIN}^2 + \text{SINH}^2) (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) S_{pc} \Big|_{z=0} \\ & \left. + \frac{2}{\beta_c} \text{SIN} \cdot \text{SINH} (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) S_{pc} \Big|_{z=\ell_m} \right\} + Q_{zpc} \Big|_{z=0}, \end{aligned}$$

$$\begin{aligned} \text{QBUA} = & \frac{2D_c\beta_c^3}{\text{DENOM}} \left[\frac{2}{\beta_c} \text{SIN} \cdot \text{SINH} (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) \right. \\ & \left. + h_{mA} (\text{SIN}^2 \cdot \text{COSH}^2 - \text{COS}^2 \cdot \text{SINH}^2) \right], \end{aligned}$$

$$\begin{aligned} \text{QBUB} = & \frac{2D_c\beta_c^3}{\text{DENOM}} \left\{ \frac{1}{\beta_c} (\text{SIN}^2 + \text{SINH}^2) (\text{SIN} \cdot \text{COSH} - \text{COS} \cdot \text{SINH}) \right. \\ & \left. + h_{mB} \left[\text{SIN} \cdot \text{SINH} (\text{SIN}^2 + \text{SINH}^2) + \text{COS} \cdot \text{COSH} (\text{SIN}^2 - \text{SINH}^2) \right] \right\}, \end{aligned}$$

$$\begin{aligned}
QBUP = & - \frac{2D_c \beta_c^3}{DENOM} \left\{ 2 (\sin^2 \cdot \cosh^2 - \cos^2 \cdot \sinh^2) u_{pc} \Big|_{z=0} \right. \\
& - 2 \left[\sin \cdot \sinh (\sin^2 + \sinh^2) \right. \\
& \left. \left. + \cos \cdot \cosh (\sin^2 - \sinh^2) \right] u_{pc} \Big|_{z=l_m} \right. \\
& + \frac{2}{\beta_c} \sin \cdot \sinh (\sin \cdot \cosh - \cos \cdot \sinh) S_{pc} \Big|_{z=0} \\
& \left. + \frac{1}{\beta_c} (\sin^2 + \sinh^2) (\sin \cdot \cosh - \cos \cdot \sinh) S_{pc} \Big|_{z=l_m} \right\} + Q_{zpc} \Big|_{z=l_m},
\end{aligned}$$

$$DENOM = (\sin^2 - \sinh^2)(\sin \cdot \cosh - \cos \cdot \sinh),$$

$$\sin = \sin(\beta_c l_m),$$

$$\cos = \cos(\beta_c l_m),$$

$$\sinh = \sinh(\beta_c l_m),$$

and

$$\cosh = \cosh(\beta_c l_m).$$

Equations 75 can be presented in the form more convenient for the matrix-inversion solution. The coefficients of the coefficient matrix and the vector for the top support to be solved in conjunction with the main equations of coupled structure, can be taken to be as follows:

Matrix Coefficients

$$\left. \begin{aligned}
C(3m-1, 1) &= AZD - BZD, \\
C(3m-1, j+1) &= AZE(j) - BZE(j) \quad (j = 1, 2, 3, \dots, m-1), \\
C(3m-1, m+i) &= AZB(i) \quad (i = 1, 2, 3, \dots, m-2), \\
C(3m-1, 2m-1) &= AZC, \\
C(3m-1, 2m) &= AZA - 1, \\
C(3m-1, 2m+i) &= BZB(i) \quad (i = 1, 2, 3, \dots, m-2), \\
C(3m-1, 3m-1) &= BZC, \\
C(3m-1, 3m) &= BZA;
\end{aligned} \right\} \begin{array}{l} (76) \\ \text{Contd.} \end{array}$$

Vector Elements

$$b_{3m-1} = AZD \cdot P_{0AU} - BZD \cdot P_{0BL} + \sum_{j=1}^{m-1} [AZE(j) P_{jAU} - BZE(j) P_{jBL}] - \sum_{i=1}^{m-2} [AZB(i) M_{PiAU} + BZB(i) M_{PiBL}] - AZC M_{P(m-1)AU} - BZC M_{P(m-1)BL} - AZF - BZF - AUL, \quad \left. \begin{array}{l} \text{Contd.} \\ (76) \end{array} \right\}$$

where

$$ZZA = -ZULA \left[ZA(m) ZC(m-1) ZA(m-1) a_{(m-1)(m-1)}^{-1} + ZE(m) \right],$$

$$ZZB(i) = ZULA \cdot ZA(m-1) \left[a_{(m-1)(i+1)}^{-1} ZMPO(i+1) + a_{(m-1)i}^{-1} ZMPl(i) \right] \quad (i = 1, 2, 3, \dots, m-2),$$

$$ZZC = ZULA \cdot ZA(m-1) \left[a_{(m-1)(m-1)}^{-1} ZMPl(m-1) + 1 \right],$$

$$ZZD = ZULA \left[ZA(m-1) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} ZPO(i) + \frac{ZG(m)}{2\pi r_m} \right],$$

$$ZZE(j) = ZULA \left[ZA(m-1) \sum_{i=j}^{m-1} a_{(m-1)i}^{-1} ZP(i, j) + \frac{r_j}{r_m} ZG(m) \right] \quad (j = 1, 2, 3, \dots, m-1),$$

$$ZZF = ZULA \left\{ ZA(m-1) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} ZMPS(i) + \frac{ZC(m)}{r_m} S_{pm} \Big|_{r_m} - ZA(m-1) M_{rpm} \Big|_{r_{m-1}} + ZE(m) M_{rpm} \Big|_{r_m} - ZG(m) \sum_{j=1}^{m-1} \frac{r_j}{r_m} \left[Q_{rpj} \Big|_{r_j} - Q_{rp(j+1)} \Big|_{r_j} \right] \right\},$$

and the first letter Z of the symbols used should be changed to A or B, when referring to the top or the bottom plate, respectively. Also,

$$\begin{aligned} AUL &= MAUP - MALP + \frac{h_m A}{2} (QALP - QAUP), \\ AULA &= \frac{r_m}{AC(m)} \left[MAUA - MALA + \frac{h_m A}{2} (QALA - QAUA) - AT \right], \\ BULA &= \frac{r_m}{BC(m)} \left[-MALB + \frac{h_m A}{2} QALB \right]. \end{aligned}$$

Similarly, for the bottom support:

Matrix Coefficients

$$\left. \begin{aligned} C(3m, 1) &= AZJ - BZJ, \\ C(3m, j+1) &= AZK(j) - DZK(j) \quad (j = 1, 2, 3, \dots, m-1), \\ C(3m, m+i) &= AZH(i) \quad (i = 1, 2, 3, \dots, m-2), \\ C(3m, 2m-1) &= AZI, \\ C(3m, 2m) &= AZG, \\ C(3m, 2m+i) &= BZH(i) \quad (i = 1, 2, 3, \dots, m-2), \\ C(3m, 3m-1) &= BZI, \\ C(3m, 3m) &= BZG - 1, \end{aligned} \right\} (77)$$

Vector Elements

$$\begin{aligned} b_{3m} &= AZJ P_0 AU - BZJ P_0 BL + \sum_{j=1}^{m-1} [AZK(j) P_j AU - BZK(j) P_j BL] \\ &\quad - \sum_{i=1}^{m-2} [AZH(i) M_{P_1 AU} + BZII(i) M_{P_i BL}] - AZI M_{P(m-1) AU} \\ &\quad - BZI M_{P(m-1) BL} - AZL - BZL - BUL, \end{aligned}$$

where

$$ZZG = -ZULB \left[ZA(m) ZC(m-1) ZA(m-1) a_{(m-1)(m-1)}^{-1} + ZE(m) \right],$$

$$\begin{aligned} \text{ZZH}(i) = & \text{ZULB} \cdot \text{ZA}(m-1) \left[a_{(m-1)(i+1)}^{-1} \text{ZMPO}(i+1) \right. \\ & \left. + a_{(m-1)i}^{-1} \text{ZMP1}(i) \right] \quad (i = 1, 2, 3, \dots, m-2), \end{aligned}$$

$$\text{ZZI} = \text{ZULB} \cdot \text{ZA}(m-1) \left[a_{(m-1)(m-1)}^{-1} \text{ZMP1}(m-1) + 1 \right],$$

$$\text{ZZJ} = \text{ZULB} \left[\text{ZA}(m-1) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} \text{ZPO}(i) + \frac{\text{ZG}(m)}{2\pi r_m} \right],$$

$$\begin{aligned} \text{ZZK}(j) = & \text{ZULB} \left[\text{ZA}(m-1) \sum_{i=j}^{m-1} a_{(m-1)i}^{-1} \text{ZP}(i, j) \right. \\ & \left. + \frac{r_j}{r_m} \text{ZG}(m) \right] \quad (j = 1, 2, 3, \dots, m-1), \end{aligned}$$

$$\begin{aligned} \text{ZZL} = & \text{ZULB} \left\{ \text{ZA}(m-1) \sum_{i=1}^{m-1} a_{(m-1)i}^{-1} \text{ZMPS}(i) + \frac{\text{ZC}(m)}{r_m} S_{\text{pm}} \Big|_{r_m} \right. \\ & - \text{ZA}(m-1) M_{\text{rpm}} \Big|_{r_{m-1}} + \text{ZE}(m) M_{\text{rpm}} \Big|_{r_m} \\ & \left. - \text{ZG}(m) \sum_{j=1}^{m-1} \frac{r_j}{r_m} \left[Q_{\text{rpj}} \Big|_{r_j} - Q_{\text{rp}(j+1)} \Big|_{r_j} \right] \right\}, \end{aligned}$$

and subscripts A and B, or substitutions of A and B for the first letter Z in the symbols, designate reference to the top and bottom plates, respectively. In addition,

$$\text{BUL} = \text{MBUP} - \text{MBLP} + \frac{h_{\text{mB}}}{2} (\text{QBLP} + \text{QBUP}),$$

$$\text{AULB} = \frac{r_m}{\text{AC}(m)} \left[\text{MBUA} + \frac{h_{\text{mB}}}{2} \text{QBUA} \right],$$

and

$$\text{BULB} = \frac{r_m}{\text{BC}(m)} \left[\text{MBUB} - \text{MBLB} + \frac{h_{\text{mB}}}{2} (\text{QBLB} + \text{QBUB}) - \text{BT} \right].$$

2. Intermediate Shell Walls

The composite structure of Fig. 1 shows intermediate circular cylindrical shells extending from the upper surface of the top plate and also from the lower surface of the bottom plate. These shells, rather than those at the edge, may sometimes be used as the vessel wall in an actual construction.

If the shells are truly to be considered in the analysis, then the external moment at the given location would depend on the characteristics of the shells and those of the composite structure as a whole. Consequently, a simultaneous analysis must be performed for all the components involved. This is accomplished by first deriving an expression that describes the behavior of the shell deformation, similar to the procedure followed for the plate-edge condition. The derived shell equation is then used to match the slope and moment at the edge of the shell and the respective quantities of the plate at the given diametral location. The described superposition procedure results in a new expression involving the unknown terms of the coupled structure. This resultant expression is then evaluated simultaneously with the other equations of the composite structure for the final solution of the intended problem.

The needed solution of the semi-infinite circular cylindrical shell problem was derived previously in this report and is given by Eqs. 43. Note, however, that a slight difference exists between the solutions of the top and the bottom shells as used in the composite structure. The difference takes place in the sign of the radial translation and rotation at the end of the shell. For example, because of identical deformation of the two plates of the structure, one shell would tend to move outward and the other inward. Because of the choice of the coordinate axes, a similar difference exists in the rotations at the edge of the shell. When the rotation of the end of the shell is matched with the slope of the plate, the given angular rotation for the top plate must be reversed. This is not the case with the bottom connection.

Consequently, the geometrical boundary conditions for the top shell of the upper plate and for the bottom shell of the lower plate can be specified as

$$\left. \begin{aligned} u_0 &= -\frac{1}{2} g_{\eta A} S_{(\eta+1)A} \Big|_{r_{\eta}}, & u_0 &= \frac{1}{2} g_{\eta B} S_{(\eta+1)B} \Big|_{r_{\eta}}, \\ S_0 &= -S_{(\eta+1)A} \Big|_{r_{\eta}}, & S_0 &= S_{(\eta+1)B} \Big|_{r_{\eta}} \end{aligned} \right\} \quad (78)$$

Incorporating conditions 78 into Eqs. 43, and setting the moment at the edge of the shell equal to the external moment at the plate surface (negative for the bottom shell), yields

$$M_{P\eta} = -2D\eta\beta_\eta^2 \left[\left(\frac{1}{2}g_\eta + \frac{1}{\beta} \right) S_{\eta+1}|_{r_\eta} \pm \left(u_P|_{z=0} + \frac{1}{\beta_\eta} S_P|_{z=0} - \frac{1}{2D\eta\beta_\eta^2} M_{zP}|_{z=0} \right) \right] \quad (79)$$

where η designates the junction at which the shell is connected, and the upper sign refers to the top shell, and the lower sign refers to the bottom shell. Using Eq. 58 in Eq. 79 and attaching the appropriate subscripts to designate the top or bottom shell by A or B, respectively, we obtain the final conditions for the intermediate shells. In line with the established procedure of making available the coefficients for the matrix inversion solution, the top-shell condition can be presented as follows:

Matrix Coefficients

$$\begin{aligned} C(3m+1, 1) &= ASP(\eta+1, 1), \\ C(3m+1, j+1) &= ASP(\eta+1, j+1) \quad (j = 1, 2, 3, \dots, m-1), \\ C(3m+1, m+i) &= ASM(\eta+1, i) \quad (i = 1, 2, 3, \dots, m), \\ C(3m+1, 2m+s) &= 0 \quad (s = 1, 2, 3, \dots, m), \\ C(3m+1, 3m+1) &= ASM(\eta+1, \eta) + AU; \end{aligned}$$

Revised Special Coefficients

$$\begin{aligned} C(s, 3m+1) &= C(s, m+\eta) \quad (s = 1, 2, 3, \dots, 3m), \\ C(m+\eta, 3m+1) &= C(m+\eta, m+\eta) - \frac{2\pi r_\eta \ell^3}{\lambda_\eta E_\eta I_\eta}; \end{aligned} \quad (80)$$

Vector Element

$$\begin{aligned} b_{3m+1} &= C(3m+1, 1) P_0 AU + \sum_{j=1}^{m-1} C(3m+1, j+1) P_j AU \\ &\quad - \sum_{i=1}^{m-1} C(3m+1, m+i) M_{Pi} AU - ASB(\eta+1) - AS, \end{aligned}$$

where the subscript and index A designates the shell associated with the top plate. Two new definitions are given also; thus,

$$AS = \frac{2\beta_A}{\beta_A g_{\eta A} + 2} \left[u_{pA}|_{z=0} + \frac{1}{\beta_A} S_{pA}|_{z=0} - \frac{1}{2D_A \beta_A^2} M_{zpA}|_{z=0} \right],$$

and

$$AU = \frac{1}{D_A \beta_A (\beta_A g_{\eta A} + 2)}.$$

The unknown associated with the condition of Eqs. 80 is the moment at the junction of the shell and the plate, i.e., $M_{P\eta AU}$. Similarly, the condition pertaining to the intermediate bottom shell becomes:

Matrix Coefficients

$$\begin{aligned} C(3m+2, 1) &= -DSP(\eta+1, 1); \\ C(3m+2, j+1) &= -BSP(\eta+1, j+1), \quad (j = 1, 2, 3, \dots, m-1), \\ C(3m+2, s) &= 0 \quad (s = 1, 2, 3, \dots, m), \\ C(3m+2, 2m+i) &= BSM(\eta+1, i) \quad (i = 1, 2, 3, \dots, m), \\ C(3m+2, 3m+2) &= BSM(\eta+1, \eta) + BU; \end{aligned}$$

Revised Special Coefficients

$$\begin{aligned} C(s, 3m+2) &= C(s, 2m+\eta), \quad (s = 1, 2, 3, \dots, 3m), \\ C(2m+n-1, 3m+2) &= C(2m+n-1, 2m+n) - \frac{2\pi r_{\eta} \ell_{\eta}^3}{\lambda_{\eta} E_{\eta} I_{\eta}}; \end{aligned} \quad (81)$$

Vector Element

$$\begin{aligned} b_{3m+2} &= -C(3m+2, 1) P_{0BL} - \sum_{j=1}^{m-1} C(3m+2, j+1) P_{jBL} \\ &\quad - \sum_{i=1}^{m-1} C(3m+2, 2m+i) M_{PiBL} - BSB(\eta+1) + BS, \end{aligned}$$

where

$$BS = \frac{2\beta_B}{\beta_B g_{\eta B} + 2} \left[u_{pB}|_{z=0} + \frac{1}{\beta_B} S_{pB}|_{z=0} - \frac{1}{2D_B \beta_B^2} M_{zpB}|_{z=0} \right],$$

$$BU = \frac{1}{D_B \beta_B (\beta_B g_{\eta B} + 2)},$$

and the subscript B refers to the intermediate shell of the bottom plate. The unknown associated with Eqs. 81 is $M_{P\eta BL}$.

Equations 80 and 81 provide the coefficients needed for introducing the intermediate top and bottom shells, respectively. These equations complete the combined matrix for yielding the bending solution of the core support structure. However, the equations have been given in terms of generalized loads, that is, the particular deflections, moments, etc. To expedite solutions of design problems, we need a more convenient load representation. This matter is discussed in Section D below.

D. Loading Representation

The bending analysis given above has been presented, among other variables, in terms of general load parameters. These appear in the above expressions as particular displacements, slopes, moments, and shear loads. Although the equations in this form have the most general representation, they are cumbersome for practical use. To alleviate this difficulty requires a more specialized representation.

Arbitrary distributions can be represented in various forms. Often certain pitfalls can be avoided by choosing for this purpose orthogonal functions. On the other hand, most practical problems possess the characteristic of relatively simple loading. Under these conditions, the most simple representation of the loads would still be sufficient and yet demand the least effort. Thus, without any specific problem in mind, the simple polynomial series representation is selected for this purpose.

1. Load Distribution on Plates

External loading of the support plates, as mentioned at the beginning of the analysis, consists of two types: transverse pressure and temperature gradients. Since only axisymmetric loading is considered, pressure on the plates in any of its regions is taken to be in the form

$$p_n(r) = \sum_{i=0}^I F_{(i+1)n} r^i, \quad (82)$$

and the temperature gradients across the plate thickness and along its radius as

$$T_n(r, z) = \sum_{i=0}^I A_{(i+1)n} r^i + \frac{z}{h_n} \sum_{j=0}^J B_{(j+1)n} r^j. \quad (83)$$

The coefficients F , A , and B , with the appropriate subscripts, are constants that must be evaluated to fit the specific design pressure and temperature distributions in question. Each series of Eqs. 82 and 83 has been arbitrarily limited to $I + 1$ or $J + 1$ terms. There is no reason for such a limitation, except when used in conjunction with a computer program; then some reasonable limit must be specified.

Equations 82 and 83 can be conveniently used to represent almost any practically occurring load distribution. In the most common case of constant pressure, for example, only the first term of Eq. 82 would be utilized, where F_{1n} would represent the pressure. Similarly, for a linear temperature distribution across the plate, just a few terms of the combined series in Eq. 83 would be needed. However, in some specific cases, such as a perforated plate, the temperature distribution may be quite complex. Whenever such detailed information becomes available, the additional terms of Eq. 83 can be used to yield an averaged representation.

The deflection, slope, moment, and shear load are functions of $p(r)$ and $T(r, z)$, as in Eq. 7, and the remaining relationships are given by Eqs. 10. Thus, when Eqs. 82 and 83 are substituted into Eqs. 7 and 10, and the series is integrated, and differentiated, the particular expressions become

$$\begin{aligned}
 w_{pn} &= \frac{1}{D_n} \sum_{i=0}^I \frac{r^{i+4}}{(i+2)^2(i+4)^2} F_{(i+1)n} - \frac{\alpha_n(1+\nu_n)}{2h_n} \\
 &\quad \sum_{j=0}^J \frac{r^{j+2}}{(j+2)^2} A_{(j+1)n}, \\
 S_{pn} &= \frac{1}{D_n} \sum_{i=0}^I \frac{r^{i+3}}{(i+2)^2(i+4)} F_{(i+1)n} - \frac{\alpha_n(1+\nu_n)}{2h_n} \\
 &\quad \sum_{j=0}^J \frac{r^{j+1}}{j+2} A_{(j+1)n}, \\
 M_{rpn} &= - \sum_{i=0}^I \frac{(i+3+\nu_n) r^{i+2}}{(i+2)^2(i+4)} F_{(i+1)n} - \frac{D_n \alpha_n(1-\nu_n^2)}{2h_n} \\
 &\quad \sum_{j=0}^J \frac{r^j}{j+2} A_{(j+1)n},
 \end{aligned}
 \tag{84}$$

Contd.

$$\left. \begin{aligned}
 M_{\theta pn} &= - \sum_{i=0}^I \frac{(i\nu_n + 1 + 3\nu_n) r_n^{i+2}}{(i+2)^2(i+4)} F_{(i+1)n} - \frac{D_n \alpha_n (1 - \nu_n^2)}{2h_n} \\
 &\quad \sum_{j=0}^J \frac{(j+1) r_n^j}{j+2} A_{(j+1)n}, \\
 Q_{rpn} &= \sum_{i=0}^I \frac{r_n^{i+1}}{i+2} F_{(i+1)n}.
 \end{aligned} \right\} \text{Contd. (84)}$$

2. Temperature along the Tube

Since the tubes are structurally connected to the plates, their temperatures must be compatible. Hence, at the ends of the tubes the temperature is the same as that of the plates at the given junction. A mathematical expression for this condition can be expressed in terms of Eq. 83 as follows:

$$\left. \begin{aligned}
 T_{nA} &= \frac{1}{2} \left[T_{nA} \left(r_n, -\frac{h_{nA}}{2} \right) + T_{(n+1)A} \left(r_n, -\frac{h_{(n+1)A}}{2} \right) \right], \\
 T_{nB} &= \frac{1}{2} \left[T_{nB} \left(r_n, \frac{h_{nA}}{2} \right) + T_{(n+1)B} \left(r_n, \frac{h_{(n+1)B}}{2} \right) \right],
 \end{aligned} \right\} \text{(85)}$$

where T_{nA} and T_{nB} are, respectively, the top and bottom end temperatures of tube in the circumferential tube distribution of radius r_n .

The temperature distribution between the limits of Eqs. 85 can be anything dictated by the case at hand. No particular representation for this is assumed here. Whenever a temperature distribution is assumed, however, care must be taken to select the proper sign for $\Delta T(x)$ of Eq. 23. The selection should be governed by the same sign convention as that used for the axial mechanical load. A linear temperature variation along the length of the tube produces no effect at the ends, as observed from Eq. 23. Thus, whenever the design conditions permit a linear temperature assumption, the second term of Eq. 23 can be neglected.

3. Load Variation in the Shells

The pressure and temperature distribution in the shells can be assumed in a similar way as that for the plates, except that the radial and axial coordinate variations need to be interchanged. Thus, for the representation of pressure on the surface of the shells, the following can be taken:

$$p(Z) = \sum_{i=0}^I G_{i+1} z^i, \quad (86)$$

and for the temperature in the wall of the shells:

$$T(\xi, Z) = \sum_{i=0}^I X_{i+1} z^i + \frac{\xi}{t} \sum_{j=0}^J Y_{j+1} z^j, \quad (87)$$

where the coefficients G , X , and Y , with the appropriate subscripts, must be evaluated for a specific distribution at hand. For this temperature distribution to be consistent with that in the plates, temperature at the edge of the shell is given by Eq. 83. A similar condition to those in Eqs. 85 should be specified for each shell.

If the upper limit of the series in Eqs. 86 and 87 is restricted, i.e., $I \leq 4$ and $J \leq 4$, then a simplified expression is obtained for the particular solution of Eq. 36, yielding

$$u_p = \frac{1}{4D\beta^4} q(r, z), \quad (88)$$

where $q(r, z)$, in terms of the coefficients of Eqs. 86 and 87, becomes

$$q(r, z) = \sum_{i=0}^I G_{i+1} z^i + \frac{E\alpha}{1-\nu} \left[\frac{(1-\nu)t}{a} \sum_{i=0}^I X_{i+1} z^i - \frac{t^2}{12} \frac{d^2}{dz^2} \sum_{j=0}^J Y_{j+1} z^j \right].$$

Hence, the particular displacement of Eq. 38 and the particular slope, moment, and shear load of Eqs. 41 can all be given in terms of the components of pressure and temperature. These, when the indicated operations are performed, yield the following expressions:

$$u_p = \frac{1}{4D\beta^4} \left\{ \sum_{i=0}^I G_{i+1} z^i + \frac{E\alpha t}{a} \left[\sum_{i=0}^I X_{i+1} z^i - \frac{at}{12(1-\nu)} \sum_{j=0}^J j(j-1) Y_{j+1} z^{j-2} \right] \right\}, \quad (89)$$

Contd.

$$\begin{aligned}
S_p &= \frac{1}{4D\beta^4} \left\{ \sum_{i=0}^I i G_{i+1} z^{i-1} + \frac{E\alpha t}{a} \left[\sum_{i=0}^I i X_{i+1} z^{i-1} \right. \right. \\
&\quad \left. \left. - \frac{at}{12(1-\nu)} \sum_{j=0}^J j(j-1)(j-2) Y_{j+1} z^{j-3} \right] \right\}, \\
M_{rp} &= -\frac{1}{4\beta^4} \left(\sum_{i=0}^I i(i-1) G_{i+1} z^{i-2} + \frac{E\alpha t}{a} \left\{ \sum_{i=0}^I i(i-1) X_{i+1} z^{i-2} \right. \right. \\
&\quad \left. \left. - \frac{at}{12(1-\nu)} \sum_{j=0}^J Y_{j+1} z^{j-4} \left[j(j-1)(j-2)(j-3) - 4\beta^4 z^4 \right] \right\} \right), \\
Q_{rp} &= -\frac{1}{4\beta^4} \left(\sum_{i=0}^I i(i-1)(i-2) G_{i+1} z^{i-3} \right. \\
&\quad \left. + \frac{E\alpha t}{a} \left\{ \sum_{i=0}^I i(i-1)(i-2) X_{i+1} z^{i-3} \right. \right. \\
&\quad \left. \left. - \frac{at}{12(1-\nu)} \sum_{j=0}^J j Y_{j+1} z^{j-5} \left[(j-1)(j-2)(j-3)(j-4) - 4\beta^4 z^4 \right] \right\} \right).
\end{aligned}$$

Contd.
(89)

E. Recapitulation of Bending Analysis

Resultant expressions given above supply the needed information for determining the bending contribution in deformation of the composite structure. Thus, to provide the solution of the problem depicted in Fig. 13, we must use the combined matrix composed of the coefficients in Eqs. 65-68. The edge conditions are provided for, by adding to the enumerated set, additional coefficients given by either Eqs. 69 and 70 or Eqs. 76 and 77, depending on the choice of the edge support. Solution of the combined matrix provides magnitudes to the unknown internal loads at the junctions, in a consecutive order as follows: $P_0, P_1, P_2, \dots, P_{m-1}, MP_{1AL}, MP_{2AL}, MP_{3AL}, \dots, MP_{(m-1)AL}, M_{mA}|_{r_m}, MP_{1BU}, MP_{2BU}, MP_{3BU}, \dots, MP_{(m-1)BU}, M_{mB}|_{r_m}$.

If the intermediate shells are to be included (i.e., the problem to be solved is depicted in Fig. 1), then Eqs. 80 and 81 must be expressed in terms of

the previous coefficients before the combined matrix is solved. Two additional unknowns, $M_{P\eta AU}$ and $M_{P\eta BL}$, would be solved for because of the incorporation of Eqs. 80 and 81, respectively.

Thus, solution of the given simultaneous equations provides values of the enumerated loads at the junctions of the structure. Further information of the structure must then be obtained in the reverse order to the procedure given earlier in this report.

Having secured the values of loads at the junctions of the structure, we can also obtain the other characteristics, such as deflection, slope, etc., at the junction. This, in turn, enables us to evaluate the constants of integration for all regions of the plates. With the availability of the constants of integration, the properties within the respective regions of the plates can be determined.

Correspondingly, having the values of loads and displacements at the junctions of the tubes, shells, and plates, we can use the equations in this report to obtain additional information between the two ends of the tubes and along the length of the shells.

Equations given thus far in this report provide the bending solution to the structure shown in Fig. 1. Since the bending component is predominant in most practical structures of this nature, the solution given would constitute the end of the analysis. It is possible, however, to include some additional effects, which at times may become appreciable. One such is the effect of in-plane stresses, which can be approximated without much difficulty. This is discussed in the following section.

VI. MEMBRANE EFFECTS IN PLATES

The allowable transverse deflection in core support structures derives from the permissible deformation of the reactor core itself. In most cases, the deflection, rather than strength of material, becomes the limiting factor in the design of such structures. The permissible plate deflection usually is in the range of thousandths of an inch, while the plate thickness is in inches. Under these conditions, the deflection of the plate would be regarded as very small. Consequently, the assumption of plane stress conditions is very nearly realized in the approximation of membrane effects.

One factor contributing to the membrane stresses is the condition of the radial edge support. It might be assumed that the radial edge restraint is proportional to the percentage of bending restraint (k). From a practical point of view, these two types of restraint would seem to be nearly equivalent. However, just in case a known radial restraint can be specified, the two restraints are distinguished in the presentation that follows.

Another contributing factor affecting the membrane stresses could be the shear loads at the end of the tubes. If the shear load is assumed to be acting in the plane of the plate, action of the ring of tubes produces a nearly uniform compressive or tensile radial load. If the transverse deflection of both plates is assumed to be positive, the resultant radial load on the two plates would be of opposite sign. Thus, if the tensile load is designated as positive, the analysis of the membrane stress approximation can be resumed.

Equations for plane stress condition were given previously, where Eqs. 17 pertain to all the regions of the combined plate shown in Fig. 13. If these are to yield numerical values, the constants of integration must be evaluated by utilization of the radial boundary conditions. This is accomplished by providing continuity, both of radial displacement and load, at each junction of the combined plate. The same compatibility conditions can be mathematically represented as follows:

$$\left. \begin{aligned} u_{n+1}|_{r_n} &= u_n|_{r_n} & (n = 1, 2, 3, \dots, m-1), \\ \sigma_{r(n+1)}|_{r_n} &= \sigma_{rn}|_{r_n} - \frac{1}{g_n} N_n & (n = 1, 2, 3, \dots, m-1), \end{aligned} \right\} \quad (90)$$

where N_n is the external axisymmetric radial load at the outer boundary of the region n , as shown in Fig. 5. Consequently, evaluating Eqs. 17 at the inner and outer boundaries, and subjecting these equations to the conditions of Eqs. 90, we obtain a set of two expressions for each internal junction and one at the edge. These yield three distinct sets of expressions: one, pertaining to the first region of the plates for which, in Eqs. 17, C_2 and r_0 had been set equal to zero; another, representing the last region of

the plates, which is directly dependent on the edge conditions; and the last, pertaining to all the remaining plate regions. These sets are identified in the subsequent listing by letters a, c, and b, respectively; thus,

$$\begin{aligned}
 & \text{a) } YA(1)C1(1) - YA(2)C1(2) + YB(2)YC(1)C2(2) = YD(1)YG(1) \\
 & \quad + YF(1), \\
 & \quad C1(1) - C1(2) - YC(1)C2(2) = YF(1)YG(1); \\
 & \text{b) } YA(n)C1(n) - YB(n)YC(n)C2(n) - YA(n+1)C1(n+1) \\
 & \quad + YB(n+1)C2(n+1) = YD(n)YG(n) \\
 & \quad + YF(n) \quad (n = 2, 3, 4, \dots, m-1), \\
 & \quad C1(n) + YC(n)C2(n) - C1(n+1) - YC(n)C2(n+1) \\
 & \quad = YE(n)YG(n) \quad (n = 2, 3, 4, \dots, m-1); \\
 & \text{c) } YA(m)C1(m) - YB(m)YC(m)C2(m) = YD(m)YG(m) \\
 & \quad \text{if edges are free,} \\
 & \quad C1(m) + YC(m)C2(m) = YE(m)YG(m) \\
 & \quad \text{if edges are fixed;}
 \end{aligned} \tag{91}$$

where*

$$YA(n) = \frac{E_n}{1 - \nu_n},$$

$$YB(n) = \frac{E_n}{1 + \nu_n},$$

$$YC(n) = \frac{1}{r_n^2},$$

$$YD(n) = \frac{E_n \alpha_n}{r_n^2},$$

*The expression $YF(n)$ for the bottom plate becomes opposite in sign to that of the top plate. This is due to the previously established sign convention.

$$YE(n) = - \frac{\alpha_n(1+\nu_n)}{r_n^2}$$

$$YF(n) = - \frac{2}{g_n(2\ell_n + g_{nA} + g_{nB})} \left[M_{P(m+n)A} \pm M_{P(2m+n)B} \right]$$

and

$$YG(n) = \int_{r_{n-1}}^{r_n} T_n(r) r dr.$$

The conditions at the edge of each plate can be related to the radial restraint coefficient, κ . We will assume that for $\kappa = 0$, the plate edge is free of radial restraint, and for $\kappa = 1$, there is no motion at the edge. In that case, the two edge conditions given in Eqs. 91 for $n = m$ can be combined into one expression and become

$$\frac{1 - \kappa\nu_m}{1 - \nu_m} C1(m) - \frac{1 - \kappa(2 + \nu_m)}{1 + \nu_m} YC(m)C2(m) = \alpha_m[1 - \kappa(2 + \nu_m)] YC(m)YG(m). \quad (92)$$

In this way, the intermediate radial edge restraint ($0 < \kappa < 1$) can also be approximated.

From Eqs. 91 and 92 it is evident that there are $2m - 1$ equations with $2m - 1$ unknown constants of integration. Simultaneous solution of these equations can again be obtained by the matrix-inversion procedure discussed previously in this report. To make this matrix solution easier, the matrix coefficients and the vectors yield, from Eqs. 91 and 92:

a) Matrix Coefficients for $n = 1$

$$\begin{aligned} C(1, 1) &= YA(1), & C(2, 1) &= 1, \\ C(1, 2) &= -YA(2), & C(2, 2) &= -1, \\ C(1, 3) &= YB(2)YC(1), & C(2, 3) &= -YC(1); \end{aligned}$$

Vector Elements for $n = 1$

$$b_1 = YD(1)YG(1) - YF(1), \quad b_2 = YE(1)YG(1)$$

b) Matrix Coefficients for $n = 2, 3, 4, \dots, m - 1$

$$\begin{aligned} C(2n - 1, 2n - 2) &= YA(n), & C(2n, 2n - 2) &= 1, \\ C(2n - 1, 2n - 1) &= -YB(n)YC(n), & C(2n, 2n - 1) &= YC(n), \\ C(2n - 1, 2n) &= -YA(n + 1), & C(2n, 2n) &= -1, \\ C(2n - 1, 2n + 1) &= YB(n + 1)YC(n), & C(2n, 2n + 1) &= -YC(n). \end{aligned}$$

(93)
Contd.

Vector Elements for $n = 2, 3, 4, \dots, m - 1$

$$b_{2n-1} = YD(n)YG(n) - YF(n), \quad b_{2n} = YE(n)YG(n)$$

c) Matrix Coefficients for $n = m$

$$C(2m-1, 2m-2) = \frac{1 - \kappa\nu_m}{1 - \nu_m},$$

$$C(2m-1, 2m-1) = -\frac{1 - \kappa(2 + \nu_m)}{1 + \nu_m} YC(m);$$

Vector Element for $n = m$

$$b_{2m-1} = \alpha_m[1 - \kappa(2 + \nu_m)] YC(m) YG(m).$$

Contd.
(93)

All the remaining coefficients in the final $2m - 1$ by $2m - 1$ matrix are to be taken as zero. The constants of integration for all regions in the combined plate are obtained from the solution of Eqs. 93. These become

$$C1(1) = \sum_{j=1}^{2m-1} a_{1j}^{-1} b_j,$$

$$C1(n) = \sum_{j=1}^{2m-1} a_{(2n-2)j}^{-1} b_j \quad (n = 2, 3, 4, \dots, m),$$

$$C2(n) = \sum_{j=1}^{2m-1} a_{(2n-1)j}^{-1} b_j \quad (n = 2, 3, 4, \dots, m),$$

(94)

where a^{-1} , with the appropriate subscript, represents the coefficients of the inverse matrix, and b_j are the vectors.

The above procedure of membrane analysis pertains to either plate and identical for both. Consequently, particularization of the analysis to each plate is unnecessary; the same procedure applies to both, provided the appropriate subscripts are attached to the symbols involved, whenever consideration is given to the top or the bottom plate. Thus, with the use of Eqs. 17 and 94 the stresses and radial displacement can be obtained for each plate separately, and the approximation of membrane effect is considered to be completed.

VII. SAMPLE RESULTS

The equations in this report have been used for the construction of a computer program. Because of lack of funds and a specific need for numerical results at present, work on the computer program has been delayed. However, for illustration, some results were secured with the help of the nearly completed program and are made available in this report.

The selected illustrative problem is based on a configuration of a core support structure which at one time had been considered for the FARET reactor. Basically, the structure consists of two perforated plates interconnected by a narrow band of tubes distributed circumferentially around the center. Consequently, this becomes a two-region problem; i.e., $m = 2$. The next paragraph describes the structure in more detail.

The two plates of radius $r_2 = 24$ in. are perforated by 301 holes, arranged in an equilateral triangular pattern. The pitch of the perforations is 2.32 in. The diameter of the holes in the top plate is 1.945 in., and that of the holes in the bottom plate is 1.883 in. There are 90 tubes (λ_1), 10 in. long (l_1), located at an average radius of $r_1 = 15.97$ in. from the center of the plates. The effective inside and outside diameters* of the tubes are 1.584 and 1.738 in., respectively. Plate thicknesses are considered the unknowns of the problem.

Loading of the structure consisted of two parts: a dead weight equivalent to 25 psi acting on the top plate, and a maximum internal differential pressure of 100 psi between the plates. The internal pressure acts only on the "solid" portion of the perforated plates. The reduced internal pressure amounts only to 50.6% of the total for the top plate, and to 53.7% for the bottom plate.

The entire structure was to be subjected to a constant ambient temperature of 1000°F. Consequently, the average material properties for Type 304 stainless steel were taken to be the following: modulus of elasticity, $E = 22.5 \times 10^6$ psi; Poisson's ratio, $\nu = 0.3$; and coefficient of thermal expansion, $\alpha = 10.3 \times 10^{-5}/^\circ\text{F}$.

We assumed in this example that bending restraints at the edges of the plates were nonexistent ($k = 0$). To make the preliminary trial more systematic, we assumed a relationship between the two plate thicknesses. Thus, primarily based on the consideration of loading, the relation was taken to be $h_B = h_A + 1$, where dimensions were in inches.

*These tube diameters were derived by assuming that cross-sectional area and moment of inertia of the tubes possessed only half their original magnitude. This was an attempt to account for the effect of holes through the walls of the tubes.

Fig. 15. Average Stress Condition in the Plates

Making the appropriate modifications in the modulus of elasticity and Poisson's ratio according to the method outlined by O'Donnell and Langer,² several problems were set up with different plate thicknesses and internal pressures. Results of the computer solution were then compiled and presented in graphic form.

The effect of plate-thickness variations is illustrated in Figs. 15-17. Figure 15 shows the effective bending stresses in the plates, Fig. 16, the maximum plate deflections, and Fig. 17, the combined stress in the tubes. The effect of variable internal pressure on the deflection of the top plate is shown in Fig. 18, and that on the bottom plate in Fig. 19.

The results of the solution are subject to assumptions enumerated at the beginning of this report, coupled with those mentioned in this section. The validity of these assumptions is discussed in the next section.

Fig. 16. Maximum Deflection of the Plates

Fig. 17. Maximum Tube Stress for Different Plate Thicknesses

Fig. 18. Top-plate Deflection

Fig. 19. Bottom-plate Deflection

VIII. DISCUSSION OF RESULTS

Any analysis is only as good as the assumptions on which it is based. It becomes necessary, therefore, to clarify the assumptions and provide justification for them. Concurrently, reference can be made to the example solution given in Section VII above.

For most engineering problems the validity of beam theory, where the deformation of the cross section has been neglected, is rarely disputed. The same is true with thin-plate analysis, where the effect of in-plane loads and shear is small. The belief of small-membrane effect is reinforced in the present analysis by the realization that usual reactor vessel walls provide little resistance to radial contraction of support plates. Thus, the membrane stresses imposed by the radial-edge restraints become minimal. The fact that the edge condition usually accounts for the largest contribution of membrane stresses in mechanical loading, and that the transverse support-plate deflections are small, provides the principal reasons for neglecting the membrane stresses in many analyses.

The support plates used in many practical problems, especially in the case of perforated plates, cannot be regarded as being thick. As a matter of fact, the shear contribution may become appreciable. The thick-plate analysis, where the shear effect is taken into account, is quite complex. Rather than proceeding with the involved analysis, we find it most expedient in engineering design problems to check for shear, whenever the bending analysis has been completed. Thus, for a simply supported plate with

uniform load, the percentage contribution of shear stress to the maximum deflection is given by Timoshenko and Woinowski-Krieger⁴ as

$$40 \frac{8 + \nu + \nu^2}{(1 - \nu)(5 + \nu)} \left(\frac{h}{a} \right)^2 \%.$$

This correction is true for homogeneous plates, but its validity becomes debatable in perforated-plate analysis. For illustration, however, the corrections of the given example become 4.4 and 5.9% for 4- and 5-in. plates, respectively. Because the two plates are actually connected, this is another reason why the corrections will not necessarily hold. In any case, the values provide some idea of the magnitude of the contribution.

According to O'Donnell and Langer,² the equivalent homogeneous-plate method of perforated plates provides reliable effective stresses for plates with numerical ratios of thickness and pitch larger than two. For the example mentioned in Section VII, the ratios for the 4- and 5-in. plates are 1.72 and 2.16, respectively. Apparently the stresses predicted for the 4-in. plate are not reliable, but because of the close proximity to the "transition point," the given results may still be within the required accuracy.

The thin-plate analysis applied to the perforated plates may also apply to a rather narrow range of plate thicknesses. On one hand, the thin-plate theory requires the plates to be thin; on the other hand, the mentioned equivalent solid-plate technique of O'Donnell and Langer applies to relatively thick plates. This dilemma is intensified when the pitch of the perforations is comparatively large.

Aside from the case covered in the enumerated assumptions and discussed here, other discrepancies will be taking place in actual problems. This is the case of the tube distribution that may be a hexagonal rather than circular pattern. Different variations will most assuredly take place in different problems.

The above discussion indicates that no single evaluation of the validity of the assumptions will be the same for two different cases. Each problem must be considered individually so that the discrepancy of the assumption can be established and the appropriate attention given.

IX. CONCLUSION

The analysis presented in this report is applicable to a variety of axisymmetric structural problems. Although the solution is primarily intended for the treatment of common reactor core support structures, it can also be used for other problems where the configurations are similar. The solution is particularly valuable for its ability to treat multiregion

plates which do not have the same properties from one region to the next. In the usual design of core support structures the outside rim of the plates would be made solid, as opposed to the interior regions. In other cases, the design requirements may dictate changing the thickness in some regions, using different-diameter tubes or holes for the perforations, and so forth. As long as these variations can be considered axisymmetrical, then the method outlined in this report seems particularly adapted to handle such problems.

The presence of exterior line loads and moments at the junctions of the plate regions provides a dual advantage. Primarily, these loads may be used to represent the actions of the fuel elements directly. Hence, there is no need to approximate an equivalent uniformly varying load across the planar area; the equivalent line loads would be used instead. Secondly, the exterior loads offer the possibility for additional superpositions, if and when this is deemed necessary.

Some indication of the worth and power of the analysis can be judged from the sample results presented in this report. Although the design configuration has partly hexagonal characteristics, the results should provide a good picture of the structure's general behavior under the load. Pertaining to the individual components in question, for more exact values of deformation or stress, some adjustments would be necessary to account for apparent discrepancies in the assumptions. Such adjustments in this example, however, would be minor and would require minimum added effort.

The series-loading representation used in this report should not be regarded as restrictive to the method in general. If some other specific loading is found to be better, that loading may be used instead of the series representation. A similar comment also should be made with regard to the type of edge supports: the analysis is not restrictive to the edge supports considered in this report. Some other more specific edge condition may be used instead, provided the needed equations are derived and substituted for those given here.

Some mention seems in order about the assumptions made concerning the junction of the tube and the plate. In the strict sense, the junction represents a point of singularity, as does the effect of concentrated normal load on a plate. Shear loads are produced here which vary along the thickness of the plate, and thus the thin-plate theory breaks down. A serious problem would probably not arise in actual designs. In any case, as long as the tubes remain flexible with respect to the plates, and the shear load produced by the tube is small, the discrepancy involved should be minor.

A better treatment with respect to the joints should have been the use of elastic rings interspersed between the plate regions. These rings would carry the end effects of the tubes, and thus the violation of Kirchhoff's

assumption would be avoided. However, the number of superpositions in this case would approximately double. Although the latter approach would undoubtedly have advantages, the result is not believed to warrant the extra effort. If it is shown, experimentally or otherwise, that the mentioned discrepancy may become appreciable, then the suggestion may be taken up in the future.

Needless to say, the analysis given in this report requires the help of a computer program. Hand calculations, although not impossible, would be impractical in design, where trial-and-error procedures must be contended with. Thus, to take full advantage of the procedure, the computer program to be published as a supplement to this report is an indispensable part of the procedure.

Regarding possible improvements, the effects of dynamics may also be incorporated into the analysis. The same procedure of superposition may be used to account for time effects, similar to the way thermal effects were incorporated in this report.

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