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STRESS ANALYSIS OF REINFORCED CIRCULAR AND RECTANGULAR FLAT PLATES

by

James L. S. Chen, Chia-shing Lai,
and Donald A. Moen

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

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NOMENCLATURE

a	Width of plate in x-direction, or radius of plate
a'	Maximum allowable distance between stiffeners
a_1	Distance between stiffeners in y-direction
b	Width of plate in y-direction
b_1	Distance between stiffeners in x-direction
C	Torsional rigidity of one stiffener ($= GJ$)
D	$\frac{Eh^3}{12(1-\nu^2)}$, flexural rigidity of isotropic plate
D'	$\frac{D_x}{a^4} + \frac{2H}{a^2b^2} + \frac{D_y}{b^4}$
D_1, D_x, D_{xy}, D_y	Rigidity constants of orthotropic plate $\left(D_1 = \frac{E_{12}h_0^3}{12}, D_x = \frac{E_{11}h_0^3}{12}, D_y = \frac{E_{22}h_0^3}{12}, D_{xy} = \frac{E_{66}h_0^3}{12} \right)$
D'_{xy}	Torsional rigidity of plate without stiffener
$E, E_{ij}(i, j = 1, 2)$	Modulus of elasticity
G	Modulus of elasticity in shear $\{= E/[2(1+\nu)]\}$
h	Thickness of isotropic plate
H	$D_1 + 2D_{xy}$
h_c	Combination height of plate and stiffener
h_0	Thickness of orthotropic plate, or "equivalent" thickness of orthotropically-stiffened plate
I	Moment of inertia of a T-section of width a_1 or b_1 (Fig. 2)
I_η, I_ξ	Moment of inertia of one stiffener about principal η -axis, ξ -axis ($I_\eta = h_c t^3/12, I_\xi = t h_c^3/12$)
J	Torsion constant
M_r, M_t, M_x, M_y	Bending moment
n	Number of stiffeners
q	Applied uniform load
t	Width of cross section of stiffener
U_b	Strain energy due to bending
V	Total potential energy of plate
W	Deflection of plate
$W_x, W_{xx}, W_{xy}, W_{rrrr}, \dots$	$\frac{\partial W}{\partial x}, \frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial x \partial y}, \frac{d^4 W}{dr^4}, \dots$
Z	Section modulus of T-beam
α	h/h_c
ϵ_i	Strain
η	Variation of a function
ν	Poisson's ratio of isotropic plate
σ_i	Stress
Ω	Potential energy of external load

STRESS ANALYSIS OF REINFORCED CIRCULAR AND RECTANGULAR FLAT PLATES

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ABSTRACT

A set of mathematical models has been constructed for applying orthotropic plate theory to the case of plates with attached ribs or stiffeners.

The procedure for obtaining an optimum design of plates has been given for a desired deflection limitation. Two sets of computer programs, one for applying simple beam theory to the deflection of plates as an approach, and the other for orthotropic plate theory, have been written in FORTRAN II language to determine the optimum cost. A numerical example has been executed by IBM-1620 computer. The results indicate that the two approaches are in reasonably good agreement.

I. INTRODUCTION

In designing plates for vacuum or pressure vessels, either stiffened or unstiffened plates may be used. It is well-known that the higher the pressure applied, the greater the thickness required. This is costly and also leads to severe thermal stress, for instance, in the case of flat plates on boilers. It is, therefore, of interest to use, in most cases, stiffened plates instead of the flat, relatively thick, unstiffened plates that would be required. Stiffened plates can be constructed by using bars or beams welded to flat plates. This, however, increases labor costs in both cutting and welding.

This report presents a mathematical analysis of stresses and deflections induced in flat plates as well as in stiffened plates for which an optimum design may be obtained.

The mathematical analysis of this problem is based upon an application of orthotropic plate theory to the case of plates with attached ribs or stiffeners. It is assumed that an orthogonally-stiffened isotropic plate is equivalent to an orthotropic plate. This type of application was introduced by Schade¹ to stiffened bottom structures in ships and by Heck² to aircraft design.

In this analysis, the small-deflection theory is used, and it is assumed that the deflection is caused by bending due to uniform load alone and that there is no bending caused by heating.

Since experiments on rectangular plates with uniform load and simply-supported edges have shown that the dangerous section is approximately a diagonal section, except for a plate in which $a \gg b$, it is more effective to stiffen the plate diagonally. However, this arrangement of stiffeners introduces the difficulty of applying the orthotropic plate theory to the problem. The method therefore recommended in this report is limited to orthogonal reinforcement. In the former case, an approximate solution may be obtained, using simple beam theory as a guide, by dividing the diagonally-stiffened plates into section and treating these plates as a T-beam. This approach, in addition to the orthotropic-theory approach, is also used in the case of unidirectionally-stiffened plates so that a comparison of these two may be made.

Circular plates are considered as well as rectangular plates. One-way reinforcement and cross reinforcement are used in both cases (Figs. 1 and 2). For a given allowable deflection because of practical reasons, there exists a set of plate-stiffener combinations that ensure the desired limitation. A digital-computer program is found to be helpful for obtaining the optimum design.

Fig. 1. One-way-stiffened Plates

Fig. 2. Cross-stiffened Plates

II. MATHEMATICAL FORMULATION

A. Rectangular Plates

To apply the analysis given for orthotropic plates to stiffened plates, the following assumptions may be made:

- (1) The stiffened plate has three planes of symmetry with respect to its elastic properties.
- (2) The stiffeners are identical and equally orthogonally spaced.
- (3) The bending and twisting stiffness factors do not depend on the boundary conditions of the stiffened plate.

Based upon Assumption (1), the stress-strain relation of the orthotropic plate for the case of plane stress can be written in the following tensor notation:

$$\left. \begin{aligned} \sigma_i &= E_{ij} \epsilon_j, \quad (i, j = 1, 2) \\ \sigma_6 &= E_{66} \epsilon_6. \end{aligned} \right\} \quad (1)$$

It is noted that $E_{ij} = E_{ji}$, due to the orthotropic properties of the plate. Therefore, there are four independent constants, E_{11} , E_{12} , E_{22} , and E_{66} , ready to be determined either theoretically³ or experimentally⁴ in terms of D_x , H , and D_y in Equation (2).

The Euler differential equation⁵ governing the deflection of an orthotropic plate in which bending action is dominant may be written in the expression (for derivation by using the variational method, see Appendix A)

$$D_x W_{xxxx} + 2H W_{xxyy} + D_y W_{yyyy} = q. \quad (2)$$

For a simply-supported rectangular plate, the forced boundary conditions are, taking the coordinate axes as shown in Fig. 1,

$$\left. \begin{aligned} W &= 0, & \text{at } x &= 0 \text{ and } x = a; \\ W &= 0, & \text{at } y &= 0 \text{ and } y = b; \end{aligned} \right\} \quad (3a)$$

and the natural boundary conditions (see Appendix A) are

$$\left. \begin{aligned} W_{xx} &= 0, & \text{at } x &= 0 \text{ and } x = a; \\ W_{yy} &= 0, & \text{at } y &= 0 \text{ and } y = b. \end{aligned} \right\} \quad (3b)$$

The Navier type solution of Equation (2) that satisfies the boundary conditions (3a) and (3b) was presented by Timoshenko and Woinowsky-krieger⁵ in the form

$$W = \frac{16q}{\pi^6} \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn \left(\frac{m^4}{a^4} D_x + \frac{2m^2 n^2}{a^2 b^2} H + \frac{n^4}{b^4} D_y \right)}, \quad (4)$$

where m and n are integer numbers. The maximum deflection of the plate occurs at its center and is found to be

$$W_{\max} = \frac{16q}{\pi^6} \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} \frac{(-1)^{\frac{1}{2}(m+n)-1}}{mn \left(\frac{m^4}{a^4} D_x + \frac{2m^2 n^2}{a^2 b^2} H + \frac{n^4}{b^4} D_y \right)}. \quad (5)$$

Observing that this is a rapidly converging series, we may obtain a high degree of accuracy by taking only the first term of the series. Thus, it follows that

$$W_{\max} = \frac{16q}{\pi^6} \frac{1}{\frac{D_x}{a^4} + \frac{2H}{a^2 b^2} + \frac{D_y}{b^4}}. \quad (6)$$

In the case of an isotropic flat plate $D_x = H = D_y = D$, Equation (6) reduces to

$$W_{\max} = \frac{16qa^4b^4}{\pi^6D(a^2+b^2)^2}. \quad (7)$$

The maximum bending stress occurs at the center of the simply-supported plate. If $a > b$, it is found that $M_y > M_x$. The use of Equation (4) and the expression,⁵

$$M_y = \int_{-h_0/2}^{h_0/2} \sigma_y z dz = -(D_y W_{yy} + D_1 W_{xx}),$$

gives the maximum bending moment at the center of the plate. This follows, after using the first one-term approximation,

$$(M_y)_{\max} = \frac{16q}{\pi^4 D'} \left(\frac{D_1}{a^2} + \frac{D_y}{b^2} \right), \quad (8a)$$

where

$$D' = \frac{D_x}{a^4} + \frac{2H}{a^2b^2} + \frac{D_y}{b^4}. \quad (8b)$$

Thus, the corresponding maximum bending stress, which must not exceed proportional limit stress, σ_p , is

$$(\sigma_y)_{\max} = \frac{6(M_y)_{\max}}{h_0^2} = \frac{96q}{\pi^4 h_0^2 D'} \left(\frac{D_1}{a^2} + \frac{D_y}{b^2} \right). \quad (9)$$

Again, in the particular case of isotropic flat plate, $h_0 = h$, $D_x = D_y = H = D$, and $D_1 = \nu D$, Equation (9) becomes

$$(\sigma_y)_{\max} = \frac{96qa^2b^2(a^2+\nu b^2)}{\pi^4 h^2(a^2+b^2)^2}. \quad (10)$$

It is known that if the elastic constant E_{ij} or the flexural rigidity constants D , D_x , H , and D_y are defined by either theoretical or experimental methods, the problem may be solved with no difficulties by using Equations (6), (7), (8), (9), and (10). Expressions⁵ for the values of the rigidities in some cases of practical interest may be taken, assuming that the stiffeners in both directions are identical and that the materials of the plating and the stiffeners are the same:

(1) Flat plate:

$$D_x = H = D_y = D = \frac{Eh^3}{12(1-\nu^2)} \quad (11)$$

(2) One-way stiffened plate (Fig. 1):

$$\left. \begin{aligned} D_x &= \frac{Ea_1h^3}{12(a_1-t+\alpha^3t)}; & D_y &= \frac{EI}{a_1}; \\ D_1 &\sim 0 \text{ (neglected the effect of the transverse contraction);} \\ H &= D_1 + 2D_{xy} = 2\left(D'_{xy} + \frac{C}{2a_1}\right) \end{aligned} \right\} \quad (12)$$

(3) Cross-stiffened plate (Fig. 2):

$$\left. \begin{aligned} D_x &= \frac{EI}{b_1}; & D_y &= \frac{EI}{a_1} \\ D_1 &\sim 0 \text{ (neglected the effect of the transverse contraction);} \\ H &= D_1 + 2D_{xy} = \frac{C}{2}\left(\frac{1}{a_1} + \frac{1}{b_1}\right) \end{aligned} \right\} \quad (13)$$

If a very narrow cross section is assigned to the stiffener, the lateral buckling must be taken into consideration. In the case of a simply-supported beam with narrow rectangular cross section, the critical value of the uniform distributed load is given by the formula⁶

$$(qa_1b_1)_{cr} = \frac{28.3 \sqrt{EI\eta C}}{b_1^2} \quad (14)$$

Note that for one-way stiffened plate, $b_1 = b$. By observing that the maximum bending moment acting on the center of the beam is $qa_1b_1^2/8$ and that the section modulus is $2I\eta/h_c$, the maximum bending stress, if $\nu = 0.3$, is

$$\sigma_{cr} = 2.198 \frac{t^2}{h_cb_1} E. \quad (15)$$

This result shows that buckling may occur only if the ratio t^2/h_cb_1 is very small for a given material, say, steel. This critical stress, σ_{cr} , must not exceed the proportional limit, σ_p ; for example, $\sigma_p = 55,000$ psi for steel. Thus, the ratio t^2/h_cb_1 is found to be 0.000834.

The maximum permissible distance, a' , between stiffeners is determined by the fact that the maximum bending stress occurring in the plate is equal to, or less than, the design (proportional limit) stress of the plate material. The case of a plate with simply-supported edges is considered since this case is known to be on the safe side, while the case of a plate with built-in edges is not. In the former case, the maximum bending stress, which occurs at the center of the plate and must not be beyond the proportional limit, σ_p , is

$$\sigma_{\max} = \frac{6M_{\max}}{h^2} \leq \sigma_p, \quad (16)$$

where

$$M_{\max} = \beta q a'^2, \quad (17)$$

in which β is a numerical factor depending on the ratio b_1/a' . Several values of this factor in the case of $b_1 \geq a'$ are given in the following table:

b_1/a'	β	b_1/a'	β
1.0	0.0479	1.7	0.0908
1.1	0.0554	1.8	0.0948
1.2	0.0627	1.9	0.0985
1.3	0.0694	2.0	0.1017
1.4	0.0755	3.0	0.1189
1.5	0.0812	4.0	0.1235
1.6	0.0862	5.0	0.1246
		>5.0	0.1250

Equating Equations (16) and (17), the maximum allowable distance between stiffeners is given in the expression

$$a' = \sqrt{\frac{\sigma_p h^2}{6\beta q}}. \quad (18)$$

The least number of stiffeners follows immediately from the information given by Equation (18). The actual distance between stiffeners, a_1 , which must not exceed a' , is given in the form

$$a_1 = \frac{a}{n+1}, \quad (19)$$

where n is the number of stiffeners.

With the mathematical models formulated in this report, the design procedures for stiffened and unstiffened plates are described in Section III.

B. Circular Plates

Since the deflection at any point of the circular plate which carries a uniformly-distributed load is a function of its radial coordinate r , a simple solution of Equation (2) can be obtained in the case of a circular plate simply supported on the edge and carrying a uniform load q . The expression of the deflection surface, for the case $D_x = D_y$, is

$$W = \frac{q(a^2 - r^2)}{64D''} \left(\frac{5D_x + D_1}{D_x + D_1} a^2 - r^2 \right), \quad (20)$$

in which

$$r^2 = x^2 + y^2$$

and

$$D'' = \frac{1}{8}(3D_x + 2H + 3D_y)$$

satisfies Equation (2) and the required boundary conditions:

$$\left. \begin{array}{l} W = 0 \quad \text{at } r = a \\ W_r = 0 \quad \text{at } r = 0 \end{array} \right\} \text{forced boundary conditions;} \quad (21a)$$

$$\left. \begin{array}{l} D_1 W_r + D_x r W_{rr} = 0 \quad \text{at } r = a \\ \lim_{r \rightarrow 0} \left(r W_{rrr} + W_{rr} - \frac{W_r}{r} \right) = 0 \end{array} \right\} \text{natural boundary conditions.} \quad (21b)$$

The maximum deflection at the center of the plate is

$$W_{\max} = \frac{qa^4}{64D''} \left(\frac{5D_x + D_1}{D_x + D_1} \right). \quad (22)$$

For an isotropic flat plate, Equation (22) reduces to

$$W_{\max} = \frac{qa^4}{64D} \left(\frac{5 + \nu}{1 + \nu} \right). \quad (23)$$

The maximum bending moment for an isotropic flat plate is

$$(M_r)_{\max} = (M_t)_{\max} = \frac{3 + \nu}{16} qa^2. \quad (24)$$

The corresponding maximum bending stress is

$$(\sigma_r)_{\max} = (\sigma_t)_{\max} = \frac{6(M_r)_{\max}}{h^2} = \frac{3(3+\nu) qa^2}{8h^2} \quad (25)$$

In designing a stiffened circular plate, Equations (11), (12), (13), (14), (15), (16), (17), (18), and (19) are unchanged and the design procedures are the same as for rectangular plate described in Section III.

III. PROCEDURES FOR OPTIMUM DESIGN

In designing a vacuum or pressure vessel, the maximum deflection of the plate or its maximum slope at the boundary is restricted by some practical reasons (e.g., to avoid a leak between the plate and the wall of the vessel). Accordingly, it is of interest to assume an allowable maximum deflection or an allowable maximum slope at the boundary. With this given information, the procedures in determining the optimum design for rectangular plates are as follows:

- (1) Use Equation (7) to determine the thickness of flat plate, h .
- (2) Substitute the value of h into Equation (10) to make sure that $(\sigma_y)_{\max} \leq \sigma_p$ ($= 55,000$ psi for steel). If $(\sigma_y)_{\max} > \sigma_p$, it is necessary to increase h accordingly.
- (3) Select, from a standard handbook, a value for h smaller than that obtained in Step (1).
- (4) Use Equation (18) to determine the least number of stiffeners.
- (5) Use Equation (19) to calculate the actual distance, a_1 , between stiffeners.
- (6) Select a set of standard stiffeners from the handbook, put those values of their dimensions into Equation (15) for buckling test, and throw away "useless" stiffeners.
- (7) Substitute the values of Steps (3), (5), and (6) into Equations (12) and (13).
- (8) Substitute the results of Steps (3), (5), and (6) into the equation $\sigma_{\max} = M_{\max}/Z \leq \sigma_p$ for maximum bending stress test on the T-beam.
- (9) Substitute the results of Steps (7) and (8) into Equation (6) to check the maximum allowable deflection.
- (10) Select various values for h , and repeat Steps (4) to (9).
- (11) From the knowledge of the above findings (a family of combinations of plate and stiffener), determine the optimum cost for a given condition of the plate according to the formula

$$\text{cost} = AX + BY + JZ,$$

where

A = unit cost of material, \$/lb,

B = unit cost of cutting, \$/in.²,

J = unit cost of welding, \$/in.,

X = weight of material used, lb,

Y = area of cutting required, in.²,

and

Z = length of welding required, in.

A digital program (Appendix C) has been set up for this problem because of its iterative nature.

The optimum design procedure for rectangular plates is applicable for circular plates, except that Equations (22), (23), and (25) are used instead of Equations (6), (7), and (10), respectively.

IV. RESULTS AND DISCUSSIONS

As an example problem, a 16- x 24-in. plate was designed to withstand a design pressure of 20 psi with a maximum allowable slope at the boundary of 0.0104. The material was taken to be Type 304 stainless steel throughout, and the design stress was set at the yield stress of 55,000 psi. For this problem, the orthotropic approach indicated that the most economical design was a unidirectionally-stiffened 0.15625-in.-thick plate with one 0.1875- x 1.5-in.-thick plate stiffener. The indicated cost of this solution was \$29.35, and the maximum bending stress was 50,343 psi. For the cross-stiffened case, the most economical solution was a 0.125-in. plate, stiffened by one stiffener in each direction. The optimum stiffener dimensions were again 0.1875 x 1.5 in., but this time the cost was \$34.35 and the maximum stress was 52,192 psi.

The simple beam theory approach for the unidirectional solution produced a 0.125-in.-thick plate stiffened by two 0.1875- x 1.5-in. stiffeners at a cost of \$31.04 and a maximum stress of 35,621 psi. These cost figures compare to an unstiffened-plate cost of \$73.70 for the 0.5-in. flat plate required.

It can be seen that the simple beam theory is more conservative, as one would expect. However, the simple beam theory is only for deflection considerations and for calculating stress in the equivalent plate and stiffener

sections, which are considered as simple T-beams. The distance between stiffeners is governed by stress in the plate itself. Thus, this distance depends on the assumptions made concerning the boundary conditions of a plate section between stiffeners. Theoretically, the assumption of two edges built in and the two edges simply supported should be the most conservative, and this theory seems to be substantiated by the results. Also, this assumption produces excellent agreement between the plate thickness at which stiffeners are no longer required and the theoretical value for an isotropic unstiffened plate. For the case of cross stiffeners, however, all edges are obviously in the same condition, and the assumption that all edges are simply supported should be made to be on the safe side. This assumption was made for the orthotropic approach in which cross stiffeners were included.

Using the orthotropic approach, it was noted that results could be obtained with no stiffeners before the plate thickness was great enough to serve as an unstiffened plate. Obviously an explanation is in order. The form of the equations for orthotropic rigidity constants changes for an unstiffened plate. Therefore, an unstiffened plate of less than the required thickness might pass the deflection test because the test was no longer working properly. The question then becomes one of explaining why the maximum distance between stiffeners became so large when it certainly appears that stiffeners are necessary. The answer is probably that a more conservative assumption concerning stress in the plate is in order. It would be interesting to try the orthotropic approach using the assumption of a plate section with two edges built in, as was done in the simple beam theory approach.

Although this group did not get to the point where much work was done with circular plates, both of the programs could quite easily be extended to circular plates as indicated in Section II of this report.

APPENDIX A

Derivations of Euler Equation and Natural Boundary
Conditions for Orthotropic Plates

The strain energy of an orthotropic plate because of bending alone is given by the expression

$$U_b = \frac{1}{2} \iint [D_x W_{xx}^2 + 2D_1 W_{xx} W_{yy} + D_y W_{yy}^2 + 4D_{xy} W_{xy}^2] dx dy \quad (A-1)$$

The potential energy of external load is

$$\Omega = -\iint q W dx dy. \quad (A-2)$$

In small deflection theory, membrane effect can be neglected; if there is no bending caused by heating, the total potential energy of the plate may be written as

$$V = U_b + \Omega.$$

Thus,

$$\left. \begin{aligned} V &= \iint \left\{ \frac{1}{2} [D_x W_{xx}^2 + 2D_1 W_{xx} W_{yy} + D_y W_{yy}^2 + 4D_{xy} W_{xy}^2] - q W \right\} dx dy \\ \text{or} \\ V &= \iint F dx dy. \end{aligned} \right\} \quad (A-3)$$

By using the procedures of the calculus of variation, the Euler equation for the double integral

$$V = \iint_R F(x, y, W, W_x, W_y, W_{xx}, W_{xy}, W_{yy}) dx dy$$

is

$$\frac{\partial F}{\partial W} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial W_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial W_y} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial W_{xx}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial W_{xy}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial W_{yy}} \right) = 0. \quad (A-4)$$

Accordingly, Equations (A-3) and (A-4) yield the Euler differential equation for the orthotropic plate,

$$D_x W_{xxxx} + 2(D_1 + 2D_{xy}) W_{xxyy} + D_y W_{yyyy} = q. \quad (A-5)$$

To determine the natural boundary conditions, the following relations which are also derived by the variational method may be used:

$$\left. \begin{aligned}
& \eta \left[\frac{\partial F}{\partial W_x} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial W_{xx}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial W_{xy}} \right) \right] = 0, \quad \text{at } x = 0 \text{ and } x = a; \\
& \eta_x \frac{\partial F}{\partial W_{xx}} = 0, \quad \text{at } x = 0 \text{ and } x = a; \\
& \eta \left[\frac{\partial F}{\partial W_y} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial W_{xy}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial W_{yy}} \right) \right] = 0, \quad \text{at } y = 0 \text{ and } y = b; \\
& \eta_y \frac{\partial F}{\partial W_{yy}} = 0, \quad \text{at } y = 0 \text{ and } y = b.
\end{aligned} \right\} \quad (\text{A-6})$$

The bracketed expressions in Equation (A-6) must vanish if the values of η , η_x , and η_y are unrestricted by forced boundary conditions. For simply-supported rectangular plates, the value of the variational function η on the boundaries is zero. Therefore, the natural boundary conditions of the plate follow from the second and fourth expressions of Equation (A-6),

$$\left. \begin{aligned}
& D_x W_{xx} + D_1 W_{yy} = 0, \quad \text{at } x = 0 \text{ and } x = a; \\
& D_y W_{yy} + D_1 W_{xx} = 0, \quad \text{at } y = 0 \text{ and } y = b;
\end{aligned} \right\} \quad (\text{A-7})$$

observing that W_{xx} and W_{yy} must vanish at $y = 0$ and $y = b$, and $x = 0$ and $x = a$, respectively. Consequently, the natural boundary conditions for the simply supported edges of the plate reduce to

$$\left. \begin{aligned}
& W_{xx} = 0, \quad \text{at } x = 0 \text{ and } x = a; \\
& W_{yy} = 0, \quad \text{at } y = 0 \text{ and } y = b.
\end{aligned} \right\} \quad (3b)$$

As might be expected, $W_{xx} = 0$ and $W_{yy} = 0$ are equivalent to having bending moments $M_x = 0$ and $M_y = 0$ along edges, respectively.

APPENDIX B

Program for Unidirectionally-stiffened Plates Using Simple Beam Theory1. Program Nomenclature

- $H(I)$ = thickness of plate (subscripted)
 $N1$ = the number of $H(I)$ values entered on the data cards
 $WIDTH(K)$ = width of rectangular stiffeners
 $N2$ = the number of $WIDTH(K)$ values entered on the data cards
 $THICK(K,L)$ = height of stiffeners
 $N3$ = the maximum number of $THICK(K,L)$ values entered on the data cards for each value of $WIDTH(K)$
 A = "x" dimension of plate
 B = "y" dimension of plate, ($A > B$)
 ANU = Poisson's ratio (ν)
 SL = maximum allowable slope of plate at the edges
 DEN = material weight density
 $VPRI$ = price of material in \$/lb
 $CPRI$ = price of cutting in \$/in.²
 $WPRI$ = price of welding in \$/in.
 Q = design pressure (or vacuum) of vessel
 $PI = \pi$
 HU = the thickness of unstiffened plate required to meet deflection criteria
 $HNEAR$ = the standard plate thickness which is nearest to HU but above it if possible
 E = Young's modulus
 $YIELD$ = design stress; this may or may not be the material yield stress depending on the factor of safety required
 $COST$ = cost of material + cost of cutting + cost of welding
 $DMAX$ = maximum allowable distance between stiffeners
 M , or AM = number of stiffeners required
 $DACT$ = actual center-to-center distance between stiffeners

RI = required moment of inertia about axis o-o to meet deflection requirements

AI = actual moment of inertia of T-beam about axis a-a

Y = distance from o-o to a-a

OI = actual moment of inertia about axis o-o

SIG = bending stress

CRIT = critical stress in buckling

2. Program

The program written in FORTRAN II language for rectangular unidirectionally-stiffened plates, using simple beam theory is as follows:

```

C      MOEN, CHEN, LAI, RECT, UNIDIR, STIFFENED PLATES
1  FORMAT (6F10,5)
2  FORMAT (2F10,0)
3  FORMAT (13H UNSTIFFENED ,3F10,5)
6  FORMAT (3I4)
7  FORMAT (7F10,5)
      DIMENSION H(30),WIDTH(20),THICK(20,30),G(7),C(7)
      READ 6, N1,N2,N3
      READ 2,E,YIELD
      READ 1,A,B,ANU,SL,DEN,VPRI,CPRI,WPRI
      READ 1, (H(I),I=1,N1)
      READ 1, (WIDTH(K),K=1,N2)
      DO 10 K=1,N2
10  READ 7,(THICK(K,L),L=1,N3)
      READ 1,Q
      PI=3.1416
      HU=(192,*(1,-ANU**2)*Q*A**4*B**3/(SL*E*(A**2+B**2)**2*PI**5))**
1(1,/3,)
      DO 9 I=1,N1
      II=I
      IF (H(I)-HU) 9,12,12
9  CONTINUE
12  HNEAR=H(II)
      COST=A*B*HNEAR*DEN*VPRI+(A+B)*HNEAR*CPRI
      PUNCH 3,HU,HNEAR,COST
      G(1)=.0833
      G(2)=.0842
      G(3)=.0822
      G(4)=.0810
      G(5)=.0794
      G(6)=.0771
      G(7)=.0739
      C(1)=4.
      C(2)=2.
      C(3)=1.5
      C(4)=1.4
      C(5)=1.3

```

```

C(6)=1.2
C(7)=1.1
DO 60 I=1,N1
DO 24 J=1,7
DMAX=(YIELD*H(I)**2/(6,*G(J)*Q))**.5
IF (C(J)=B/DMAX)25,25,24
24 CONTINUE
25 M=A/DMAX
IF(0=M)26,61,61
26 AM=M
DACT=A/(AM+1.)
RI=Q*DACT*B**3/(E*SL*24.)
DO 60 K=1,N2
DO 60 L=1,N3
IF(0.=THICK(K,L))27,60,60
27 AI=DACT*H(I)**3/3.*WIDTH(K)*THICK(K,L)**3/3.
Y=(WIDTH(K)*THICK(K,L)**2/2.=DACT*H(I)**2/2.)/(WIDTH(K)*THICK(K,L)
1+DACT*H(I))
OI=AI-(WIDTH(K)*THICK(K,L)+DACT*H(I))*Y**2
IF (OI=RI) 60,40,40
40 SIG=Q*DACT*B**2*(THICK(K,L)-Y)/(8.*OI)
IF (YIELD-SIG) 60,42,42
42 CRIT=2.198*WIDTH(K)**2*E/((THICK(K,L)+H(I))*B)
IF (CRIT-SIG) 60,60,45
45 COST=(A*B*H(I)*DEN+WIDTH(K)*THICK(K,L)*H*AM*DEN)*VPRI+((A+B)*
1H(I)+WIDTH(K)*THICK(K,L)*AM)*CPRI+2.*B*AM*WPRI
PUNCH 5, H(I),WIDTH(K),THICK(K,L),COST,AM,SIG,DACT
5 FORMAT (11H UNI STIFF ,3F10.5,2F8.2,F8.0,F7.3)
60 CONTINUE
61 STOP
END

```

3. Flow Diagram

4. Discussion of Program

This program was written for the IBM-1620 computer. For the sake of speed, the output is on punched cards.

The method used in this program is to assume a combination of plate and stiffener dimensions and then to test to see if the combination is satisfactory with regard to deflection, stress, and buckling considerations. If a given combination is found to be satisfactory, $H(I)$, $WIDTH(K)$, $THICK(K,L)$, AM , $COST$, SIG , and $DACT$ are punched on an output card, and the computer assumes a new combination and repeats the sequence. If the given combination is not satisfactory, the machine immediately discards it and tries again. Thus, the output includes a cost figure for every satisfactory combination and the program user can optimize the design simply by looking at the $COST$ column and picking the smallest figure.

To use the program, a designer must select several standard values of $H(I)$, $WIDTH(K)$, and $THICK(K,L)$ from his catalog and put them on data cards along with the other indicated input data. For each stiffener $WIDTH(K)$, a series of stiffener heights is used so that the assumed solutions can consist entirely of standard-sized materials. The $H(I)$ values must be arranged in increasing order, with $H(I)$ being the smallest and $H(N1)$ being the largest. It is suggested that a similar procedure be followed for $WIDTH(K)$ and $THICK(K,L)$. (Assume, in the following discussion, that this has been done.)

The computer will first calculate and punch out H_U , H_{NEAR} , and $COST$, where $COST$, at this point, is for an unstiffened plate of thickness H_{NEAR} . The user is cautioned to note whether H_{NEAR} is greater or less than H_U . If H_{NEAR} is greater, the solution is valid, but if H_U is greater, this means that there was no value of $H(I)$ greater than H_U and the cost figure for the unstiffened case should be disregarded.

The computer will now go on to the unidirectionally-stiffened situation. It first assumes the smallest plate thickness $H(I)$ and calculates the corresponding D_{MAX} , AM , $DACT$, and RI . D_{MAX} , AM , and $DACT$ are calculated by considering a section of plate between stiffeners to have two edges simply supported and two edges built in, as illustrated below.

D_{MAX} is calculated from the expression

$$D_{MAX} = \frac{(YIELD)(H(I)^2)}{(6G(J)Q)},$$

where $G(J)$ has been used to represent γ and comes from Table 29 in Timoshenko and Woinowsky-krieger.⁵

Table 29 has been built into the program so that the computer automatically selects the proper value of $G(J)$. RI is calculated by considering the stiffened plate to be made up of a series of simple T-beams and applying simple beam theory.⁷ By considering maximum allowable deflection to be $SL(B/2)$, RI is calculated from the equation

$$RI = \sqrt{\frac{Q(DACT)B^3}{E(SL)24}},$$

where

$$DACT = A/(AM + 1).$$

The machine next assumes the smallest values of $WIDTH(K)$ and $THICK(K,L)$, respectively, and proceeds to test the new complete set of dimensions. It first calculates OI and compares its value to RI . OI is calculated from the equations

$$AI = (DACT) H(I)^3/3 + WIDTH(K) THICK(K,L)^3/3,$$

$$Y = [WIDTH(K) THICK(K,L)^2/2 - (DACT) H(I)^2/2] / [WIDTH(K) THICK(K,L) + (DACT) H(I)],$$

and

$$OI = AI - [WIDTH(K) THICK(K,L) + (DACT) H(I)] Y^2.$$

If OI is greater than RI , the deflection test is satisfied and the machine goes on to test for bending stress. Again, using simple beam theory, maximum bending stress is calculated from the equation

$$SIG = Q(DACT) B^2(THICK(K,L) - Y)/8(OI).$$

If SIG is less than $YIELD$, the bending stress test is satisfied and the computer goes on to the buckling test. $CRIT$ is calculated from the equation

$$CRIT = 2.189 WIDTH(K)^2 E / (THICK(K,L) + H(I)B)$$

and is also compared with $YIELD$. If $CRIT$ is greater than $YIELD$, the assumed set of dimensions has been found satisfactory. This being the case, the quantities to be output are punched on a card and the machine assumes a new combination by going to the next value of $THICK(K,L)$. If the given set of dimensions fails one of the above tests, the machine immediately discards it and goes directly to the next greater value of $THICK(K,L)$. This scheme continues until all the values of $THICK(K,L)$ corresponding to the initial value of $WIDTH(K)$ have been tried. The machine then

increases the value of $WIDTH(K)$ and moves to a new sequence of $THICK(K, L)$ values. This procedure is repeated until all values of $WIDTH(K)$ have been tried, at which time the computer increases the value of $H(I)$ and the entire sequence is repeated. When all possible combinations of dimensions have been tried, or when M becomes zero, processing is complete.

APPENDIX C

Program for Unidirectional and Cross-stiffened Plates
Using Orthotropic Theory

1. Program Nomenclature

The nomenclature used in this program is similar to that used in the program described in Appendix B. Hence, only those symbols which have been changed or added are listed here.

MA or AM = number of short stiffeners (stiffeners parallel to the B dimension)

MB or BM = number of long stiffeners (stiffeners parallel to the A dimension)

DACTA = actual center-to-center distance between short stiffeners

DACTB = actual center-to-center distance between long stiffeners

WMAX = maximum allowable deflection at the center of the plate

G = modulus of elasticity in shear

DX = D_x (Equations 11, 12, and 13)

DY = D_y (Equations 12 and 13)

HCON = H (Equations 12 and 13)

DPRIM = D' (Equation 8b)

W = actual deflection at the center of the plate

2. Program

The program written in FORTRAN II language for unidirectionally- and cross-stiffened rectangular plates, using orthotropic plate theory, is as follows:

```

C      MOEN, CHEN, LAI, RECTANGULAR PLATES
      DIMENSION H(30),WIDTH(20),THICK(20,30),BETA(15),C(15)
1  FORMAT (6F10.5)
2  FORMAT (2F10.0)
3  FORMAT (13H UNSTIFFENED ,3F10.5)
6  FORMAT (3I4)
7  FORMAT (7F10.5)
      READ 6,N1,N2,N3
      READ 2,E,YIELD
      READ 1,A,B,ANU,SL,DEN,VPRI,CPRI,WPRI
      READ 1,(H(I),I=1,N1)
      READ 1,(WIDTH(K),K=1,N2)
      DO 10 K=1,N2
10  READ 7,(THICK(K,L),L=1,N3)
      READ 1,G

```

```

PI=3.1416
HU=(192.*(1.-ANU**2)*Q*A**4*B**3/(SL*E*(A**2+B**2)**2*PI**5))**
1(1./3.)
DO 9 I=1,N1
  II=I
  IF (H(I)=HU) 9,12,12
9 CONTINUE
12 HNEAR=H(II)
  COST=A*B*HNEAR*DEN*VPRI+(A+B)*HNEAR*CPRI
  PUNCH 3,HU,HNEAR,COST
  WMAX=SL*B/2.
  G=E/(2.*(1.+ANU))
  BETA(1)=.125
  BETA(2)=.1246
  BETA(3)=.1235
  BETA(4)=.1189
  BETA(5)=.1017
  BETA(6)=.0985
  BETA(7)=.0940
  BETA(8)=.0908
  BETA(9)=.0862
  BETA(10)=.0812
  BETA(11)=.0755
  BETA(12)=.0694
  BETA(13)=.0627
  BETA(14)=.0554
  BETA(15)=.0479
  C(1)=10.
  C(2)=5.
  C(3)=4.
  C(4)=3.
  C(5)=2.
  C(6)=1.9
  C(7)=1.8
  C(8)=1.7
  C(9)=1.6
  C(10)=1.5
  C(11)=1.4
  C(12)=1.3
  C(13)=1.2
  C(14)=1.1
  C(15)=1.
DO 60 I=1,N1
DO 24 J=1,15
  DMAX=(YIELD*H(I)**2/(6.*BETA(J)*Q))**.5
  IF (C(J)=B/DMAX)25,25,24
24 CONTINUE
25 MA=A/DMAX
  IF (0=MA)26,61,61
26 AM=MA
  DACTA=A/(AM+1.)
DO 60 K=1,N2
DO 60 L=1,N3
  IF (0.=THICK(K,L))27,60,60
27 AI=DACTA*H(I)**3/3.+WIDTH(K)*THICK(K,L)**3/3.
  Y=(WIDTH(K)*THICK(K,L)**2/2.-DACTA*H(I)**2/2.)/
1(WIDTH(K)*THICK(K,L)+DACTA*H(I))

```

```

OI=AI-(WIDTH(K)*THICK(K,L)+DACTA*H(I))*Y**2
SIG=Q*DACTA*R**2*(THICK(K,L)-Y)/(8.*OI)
IF (YIELD-SIG)60,42,42
42 CRIT=2.198*WIDTH(K)**2*E/((THICK(K,L)+H(I))*B)
IF (CRIT-SIG)60,45,45
45 DX=E*DACTA*H(I)**3/(12.*(DACTA-WIDTH(K)+(H(I)/(H(I)+THICK(K,L)))*
13*WIDTH(K)))
DY=E*WIDTH(K)*THICK(K,L)**3/(12.*DACTA)
HCON=2.*H(I)**3*R*G/3.+G*THICK(K,L)*WIDTH(K)**3/(3.*DACTA)
DPRIM=DX/A**4+2.*HCON/(A**2*R**2)+DY/B**4
W=16.*Q/(PI**6*DPRIM)
IF (WMAX=W)60,50,50
50 COST=(A*B*H(I)*DEN+WIDTH(K)*THICK(K,L)*B*AM*DEN)*VPRI+((A+B)*H(I)+
1*WIDTH(K)*THICK(K,L)*AM)*CPRI+2.*R*AM*WPRI
5 FORMAT(11H UNI STIFF ,3F10.5,2F8.2,F8.0,F7.3)
PUNCH 5, H(I),WIDTH(K),THICK(K,L),COST,AM,SIG,DACTA
60 CONTINUE
61 DO 120 I=1,N1
DMAX=(YIELD*H(I)**2/(6.*BETA(15)*Q))**.5
64 MA=A/DMAX
MB=B/DMAX
IF (0-MB)63,121,121
63 AM=MA
BM=MB
DACTA=A/(AM+1.)
DACTB=B/(BM+1.)
DO 120 K=1,N2
DO 120 L=1,N3
IF (0.=THICK(K,L))62,120,120
62 AI=DACTA*H(I)**3/3.+WIDTH(K)*THICK(K,L)**3/3.
Y=(WIDTH(K)*THICK(K,L)**2/2.-DACTA*H(I)**2/2.)/
1(WIDTH(K)*THICK(K,L)+DACTA*H(I))
OI=AI-(WIDTH(K)*THICK(K,L)+DACTA*H(I))*Y**2
SIG=Q*DACTA*R**2*(THICK(K,L)-Y)/(8.*OI)
IF (YIELD-SIG)120,65,65
65 CRIT=2.198*WIDTH(K)**2*E/((THICK(K,L)+H(I))*B)
IF (CRIT-SIG)120,70,70
70 DX=E*WIDTH(K)*THICK(K,L)**3/(12.*BM)
DY=E*WIDTH(K)*THICK(K,L)**3/(12.*AM)
HCON=(1./6.)*(THICK(K,L)*WIDTH(K)**3/BM+THICK(K,L)*WIDTH(K)**3/AM)
DPRIM=DX/A**4+2.*HCON/(A**2+B**2)+DY/B**4
W=16.*Q/(PI**6*DPRIM)
IF (WMAX=W)120,75,75
75 COST=(A*B*H(I)*DEN+WIDTH(K)*THICK(K,L)*(B*AM+(A=AM*WIDTH(K))*BM)*
1DEN)*VPRI+((A+B)*H(I)+WIDTH(K)*THICK(K,L)*(AM+BM*(AM+1.)))*CPRI+(B
1*AM+A*BM)*WPRI*2.
8 FORMAT (7H CROSS ,3F10.5,2F8.2,F8.0,2F7.3)
PUNCH 8,H(I),WIDTH(K),THICK(K,L),COST,AM,SIG,DACTA,DACTB
120 CONTINUE
121 STOP
END

```

3. Flow Pattern

The organization of this program is nearly identical to that of the program described in Appendix B. The differences are:

- (1) A slightly different assumption is used in calculating DMAX.

- (2) Orthotropic theory is used in establishing deflection criteria instead of simple beam theory.
- (3) An additional section for the cross-stiffened case has been added to the unidirectionally-stiffened section.

Since the flow pattern of the cross-stiffened section is the same as that of the unidirectional section, the flow diagram of Appendix B also applies to this program if one realizes that starting with the assumption of $H(I)$, the entire pattern is simply repeated for cross-stiffened plates.

4. Discussion of Program

As previously indicated, the organization and sequencing used in this program are almost identical to those described in Appendix B. Hence, the description will not be repeated here.

The first difference one encounters is in the calculation of DMAX. Instead of considering a plate section to be built in on two edges, the plate section is now assumed to be simply supported on all edges. The method of calculation remains unchanged, but the constants BETA(J) are now taken from Table 8 in Timoshenko and Woinowsky-krieger.⁵ The table has been built into the program so that for the unidirectional case, the computer automatically selects the proper value of BETA(J). For the cross-stiffened case, the plate section is assumed to be nearly square. Hence, BETA(J) becomes simply BETA(15).

The major difference between the two programs is that in this program, orthotropic theory is used to calculate deflection at the center of the plate. Deflection is calculated by using the equation

$$W = \frac{16Q}{(PI)^6 (DPRIM)}$$

and is compared to WMAX. If WMAX is greater than W, the deflection test has been passed.

Simple beam theory is still used, however, in the bending stress test. For the cross-stiffened case, one must realize that this theory might be significantly in error. The error, however should be on the safe side and hence is not considered to be serious.

5. Recommendations for Improving the Program

The results of the two programs indicate that the assumptions concerning boundary conditions may have a considerable effect when calculating

DMAX. Thus, it is thought that the program could be improved by putting the BETA(J) table on data cards instead of writing it into the source program. With this change, the user could select different boundary conditions simply by putting in the proper values of BETA(J).

There should be no problems associated with running this program on another computer. For example, to run the program on the CDC-160A, the only change should be to change all PUNCH statements to PRINT statements.

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