COO-2118-0026

UIUCDCS-R-71-489

AN APPROXIMATION METHOD FOR SOLVING THE SOFA PROBLEM

By

Kiyoshi Maruyama

October, 1971

Department of Computer Science
University of Illinois at Urbana-Chanpaign
Urbana, Illinois 61801

This work was supported in part by Contract AT(11-1)-2118 with the U.S. Atomic Energy Commission.

AN APPROXIMATE METHOD FOR SOLVING THE SOFA PROBLEM

Abstract

A procedure for the solution of the two-dimensional sofa problem is described. A new class of polygons, angularly simple polygons, is defined as a class of permissible sofas. The pattern representation, $S_r(x_0)$, developed for this class of polygons has the advantage of allowing easy polygonal transformations. The procedure called GSPS, described herein, gives a good approximate solution to the sofa problem in reasonable time. Slight modification of the procedure leads to an algorithm for the solution of the general sofa problem.

Index terms--sofa problem, hallway, objective function, polygonal Jordan curve, polygon, angularly simple, pattern sequence.

ACKNOWLEDGMENT

The author wishes to thank Dr. B. H. McCormick, Professor of Computer Science and Physics at the University of Illinois for his helpful comments. Thanks are also extended to Mrs. Judy Arter who both typed and helped edit this report.

I. INTRODUCTION

The sofa problem - determining the largest region (or sofa) which can be moved through a two-dimensional hallway of width 1 (See Figure 1) - was originally proposed by Leo Moser [11]. Some analytical solutions for the sofa problem of Moser's hallway have been given by Goldberg [5] and Hammersley and Sebastian [15]. These solutions by Goldberg and Hammersley are illustrated in Figures 2 and 3, respectively. To the author's knowledge, $\pi/2 + 2/\pi$ is the known lower bound for Moser's problem and the upper bound for the problem is given by Sebastian as $2\sqrt{2}$, which is illustrated in Figure 4.

The lower bound sofa is that sofa which can be moved through the hallway with continuous transformations, while the upper bound sofa cannot be moved through the hallway. That is to say that these sofas bound the area size of the maximal achievable sofa for a given hallway.

A computer approach for the solution of Moser's hallway with an objective function, here selected to be largest rectangle for a given width $w<1$, has been studied by Howden [6]. He used a chain representation [4] for his rectangular sofa and his search strategy evoked straightforward exhaustive trials for a given width, w, of the rectangle. By increasing the length, ℓ, of his sofa, he found what maximal rectangular sofa could go through the hallway, where the theoretical upper bound, ℓ_{max}, is given by $2(\sqrt{2} - w)$. Howden showed by his approach that 84% of the bound, ℓ_{max}, could be moved through the hallway. He also pointed out that the accuracy is more dependent on the size of an unit translation, λ, than on the size of an unit rotation angle, δ.

Howden's approach can be applied if the given objective function

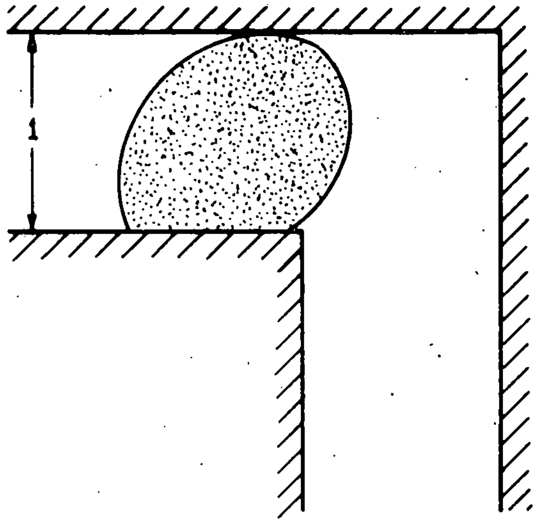


Figure 1. Moser's Hallway (from [11])

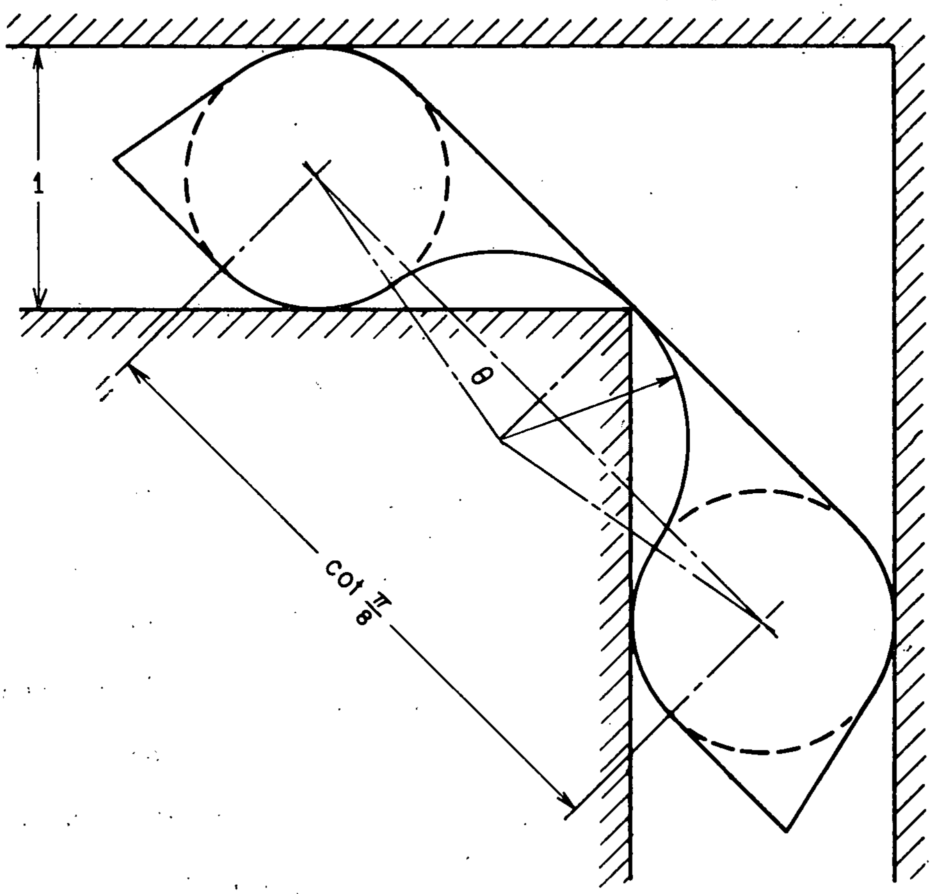


Figure 2. A Solution By Goldberg: the Lower Bound is 2.044

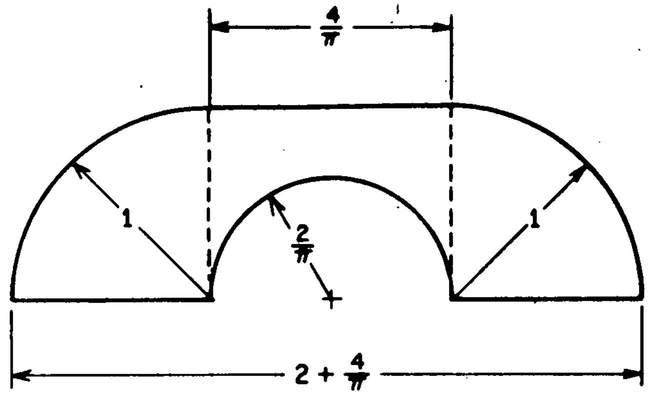


Figure 3. A Solution By Hammersley:
the Lower Bound is $\pi/2 + 2/\pi$

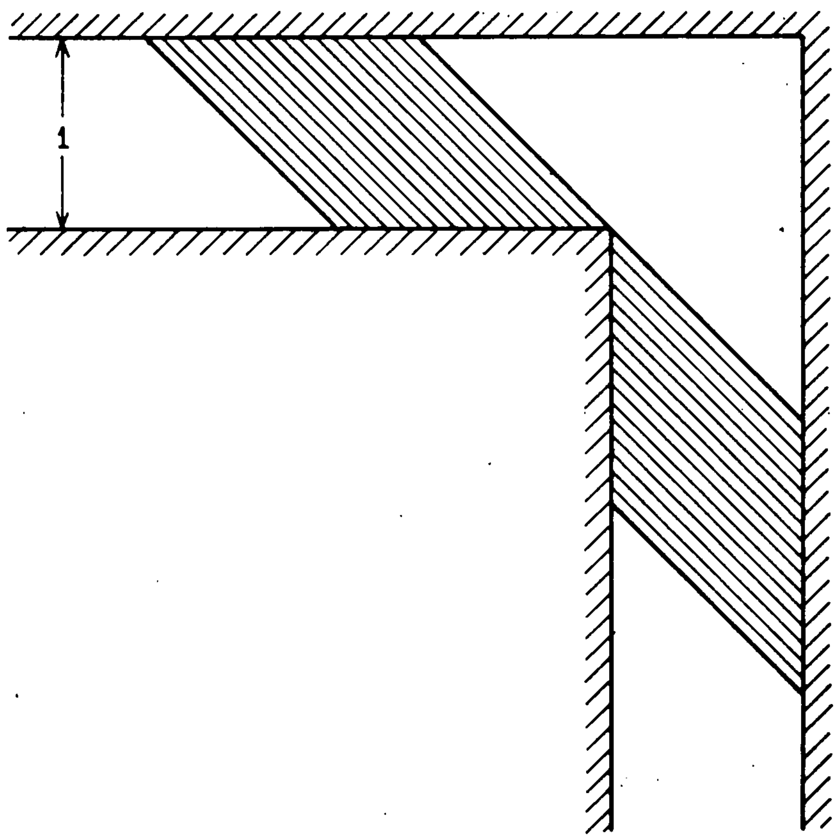


Figure 4. A Solution By Sebastian:
the Upper Bound is $2\sqrt{2}$

specifies the generic shape, e.g. the largest square, the widest rectangle for a given length, the longest rectangle for a given width, etc. However, it cannot be applied if the objective function is the largest area sofa, i.e. Moser's objective function, since the shape of the solution for such a problem is unknown. Accordingly the solution of the sofa problem with Moser's objective function is, in general, non-trivial.

Our computer algorithm for the solution of a two-dimensional sofa problem generalizes to a procedure for almost any hallway (or for sequentially connected hallways called 'composite hallways') as well as for almost any objective function. However, the technique is especially good for the sofa problem with Moser's objective function. We introduce a new class of sofas which will restrict the shape of sofas to <u>angularly simple polygons</u>, yet will give a good approximate solution to the shape of the sofa, the size of the sofa and the sequence of the transformations required to move the sofa through the given hallway, all computed within a reasonable amount of time.

In Table I, we show analytical solutions for some hallways with Moser's objective function and these hallways are illustrated in Figure 5. We show these lower and upper bounds for some hallways to get some idea of what size sofa can move through and what size cannot. It is quite possible that someone may improve these bounds, but this is not the purpose of this paper.

Table I Analytical Solutions for Hallways of Figure 5 With Moser's Objective Solution

Hallways \ Bounds	A_r	A_u	A_ℓ	K_u	K_ℓ
Moser's Hallway of Figure 1	1	$2\sqrt{2}$	$\pi/2+2/\pi$	$2\sqrt{2}$	$\pi/2+2/\pi$
Inverse L and L Hallway, Fig. 5(a)	1	$2\sqrt{2}-1$	$\pi/2$	$2\sqrt{2}-1$	$\pi/2$
U-shape Hallway, Fig. 5(b)	$\pi/4$	2.381	2.324	3.03	2.96
S-shape Hallway, Fig. 5(c)	$\pi/4$	1.95	1.82	2.48	2.32
Hallway of Figure 5(d)	$\pi/4$	1.82	$\pi/2$	2.32	2.00

A: denotes area

A_r: Maximum area which can go through without rotation.

A_u: An upper bound of area with rotation.

A_ℓ: A Lower bound of area with rotation.

K: denotes ratio

K_r: $A_r/A_r = 1$

K_u: A_u/A_r

K_ℓ: A /A_r

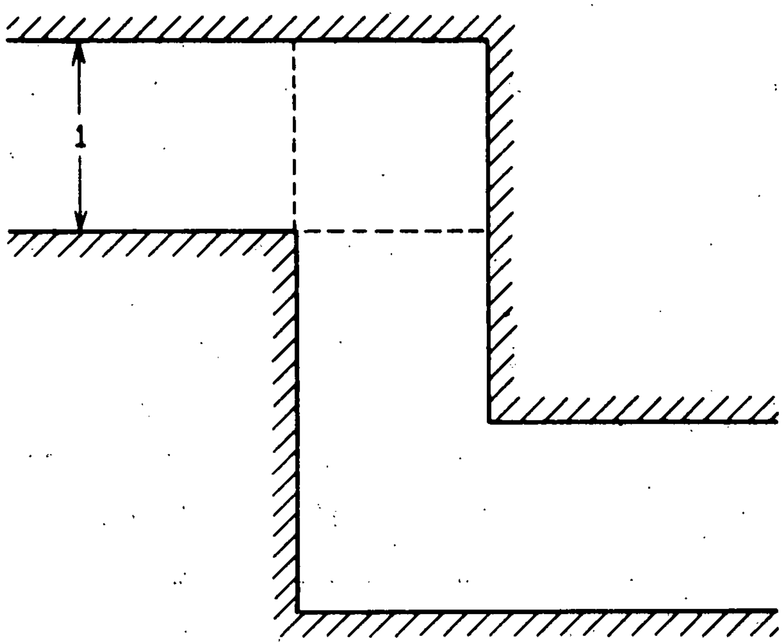


Figure 5. (a) A Hallway of Inverse L and L: $A_r = 1$

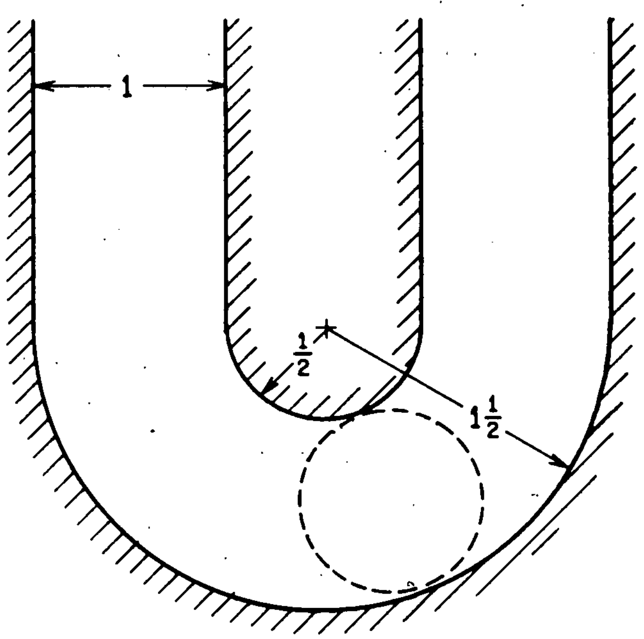


Figure 5. (b) A U-shape Hallway: $A_r = \pi/4$

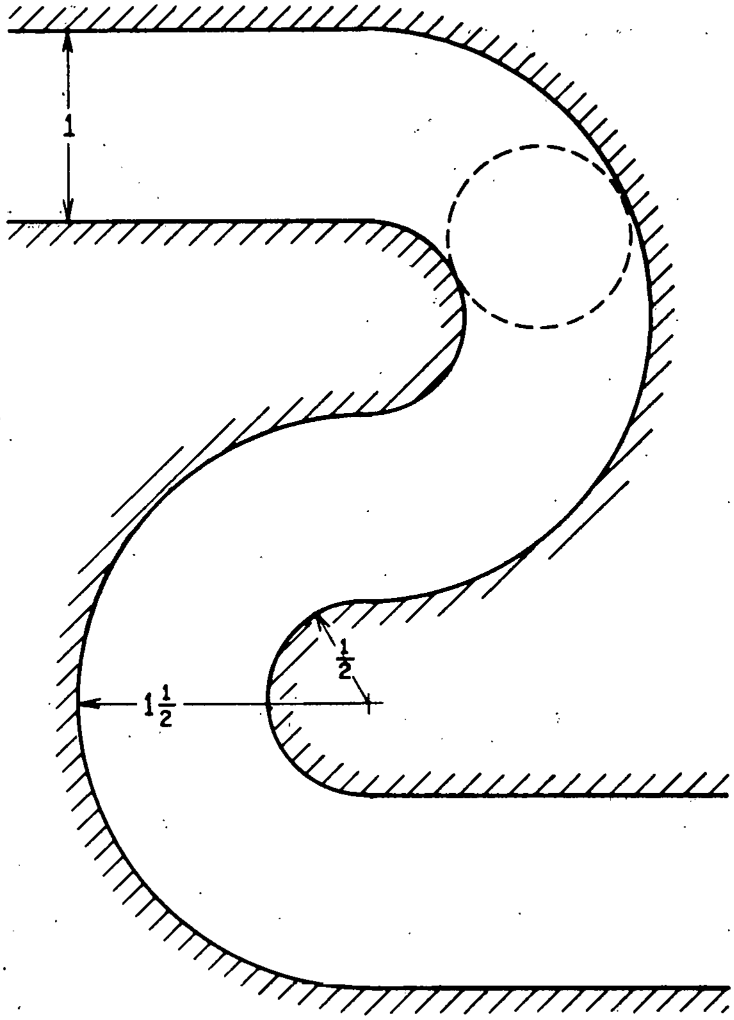


Figure 5. (c) A S-shape Hallway: $A_r = \pi/4$

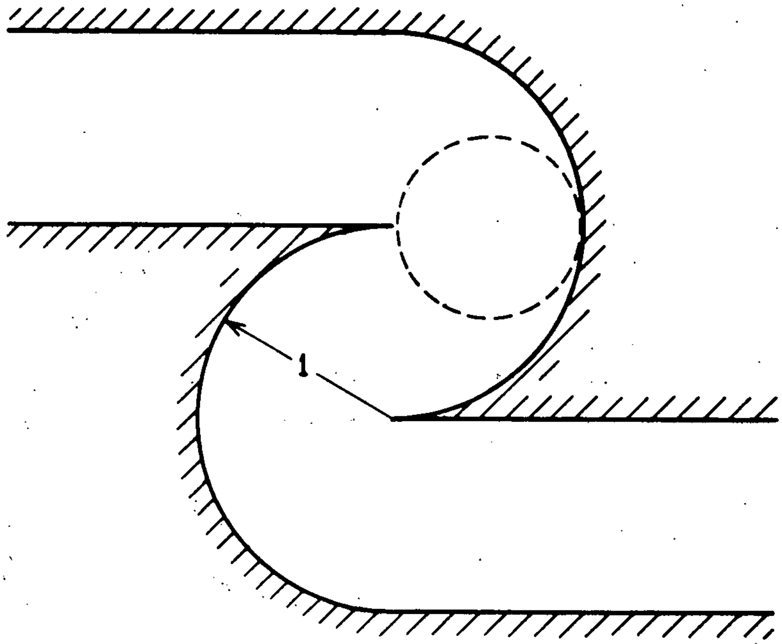


Figure 5. (d) A Hallway: $A_r = \pi/4$

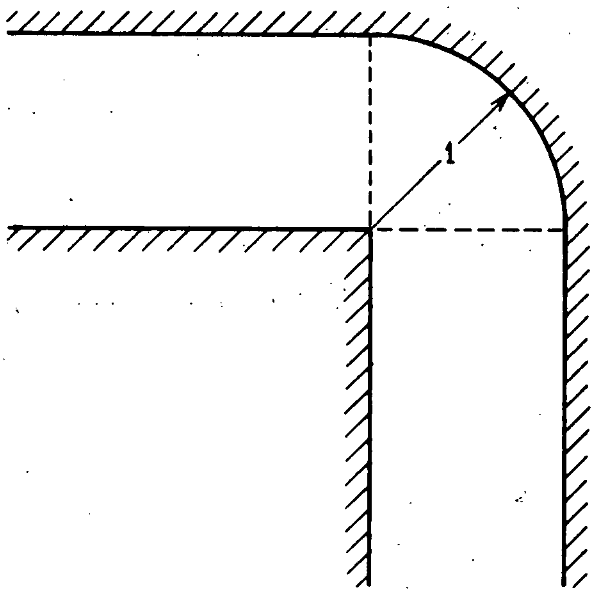


Figure 5. (e) A Hallway: $A_r = \pi/4$

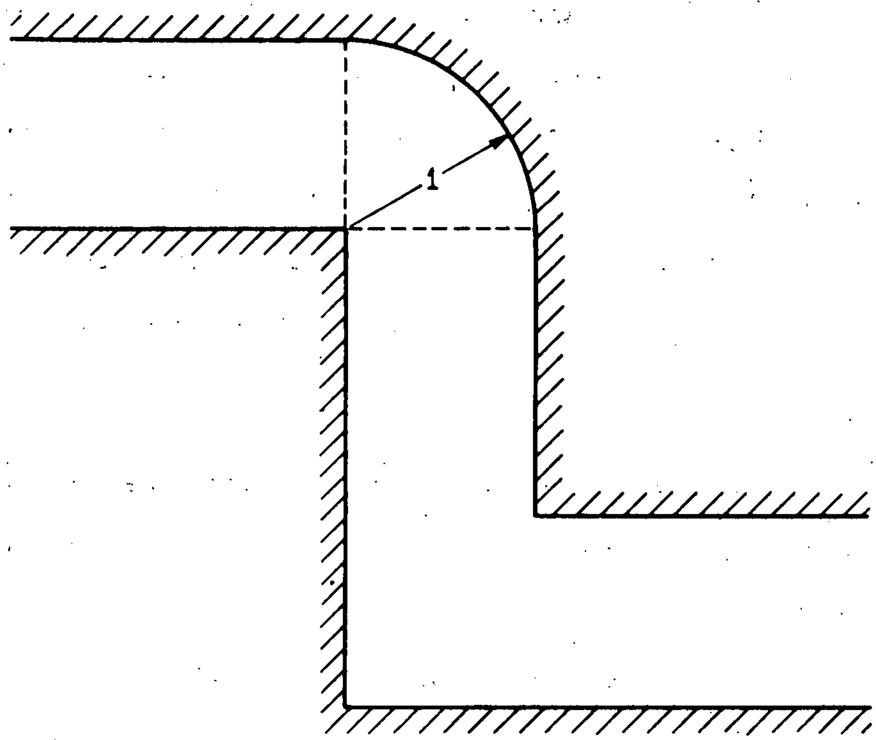


Figure 5. (f) A hallway: $A_r = \pi/4$

II. SEQUENTIAL REPRESENTATION OF SOFA

Let us first describe a class of sofas and then we will introduce the representation of this class.

A polygonal Jordan curve, γ, is a simple closed curve consisting of a finite number of line segments. Its inside region is called the polygonal region or polygon and is denoted by Ω_γ. A polygonal Jordan curve, γ (or a polygon Ω_γ), with basic points p_i, $i = 1, 2, \ldots, n$, is said to be angularly simple if there exists a point x_0 in Ω_γ such that the line segment between x_0 and p_i does not intersect any edge of Ω_γ for all i. In other words, Ω_γ is angularly simple if there exists an x_0 in Ω_γ such that at x_0, γ is totally visible (See Figure 6).

Lemma 1:

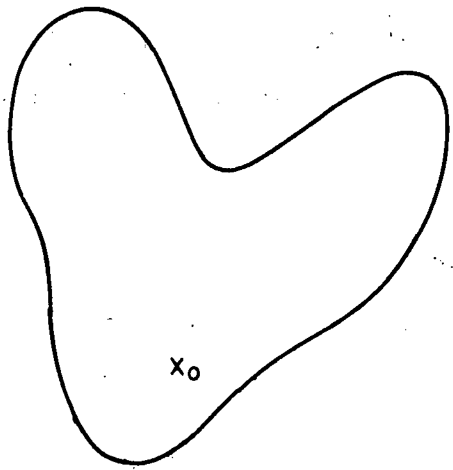
Let Ω_γ be an angularly simple polygon. Let us also assume that Ω_γ is angularly simple at x_0 and x_0', $x_0 \neq x_0'$. Then Ω_γ is angularly simple at x_0'', where

$$x_0'' = \omega x_0 + (1 - \omega)x_0' \quad , \quad 0 \leq \omega \leq 1.$$

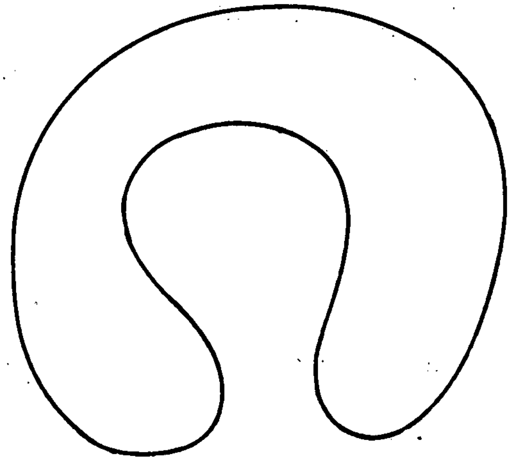
In other words, Ω_γ is totally visible at any point on the line segment $\overline{x_0, x_0'}$.

Proof:

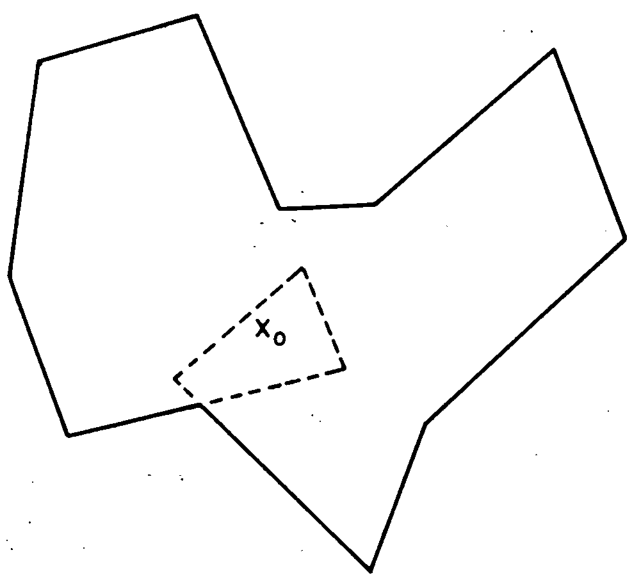
Let us assume that there exists a point x_0'' on the line segment $\overline{x_0, x_0'}$ such that x_0'' does not define Ω_γ to be angularly simple. Then by the definition of angular simplicity, the line segment $\overline{x_0'', p_i}$, intersects at least one edge of γ for some i. This implies that the line segment $\overline{x_0, x_0'}$ intersects at least twice with edges of γ, and since Ω_γ is angularly simple at x_0 and x_0', this implies that the line segments $\overline{x_0, p_i}$ and $\overline{x_0', p_i}$ have no intersection with γ except at p_i, for all i. Let e_j be an edge of γ



(a) An Angularly Simple Curve



(b) A Curve That Is Not Angularly Simple



(c) An Angularly Simple Polygon

Figure 6. The Definition of a New Class of Objects

which intersects $\overline{x_0, x_0'}$ and let p_j and p_{j+1} be two end points of e_j. Without loss of generality, let us assume that p_j is contained in the triangle of points x_0, x_0' and p_i. Then it is clear that either $\overline{x_0, p_{j+1}}$ or $\overline{x_0', p_{j+1}}$ intersects an edge of γ. This implies that at least one of x_0 or x_0' does not define Ω_γ to be angularly simple, which is a contradiction.

Q.E.D.

Lemma 1 implies that if Ω_γ is angularly simple at points x_i, $i = 1, 2, \ldots, m$, then Ω_γ is angularly simple at any point which is the linear convex sum of the x_i's, i.e. at

$$\sum_{i=1}^{m} \omega_i x_i \text{ , where } \sum_{i=1}^{m} \omega_i = 1.$$

Theorem 1:

Let Ω_γ be an angularly simple polygon. Then the set of points which defines Ω_γ to be angularly simple is a convex set.

Proof:

Let us assume that such a set is not convex. Then there exists two points, x_j and x_k in the set such that some points on the line segment $\overline{x_j, x_k}$ are not contained in the set. Any such point is obviously defined by the linear convex sum of x_j and x_k. Since Ω_γ is angularly simple at x_j and x_k, by our assumption, any point on such a line segment defines Ω_γ to be angularly simple by Lemma 1, which is a contradiction.

Q.E.D.

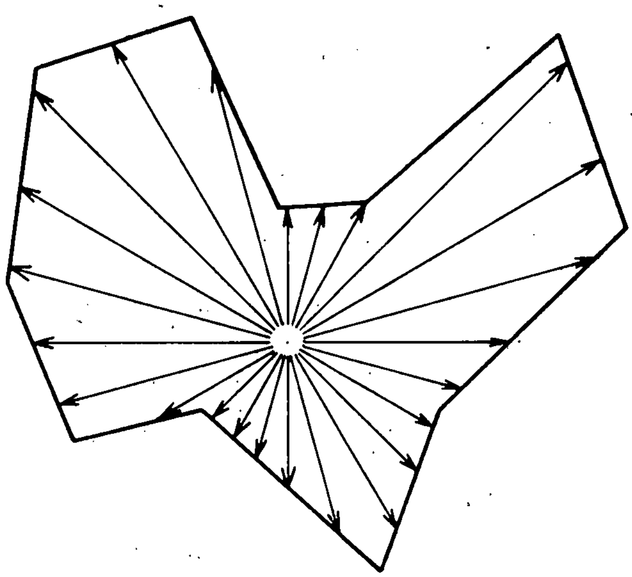
Hereafter we consider our class of sofas to be angularly simple polygons. Other properties of the angularly simple polygons and the generalization of angularly simple polygons to cover any polygonal regions have been studied by Maruyama [9].

Since any angularly simple polygon, Ω_γ, contains a point (or a set of points) x_0 such that a vector from x_0 to a point on γ can trace γ in one direction, we can use such a vector sequence to denote angularly simple curves as well. Such a vector sequence is illustrated in Figure 7, where the region covered by the vectors shows the polygon and the curvature formed by connecting the tops of vectors indicates the polygonal Jordan curve. Let us define such a sequence as follows: A pattern sequence, S, is an ordered set of elementary patterns,

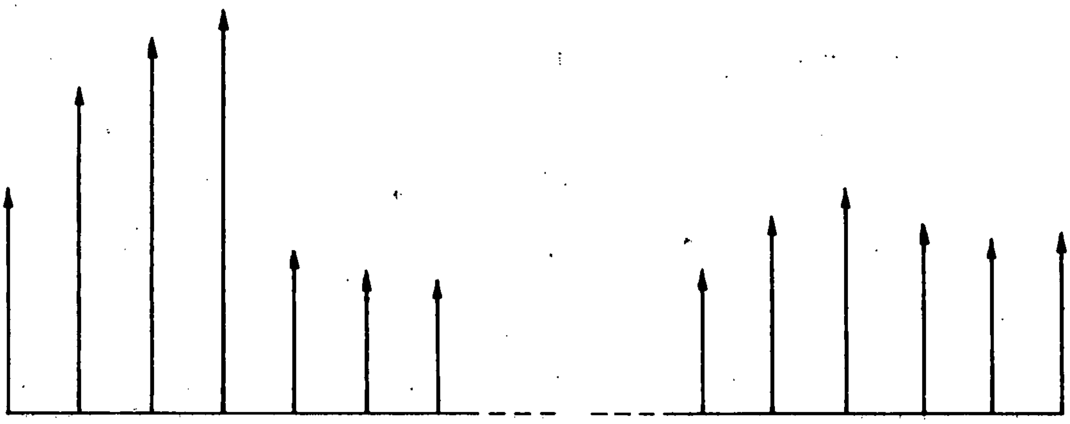
$$S = s_0, s_1, s_2, \ldots, s_i, \ldots, s_{n-1} \qquad s_i \varepsilon R^m$$

where m denotes the dimension of the elementary patterns and n is called the circularity of S. To make S denote an unique polygon we assume that any two adjacent elementary patterns, s_i and s_{i+1}, have an angular difference of $2\pi/n$. In practice, we choose $n \geq 48$. For our present purpose of describing Ω_γ, it is sufficient to consider that the dimension, m, of s_i is one, since we assume that each s_i denotes the distance between x_0 and the intersection point of γ and the vector whose direction corresponds to $2\pi i/n$, for $i = 0, 1, 2, \ldots, n-1$.

When we consider rotations and translations of a polygon whose representation is in basic point coordinates (possibly with line equalities), it is usually necessary to change the point coordinates (and line equalities). However, a transformation of a polygon which is



(a) An Angularly Simple Polygon



(b) A Pattern Sequence Corresponding to Polygon (a)

Figure 7. An Angularly Simple Polygon and Its Pattern Sequence

denoted by a pattern sequence, S, is simple;

(i) a translation of an angularly simple polygon, Ω_γ, corresponds simply to a translation of x_0 of the pattern sequence, S, and

(ii) a rotation of Ω_γ at x_0 corresponds to circular shifting of indices of elementary patterns, i.e. it is adequate to consider the <u>rotation index</u>, r, which will be defined later.

Before we define the rotational transformation of S, let us define the <u>canonical</u> pattern sequence. A pattern sequence, S, is called a canonical pattern sequence if the first elementary pattern corresponds to the direction of the X-coordinate, and the ordering of the elementary patterns corresponds to the counter-clockwise rotation of corresponding vectors. Henceforth, we assume that each pattern sequence is canonical. This assigns the orientation of the corresponding polygon (without confusion, we sometimes use polygon when refering to angularly simple polygons).

For the rotation of a pattern sequence, S, at a point x_0, it is convenient to assume that the unit rotation angle, δ, is $2\pi/n$ (or possibly, an integer multiple of $2\pi/n$). For example, if

$$S(x_0) = s_0, s_1, s_2, \ldots, s_i, \ldots, s_{n-2}, s_{n-1}$$

then the clockwise rotation of $S(x_0)$ through δ degrees is given by

$$s_1, s_2, \ldots, s_i, \ldots, s_{n-1}, s_0$$

and the counter-clockwise rotation of $S(x_0)$ through δ degrees is given by

$$s_{n-1}, s_0, s_1, \ldots, s_i, \ldots, s_{n-2}.$$

We will define r to be the rotation index. If $r>0$ $S(x_0)$ has been rotated in a clockwise direction through an angle of r degrees. If $r<0$, $S(x_0)$

has been rotated in a counter-clockwise direction through an angle of $r\delta$ degrees. Thus, in general, we have the following expression for a pattern sequence:

$$S_r(x_0) = s_r, s_{r+1}, s_{r+2}, \ldots, s_{r+i}, \ldots, s_{r+n-1}$$

if $r + i < 0$ then $r + i$ becomes $n + r + i$
($r + i \pmod n$ for all i.)

Thus far we have described rotation of $S(x_0)$ around x_0. Rotation around any point is accomplished simply by the change of the rotation index with an appropriate translation of x_0 of $S(x_0)$.

To generate a sequence, S, for a specified x_0, we project a ray starting from x_0 along each direction $2\pi i/n$, for $i = 0, 1, \ldots, n-1$. Then we measure the distance, s_i, by detecting the intersection between the ray and an edge of the given hallway, if any intersection exists within the distance V, called the visibility distance from x_0. For the representation of hallways we use the usual chain representation, i.e. points and line segments are connected in such a way that the clockwise sequence describes the free space as its right hand side.

III. STRATEGY OF THE SOFA PROGRAM AND GSPS

The procedure for the solution of the arbitrary two-dimensional sofa problem, which will be described, is intuitive. It is as simple as the "paper-cut" approach in which one takes a sufficiently large round paper[1], $S_r(x_0)$ (r and x_0 not important), and cuts away the minimum amount of paper necessary to enable the paper to move through the hallway. In other words one trys to maximize the remaining paper area which can still go through the hallway.

To get an idea of this procedure, let us consider an example which is illustrated in Figure 8. Two canonical pattern sequences, $S_0(x_1)$ and $S_0(x_2)$, denote papers which are possibly maximal at locations x_1 and x_2, respectively, where the circularity, n, is 24.

$$S_0(x_1) = s_0, s_1, s_2, \ldots, s_i, \ldots, s_{23}$$

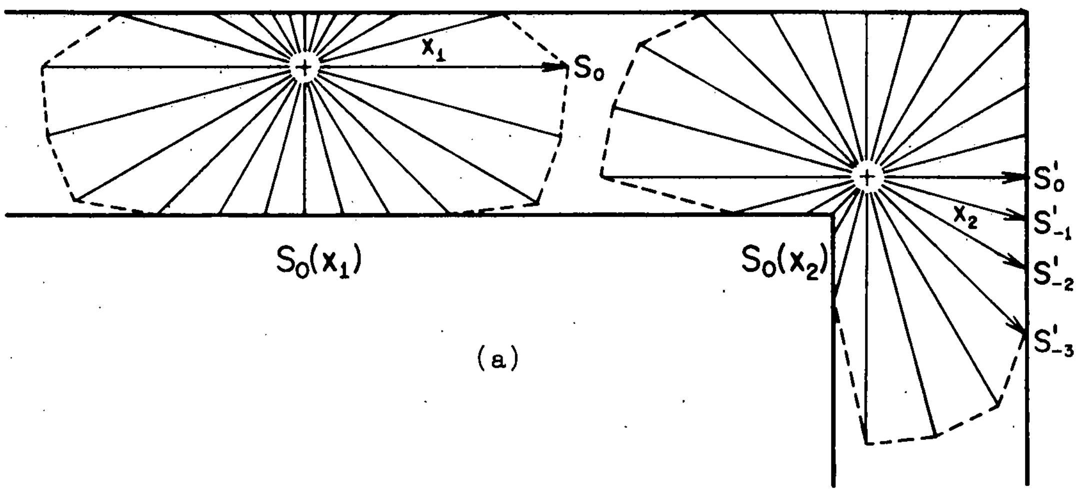
$$S_0(x_2) = s'_0, s'_1, s'_2, \ldots, s'_i, \ldots, s'_{23}$$

The paper, $S_r(\omega x_1 + (1-\omega)x_2)$, which can be located at both x_1 and x_2 and whose area becomes maximal, is obtained by intersecting the paper $S_0(x_1)$ and the reoriented paper $S_{-3}(x_2)$ of $S_0(x_2)$, namely:

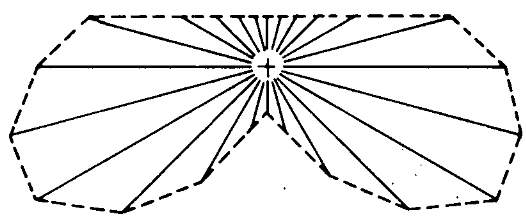
$$\begin{aligned} S_{-3}(x_2) &= s'_{-3}, s'_{-2}, s'_{-1}, s'_0, \ldots, s'_{i-3}, \ldots, s'_{20} \\ &= s'_{21}, s'_{22}, s'_{23}, s'_0, \ldots, s'_{i-3}, \ldots, s'_{20} \\ &= s''_0, s''_1, s''_2, s''_3, \ldots, s''_i, \ldots, s''_{23} \quad . \end{aligned}$$

---

1. The term "a paper" is used to reference a two-dimensional object which may or may not go through a given hallway. An edge trimmed paper which can go through a given hallway is called a sofa for the given hallway.



(a)



(b) $S_0(wx_1 + (1-w)x_2) = S_0(x_1) \cap S_{-3}(x_2)$

Figure 8. Sofas at Point x_1 and x_2 and Their Intersection, $S_0\ (\omega x_1 + (1 - \omega)x_2)$

Thus

$$S_r(\omega x_1 + (1-\omega)x_2) = S_0(x_1) \cap S_{-3}(x_2)$$

$$= \min(s_i, s_i'') \text{ for } i = 0, 1, \ldots, 23$$

where $\omega = 0$ or 1, and $r=0$ if $\omega=1$ and $r=-3$ if $\omega=0$.

Let us consider the case where the distance between x_1 and x_2, denoted by $d(x_1,x_2) = \lambda$ (where λ is a unit translation distance), is small and the circularity, n, of the pattern sequence is sufficiently large. Then both the unit rotation angle, δ, and the unit translation distance, λ, are sufficiently small. Hence, application of such transformations can be thought of as "continuous" transformations of S from x_1 to x_2. So, in general, we have:

$$S_r(\omega x_1 + (1-\omega)x_2) = S_{r_1}(x_1) \cap S_{r_2}(x_2)$$

where $|r_1 - r_2| \leqq 1$, $r = r_1\omega + r_2(1-\omega)$,

$d(x_1,x_2) = \lambda$ and $0 \leqq \omega \leqq 1$.

The above intersection operation may be interpreted as "min" operation and the following tree search strategy may be thought of as the "max" operation.

Many studies have been done on both combinatorial and heuristic search algorithms [2, 10, 12, 14, etc.], and a comprehensive survey of them has most recently been done by Pohl [13]. While Howden [6] used a straight-forward exhaustive search strategy for the solution of the sofa problem by computer, we use the following heuristic tree search strategy.

Our partial ternary tree (sometimes 5-ary is required depending on the complexity of the given hallway) is developed to a depth of L levels in the following way. Nodes are divided into two classes: active and terminal. Nodes coming from an active node are examined for bounding. This stops further wasteful exploration by using the property that the area of paper is monotonically non-increasing (because of the intersection operation which was defined above). As soon as the paper area becomes smaller than the bound B at a node v_r, the tree exploration from such a node is terminated. Then the path from such a terminal node to the root node is eliminated (or pruned) from the partial tree which is currently being developed. When the partial search tree is completed to L levels by the repetition of the above generation and pruning, the paper will be moved down in the tree until the paper encounters a node that leads to more than two active nodes. It is possible that the paper cannot be moved down in the tree by the above process. In such a case the paper will be moved one level down the tree in such a way that the next node which has been chosen leads to a better solution. If the developed L level partial search tree has no active nodes to be explored in the next, then our procedure will stop and we conclude that a "sofa" which is larger than the present bound cannot be moved through the given hallway.

To expand an active node, v_r, whose paper orientation is δr, where $\delta = 2\pi/n$, we attach to v_r three successor nodes, v'_{r-1}, v'_r and v'_{r+1} whose orientations correspond to rotation indices r-1, r and r+1, respectively. Here the distance between v_r and any of v'_{r-1}, v'_r or v'_{r+1} is the unit translation distance λ.[2] Thus the node v'_r of the rotation index r means simply the

2 Some successor nodes can have only rotational transformations.

unit translation λ of a paper at node v_r in the direction δr coupled with the intersection operation. The nodes v'_{r-1} and v'_{r+1} indicate the unit translation of the paper at v_r in the directions $\delta(r-1)$ and $\delta(r+1)$, respectively. That is, the former contains the unit angle rotation of the paper in the clockwise direction and the latter rotates the paper through the unit angle in the counter-clockwise direction. Thus by our approach, we treat translation and rotation of a paper simultaneously. This is the major advantage of our representation of the paper. (With a slight change of the above strategy, one can deal with rotation independently.)

An example of a 4-level ternary search tree with bounding is shown in Figure 9. Those doubly circled nodes are terminal nodes and others are active nodes. Those marked $A_r(x_{17})$ and $A_{r+2}(x_{19})$ are actually active nodes for the expansion of the next search tree whose next root node will be $A_{r+1}(x_{12})$. Figure 10 shows the data structure of our tree search approach which corresponds to the example of Figure 9.

If the level of the partially developed tree is one, $L = 1$, then the procedure discussed above is simple a "mini-max" strategy which turns out to be strictly a local optimization. If $L > 1$ then the procedure contains some global optimization as well as local optimization.

We repeat the above L-level partial search tree generation and pruning process until the paper reaches the other end of the given hallway. Then the resulting paper, $S^{(k)}$, is stored as the present maximal "sofa" for the given problem as well as the new bound $B^{(k)}$. A slightly larger paper than $S^{(k)}$, $S'^{(k)}$, is fed into the hallway next and the paper-cut process is repeated unless there is no gain of the sofa obtained since the previous iteration. Because we use a heuristic search strategy rather than an exhaustive type strategy, the iteration of the paper-cut process as well

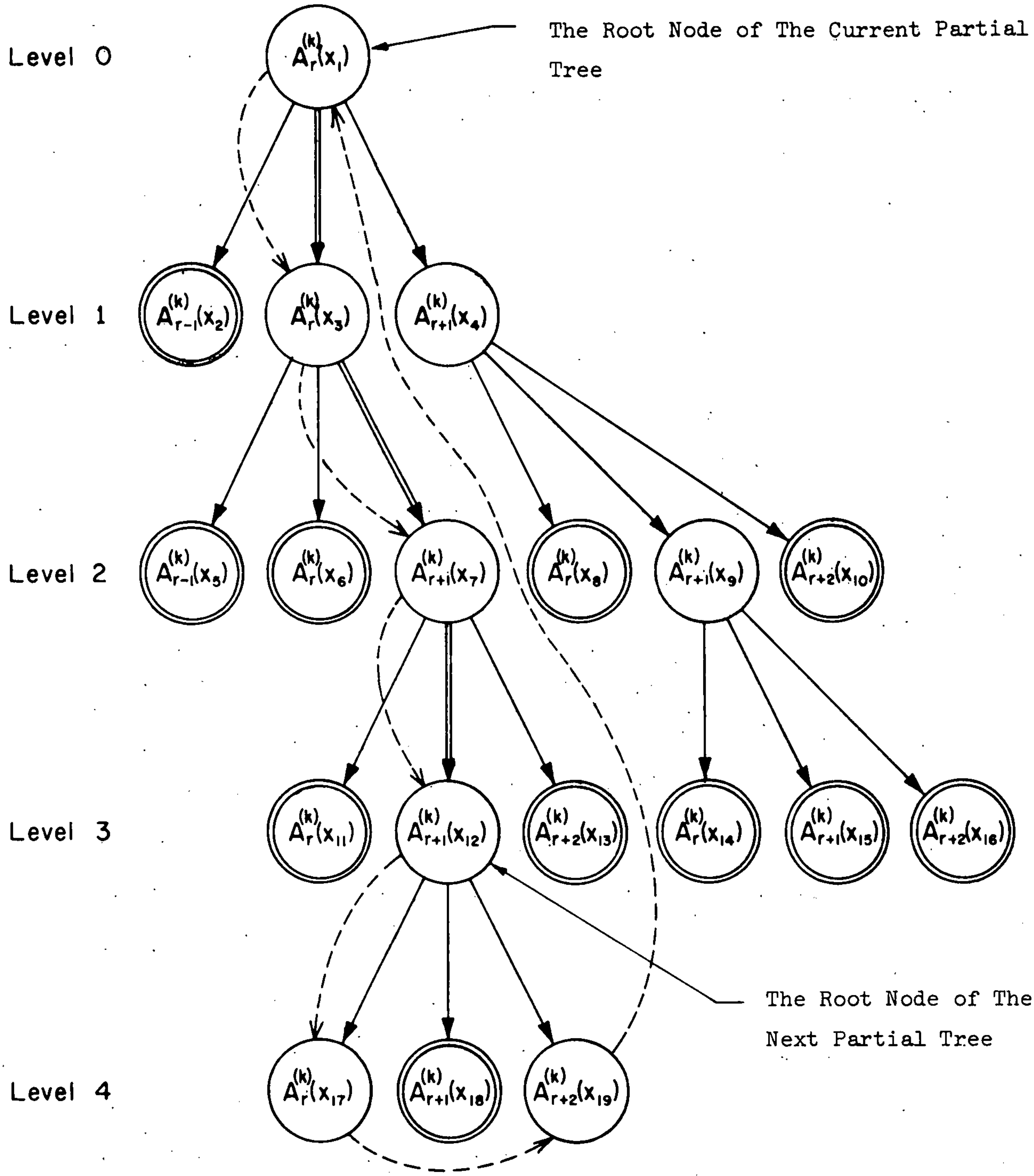


Figure 9. Four-level Ternary Search Tree With Bounding: those Doubly Circled Nodes are Terminal Nodes and Others are Active Nodes

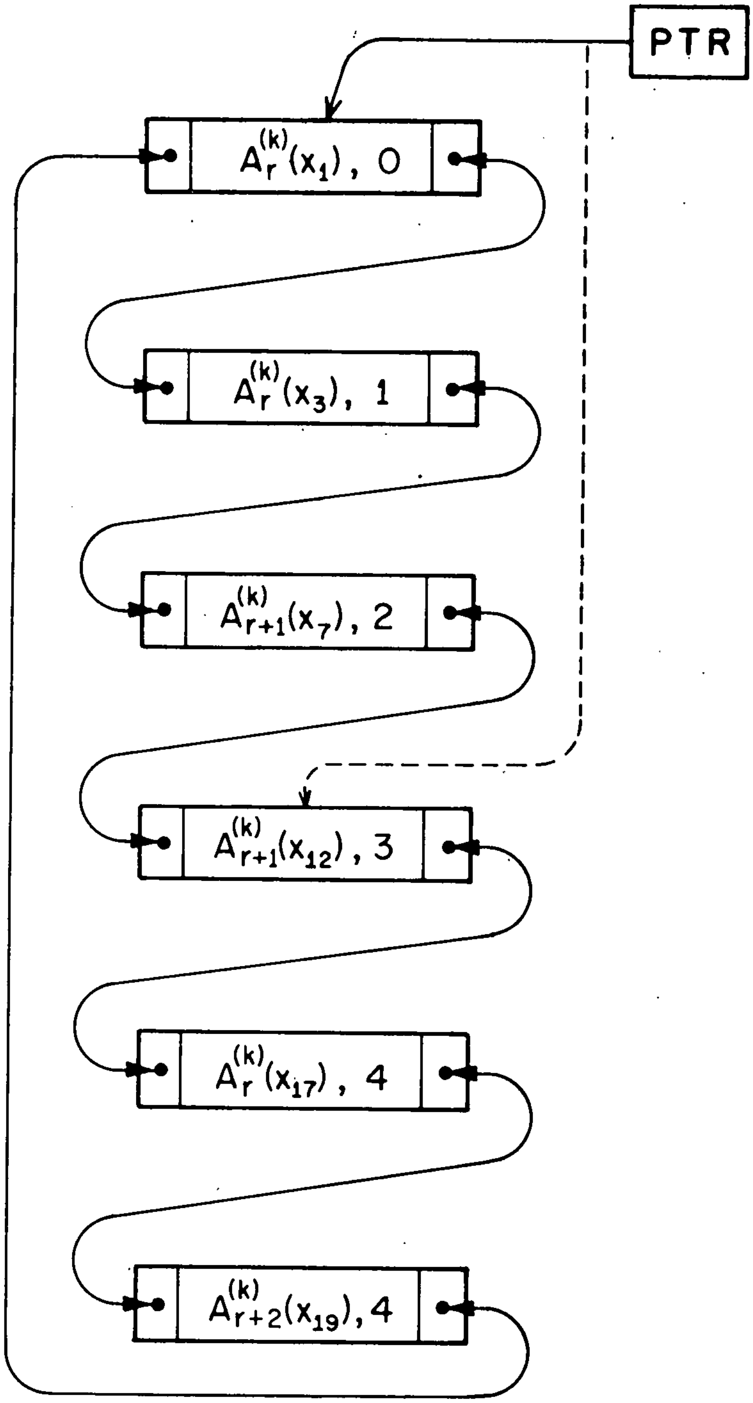


Figure 10. Double Linked List Used for the Search Strategy (the state corresponds to the partial search tree of Fig. 9).

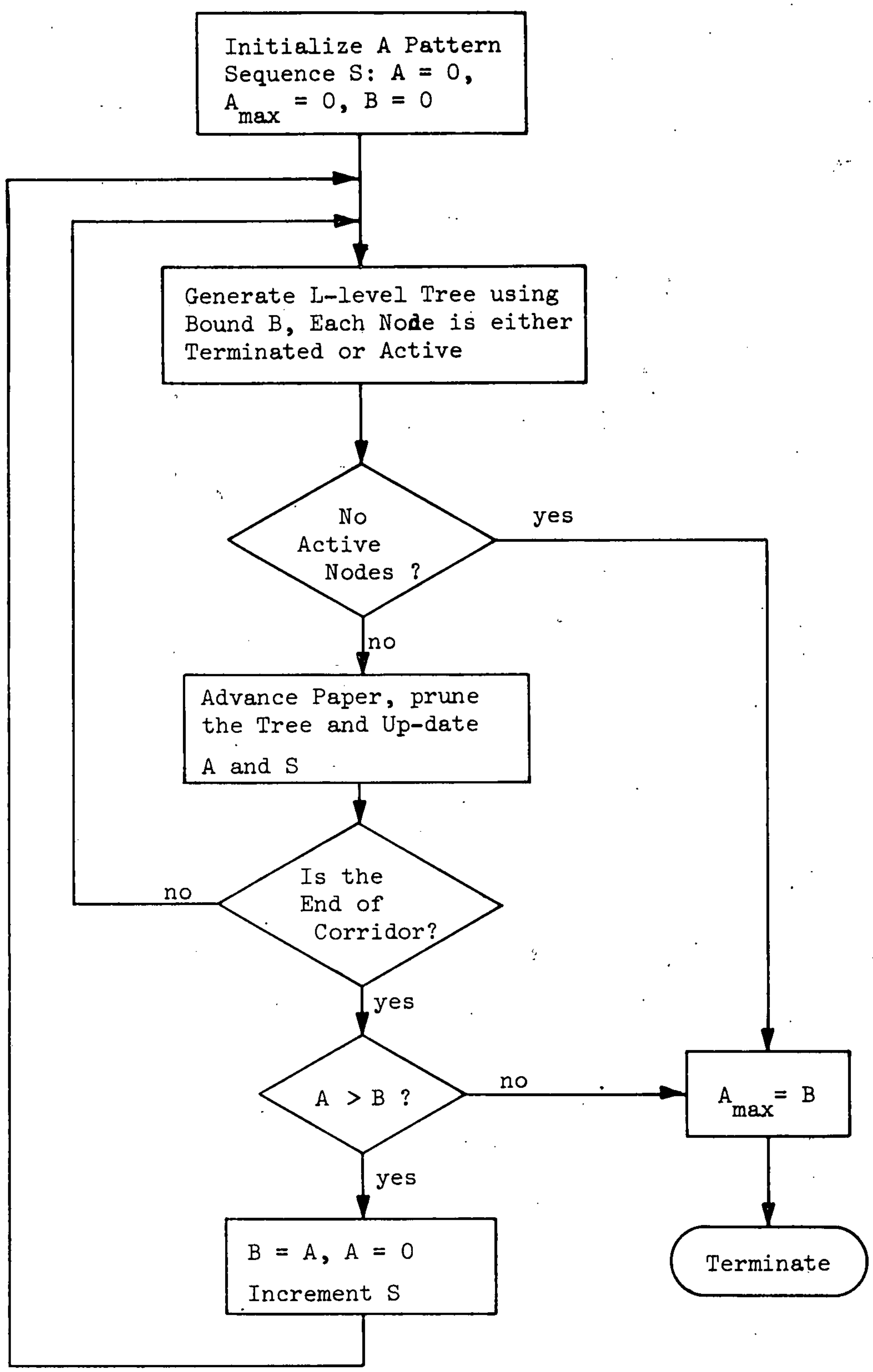


Figure 11. A Simplified Flow Chart of GSPS

as the incrementation of the obtained sofa for the next iteration become quite important in finding an optimal trajectory.[3]

From the above argument, we have the simplified flow chart of GSPS[4] (General Sofa Problem Solver) which is illustrated in Figure 11. The following is a description of the flow chart.

Initialization of the Pattern Sequence (or Paper)

We can choose any one of the following starting papers:

(i) A sufficiently large sofa.

(ii) The lower or the upper bound sofa.

(iii) A sofa which can go through the given hallway without rotation (this can be found easily).

(iv) A sufficiently small sofa.

Of course the number of iterations required for the convergence to the solution by GSPS depends upon which of the papers we choose as the initial paper, upon the initial bound for the sofa area, and upon the means to increment the sofa for the next iteration (this will be discussed next). To reduce the number of iterations, it is preferable to choose a smaller, lower bound sofa as an initial paper, if such a shape of the bound sofa is easily estimatable.

---

3. An optimal trajectory T is described by a sequence of pairs of elements, x and r:

$$T = (x_{i_1}, r_{i_1}), \ldots, (x_{i_m}, r_{i_m})$$

Or it is determined by a pair of sequences of x and r.

4. GSPS will find both an optimal trajectory (or path) and an optimal shape of a sofa. However, if a trajectory is given, then GSPS will find an optimal shape of a sofa for a given hallway.

Incrementation of Sofa

Let us assume that after the k-th iteration we have a pattern sequence:

$$S^{(k)} = s_0^{(k)}, \ldots, s_i^{(k)}, \ldots, s_{n-1}^{(k)}$$

whose area is denoted by $A^{(k)}$:

$$A^{(k)} = \sin(2\pi/n) \cdot (s_0^{(k)} s_{n-1}^{(k)} + \sum_{i=0}^{n-2} s_i^{(k)} s_{i+1}^{(k)}) \;/\; 2 \quad .$$

Then the paper which will be provided for the (k+1)st iteration is the one whose area is slightly larger that $A^{(k)}$. For such an incrementation of the sofa, we may consider the following approaches.

(i) Equi-increment:

$$s_i'^{(k)} \leftarrow s_i^{(k)} + \varepsilon \qquad , \text{ for all } i \ (\varepsilon \text{ small}).$$

(ii) Isomorphic-increment:

$$s_i'^{(k)} \leftarrow c \cdot s_i^{(k)}, \text{ for all } i,\ c > 1.$$

(iii) Differential-increment:

$$s_i'^{(k)} \leftarrow s_i^{(k)} + c(s_i^{(k)} - s_i^{(k-1)}), \text{ for all } i,\ 0 < c \leqq 1.$$

One may consider some other incrementation approaches as well as the mixed approaches of the above three. If we know the lower bound sofa and if we have chosen our starting paper relatively far from the bound sofa, then it seems that the best incrementation approach is to consider the difference between the lower bound sofa and the present paper. However, if the lower bound sofa is unknown or not accurately estimatable, then this approach be used. We will see that any one of the above three can be used satisfactorily, and we will also consider the combination of the above three, namely:

$$s_i'^{(k)} \leftarrow s_i^{(k)}(1 + c_1) - c_2 s_i^{(k-1)} + \varepsilon \quad .$$

For the new bound $B^{(k)}$ of the (k+1)st tree search interation, the present $A^{(k)}$, which is not incremented, is used.

Paper-cut Method

We apply the following conjecture for our paper-cut process. If the given hallway is symmetric, then the solution for the sofa problem with Moser's objective function, i.e. the maximal region which can go through the hallway, is also symmetric. This conjecture gives us a little gimmic to simplify our GSPS and makes it easier to implement as well as enabling a faster convergence of the solution.

IV. SOME COMPUTATIONAL RESULTS

Solutions for those hallways which are illustrated in Figure 5, as well as a solution to Moser's hallway of Figure 1, by GSPS are illustrated in Figures 12 through 16. Since the solutions for the sofa problems with Moser's objective function are non-trivial, it may be preferable to indicate the obtained solution as the ratio between the solution area, A, and A_r, the sofa area which can go through the given hallway without an application of any rotational transformation (this is also the obtained area).

Table II shows the solutions (the unit translation, λ, and the unit rotation $\delta = 2\pi/n$) for Moser's sofa problem using Mini-Max strategy (L = 1). The table shows that the larger n is, the larger A we get; this agrees with our intuitive knowledge since with larger n we get a more accurate representation of angularly simple polygons by a pattern sequence, especially if some elementary patterns are huge. Also for larger λ we get larger A. This result seems to contradict Howden's statement ([6] p. 300): "indicating that accuracy (of approximately three units) is more dependent on x, the fineness G, than on the size of $\Delta\theta$". However, Howden's method and ours are quite different - the sofa will be operated on in such a way so that it can move through the given hallway and will therefore have less constraints from the hallway for larger λ. Of course λ should be less than a certain amount, e.g. 8/20, otherwise the translation of the sofa becomes so discrete that a solution by our GSPS does not make sense.

After testing our GSPS for different $L \geq 1$, we found that L = 4 is enough for iterating the paper-cut prpcess. Thus we set L = 4, n = 48, and $\lambda = 1/10$.

A solution for Moser's sofa problem with an equi-incrementation of

Table II. Results Obtained by Mini-Max Strategy

For Moser's Sofa Problem

λ \ n	24	48
1/20	1.1	1.38
2/20	1.35	1.61
4/20	1.38	1.73
6/20	1.75	1.86

λ: Unit translation distance

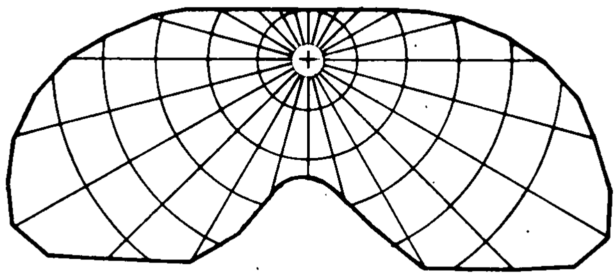
n: Circularity (the number of elementary patterns). Thus the unit rotation angle is $\delta = 2\pi/n$.

K: Computer area/A_r , $A_r = 1$
Upper bound = 2.828
Lower bound = 2.207

feed-back for the sofa is shown in Figure 12. Figure 12(a) shows the shape of the sofa after the first iteration. $A^{(1)} = 1.88$ which is about 85% of the lower bound, $A_{\ell} = \pi/2 + 2/\pi$, indicated in Table I. After the 4th iteration, the sofa is about 90% of A_{ℓ}, which is a good approximate solution for the given problem. Figure 13 shows the solution for the hallway of Figure 5(a). After the 3rd iteration we get $A^{(3)}=K^{(3)}= 1.27$ -- 81% of the lower bound, $A_{\ell} = \pi/2$. A solution for the hallway of Figure 5(b) is illustrated is Figure 14, and we get about 95% of the lower bound $A_{\ell} = 2.324$ (2.8 times the area of the sofa which can go through the hallway without rotation). From these results we may conclude that for a smooth hallway GSPS works very well. Figure 15 shows the solution for the hallway of Figure 5(c) in which we achieved 88% of the lower bound, $A_{\ell} = 1.82$. Figure 16 shows the solution for the hallway of Figure 5(d) whose lower bound is $A_{\ell} = \pi/2$, and we get 77% of the lower bound. This percentage sounds low, but it is fairly good considering the severe constraints. The shape is, still, quite similar to the lower bound sofa which consists of two connected circles.

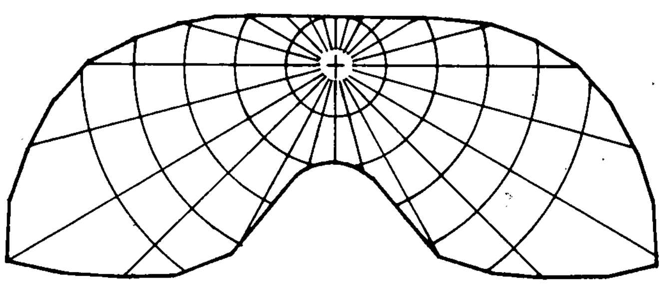
Solutions for the sofa problems in Figures 5(e) and (f) are not illustrated since their solutions are quite similar to those of Figures 12 and 13, respectively. The ratio between the area obtained and the sofa which can move through the hallway without rotational transformations are different, as are the areas.

From these solutions we conclude that GSPS gives a fairly good approximate solution for two-dimensional sofa problems (including an optimal trajectory for such a sofa) with Moser's objective function in a reasonable amount of time. The average run time for a single hallway is 10 to 15 seconds per iteration and about 1.5 times this for a doubly connected hallway. The procedure was written in PL/1 language and implemented on an IBM 360/75 at the University of Illinois.



(a) After the First Iteration

$A^{(1)} = K^{(1)} = 1.88$



(b) After the Fourth Iteration

$A^{(4)} = K^{(4)} = 1.98$

Figure 12. A Solution for Moser's Sofa Problem With Moser's Objective Function By GSPS Equi-Increment

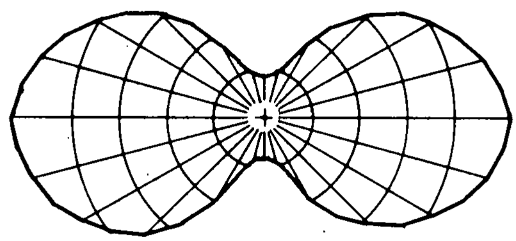
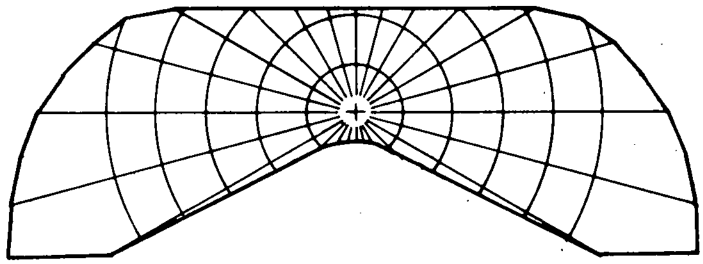
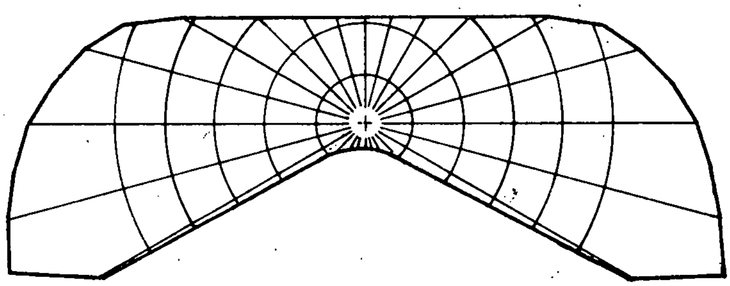


Figure 13 A Solution for the Hallway of Figure 5(a) With Moser's Objective Function By GSPS; After the Third Iteration $A^{(3)}=K^{(3)}= 1.27$. Isomorphic Increment

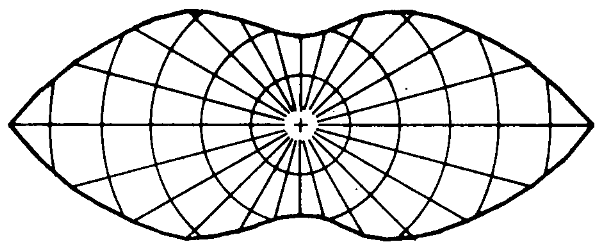


(a) After the First Iteration
$A^{(1)} = 2.09$, $K^{(1)} = 2.66$

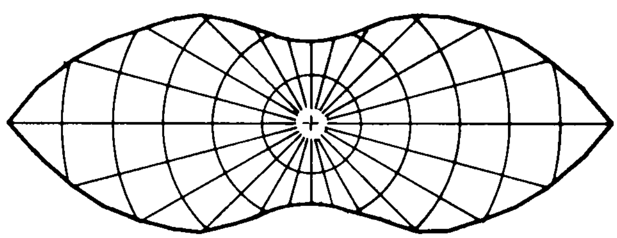


(b) After the Second Iteration
$A^{(2)} = 2.2$, $K^{(2)} = 2.8$

Figure 14. A Solution for the Hallway of Figure 5(b) With Moser's Objective Function By GSPS Mixed (Equi-Isomorphic) Increment

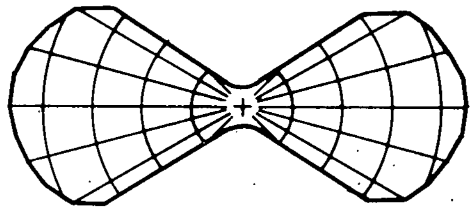


(a) After the First Iteration
$A^{(1)} = 1.57,\ K^{(1)} = 2.0$

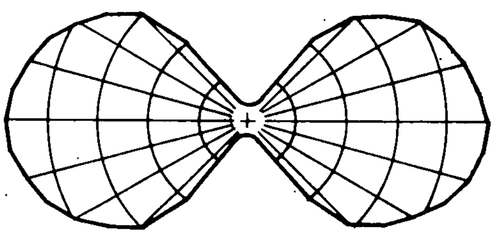


(b) After the Second Iteration
$A^{(2)} = 1.60,\ K^{(2)} = 2.03$

Figure 15. A Solution for the Hallway of Figure 5(c) With Moser's Objective Function By GSPS Differential Increment



(a) After the First Iteration
$A^{(1)} = 0.97,\ K^{(1)} = 1.23$



(b) After the Third Iteration
$A^{(3)} = 1.21,\ K^{(3)} = 1.54$

Figure 16. A Solution for the Hallway of Figure 5(b) With Moser's Objective Function By GSPS Mixed (Equi-Isomorphic) Increment

V. CONCLUSION

By restricting the class of sofas to a class of angularly simple polygons, we have developed the most easily transformable representation of such a polygon, called a sequential pattern sequence. However, as we have pointed out, the restriction of objects to a class of angularly simple polygons is the strongest restriction and such a polygon may not represent exactly a solution for some sofa problems. Still, the shapes of the sofas obtained are quite similar to those of the lower bound sofas found analytically.

Through the restriction of sofas to angularly simple polygons and the heuristic tree search strategy which is applied by the character of non-increasing sofas, we have developed the two-dimensional sofa problem solver, GSPS. As we can see from our computation examples, the system is fast enough to give us "good" approximate, or near optimal, solutions for the sofa problem. A little modification of GSPS leads to the most generalized sofa problem solver, with not only Moser's objective function but with some other predefined objective function, such as Howden's objective function.

The idea of angularly simple polygons and their representation leads us not only to the computer solution of the sofa problem but also to the solution of the two-dimensional hiden line problem, the path finding problem [8] in a geometrically constrained space with limited sight, the dynamic sofa problem [9], and form perception in psychology [1], among others. We feel positive that there are more applications of this class of objects as well as applications of the generalized angularly simple polygons [9] with a combination of artificial intelligence. Finally, we feel that "analog" approaches rather than numerical approaches for solving problems gives some clues to solving other problems by computer.

REFERENCES

[1] F. Attneave and M. D. Arnoult, "The Quantitative Study of Shape and Pattern Perception", Psychological Bulletin, Vol. 53, No. 6, November 1956, PP. 452-471.

[2] J. E. Doran and D. Michie, "Experiments with Graph Traverser Programs", Proceedings of the Royal Society (A), Vol. 294, September 1966, PP. 235-259.

[3] H. Freeman, "On the Encoding of Arbitrary Geometric Configurations", IRE Transactions on Electronic Computers, Vol. EC-10, June 1961, PP. 260-268.

[4] H. Freeman and L. Garder, "A Pictorial Jigsaw Puzzle: The Computer Solution of a Problem in Pattern Recognition", IEEE Transactions on Electronic Computers, Vol. EC-13, April 1964, PP. 118-127.

[5] M. Goldberg, "A Solution of Problem 66-11: Moving Furniture Through a Hallway", SIAM Review, Vol. 11, No. 1, January 1969, PP. 118-127.

[6] W. E. Howden, "The Sofa Problem", Computer Journal, Vol. 11, No. 3, November 1968, PP. 299-301.

[7] A. Kaufmann, Graphs, Dynamic Programming and Finite Games, Academic Press, New York, 1967.

[8] K. Maruyama, "A Procedure for Detecting Intersections and Its Application", Department of Computer Science, University of Illinois, Urbana, Illinois Report No. 449, May 1971.

[9] K. Maruyama, unpublished paper, May 1971.

[10] D. Michie, "Strategy-Building with Graph Traversers", Machine Intelligence, Vol. 1, N. L. Collins and D. Michie (Eds.), 1967, PP. 3-15.

[11] L. Moser, "Problem 66-11: Moving Furniture Through a Hallway", SIAM Review, Vol. 8, No. 3, July 1966, PP. 381.

[12] A. Newell and G. Ernst, "The Search for Generality", Proceedings of the IFIP Congress, 1965, PP. 17-22.

[13] I. Pohl, "Bi-Directional and Heuristic Search in Path Problems", SLAC Report, No. 104, May 1969.

[14] I. Scoins, "Linear Graphs and Trees", Machine Intelligence, Vol. 1, N. L. Collins and D. Michie (Eds.), 1967, PP. 3-15.

[15] J. Sebastian, "A Solution of Problem 66-11: Moving Furniture Through a Hallway", SIAM Review, Vol. 12, 1970, PP. 582-586.

[16] I. E. Sutherland, "A Method for Solving Arbitrary Wall Mazes by Computer", IEEE Transactions on Computers, Vol. C-18, No. 12, December 1969, PP. 1092-1097.

Form AEC–427
(6/68)
AECM 3201

U. S. ATOMIC ENERGY COMMISSION
UNIVERSITY–TYPE CONTRACTOR'S RECOMMENDATION FOR DISPOSITION OF SCIENTIFIC AND TECHNICAL DOCUMENT

(See Instructions on Reverse Side)

1. AEC REPORT NO.	2. TITLE
COO-2118-0026 UIUCDCS-R-71-489	AN APPROXIMATION METHOD FOR SOLVING THE SOFA PROBLEM

3. TYPE OF DOCUMENT (Check one):

- [X] a. Scientific and technical report
- [ ] b. Conference paper not to be published in a journal:
  - Title of conference ______
  - Date of conference ______
  - Exact location of conference ______
  - Sponsoring organization ______
- [ ] c. Other (Specify) ______

4. RECOMMENDED ANNOUNCEMENT AND DISTRIBUTION (Check one):

- [X] a. AEC's normal announcement and distribution procedures may be followed.
- [ ] b. Make available only within AEC and to AEC contractors and other U.S. Government agencies and their contractors.
- [ ] c. Make no announcement or distrubution.

5. REASON FOR RECOMMENDED RESTRICTIONS:

6. SUBMITTED BY: NAME AND POSITION (Please print or type)

Kiyoshi Maruyama
Research Assistant

Organization

Department of Computer Science
University of Illinois
Urbana, Illinois 61801

Signature	Date
	December 17, 1971

FOR AEC USE ONLY

7. AEC CONTRACT ADMINISTRATOR'S COMMENTS, IF ANY, ON ABOVE ANNOUNCEMENT AND DISTRIBUTION RECOMMENDATION:

8. PATENT CLEARANCE:

- [ ] a. AEC patent clearance has been granted by responsible AEC patent group.
- [ ] b. Report has been sent to responsible AEC patent group for clearance.
- [ ] c. Patent clearance not required.

BIBLIOGRAPHIC DATA SHEET	1. Report No. COO-2118-0026 UIUCDCS-R-71-489	2.	3. Recipient's Accession No.

4. Title and Subtitle
AN APPROXIMATION METHOD FOR SOLVING THE SOFA PROBLEM

5. Report Date
Oct. 1971

6.

7. Author(s)
Kiyoshi Maruyama

8. Performing Organization Rept. No.

9. Performing Organization Name and Address
Department of Computer Science
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

10. Project/Task/Work Unit No.
Illiac IV

11. Contract/Grant No.
AT(11-1)-2118

12. Sponsoring Organization Name and Address
U.S. Atomic Energy Commission

13. Type of Report & Period Covered
March 1971-June 1971

14.

15. Supplementary Notes

16. Abstracts

A procedure for the solution of the two-dimensional sofa problem is described. A new class of polygons, angularly simple polygons, is defined as a class of permissible sofas. The pattern representation, $S_r(x_o)$, developed for this class of polygons has the advantage of allowing easy polygonal transformations. The procedure called GSPS, described herein, gives a good approximate solution to the sofa problem in reasonable time. Slight modification of the procedure leads to an algorithm for the solution of the general sofa problem.

17. Key Words and Document Analysis. 17a. Descriptors

Sofa problem, hallway, objective function, polygonal Jordan curve, polygon, angularly simple, pattern sequence.

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

18. Availability Statement	19. Security Class (This Report)	21. No. of Pages
Releasable to the public	UNCLASSIFIED	44
	20. Security Class (This Page) UNCLASSIFIED	22. Price