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**ON GENERAL ELECTRODYNAMIC RECIPROCITY,
WITH APPLICATION TO THE ANALYSIS OF EMP PROBLEMS**

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Abstract

A general reciprocity theorem is developed for a volume of space containing closed regions of matter, on the surfaces of which it may be difficult to satisfy boundary conditions. The media everywhere are assumed to be linear and inhomogeneous and perhaps anisotropic (but not gyrotropic) in general. The reaction of the interior closed regions upon the outer volume is represented by (self-consistent) surface distributions of electric and magnetic surface currents just outside the volume. The formalism is applied to scattering by a perfectly conducting obstacle, and scattered fields are expressed in terms of incident fields of sources and electric surface currents generated outside the volume of interest. However, we show how both electric and magnetic surface current distributions are required for the exact solution. We then show how a well known variational formula for a scattered electric field (in terms of electric surface currents alone) is correct despite the presence of magnetic surface currents. Finally, a procedure is given for calculating EMP-induced voltages within a well shielded enclosure of general shape containing occasional apertures. By reciprocity, this approach relates those voltages to measured and calculated fields on the apertures. To obtain the approximate EMP-induced aperture fields, we first calculate from a variational expression the fields which would be generated on electrically shorted apertures and then deduce, by perturbation theory, the fields present after the apertures are cut.

1. Introduction

Reciprocity theorems for the solution of electromagnetic fields in regions containing linear media are well known.¹⁻⁸ A reciprocity theorem is employed to obtain a point field due to a given excitation by inserting a "test generator" at that point and computing the interaction of the generator fields with the given excitation. The advantages of such an approach in determining fields scattered by obstacles, especially when variational principles are invoked, are apparent from the literature. Reciprocity is particularly useful for computing the scattering of a wire-like circuit, for which the response to a "test generator" can usually be computed accurately by lumped-circuit theory.

Reciprocity is conceptually simple to understand but may become confusing when considering the details. For example, in the solution of boundary-value problems with

perfectly conducting surfaces, we may wish to insert electric and/or magnetic (fictitious) surface currents on those surfaces in order to force the condition of zero tangential electric field. Should we regard these surface currents strictly as volume currents within the volume of space under consideration, as surface currents right on the surface enclosing the volume, or as excitations just outside the volume? Confusion about this could cause errors in the signs of these surface currents. Scattering problems are usually formulated in terms of self-consistent electric surface currents on perfectly conducting obstacles. Does this mean there is no magnetic surface current required, and if it is required, how does it help determine a self-consistent solution to the problem? Then, too, the question often arises of whether or not to double a surface current caused by an incident field on a perfectly conducting obstacle. Such an electric current is usually twice the incident tangential magnetic field in amplitude in the physical optics approximation but it is not doubled in the longer wavelength approximation. Should we then deliberately take an intermediate value of surface current at intermediate frequencies? And how do the surface electric and/or magnetic currents determine rigorously a self-consistent solution to a scattering problem?

In order to answer such questions and develop a reciprocity theorem unambiguously for correct application we first define the volume of space, V_0 , under discussion, enclosed by two surfaces; the outer surface is often put at infinity and the inner one, S_i , excludes all obstacles or scattering bodies. The general reciprocity theorem is derived for this volume, which contains both arbitrary electric and magnetic current sources. The theorem is stated for two solutions of Maxwell's equations within V_0 , neither of which need necessarily satisfy the correct boundary conditions on S_i . All effects on the inner surface S_i are represented by surface integrals involving fields only. These fields are expressed self-consistently in terms of self-consistent currents—electric and/or magnetic—inside the inner surface. For example, a perfectly conducting body would have magnetic surface currents placed in it which are just outside the volume V_0 , and which unambiguously determine the tangential fields on S_i . These magnetic surface currents and total tangential magnetic field (equivalent to an electric surface current) on S_i and outside V_0 must then be determined self-consistently to satisfy the boundary condition of zero tangential electric field on S_i .

After deriving a general, rigorous reciprocity theorem in Sec. 2 we shall apply it in Secs. 3 and 4 to verify results obtained by other workers.

2. Formulation of a General Temporal Reciprocity Theorem

We consider the volume of space, V_0 , surrounded by closed surfaces, S_0 and S_i , shown in Fig. 1. The electric current sources $\vec{J}(\vec{r}, t)$ (the bar denotes a vector) are confined to a finite volume within V_0 which is characterized by the linear parameters[†]

[†]The parameters are scalar but could be tensor; however not gyrotropic or gyroelectric.

Fig. 1. Linear region of space V_0 surrounded by outer surface S_0 and inner surface S_i between which lie all volume electric current sources $\vec{J}(\vec{r}, t)$ and magnetic current sources $\vec{J}^*(\vec{r}, t)$. Within S_i in V_1 there are shown equivalent volume currents $\vec{J}_1(\vec{r}, t)$ and $\vec{J}_1^*(\vec{r}, t)$ as well as equivalent surface currents $\vec{K}_1$ and $\vec{K}_1^*$, which represent the effect of the inner region upon S_i and upon V_0 .

of μ (permeability), ϵ (permittivity) and σ (conductivity). Inner surface S_i lies close (with respect to all wavelengths of interest) to another surface S_1 enclosing an inner volume V_1 which is regarded as a perturbation upon the electromagnetic field solution within V_0 . For example, V_1 may be a perfectly conducting obstacle or a dielectric body. To represent the electrical effect of V_1 upon V_0 , we postulate the presence of equivalent volume electric current density $\vec{J}(\vec{r}, t)$ and magnetic density $\vec{J}^*(\vec{r}, t)$ [†] and/or surface densities $\vec{K}$ and $\vec{K}^*$ on S_1 inside S_i . All these currents within S_i must be determined self-consistently so the tangential electric and magnetic fields on S_i are correct; i.e., they satisfy the correct boundary conditions on S_i (which is essentially zero electrical distance from S_i). By being so precise about S_i and S_1 just inside it we clearly separate the "induction" of region V_1 upon S_i and therefore V_0 —which is conveniently approximated, often in a variational sense—from the precise solution of Maxwell's

[†]The superscript $*$ represents a magnetic current or charge, not a complex conjugate. There are no complex conjugates in reciprocity theorems.

equations with exact boundary conditions on S_1 and within V_1 . Subsequent examples will illustrate computational advantages in this point of view.

We proceed to develop a reciprocity theorem for V_0 (following Choe⁷) in which there are no surface currents on S_1 —only fields. The basic equations are:

$$\nabla \times \vec{E}(\vec{r}, t) = -\dot{\vec{B}}(\vec{r}, t) - \vec{J}(\vec{r}, t) \quad (1a)$$

$$\nabla \times \vec{H}(\vec{r}, t) = \vec{J}(\vec{r}, t) + \vec{J}_{\text{ind}}(\vec{r}, t) + \dot{\vec{D}}(\vec{r}, t), \quad (1b)$$

in which $\vec{B}$, $\vec{D}$, and $\vec{J}_{\text{ind}}$ are related to $\vec{H}$ and $\vec{E}$ at every point $\vec{r}$ as

$$\vec{B}(\vec{r}, t) = \int_{-\infty}^{\infty} \mu(\vec{r}, \beta) \vec{H}(\vec{r}, t - \beta) d\beta \quad (2a)$$

$$\vec{D}(\vec{r}, t) = \int \epsilon(\vec{r}, \beta) \vec{E}(\vec{r}, t - \beta) d\beta \quad (2b)$$

$$\vec{J}_{\text{ind}}(\vec{r}, t) = \int \sigma(\vec{r}, \beta) \vec{E}(\vec{r}, t - \beta) d\beta. \quad (2c)$$

Here μ , ϵ , and σ , independent of time, are the Fourier transforms of those parameters in the frequency domain, where they are usually measured. $\vec{J}(\vec{r}, t)$ in Eq. (1b) is the driving generator, presumed known, and $\vec{J}^*$ might exist for test purposes. Eq. (1a) implies $\nabla \cdot \vec{B}(\vec{r}, t) = 0$ for all time if initially so, in a physical problem where $\vec{J}^* = 0$. The divergences of Eq. (1) imply generally, in the frequency domain:

$$\nabla \cdot \vec{J}^*(\vec{r}, \omega) = -\nabla \cdot \dot{\vec{B}} = -\nabla \cdot [\mu \dot{\vec{H}}] = -\dot{\rho}(\vec{r}, \omega) \quad (3a)$$

$$\nabla \cdot \vec{J}(\vec{r}, \omega) = -\nabla \cdot (\sigma \vec{E}) - \nabla \cdot \dot{\vec{D}} = -\nabla \cdot [\sigma \vec{E} + \epsilon \dot{\vec{E}}] = -\dot{\rho}(\vec{r}, \omega) \quad (3b)$$

which serve to relate charge densities $\rho(\vec{r}, \omega)$ and $\rho^*(\vec{r}, \omega)$ and their transforms $\rho(\vec{r}, t)$ and $\rho^*(\vec{r}, t)$ to the currents and fields in a consistent manner.

We next postulate two solutions of Maxwell's equations within V_0 , denoted the a-solution due to $\vec{J}_a$ and the b-solution due to $\vec{J}_b$. Neither need necessarily satisfy the correct boundary conditions on S_1 . By manipulating Maxwell's equations and interpreting them, Choe derives a reciprocity theorem in the form:

$$\int_{-\infty}^{\infty} dt \oint_{S_0 + S_1} d\vec{S} \cdot [\vec{E}_a(\vec{r}, \tau - t) \times \vec{H}_b(\vec{r}, t) - \vec{E}_b(\vec{r}, t) \times \vec{H}_a(\vec{r}, \tau - t)]$$

$$= \int_{-\infty}^{\infty} dt \int_{V_0} [\bar{J}_a(\bar{r}, \tau - t) \cdot \bar{E}_b(\bar{r}, t) - \bar{J}_b(\bar{r}, t) \cdot \bar{E}_a(\bar{r}, \tau - t) - \bar{J}_a(\bar{r}, \tau - t) \cdot \bar{H}_b(\bar{r}, t) + \bar{J}_b(\bar{r}, t) \cdot \bar{H}_a(\bar{r}, \tau - t)] dV, \quad (4)$$

Vector $d\bar{S}$ points out of volume V_0 on S_0 and S_1 . In this form all effects of matter within volume V_1 inside S_1 are represented by $\oint_{S_1}$. It is interesting to observe that the derivation of Eq. (4) allows $\bar{J}_a(\bar{r}, t)$, $\bar{J}_b$, $\bar{J}_a$, and $\bar{J}_b$ to be of infinite duration, starting from finite times. If the currents are Fourier transformable (as they invariably are), the time convolution in Eq. (4) implies corresponding cross- and dot-products in the frequency domain.

It is convenient to abbreviate the notation of Eq. (4) by symbolizing the volume integration by $\langle \rangle$, and by omitting arguments of surface fields. In a time convolution of two excitations it is irrelevant which has argument t and which has $(\tau - t)$. Hence we write our general reciprocity relation in the abbreviated form:

$$\oint_{S_0+S_1} d\bar{S} \cdot [\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a] = -\langle \bar{E}_a, \bar{J}_b \rangle + \langle \bar{E}_b, \bar{J}_a \rangle + \langle \bar{H}_a, \bar{J}_b \rangle - \langle \bar{H}_b, \bar{J}_a \rangle, \quad (5)$$

which implies either a further convolution in the time domain or else products of excitations in the frequency domain.

Harrington⁸ shows that $\oint_{S_0} = 0$ in Eq. (5) if $R_0 = \infty$, where the a and b fields assume their radiation-field forms.

3. Scattering by a Perfectly Conducting Obstacle

We are interested in the field scattered from the obstacle to a receiver point $\bar{r}_R$ with a transmitting electric dipole at point $\bar{r}_T$ (see Fig. 2a). For our a -system we take $R_0 = \infty$ as in Fig. 1 with $\bar{J}(\bar{r}, t) = \bar{J}_a(\bar{r}_T, t)$ within V_0 and an initially unknown distribution of surface currents $\bar{K}_a$ and $\bar{K}_a^*$ on S_1 inside S_1 . These create fields $\bar{E}_a$ and $\bar{H}_a$ on S_1 . For our b -system we take a unit current dipole $\bar{J}_b(\bar{r}, t) = \bar{J}_b(\bar{r}_R, t)$ at the position of the receiver and no currents outside of S_1 , i.e., $\bar{J}_b$ radiates in free space. The two systems are shown in Fig. 2.

Applying reciprocity theorem Eq. (5) we have:

$$\oint_{S_1} d\bar{S}_1 \cdot [\bar{E}_a \times \bar{H}_b^i - \bar{E}_b^i \times \bar{H}_a] = -\bar{E}_a \cdot \bar{J}_b(\bar{r}_R) + \bar{E}_b^i \cdot \bar{J}_a(\bar{r}_T). \quad (6)$$

Fig. 2. Scattering by a perfectly conducting obstacle. (a) The transmitter current scatters off the obstacle, to create $\bar{E}_a$ at the receiver, which is the sum of incident and scattered components, (b) a unit current $\bar{I}_b$ at the receiver point radiates a plane wave $\bar{E}_b = \bar{E}_b^i$ in empty space.

Here superscript i denotes an incident or free-space field. Because $\bar{E}_a$ is the total field in $S_1 \approx S_1$, then for a perfectly conducting surface $\bar{dS}_1 \times \bar{E}_a = 0$ and the first $\oint_{S_1}$ in Eq. (6) vanishes. We can further simplify Eq. (6) by observing that if we had chosen our a -system to be the transmitter radiating in free space (volume V_1 shrunk to zero), in place of Eq. (6) (in which $\bar{E}_a \rightarrow \bar{E}_a^i$, $\bar{H}_a \rightarrow \bar{H}_a^i$) we would have obtained:

$$0 = -\bar{E}_a^i \cdot \bar{I}_b(\bar{r}_R) + \bar{E}_b^i \cdot \bar{J}_a(\bar{r}_T). \quad (7)$$

Because $\bar{E}_a = \bar{E}_a^i + \bar{E}_a^s$ by definition of the scattered component $\bar{E}_a^s$, subtraction of Eq. (7) from Eq. (6) cancels the $\bar{E}_b^i \cdot \bar{J}_a$ term. Rearrangement then yields:

$$\bar{E}_a^s \cdot \bar{I}_b(\bar{r}_R) = \oint_{S_1} \bar{E}_b^i \cdot [-\bar{dS}_1 \times \bar{H}_a], \quad \bar{K}_a \triangleq -\bar{n}_1 \times \bar{H}_a \text{ on } S_1, \quad (8)$$

which is precisely Harrington's form⁹ for the scattered field at the receiver, $\bar{r}_R$, in terms of known incident receiver field and a total, self-consistent tangential $\bar{H}_a$ created by the transmitter on S_1 . We can, if we wish, regard $-\bar{n}_1 \times \bar{H}_a$ on S_1 as an equivalent electric surface current $\bar{K}_a$ according to Stratton,¹⁰ but it is irrelevant. The important point is that $\bar{H}_a$ (or the equivalent $\bar{K}_a$) on S_1 will generate tangential magnetic field on S_1 which must be self-consistent by Stratton's formula (written in the frequency domain with $e^{i\omega t}$ - time-dependence):

$$\bar{H}_a(\bar{r}', \omega)|_{\tan} = \left\{ \bar{H}_a^i(\bar{r}', \omega) + \frac{1}{4\pi} \oint_{S_1} \left[-i\omega\epsilon \bar{K}_a^* \phi - (\bar{n}_1 \times \bar{H}_a) \times \nabla \phi - (\bar{n}_1 \cdot \bar{H}_a^S) \nabla \phi \right] dS_1 \right\}_{\tan},$$

$$\phi = \exp(-ik|\bar{r} - \bar{r}'|)/|\bar{r} - \bar{r}'|, \quad \bar{r} \text{ and } \bar{r}' \text{ on } S_1, \quad e^{i\omega t} \text{ - variation.} \quad (9)$$

Here $\bar{E}_a = \bar{E}_a^i + \bar{E}_a^S = 0$ on S_1 , but $\bar{E}_a^i$ due to volume current source $\neq 0$, and so we should choose a distribution of magnetic surface current $\bar{K}_a^*$ on S_1 which will be a source of $\bar{E}_a^S$ and $\bar{H}_a^S$ fields and cancel $\bar{E}_a^i$ on S_1 ,

$$\begin{aligned} \bar{E}_a^S(\bar{r}', \omega)|_{\tan} = -\bar{E}_a^i(\bar{r}', \omega)|_{\tan} = & \left\{ -\frac{1}{4\pi} \oint_{S_1} \left[-i\omega\mu (\bar{n}_1 \times \bar{H}_a) \phi \right. \right. \\ & \left. \left. + \bar{K}_a^* \times \nabla \phi + (\bar{n}_1 \cdot \bar{E}_a^S) \nabla \phi \right] dS_1 \right\}_{\tan}. \end{aligned} \quad (10)$$

In Eqs. (9) and (10) the primary excitations to be determined self-consistently on S_1 are regarded as $-\bar{n}_1 \times \bar{H}_a = -\bar{n}_1 \times \bar{H}_a = \bar{K}_a^*$, and $\bar{K}_a^*$. These must be consistent with $\bar{n}_1 \cdot \bar{E}_a = \bar{n}_1 \cdot \bar{E}_a^S$ and $\bar{n}_1 \cdot \bar{H}_a^S$ (needed to cancel $\bar{n}_1 \cdot \bar{H}_a^i$ on S_1), respectively, as follows:

$$\nabla_T \cdot [-\bar{n}_1 \times \bar{H}_a] = \nabla_T \cdot \bar{K}_a^* = -i\omega\eta = +i\omega\epsilon (\bar{n}_1 \cdot \bar{E}_a^S), \quad (11)$$

where

$$\nabla_T \cdot \bar{K}_a^* = +i\omega\mu (\bar{n}_1 \cdot \bar{H}_a^S)$$

and $\eta =$ charge density. Insertion of these expressions for $(\bar{n}_1 \cdot \bar{E}_a^S)$ and $(\bar{n}_1 \cdot \bar{H}_a^S)$ in Eqs. (9) and (10) enables us to write them as

$$\bar{H}_a(\bar{r}', \omega)|_{\tan} = \left\{ \bar{H}_a^i + \frac{1}{4\pi} \oint_{S_1} \left[-i\omega\epsilon \bar{K}_a^* \phi + (-\bar{n}_1 \times \bar{H}_a) \times \nabla \phi - \frac{\nabla_T \cdot \bar{K}_a^*}{i\omega\mu} \nabla \phi \right] dS_1 \right\}_{\tan}, \quad (12)$$

and

$$\begin{aligned} \bar{E}_a^S(\bar{r}', \omega)|_{\tan} = -\bar{E}_a^i(\bar{r}', \omega)|_{\tan} = & \left\{ \frac{1}{4\pi} \oint_{S_1} \left[+i\omega\mu (-\bar{n}_1 \times \bar{H}_a) \phi \right. \right. \\ & \left. \left. + \bar{K}_a^* \times \nabla \phi + \frac{\nabla_T \cdot (-\bar{n}_1 \times \bar{H}_a)}{i\omega\epsilon} \nabla \phi \right] dS_1 \right\}_{\tan}. \end{aligned} \quad (13)$$

Equations (12) and (13) are of the form

$$(\vec{H}_a)_{\text{tan}} - (\vec{H}_a^i)_{\text{tan}} = \oint_{S_1} [(\vec{H}_a)_{\text{tan}}, \vec{K}_a] dS_1, \quad (12')$$

and

$$(\vec{E}_a^s)_{\text{tan}} = -(\vec{E}_a^i)_{\text{tan}} - \oint_{S_1} [(\vec{H}_a)_{\text{tan}}, \vec{K}_a] dS_1, \quad (13')$$

and can be solved, in principle, for the self-consistent $(-\vec{n}_1 \times \vec{H}_a)_r$ and $\vec{K}_a(\vec{r})$ on S_1 . Satisfaction of the tangential components of Eqs. (12) and (13) will imply satisfaction of the normal boundary conditions as well, since the fields satisfy Maxwell's equations in V_1 right up to S_1 .

We see just how involved the complete self-consistent solution is for $-\vec{n}_1 \times \vec{H}_a = \vec{K}_a$ and $\vec{K}_a$ on S_1 . Although the scattered field $\vec{E}_a^s$ is determined by the former surface excitation in Eq. (3), in fact $-\vec{n}_1 \times \vec{H}_a$ is partially determined by $\vec{K}_a$, and both must be determined self-consistently in an exact solution. Often one ignores $\vec{K}_a$ and merely determines $-\vec{n}_1 \times \vec{H}_a$ on S_1 to be consistent with Eq. (12) alone, minimizing the errors by using a variational expression for the scattered field in terms of the unknown surface current $\vec{K}_a$. This is done, for example, when solving for the scattering from a long thin wire.

It is instructive now to prove by reciprocity arguments a result which will be needed later,¹¹ namely

$$\oint_{S_1} \vec{E}_b^i \cdot [-d\vec{S}_1 \times \vec{H}_a] = \oint_{S_1} \vec{E}_a^i \cdot [-d\vec{S}_1 \times \vec{H}_b], \quad (14)$$

in which $\vec{H}_a$ is the total magnetic field induced by a unit electric current at $\vec{r}_T$ of Fig. 2a and $\vec{H}_b$ is induced on S_1 by a unit electric current at $\vec{r}_R$ in Fig. 2b. Using the bracket notation, Eq. (14) becomes:

$$\langle \vec{E}_b^i, \vec{K}_a \rangle = \langle \vec{E}_a^i, \vec{K}_b \rangle, \quad (14')$$

or, in terms of scattered fields $\vec{E}_b^S = -\vec{E}_b^i$, $\vec{E}_a^S = -\vec{E}_a^i$ on S_1 ,

$$\langle \vec{E}_b^S, \vec{K}_a \rangle = \langle \vec{E}_a^S, \vec{K}_b \rangle. \quad (14'')$$

This statement, showing a reciprocity between $\vec{K}_{b,a}$ and their scattered fields $\vec{E}_{b,a}^S$ (generated as if the obstacle were not present) is plausible but not entirely obvious: after all, (fictitious) magnetic current densities $\vec{K}_{a,b}$ on S_1 also exist self-consistently with $\vec{K}_{a,b}$ and generally will contribute to $\vec{E}_{a,b}^S$.

Proceeding with the proof of Eqs. (14)-(14''), elementary reciprocity holds by Eq. (5), with $\bar{E}_a$ and $\bar{E}_b$ purely normal to S_i , with an interchange of the unit current generators at $\bar{r}_R$ and $\bar{r}_T$ in the presence of the obstacle, i.e.,

$$\bar{E}_a \cdot \bar{I}_b(\bar{r}_R) = \bar{E}_b \cdot \bar{I}_a(\bar{r}_T), \quad \bar{E}_{a,b} = \bar{E}_{a,b}^i + \bar{E}_{a,b}^s. \quad (15)$$

In the absence of the obstacle, Eq. (15) would say $\bar{E}_a^i \cdot \bar{I}_b(\bar{r}_R) = \bar{E}_b^i \cdot \bar{I}_a(\bar{r}_T)$. Subtraction of this from Eq. (15) yields:

$$\bar{E}_a^s \cdot \bar{I}_b(\bar{r}_R) = \bar{E}_b^s \cdot \bar{I}_a(\bar{r}_T) \quad (16)$$

in which the arrows indicate that $\bar{I}_a(\bar{r}_T)$ creates $\bar{E}_a^s(\bar{r}_R)$ and $\bar{I}_b(\bar{r}_R)$ creates $\bar{E}_b^s(\bar{r}_T)$. Now, Eq. (8) is valid for $\bar{I}_a = \bar{I}$ at $\bar{r}_T$, and by interchanging b and a , and $\bar{r}_R$ and $\bar{r}_T$, we get

$$\bar{E}_b^s \cdot \bar{I}_a(\bar{r}_T) = \oint_{S_i} \bar{E}_a^i \cdot [-d\bar{S}_i \times \bar{H}_b] = \oint_{S_i} \bar{E}_a^i \cdot \bar{K}_b dS_i. \quad (17)$$

Hence, Eqs. (16) and (17) do verify Eqs. (14)-(14''). In Eq. (17), $\bar{I}_a$ is the direct cause of $\bar{E}_a^i$ (i.e., generates it in free space), and $(-d\bar{S}_i \times \bar{H}_b) = \bar{K}_b dS_i$, the only physical surface current on S_i , creates $\bar{E}_b^s$ directly.

Suppose we wish to measure the scattered magnetic $\bar{H}^s(\bar{r}_R)$ -field at the receiver of Fig. 2 due to the same transmitter current $\bar{J}_a$ at $\bar{r}_T$. We can deduce quickly that it is just $-\bar{E}_a^s \times \bar{a}_r / \eta_0$, $\eta_0 \approx \sqrt{\mu_0 / \epsilon_0}$, in the radiation region, as follows. We leave the a -excitations unchanged but convert the electric current $\bar{I}_b(\bar{r}_R)$ of the b -excitation into a magnetic current $\bar{I}_b^*(\bar{r}_R)$. Suppose $\bar{I}_b$ is perpendicular to $\bar{a}_r$ in the plane of the paper (the orthogonal case of $\bar{I}_b$ perpendicular to the plane of the paper is treated separately) as shown, then let $\bar{I}_b^*$ be $\bar{I}_b \times \bar{a}_r$ out of the paper. By duality we know $\bar{I}_b^*$ will establish a free-space $\bar{E}_b^{iy}$ on the obstacle in the same direction as did $\bar{I}_b$ but numerically equal to $\bar{H}_b^i$ (i.e., $\bar{E}_b^{iy} = \bar{E}_b^i / \eta_0$). Thus $\eta_0 \bar{I}_b^*$ will create $\bar{E}_b^i$ again on the obstacle. The reciprocity relation of Eq. (5) in terms of $\bar{J}_a = \bar{I}_a$ and $\bar{J}_b^* = \eta_0 \bar{I}_b^*$ now reads:

$$\oint d\bar{S}_i \cdot [-\bar{E}_b^i \times \bar{H}_a] = \bar{E}_b^i \cdot \bar{I}_a(\bar{r}_T) + (\bar{H}_a \cdot \eta_0 \bar{I}_b^*(\bar{r}_R)). \quad (18)$$

If we had chosen our a -system to be the transmitter radiating in free-space we would obtain, instead of Eq. (7), with $\bar{E}_a \rightarrow \bar{E}_a^i$:

$$0 = \bar{E}_b^i \cdot \bar{I}_a(\bar{r}_T) + \bar{H}_a \cdot \eta_0 \bar{I}_b^*(\bar{r}_R). \quad (19)$$

Subtraction of Eq. (19) from Eq. (18) and rearrangement gives:

$$-\vec{H}_a^S \cdot \eta_0 \vec{l}_b^* (\vec{r}_R) = \oint_{S_i} \vec{E}_b^i \cdot \{-\vec{dS}_i \times \vec{H}_a\}. \quad (20)$$

Compare this with Eq. (8), which has the same integral. We conclude $\vec{H}_a^S$ is numerically $\vec{E}_a^S/\eta_0$, since $\vec{l}_b^*$ points out of the paper, and that $\vec{E}_a^S \times \vec{H}_a^S$ points in the $+\vec{a}_r$ direction in Fig. 2.

4. Variational Expression for the Field Scattered from a Perfectly Conducting Obstacle

In Sec. 2 we outlined an exact solution for the scattering problem, in which the transmitter induced on the obstacle a total tangential magnetic field $(-\vec{n}_1 \times \vec{H}_a)$, which equal $\vec{K}_a$, and a magnetic surface current source $\vec{K}_a^*$ was introduced just outside the volume V_0 where the scattered field was computed. Strictly speaking both $\vec{K}_a$ and $\vec{K}_a^*$ must be determined self-consistently according to remarks following Eq. (13). We shall now discuss a variational procedure to follow when computing the field $\vec{E}_a^S$ scattered by the obstacle from the transmitter when estimates are made in $\vec{K}_a$.

The problem is to compute $\vec{E}_a^S(\vec{r}_R)$ in the direction $\vec{l}_b$ given by Eq. (8), where "a" denotes the system in which the transmitter radiates toward the receiver, and "b" denotes the reciprocal system in which the receiver current $\vec{l}_b$ excites the transmitter.

Abbreviating $(-\vec{n}_1 \times \vec{H}_a)$ by $\vec{K}_a$, the equivalent electric surface current on the obstacle, we observe from Eqs. (8) and (14):

$$E_a^S(\vec{r}_R) = \oint_{S_i} \vec{E}_b^i \cdot \vec{K}_a dS_i = \oint_{S_i} \vec{E}_a^i \cdot \vec{K}_b dS_i, \quad (21)$$

$\vec{K}_b$ being the equivalent electric surface current induced by the receiver dipole $\vec{l}_b(\vec{r}_R)$. Suppose we abbreviate Eq. (21) with the bracket notation as:

$$E_a^S(\vec{r}_R) = \langle \vec{E}_b^i, \vec{K}_a \rangle = \langle \vec{E}_a^i, \vec{K}_b \rangle, \quad (21')$$

$\vec{K}_{a,b}$ are the correct surface currents. Suppose we approximate them by $\vec{K}_{\alpha,\beta}$; $\vec{K}_\alpha = \vec{K}_a + \delta\vec{K}_\alpha$, $\vec{K}_\beta = \vec{K}_b + \delta\vec{K}_\beta$, $\delta\vec{K}_{\alpha,b}$ being errors. Now consider the ratio

$$E_\alpha^S = -\langle \vec{E}_b^i, \vec{K}_\alpha \rangle \langle \vec{E}_a^i, \vec{K}_\beta \rangle / \langle \vec{E}_\beta^S, \vec{K}_\alpha \rangle, \quad (22)$$

where, by Eq. (17),

$$\begin{aligned}\bar{\mathbf{E}}_b^S(\bar{\mathbf{r}}') \cdot \bar{\mathbf{I}}_a(\bar{\mathbf{r}}') &= \oint_{S_i} \bar{\mathbf{E}}_a^i(\bar{\mathbf{r}}) \cdot \bar{\mathbf{K}}_b dS_i, \\ &= - \oint_{S_i} \bar{\mathbf{E}}_a^S(\bar{\mathbf{r}}) \cdot \bar{\mathbf{K}}_b dS_i, \quad \bar{\mathbf{K}}_b = -\hat{\mathbf{n}}_i \times \bar{\mathbf{H}}_b.\end{aligned}$$

Equation (22) is correct if all the fields and currents are correct, for then

$$-\langle \bar{\mathbf{E}}_b^i, \bar{\mathbf{K}}_a \rangle / \langle \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle \rightarrow -\langle \bar{\mathbf{E}}_b^i, \bar{\mathbf{K}}_a \rangle / \langle \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle = +1$$

and Eq. (22) reduces to Eq. (17). We are also implying the estimated current density $\bar{\mathbf{K}}_b$ generates the scattered estimated field $\bar{\mathbf{E}}_b^S$ by the correct electrodynamic formula, viz. Eq. (13), with $\bar{\mathbf{K}}_a^*$ present (though neglected in many simplified treatments of scattering).

We can show the error in $\bar{\mathbf{E}}_a^S, \delta \bar{\mathbf{E}}_a^S$, in quadratic in the errors $\delta \bar{\mathbf{K}}_a, \delta \bar{\mathbf{K}}_b$, and $\delta \bar{\mathbf{E}}_b^S$ as generated by $\delta \bar{\mathbf{K}}_b$, as follows. Substitution of $\bar{\mathbf{E}}_a^S + \delta \bar{\mathbf{E}}_a^S$ for $\bar{\mathbf{E}}_a^S$, etc., in Eq. (22), multiplication through by the denominator, and cancellation of the non- δ terms yields:

$$\begin{aligned}\bar{\mathbf{E}}_a^S \left[\langle \delta \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle + \langle \bar{\mathbf{E}}_b^S, \delta \bar{\mathbf{K}}_a \rangle \right] + \delta \bar{\mathbf{E}}_a^S \langle \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle + \delta \delta\text{-terms} &= -\langle \bar{\mathbf{E}}_b^i, \delta \bar{\mathbf{K}}_a \rangle \langle \bar{\mathbf{E}}_a^i, \bar{\mathbf{K}}_b \rangle \\ &\quad - \langle \bar{\mathbf{E}}_b^i, \bar{\mathbf{K}}_a \rangle \langle \bar{\mathbf{E}}_a^i, \delta \bar{\mathbf{K}}_b \rangle + \delta \delta\text{-terms.} \quad (23)\end{aligned}$$

If we can show the first bracketed expression cancels the first two terms on the right we will have shown $\delta \bar{\mathbf{E}}_a^S \propto \delta \delta\text{-terms}$, i.e., Eq. (22) is variational for $\bar{\mathbf{E}}_a^S$. Since $\bar{\mathbf{E}}_a^S = \langle \bar{\mathbf{E}}_a^i, \bar{\mathbf{K}}_b \rangle = \langle \bar{\mathbf{E}}_b^i, \bar{\mathbf{K}}_a \rangle$ this requires showing:

$$\langle \delta \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle + \langle \bar{\mathbf{E}}_b^S, \delta \bar{\mathbf{K}}_a \rangle = -\langle \bar{\mathbf{E}}_b^i, \delta \bar{\mathbf{K}}_a \rangle - \langle \bar{\mathbf{E}}_a^i, \delta \bar{\mathbf{K}}_b \rangle. \quad (24)$$

With $\bar{\mathbf{E}}_b^i = -\bar{\mathbf{E}}_b^S$ on S_i , two of these terms cancel so we only have to show:

$$\langle \delta \bar{\mathbf{E}}_b^S, \bar{\mathbf{K}}_a \rangle = \langle \bar{\mathbf{E}}_a^S, \delta \bar{\mathbf{K}}_b \rangle. \quad (25)$$

To prove Eq. (25) we observe that $\delta \bar{\mathbf{E}}_b^S$ and $\delta \bar{\mathbf{K}}_b$ are errors and so we must state how they are related. We start from Eq. (8) and move the point $\bar{\mathbf{r}}_R$ to $\bar{\mathbf{r}}'$ on S_i ; it also gives us a change $\delta \bar{\mathbf{E}}_a^S$ in terms of a change $\delta \bar{\mathbf{K}}_a$, so that:

$$\bar{\mathbf{E}}_a^S(\bar{\mathbf{r}}') \cdot \delta \bar{\mathbf{K}}_b(\bar{\mathbf{r}}') = \oint_{S_i} \delta \bar{\mathbf{E}}_b^i \cdot \bar{\mathbf{K}}_a(\bar{\mathbf{r}}) dS, \quad \delta \bar{\mathbf{E}}_b^i \text{ due to } \delta \bar{\mathbf{K}}_b(\bar{\mathbf{r}}'). \quad (26)$$

We can write¹¹ $\delta \bar{E}_b^i(\bar{r}) = \tilde{\Gamma}(\bar{r}, \bar{r}') \cdot \delta \bar{K}_b(\bar{r}')$ so Eq. (26) reads, after integrating over $\bar{r}'$ on S_i :

$$\langle \bar{E}_a^S, \delta \bar{K}_b \rangle = \oint_{S_i} \oint_{S_i} dS' \bar{K}_a(\bar{r}) \cdot \tilde{\Gamma}(\bar{r}, \bar{r}') \cdot \delta \bar{K}_b(\bar{r}') dS. \quad (27)$$

Interchanging the transmitter and receiver positions in Eq. (8), i.e., $a \leftrightarrow b$, we get $\delta \bar{E}_b^S(\bar{r}')$ due to a change $\delta \bar{K}_b(\bar{r})$ as:

$$\delta \bar{E}_b^S \cdot \bar{K}_a(\bar{r}') = \oint_{S_i} \bar{E}_a^i(\bar{r}) \Big|_{\bar{K}_a(\bar{r}')} \cdot \delta \bar{K}_b(\bar{r}) dS = - \oint_{S_i} \bar{E}_a^S(\bar{r}) \Big|_{\bar{K}_a} \cdot \delta \bar{K}_b(\bar{r}) dS. \quad (28)$$

Again $\bar{E}_a^i(\bar{r}) = \tilde{\Gamma}(\bar{r}, \bar{r}') \cdot \bar{K}_a(\bar{r}')$, so after integrating over S_i , Eq. (28) yields:

$$\langle \delta \bar{E}_b^S, \bar{K}_a \rangle = \oint_{S_i} \oint_{S_i} dS' \delta \bar{K}_b(\bar{r}) \cdot \tilde{\Gamma}(\bar{r}, \bar{r}') \cdot \bar{K}_a(\bar{r}') dS. \quad (29)$$

It has been shown¹¹ that $\tilde{\Gamma}$ is symmetric in $\bar{r}, \bar{r}'$ in the sense that the right sides of Eqs. (27) and (29) are equal. Hence:

$$\langle \delta \bar{E}_b^S, \bar{K}_a \rangle = \langle \bar{E}_a^S, \delta \bar{K}_b \rangle, \text{ regardless of } \bar{K}_{a,b}^{\pm}, \bar{K}_{a,\beta}^{\pm}. \quad (30)$$

The arrows show the cause and effect. Equation (30) is the same as Eq. (25) and we have thereby shown that Eq. (22) is variational for $\bar{E}_a^S$ even though we have imposed no constraint conditions on $\bar{K}_\alpha$ and $\bar{K}_\beta$ and have merely implied their consistency with $\bar{K}_\alpha^{\pm}$ and $\bar{K}_\beta^{\pm}$.

We can obtain additional relations from Eqs. (30) and (14''). Equation 14'' is:

$$\langle \bar{E}_a^S, \bar{K}_b \rangle = \langle \bar{E}_b^S, \bar{K}_a \rangle. \quad (14'')$$

By adding this to Eq. (30) we obtain:

$$\langle \bar{E}_a^S, \bar{K}_\beta \rangle = \langle \bar{E}_\beta^S, \bar{K}_a \rangle, \quad (31)$$

In the manner of deriving Eq. (30) we can show that:

$$\langle \delta \bar{E}_a^S, \bar{K}_\beta \rangle = \langle \bar{E}_\beta^S, \delta \bar{K}_a \rangle \quad (32)$$

and when we add this to Eq. (31) we get:

$$\langle \bar{E}_\alpha^S, \bar{K}_\beta \rangle = \langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle, \quad (33)$$

which completes the list of all the reciprocity relations of interest. Observe that Eq. (33) does not follow obviously from simple reciprocity considerations because both $\bar{K}_{\alpha,\beta}$ and $\bar{K}_{\alpha,\beta}$ determine $\bar{E}_{\alpha,\beta}^S$.

Harrington¹² derives the equivalent of Eq. (22) and constrains it to yield the correct $\bar{E}_\alpha^S$ with second-order error $\delta \bar{E}_\alpha^S$ by imposing the constraints of Eq. (34) on $\bar{K}_\alpha$ and $\bar{K}_\beta$ (in our notation):

$$\langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle = \langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle = \langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle, \quad (34)$$

in which $\bar{E}_\beta^S = -\bar{E}_\beta^i$, is known. Since $\bar{K}_\alpha$ (the correct transmitter-induced current) is not known, this last equality must be rewritten with the aid of Eq. (31):

$$\langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle = \langle \bar{E}_\alpha^S, \bar{K}_\beta \rangle = -\langle \bar{E}_\alpha^i, \bar{K}_\beta \rangle.$$

Then it may be shown that Harrington's procedure of constraining the equalities:

$$\langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle = -\langle \bar{E}_\beta^i, \bar{K}_\alpha \rangle = -\langle \bar{E}_\alpha^i, \bar{K}_\beta \rangle \quad (34')$$

by adjustment of $\bar{K}_\alpha, \bar{K}_\beta$ does force $\bar{E}_\alpha^S = \bar{E}_\alpha^S + \delta^2$ -terms in Eq. (22); i.e., it is correct within first-order δ -terms. In this proof we use Eq. (30). However, Harrington need not have explicitly constrained all of Eqs. (34) and (34') by adjustment of $\bar{K}_{\alpha,\beta}$; he needed only to demand that:

$$\langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle = -\langle \bar{E}_\beta^i, \bar{K}_\alpha \rangle \quad (35)$$

by proper adjustment of $\bar{K}_\beta$ (which determines $\bar{E}_\beta^S$) and $\bar{K}_\alpha$. Then we can show the additional equalities of Eqs. (21) and (30) make $\bar{E}_\alpha^S = \bar{E}_\alpha^S + \delta^2$ -terms in Eq. (22).

We have obtained a variational expression for the electric field scattered to the receiver point by a perfectly conducting obstacle intercepting and reradiating transmitter field, in terms of estimated equivalent surface electric currents $\bar{K}_\alpha$ and $\bar{K}_\beta$ induced by the transmitter and receiver, respectively. We have only specified that they be close to the correct currents so that $\bar{E}_\alpha^S$ will be accurate. To improve our calculation of $\bar{K}_\alpha \equiv (-\bar{n}_1 \times \bar{H}_\alpha)$ on S_1 , we might insert a magnetic surface current $\bar{K}_\alpha^*$ on S_1 to reduce partially the incoming tangential component of $\bar{E}_\alpha^i$ to zero. $\bar{K}_\alpha^*$ can be chosen¹³ to equal $(-\bar{n}_1 \times \bar{E}_\alpha^i)$, and this will induce a field $\bar{E}(\bar{K}_\alpha^*)$ on S_1 tending to cancel $\bar{E}_\alpha^i$. We say "tending" because, in fact, if Stratton's formula¹⁰ Eq. (13)—with $\bar{K}_\alpha^* = -\bar{n}_1$

$\times \bar{E}_a^i$ —is used to compute $\bar{E}(\bar{K}_a^*)$, the near-current elements will contribute precisely $-\frac{1}{2}(\bar{E}_a^i)_{\tan}$. If $\bar{K}_a^*$ and K_a were entirely correct (equal to $\bar{K}_a^*$, $\bar{K}_a$) their far-current contributions would furnish the remaining $-\frac{1}{2}(\bar{E}_a^i)_{\tan}$. Still it is reasonable to insert $\bar{K}_a^* = -\bar{n}_i \times \bar{E}_a^i$ on S_i , in which case it would help to determine self-consistently a trial field $[-\bar{n}_i \times \bar{H}_a] \equiv \bar{K}_a$ on S_i for a variational calculation of the scattered $\bar{E}_a^s$ in Eq. (22) [see remarks following Eq. (13)]. Analogously, we could compute a trial $(-\bar{n}_i \times \bar{H}_b) \equiv \bar{K}_b$ for the obstacle illuminated by the receiver after depositing a $\bar{K}_b^* = -\bar{n}_i \times \bar{E}_b^i$ given by the incident field from the receiver and then calculate self-consistently the $\bar{K}_b$. Of course, the best trial current $\bar{K}_a$ would be found by solving Eqs. (12) and (13) approximately for the self-consistent $\bar{K}_a^*$ and $(-\bar{n}_i \times \bar{H}_a)$, and similarly for $\bar{K}_b$.

In the geometrical optics approximation,¹⁴ the wavelength is presumed so short on each essentially flat area of obstacle surface that only the near-current elements of $\bar{K}$ contribute to the scattered field. In this case, if $\bar{K} = -\bar{n}_i \times (\bar{H}^s + \bar{H}^i)$ is assumed to radiate scattered field from a unit area of the obstacle, Stratton's formula would yield $\bar{H}_{\tan}^s = \frac{1}{2}\bar{H}_{\tan} = \frac{1}{2}(\bar{H}^s + \bar{H}^i)_{\tan}$, or $\bar{H}^s = \bar{H}^i$, which implies $\bar{K} = (-\bar{n}_i \times 2\bar{H}^i)$ in computing the scattered fields by Eq. (21) for example. Harrington attributes this "doubling of incident $\bar{H}^i$ " to get the correct $\bar{n}_i \times \bar{H}$ on the obstacle for calculation of the scattered field to image theory, but it is due fundamentally to two approximations: (1) the very small wavelength assumption, and (2) neglect of the influence of $\bar{K}^*$ required for boundary matching upon $\bar{K}$.

This curious fact that the near-field contributions of Stratton's¹⁰ Eqs. (19) and (20) yield precisely half the total tangential field values in the sense that

$$\bar{r}' \rightarrow \bar{r}_s \text{ on } S_i \quad - \frac{1}{4\pi} \int_{\Delta S_i} (\bar{n} \times \bar{E})_{\bar{r}} \times \nabla \phi(\bar{r}', \bar{r}) dS = \frac{1}{2} (\bar{n}_i \times \bar{E})_{\bar{r}_s} \times \bar{n}_i \quad (36a)$$

$$\bar{r}' \rightarrow \bar{r}_s \text{ on } S_i \quad - \frac{1}{4\pi} \int_{\Delta S_i} (\bar{n} \times \bar{H})_{\bar{r}} \times \nabla \phi(\bar{r}', \bar{r}) dS = \frac{1}{2} (\bar{n}_i \times \bar{H})_{\bar{r}_s} \times \bar{n}_i \quad (36b)$$

may be verified by direct calculation. The remaining half of the tangential field component at a point on S_i is furnished by "distant" fields on S_i and perhaps currents and charges within the volume. For example, one may verify that when the integral of Eq. (36b) is carried out for the entire surface of an infinitely long straight circular wire carrying uniform current on its surface, in the zero frequency approximation it yields $H_{\tan} = I/(2\pi \text{ radius})$ and not half this value.

5. Procedure for Calculating EMP-Induced Voltages within Well Shielded Enclosures

We now outline an efficient, accurate procedure for calculating an EMP-induced voltage across a circuit element within the well-shielded enclosure of Fig. 3. The procedure will invoke both the results of measurement and calculation in order to be realistic and to simplify the computational procedure.

Fig. 3. A circuit element Z inside a well shielded enclosure, excited by an EMP dipole $\vec{E}$ through the dashed aperture on surface S_i .

In this situation, with the EMP field penetrating essentially only through the aperture shown, it is convenient to solve the problem in three steps: (1) use reciprocity to express E_z in terms of the EMP-induced tangential fields on S_i and also the measured fields there which are due to current generator I_b across the element of interest; (2) calculate the

EMP-induced normal-electric and tangential-magnetic fields on a short-circuited S_i , due to scattering by a perfectly conducting solid shield and ground plane; (3) compute the tangential E- and H-fields induced on S_i when the aperture is opened, by a perturbation technique, for insertion into the formula for E_z derived in step (1).

The advantages of this procedure are twofold; we relate E_z , across the circuit Z of interest, directly to the EMP fields on S_i without the need for laborious calculations within volume V , and in addition we compute the fields on S_i accurately according to the properties of the aperture, independent of the shape of the shield at distant points. Of course we must accurately compute the normal-E and tangential-H fields on the S_i -surface before the aperture is cut, and these fields depend strongly on the shape of the entire shield as well as reflections from the ground plane. We shall specify the determination of these fields on the shorted aperture by variational means.

Our first step is to relate E_z across Z inside V (Fig. 3) to the aperture fields on S_i . Volume V corresponds to V_0 of Fig. 1, in which there is now no inner region V_1 , and surface S_0 corresponds to S_i plus the perfectly conducting shield and ground surface. In Eq. (5) we let the "a" excitations correspond to the distant EMP current dipole (E), and the "b" excitations be due to current generator I_b across Z . Then Eq. (5) reads:

$$\int_{S_i} dS_i \vec{n}_s \cdot \{ \vec{E}_a \times \vec{H}_b - \vec{E}_b \times \vec{H}_a \} = -(E_a)_Z I_b. \quad (37)$$

Denoting, for emphasis, the "a" excitations by superscript E for EMP and the "b" excitations by superscript m for measured, and setting $I_b = 1$ for E_z at Z , we have:

$$E_Z^{(E)} = \int_{S_i} dS_i \hat{n}_s \cdot \{ \bar{E}^{(m)} \times \bar{H}^{(E)} - \bar{E}^{(E)} \times \bar{H}^{(m)} \}, \quad (37')$$

in which tangential $\bar{E}^{(m)}$ may be conveniently measured over the hole by a small electric dipole probe, and tangential $\bar{H}^{(m)}$ by a small loop.

Our next step is to calculate, reasonably accurately, the EMP-induced tangential $-\bar{H}^{(E)}$ and tangential $-\bar{E}^{(E)}$ fields on S_i . As mentioned earlier, it is convenient to calculate them as perturbations of the tangential- $\bar{H}$ and normal- $\bar{E}$ present on the aperture before the hole is cut. It will suffice to compute these fields (denoted $\bar{H}_0$ and $\bar{E}_0$) at the geometrical center of the aperture, and we insert magnetic and electric currents, respectively, as shown in Fig. 4 to measure these fields. All excitations now lie in the

Fig. 4. Test generators $\bar{I}_b$ and $\bar{I}_b^*$ on the closed aperture, inserted to measure the EMP-induced fields $\bar{E}_0$ and $\bar{H}_0$ there.

volume V_0 of Fig. 1, outside the shield and also above the ground plane. $I_b = 1$, now generating "b" fields in V_0 , measures $\bar{E}_0 \equiv \bar{E}_a$ due to the EMP dipole $\bar{I}^E$ (which generates the "a" fields), and since S_i is everywhere a perfect conductor, Eq. (21) now yields[†]:

$$\begin{aligned} E_0^S \big|_{I_b=1} &= \bar{n}_i \cdot (\bar{E}_0 - \bar{E}_0^i) \\ &= \int_{S_i} \bar{E}_b^i \cdot \bar{K}_a dS = \langle \bar{E}_b^i, \bar{K}_a \rangle. \end{aligned} \quad (38)$$

For accuracy, we can regard the enclosure and ground as a perfectly conducting scatterer corresponding to volume V_1 of Fig. 2 and write the variational expression Eq. (22) for $\bar{E}_0^S$, the scattered component of $\bar{E}_0$ on the aperture:

$$E_0^S = -\langle \bar{E}_b^i, \bar{K}_a \rangle \langle \bar{E}_a^i, \bar{K}_\beta \rangle \langle \bar{E}_\beta^S, \bar{K}_a \rangle \quad (39)$$

where

$\bar{E}_b^i$ is generated by $I_b = 1$ (perpendicular to the aperture) directly in free space,

$\bar{K}_a$ is the estimated electric surface current generated by $\bar{I}^E$,

$\bar{E}_a^i = \bar{E}_0^i$ in free space,

[†]Eq. (5), which says $E_0 \big|_{I_b=1} = \langle \bar{E}_b, \bar{I}^E \rangle$ is not convenient because we don't know accurately the net $\bar{E}_b$ radiated by $\bar{I}_b$ at the aperture.

$\bar{K}_\beta$ = estimated electric surface current generated by $\bar{I}_b = \bar{I}$, and

$\bar{E}_\beta^S$ = electric field generated directly by $\bar{K}_\beta$.

The details of the best procedure for evaluating Eq. (36) conveniently and accurately will be discussed at a later date.

We must also insert magnetic current $\bar{I}_b^* = \bar{I}^*$ at the center of the aperture to measure the magnetic field present before cutting the hole. (If this is placed on the aperture it creates only a tangential magnetic field and a normal electric field in its vicinity and upsets the boundary conditions only slightly on an essentially flat aperture). The only essential difference between magnetic-current excitation of the "b" fields in Fig. 4 and the foregoing electric current excitation is in the field being sampled; i.e., $\bar{H}_{\text{tan}}$ on the aperture instead of $\bar{E}_{\text{normal}}$. Consequently, the following changes are made in the preceding formulas:

In Eq. (5), $-\langle \bar{E}_a, \bar{J}_b \rangle$ becomes $+\langle \bar{H}_a, \bar{J}_b^* \rangle$.

In Eq. (6), $-\bar{E}_a \cdot \bar{I}_b(\bar{r}_R)$ becomes $+\bar{H}_a \cdot \bar{I}_b^*(\bar{r}_R)$.

In Eq. (7), $-\bar{E}_a^i \cdot \bar{I}_b$ becomes $+\bar{H}_a^i \cdot \bar{I}_b^*(\bar{r}_R)$.

In Eq. (8), $\bar{E}_a^S \cdot \bar{I}_b$ becomes $-\bar{H}_a^S \cdot \bar{I}_b^*(\bar{r}_R) = \langle \bar{E}_b^i, \bar{K}_a \rangle$.

Equation (14) remains valid so $\langle \bar{E}_b^i, \bar{K}_a \rangle = \langle \bar{E}_a^i, \bar{K}_b \rangle$.

Then Eq. (17), $\bar{E}_b^S \cdot \bar{I}_a$ becomes $-\bar{H}_b^S \cdot \bar{I}_a^*(\bar{r}_T) = \langle \bar{E}_a^i, \bar{K}_b \rangle$.

In Eq. (21), $\bar{E}_a^S$ becomes $-\bar{H}_a^S = \langle \bar{E}_b^i, \bar{K}_a \rangle$.

In Eq. (22), $\bar{E}_\alpha^S$ becomes $-\bar{H}_\alpha^S$ on the left side of the variational expression.

In Eq. (23), $\bar{E}_a^S, \delta \bar{E}_a^S$ becomes $-\bar{H}_a^S, -\delta \bar{H}_a^S$.

Equations (24) through (33) remain valid because they are independent of the manner in which the responses $\bar{E}_b$, $\bar{H}_b$, $\bar{K}_b$ (and $\bar{K}_b^*$) are generated; i.e., whether by electric source current $\bar{I}_b$ or magnetic source current $\bar{I}_b^*$.

Therefore we have deduced that Eq. (39) is a variational expression for $-\bar{H}_0^S$ instead of $\bar{E}_0^S$, thus giving:

$$-\bar{H}_0^S \cdot \bar{I}_b^*(\bar{r}_R) = -\langle \bar{E}_b^i, \bar{K}_\alpha \rangle \langle \bar{E}_a^i, \bar{K}_\beta \rangle / \langle \bar{E}_\beta^S, \bar{K}_\alpha \rangle \quad (40)$$

where

$\vec{E}_b^i$ is generated by $\vec{I}_b^*$ at the aperture, into free space,

$\vec{I}_a$ and $\vec{E}_a^i$ are the same as for Eq. (39),

$\vec{K}_\beta$ is the estimated electric surface current generated by $\vec{I}_b^*$ lying on S_i , and

$\vec{E}_\beta^s$ is the electric field generated directly by this $\vec{K}_\beta$.

The details of calculating efficiently two orthogonal components of $\vec{H}_b^s$ on the aperture, corresponding to two directions of $\vec{I}_b^*$, will be discussed in the future.

After we complete step (2) of the procedure and compute $\vec{E}_o$ and $\vec{H}_o$ on the aperture before it is cut, we can compute the tangential components $\vec{H}^{(E)}$ and $\vec{E}^{(E)}$ due to the EMP after the aperture is cut [for Eq. (37')] by which the internal EMP field $\vec{E}_Z^{(E)}$ is evaluated. The evaluation of the first portion of Eq. (37') by the integral

$$\mathcal{J}_1 = \int_{ap} dS_i \vec{n}_s \cdot [\vec{E}^{(m)} \times \vec{H}^{(E)}] \quad (41)$$

is simple because, if the aperture is small compared to a wavelength, $\vec{H}^{(E)}$ at the hole is cut is nearly $\frac{1}{2} \vec{H}_o$ present before it is cut. Hence:

$$\mathcal{J}_1 \simeq \frac{1}{2} \int_{ap} dS_i \vec{n}_s \cdot [\vec{E}^{(m)} \times \vec{H}_o]. \quad (41')$$

The second portion of Eq. (37') may be expressed in terms of the polarizations $\vec{M}(\vec{H}_o)$ and $\vec{P}(\vec{E}_o)$ for the aperture; these are known for small circular and elliptical apertures and may be reasonably approximated for other shapes. It may be shown¹⁵ that the second integral of Eq. (37') may be written as

$$\mathcal{J}_2 = \int_{ap} dS_i \vec{n}_s \cdot [\vec{E}^{(E)} \times \vec{H}^{(m)}] = \frac{1}{2} i\omega \left[\mu_o \vec{M} \cdot \vec{H}^{(m)}(0) - \vec{P} \cdot \vec{E}^{(m)}(0) \right], \quad (42)$$

where $\vec{H}^{(m)}(0)$ and $\vec{E}^{(m)}(0)$ are measured at the geometrical center of the aperture and are assumed to be uniform over it (long wavelength approximation).

6. Conclusion

Using a somewhat more rigorous analysis than is usually employed, we have verified in Sec. 3 many of the common expressions for an electric field scattered from a perfectly conducting obstacle. The scattered field is expressed by an integral of an

incident field times the correct surface electric current over the obstacle, despite the presence of a magnetic surface current. In Sec. 4 we verified a common variational expression for scattered electric field in terms of electric surface currents only. Those currents should be self-consistent with the associated surface magnetic currents also required to satisfy proper surface boundary conditions.

We have outlined, in Sec. 5, an efficient program for closely estimating the EMP voltages generated across sensitive circuit elements within well-shielded enclosures using reciprocity and variational formulas for scattered fields. The computational problem of relating an EMP circuit voltage to EMP fields on the surfaces of apertures in the shield has been obviated by expressing that voltage as an integral over the apertures of measured test-generator fields times the EMP fields. The EMP fields in the open apertures may be closely estimated from the EMP fields present in the electrically shorted apertures. These latter fields may be calculated from the variational expression for an electric field scattered on each aperture by an EMP-equivalent dipole, in terms of self-consistent electric surface currents on the shield, and from the analogous variational expression for a magnetic field scattered on each (electrically shorted) aperture.

The details of calculating the EMP fields induced on electrically shorted apertures by placing normal electric and tangential magnetic current test generators on them are not included herein but have been considered by other, notably J. R. Wait and R. F. Harrington.

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