The Pennsylvania State University

The Graduate School

Department of Nuclear Engineering

Steady State Solutions of an Ion Injection
Spherical Electrostatic Confinement Device
with Coexisting Beam and Trapped Particles

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LIST OF SYMBOLS AND ABBREVIATIONS

Subscript

bi = beam ion
 be = beam electron
 ti = trapped ion
 te = trapped electron

R.A. = Real Anode

R.C. = Real Cathode

V.A. = Virtual Anode

V.C. = Virtual Cathode

T.I. = Trapped ion

T.E. = Trapped Electron

B.I. = Beam Ion

B.E. = Beam Electron

Region I = region between the virtual anode and real cathode

Region II = region between the virtual cathode and virtual anode

K_+ : normalized beam ion current parameter = $\frac{I_{bi}}{V_0^{3/2}} \left(\frac{2m_i}{e} \right)^{1/2}$
 K_- : normalized beam electron current parameter = $\frac{I_{be}}{V_0^{3/2}} \left(\frac{2m_e}{e} \right)^{1/2}$
 λ_+ : mass weighted beam electron to beam ion current ratio = $\frac{K_-}{K_+} = \frac{I_{be}}{I_{bi}} \left(\frac{m_e}{m_i} \right)^{1/2}$
 C_+ : normalized trapped ion parameter
 C_- : normalized trapped electron parameter
 $V(r)$: electrostatic potential
 $\phi(r)$: normalized electrostatic potential = $|V(r)/V_0|$
 α : ion current amplification factor = $(I_{bi} + I_{ti})_{v.c.} / (I_{bi})_{r.c.}$
 J_{-s} : inward current density of species s
 J_{+s} : outward current density of species s
 M_- : trapped electron parameter for cut-off Maxwellian distribution

ABSTRACT

Theoretical solutions for the potential, density and pressure are obtained for a spherical electrostatic fusion confinement device using ion injection. The charged particles in the device are assumed to consist of monoenergetic beam particles and distributed trapped particles, both with purely radial motions. Self-developed potential wells, as deep as the negative voltage applied to the cathode, exist for various combinations of beam particles and trapped particles. The self-developed well of primary interest contains a virtual anode (i.e., a point at which the potential equals that of the grounded anode) inside the cathode, and a virtual cathode (where the potential equals that of the real cathode) inside the virtual anode. Beam, but not trapped, particles establish these virtual electrodes. Ions can be trapped in the potential wells on either side of the virtual anode while electrons are trapped between the real cathode and the virtual cathode. The trapped particle distributions are taken as uniform in speed space. Various ratios of trapped to beam particle currents are studied to determine the effects of trapped particles on potential shape and on ion current amplification (i.e., the ratio of the ion current at the virtual cathode to the input ion current.) The following effects are noted: (1) Trapped electrons cause the virtual anode to move toward the center; (2) Trapped ions inside the virtual anode cause the virtual cathode to move

toward the center; (3) Trapped ions outside the virtual anode cause the virtual anode to move outward; (4) The latter effect provides a basis for explaining a possible virtual anode found experimentally by Hirsch; and (5) Current amplification is less than 1.5 if there are no trapped ions outside the virtual anode, but can be large when such ions are present.

1. INTRODUCTION

Hirsch¹⁾, using a spherical inertial electrostatic confinement device with ion injection, detected appreciable neutron fluxes from the central region of that device. These neutrons suggested that a significant fusion rate was occurring that required ion densities in the center of approximately 10^{13} ions/cm³ or larger. It also suggested that the ion current in the center was at least 10^4 times the input ion current (i.e., there was a current amplification of at least 10^4). How could such current be present? One possibility is that a deep potential well developed in the center and there was large ion trapping in this well. Since no external magnetic field is applied, this confinement will automatically be exempted from the magnetic instabilities, and hence is a very desirable fusion confinement device. Let us consider how such a well might be formed. A simplified model will be considered:

Ions are released uniformly with zero kinetic energy from the grounded outer radius, i.e., the anode (see Fig. 1). They are then accelerated toward the center by the highly negatively biased inner sphere, the cathode, which is assumed permeable to ions. Space charges are thus accumulating around the center until further incoming ions are stopped at some radius r_a by the big repulsive force due to the ion space charges. Since the ions have no kinetic

Figure 1 Operation of an Electrostatic Confinement Device

energy at r_a , the electric potential at r_a must be the same as the anode. This is hence called a virtual anode. If electrons, at this stage, are released from the cathode, they will be accelerated by the ion space charges and penetrate through them to form a virtual cathode by the same arguments. (See Figure 2).

During the build-up of the virtual cathode, according to Barnes' computer simulation²⁾, an electron instability develops which greatly randomizes and traps the beam electrons while leaving the ion beam only slightly affected. This is because the instability occurs during a much shorter time scale (the electron transit time) than the ion time scale (the ion transit time). This process is believed the main process for electron trapping, although we neglect those possibly important processes like charge exchange, ionization and collisions.

Barnes found that under certain conditions quasi-stationary ion-trapping deep wells were produced. However, substantial ion trapping did not occur and the deep well ultimately washed out. If trapping did occur, possibly as a result of collisions and charge exchange, which were not included in the computer simulation, a potential structure like that shown in Fig. 2 might result. Hirsch's neutron and X ray data suggest this type of potential well.

Theoretical studies on this device^{1),3),4)} are either for the pure beam case^{1),3)}, i.e., assuming

Figure 2 Potential Profile Showing Trapped Particles

electron and ion distributions are radial and monoenergetic, or for trapped particles without beams⁴⁾. No big current amplification, which is essential to the feasibility as a fusion device, was found. A more reasonable choice based on electron instabilities during start-up, as well as collisions, charge exchange, etc., would be a solution in which there was co-existence of beam particles and trapped particles. The present work will be devoted to the study of this case.

In Chapter II we develop the concept of expressing a distribution function in terms of constants of motion, and its application to the collisionless Boltzmann Equation and Poisson's Equation.

In Chapter III we set up the model for a collisionless, steady state electrostatic confinement device with purely radially trapped charged particles present. Also discussed are all parameters of interest and various derived quantities.

In Chapter IV, the results of numerical calculations are reported. Chapter V bears the discussion of the results. Chapter VI, an essential part of the present study, contains a theoretical investigation of whether an upper bound for ion current amplification exists. Also included in this chapter is a theoretical model which explains for the first time certain features of Hirsch's experimental results. Chapter VII contains a summary of the whole study and suggestions for further work.

II. THEORIES

In this chapter the constants of the motion associated with the electrostatic device are obtained and the electron and ion distribution functions are written in terms of these constants of the motion. Density and current are then obtained as integrals over the constants of the motion. The method is well developed by Bernstein⁶⁾.

Consider the specific but rather general case, where there are three constants of the particle motion E_1 , E_2 , E_3 and the constants of motion are such that the velocity components V_1 , V_2 , V_3 can be expressed in terms of them, i.e.,

$$V_i = V_i(E_1, E_2, E_3) \quad i=1,2,3$$

Then the usual distribution function $f(\underline{r}, \underline{v}, t)$ can be transformed to $F(\underline{r}, E_1, E_2, E_3)$ by

$$F(\underline{r}, E_1, E_2, E_3, t) \equiv f(\underline{r}, \underline{V}(E_1, E_2, E_3), t)$$

where f and F satisfy

$$N(\underline{r}, t) = \int f(\underline{r}, \underline{v}, t) d^3v \quad (2.1a)$$

$$= \int F(\underline{r}, E_1, E_2, E_3, t) \frac{\partial(V_1, V_2, V_3)}{\partial(E_1, E_2, E_3)} dE_1 dE_2 dE_3 \quad (2.1b)$$

This equation for $n(\underline{r}, t)$ is used in Poisson's equation

$$\nabla^2 V(\underline{r}, t) = -4\pi \sum_s N_s(\underline{r}, t) q_s \quad (2.2)$$

to form a self-consistent problem.

Assume the mean free path for collisions is long so that the collisionless Boltzmann equation is applicable. At steady state, f must satisfy

$$\frac{Df}{Dt} = (\underline{v} \cdot \nabla_{\underline{r}})f + (\underline{a} \cdot \nabla_{\underline{v}})f = 0 \quad (2.3)$$

where $\frac{D}{Dt}$ is the substantial derivative, i.e., the differential along the particle path. Equivalently, we require

$$\frac{DF}{Dt} = (\underline{v} \cdot \nabla_{\underline{r}})F + \sum_{i=1}^3 \frac{\partial f}{\partial E_i} \dot{E}_i = 0 \quad (2.4)$$

To solve the self-consistent problem (2.1a), (2.2), (2.3) with appropriate boundary conditions, we need first to solve (2.1a) for f . It is this first step that is difficult or even unsolvable in most cases. However the same problem can be solved by the set (2.1b), (2.2) and (2.4). The first step in this case is much easier.

Notice in Eq. (2.4), the $\dot{E}_i$'s are identically zero, hence the only requirement for F is

$$(\underline{v} \cdot \nabla_{\underline{r}})F(\underline{r}, E_1, E_2, E_3) = 0$$

This is required at every $\underline{r}$, hence the only requirement becomes that F be explicitly independent of $\underline{r}$.

We can thus start with an arbitrary set of functions $F_s(E_1, E_2, E_3)$ (of course with some physical intuition) to calculate the number densities (2.1b), then substitute them into Poisson's equation to solve for $V(\underline{r})$ with the appropriate boundary conditions. Once $V(\underline{r})$ can be solved and all the boundary conditions can be satisfied, the whole problem is self-consistent, hence the F 's so chosen will be an acceptable set. We can thus calculate all the derived quantities of F 's, such as particle densities, particle pressures, and current densities.

2-1. DISTRIBUTION FUNCTION IN TERMS OF CONSTANTS OF MOTION

In a sphere with radially symmetric electric potential, the constants of motion satisfied by charged particles are:

i. total energy	$E = \frac{1}{2}mv^2 + qV(r)$
ii. angular momentum Z-axis	$Q = mr^2 \sin^2 \theta \dot{\phi}$
iii. total angular momentum	$J = mr^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)^{1/2}$

which implies:

$$\begin{aligned}\dot{\phi} &= Q / mr^2 \sin^2 \theta \\ \dot{\theta} &= \pm \left[\left(\frac{J}{mr^2} \right)^2 - \left(\frac{Q}{mr^2 \sin^2 \theta} \right)^2 \right]^{1/2} \\ \dot{r} &= \pm \left[\frac{2(E - qV(r))}{m} - \frac{J^2}{m^2 r^2} \right]^{1/2}\end{aligned}\quad (2.5)$$

Thus, once E , Q , J and $V(r)$ are specified at a point, the trajectory of that particle is completely determined.

The distribution function $f(\underline{r}, \underline{v}, t)$ is generally a function of time and six other independent variables. At steady state, if it is assumed that f is spherically symmetric in configuration space and azimuthally symmetric in velocity space about the position vector $\underline{r}$, then $f(\underline{r}, \underline{v}, t)$ reduces to a function of 3 variables, viz, r , v , θ_v , where θ_v is the angle between $\underline{r}$ and $\underline{v}$, (see Fig. 3). It should be noted that on Fig. 3 θ , ϕ and μ ($\mu = \cos\theta$) refer to coordinates in configuration space, while θ_v , ϕ_v and μ_v ($\mu_v = \cos\theta_v$) refer to velocity space.

For the present case, v and μ_v can be replaced by the two constants of motion: total energy E and angular momentum J , where

$$\begin{aligned} E &= \frac{1}{2} m v^2 + q V(r) \\ J &= |m \underline{v} \times \underline{r}| = m r v (1 - \mu_v^2)^{1/2} \end{aligned} \quad (2.6)$$

We can hence do the following transformation:

$$F(E, J) = f(r, v(E, J), \mu_v(E, J))$$

Notice that $\pm \mu_v$ give the same J , so we should express F as:

$$F = F_+(E, J) h(\mu_v) + F_-(E, J) h(-\mu_v) \quad (2.7)$$

Figure 3 Coordinate System in Configuration Space and Velocity Space

where $h(\mu_v)$ is the Heaviside step function, and $\pm$ corresponds to particles with same J but $\pm\mu_v$, i.e., particles moving outward and inward respectively.

Now we can calculate the inward and outward current densities $J_-(r)$, and $J_+(r)$:

$$J_-(r) \equiv \int_{\mu < 0} q f(r, \underline{v}) \underline{v} \cdot \underline{\mu} d^3v = \frac{2\pi q}{m^3 r^2} \int_{g_V(r)}^{\infty} \int_0^{mrv(E)} J F_-(E, J) dE dJ \quad (2.8)$$

$$J_+(r) \equiv \int_{\mu > 0} q f(r, \underline{v}) \underline{v} \cdot \underline{\mu} d^3v = \frac{2\pi q}{m^3 r^2} \int_{g_V(r)}^{\infty} \int_0^{mrv(E)} J F_+(E, J) dE dJ \quad (2.9)$$

For steady state

$$J_+(r) = J_-(r) \quad , \quad \text{i.e.,} \quad \iint (F_+ - F_-) dE dJ = 0$$

which holds at any r , implying that

$$F_+(E, J) = F_-(E, J)$$

Thus

$$F_+ = F_- = F/2 \quad (2.10)$$

With Eq. (2.10), we can calculate the number density:

$$\begin{aligned} N(r) &= \int f(r, \underline{v}) d^3v = 2\pi \int_0^{\infty} dv \int_{-1}^{+1} f(r, v, \mu_v) v^2 dv d\mu_v \\ &= 2 \cdot \left[2\pi \int_0^{\infty} dv \left(\int_0^1 f(r, v, \mu_v) v^2 d\mu_v \right) \right] \\ &= 4\pi \iint F_+(E, J) \frac{\partial(v, \mu_v)}{\partial(E, J)} v^2(E) dE dJ \\ &= \frac{4\pi}{m^{5/2} r^2} \int_{g_V(r)}^{\infty} \int_0^{mrv(E)} \frac{F_+(E, J) J dE dJ}{\left(1 - \frac{J^2}{m^2 r^2 v^2}\right)^{1/2} (2(E - g_V(r)))^{1/2}} \quad (2.11) \end{aligned}$$

The distribution function F is contributed by two parts; viz., F_b , representing monoenergetic beam particles, and F_t , representing trapped charged particle in the potential well. The beam part satisfies the boundary condition

$$I_s = \lim_{r \rightarrow r_c} 4\pi r^2 J_-(r)$$

where r_c is the radius of the real cathode, and I_s is the inward current of species s at the real cathode.

Let $F_b(E, J) = B_{bs} \delta(E - E_0)$, in which E_0 is the initial releasing energy. Using the definition of current (2.8), we find

$$B_{bs} = \frac{2 m_s^3 I_s}{8\pi^2 |q|} \quad (2.12)$$

2-2. APPLICATION TO POISSON'S EQUATION

With the particle densities obtained in Section 2-1, Poisson's equation reads:

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} V(r) = -4\pi e (N_{bi} + N_{ti} - N_{be} - N_{te}) \quad (2.13)$$

where

$$N_{bi}(r) = \frac{2 I_i m_i}{4\pi r^2 e} \frac{1}{[2(E_i - eV(r))]^{1/2}} \quad (2.14a)$$

$$N_{ti}(r) = \frac{2\pi}{m_i^{1/2} r^2} \int_{eV(r)}^{\infty} \int_0^{m_i r V(E)} \frac{F_{ti}(E, J) dE dJ}{(1 - \frac{J^2}{m_i^2 r^2 V^2})^{1/2} (2(E_i - eV))^{1/2}} \quad (2.14b)$$

$$N_{be}(r) = \frac{2I_e m_e}{4\pi r^2 e} \frac{1}{[2(E_e + eV(r))]^{1/2}} \quad (2.14c)$$

$$N_{te}(r) = \frac{2\pi}{m_e^{1/2} r^2} \int_{eV(r)}^{\infty} \int_0^{m_e r V(E)} \frac{F_{te}(E, J) dE dJ}{\left(1 - \frac{J^2}{m_e^2 r^2 v^2}\right)^{1/2} (2(E_e + eV))^{1/2}} \quad (2.14d)$$

The first and third terms represent the beam ion and electron densities, while the second and fourth represent the trapped ion and electron densities. Equation (2.13) is a very complicated differential-integral equation that can only be solved numerically.

Bernstein has pointed out⁶⁾ that for a series of potential wells (Fig. 4), the functional forms of the same species of trapped particles in the different wells are independent; however, in a single well it must be unchanged. Since the potential energies experienced by ions and electrons are just the reverse versions of each other, the potential wells that trap electrons and ions are different. Hence the changes of functional form for ions and electrons occur at different places, as shown in Fig. 4. For example, the functional form for trapped ions is unchanged in region A-C, while the electron distribution is unchanged in region B-D. The ion distribution can change at point C, while the electron distribution can be different on either side of point D. Thus in region A-B, if we specify the shape of the potential profile, the amount of beam ions, beam electrons and

Figure 4 Potential Profile Showing Potential Wells for Electrons and Ions

trapped electrons, the functional form for trapped ions in this region can be found. This ion distribution can in turn be extended to region B-C to find the trapped electrons, and so forth.

Alternatively if one specifies the amount of beam ions, beam electrons, trapped ions, and trapped electrons, the shape of each well can be calculated. The latter technique was used throughout the present calculations.

2-3. SUMMARY OF THE PURE BEAM CASE

(1). The pure beam case, i.e., $N_{ti} = N_{te} = 0$, has previously been studied^{1),3)}. The general characteristics are summarized as follows:

For a given input ion current, there is a maximum allowed electron current corresponding to the space-charge-limited case. This implies the condition $\frac{dV}{dr} = 0$ at the cathode. Figure 5 shows the region of allowed electron currents for given ion current. The results are presented in terms of the normalized parameters K_+ , λ_+ which are defined by

$$K_+ \equiv \frac{I_i}{V_0^{3/2}} \left(\frac{2m_i}{e} \right)^{1/2}, \quad \lambda_+ \equiv \frac{I_e}{I_i} \left(\frac{m_e}{m_i} \right)^{1/2}$$

The shaded area is the allowed region for forming a virtual anode and the curve separating the regions is the space-charge-limited solution.

(2). A virtual cathode is formed inside the virtual anode. The formation of the virtual cathode draws part of the input ion current across the virtual anode. The amount drawn is unique and is determined by choosing the correct ion current K_+^i so that the boundary conditions

$$\begin{aligned} V(r_{v.a.}) &= 0 & \left. \frac{dV}{dr} \right|_{r_{v.a.}} &= 0 \\ V(r_{v.c.}) &= 0 & \left. \frac{dV}{dr} \right|_{r_{v.c.}} &= 0 \end{aligned}$$

can be satisfied. Porter ³⁾ showed that K_+^i must be smaller than K_+ , hence there must be an ion current reduction across the virtual anode. The result is shown in Fig. 6. Hirsch estimated an ion current amplification¹⁾ of at least 10^4 is required to make this device feasible for fusion purposes. Figure 6 indicates that the ion current, in fact, greatly decreases, so that this pure beam model is very unsatisfactory for explaining the experimental results.

Figure 5 Electron Current vs Ion Current for Pure Beam Case

Figure 6 Transmitted Ion Beam across the Virtual Anode for Pure Beam Case

III. PRELIMINARIES FOR NUMERICAL CALCULATIONS

3-1. MODEL

Barnes' computer simulations²⁾ indicate that electrons are approximately uniformly trapped in speed space between the real cathode and the first virtual cathode while a negligible amount of ions are trapped between the real cathode and the virtual anode. As we will show in Chapter VI, large ion trapping between the real cathode and virtual anode has a profound effect on the operation of the device. However, we will build up to this case systematically and start with the simpler case of no ion trapping outside the virtual anode. Specifically, the model, with the help of Fig. 7, can be stated as follows:

- (1). The radius $r_{R.A.}$ is the grounded anode, where the ion beam is uniformly released with zero kinetic energy.
- (2). The radius $r_{R.C.}$ is the highly negatively biased cathode where the electron beam is uniformly released with zero kinetic energy.
- (3). Inside the real cathode, a virtual anode is formed at $r_{V.A.}$.
- (4). Inside the virtual anode, a virtual cathode is formed at $r_{V.C.}$.
- (5). Electrons are uniformly trapped in speed space between the real cathode and the virtual cathode.
- (6). Ions are only trapped between the virtual anode

Figure 7 Model for Theoretical Analysis and Numerical Calculations

and the virtual cathode. (After investigating this case, we will study the case when the trapped ions are also added in this region.)

- (7). All particles move radially.
- (8). The cathode is transparent to the ions.
- (9). At the virtual electrodes, space charge limited conditions are satisfied, hence the electric field there must vanish.
- (10). The potential is radially symmetric.
- (11). The device is in steady state.

3-2. FORMULATION OF PROBLEM

The general form for the density of an azimuthally symmetric trapped particle distribution is:

$$N_t(r) = \int f_t(r, \underline{v}) d^3\underline{v} = 2\pi \int_{\mu_v=-1}^{+1} d\mu_v \int_0^\infty f_t(r, v, \mu_v) v^2 dv$$

The notion "uniform in speed space" means the integrand in front of dv is constant, while "radial" means f_t is described by $\delta(\mu_v \pm 1)$. So the distribution which is radial and uniform in speed space means

$$v^2 f_t(r, v, \mu_v) = \frac{\beta}{r^2} \left(\frac{\delta(\mu_v + 1)}{2} + \frac{\delta(\mu_v - 1)}{2} \right)$$

where β is some constant and $\frac{1}{r^2}$ is added for the common spherical effect. It can be shown that this distribution

function satisfies the Vlasov equation.

Let us now consider the transformation to the constants of the motion. In order to express $\delta(\mu_v \pm 1)$ in terms of $\delta(J)$ consider an integral I of the form

$$I = \int J \delta(\mu_v - 1) g(J) dJ$$

where μ_v and J are related by Eq. (2.6) i.e. $\mu_v = (1 - \frac{J^2}{m^2 r^2 v^2})^{1/2}$

If we define $y \equiv 1 - \mu_v$, then

$$\begin{aligned} I &= \int J(y) \delta(y) g(J(y)) \frac{dJ}{dy} dy \\ &= \int J(y) m^2 r^2 v^2 g(J(y)) \frac{\delta(y)}{J(y)} dy = m^2 r^2 v^2 g(0) \end{aligned}$$

Hence the operation of $\delta(\mu_v - 1)$ is equivalent to $\frac{\delta(J)}{J} m^2 r^2 v^2$.

Thus the transformation will be

$$\delta(\mu_v \pm 1) = \frac{\delta(J)}{J} m^2 r^2 v^2 \quad (3.2)$$

As discussed in chapter II, we can thus write

$$F_{\pm}(E, J) = \frac{\beta m^2}{2} \frac{\delta(J)}{J}$$

which is explicitly independent of r as it should be, and is automatically a solution to the Vlasov equation. It has the property of describing a trapped distribution that is radial and uniform in speed space. Hence, in later calculations, we use

$$F_{\pm}(E, J) = \frac{\beta}{2} \frac{\delta(J)}{J} [h(E + e|V_0|) - h(E + e|V|)] \quad (3.3)$$

where $V_0 = -|V_0|$ is the applied voltage, and $V = -|V|$ is the voltage at a position r .

Substituting Eq. (3.3) into Eqs. (2.7), (2.8) and (2.9), we obtain the density and current density for trapped electrons:

$$\begin{aligned} N_{te}(r) &= \frac{2\pi\beta_e}{m_e^{5/2}r^2} \int_{e|V|}^{e|V_0|} \frac{dE}{[2(E+eV)]^{1/2}} \\ &= \frac{2\pi\beta_e}{m_e^{5/2}r^2} [2e(|V_0|-|V|)]^{1/2} \end{aligned} \quad (3.4)$$

$$J_{+te} = J_{-te} = \frac{\pi e^2 \beta_e}{m_e^3 r^2} [|V_0| - |V|] \quad (3.5)$$

With the density so derived, we can start to investigate the region between the virtual anode and the real cathode.

From now on, we will use the following notations for simplicity:

R.A. = real anode
 R.C. = real cathode
 V.A. = virtual anode
 V.C. = virtual cathode
 T.I. = trapped ions
 T.E. = trapped electrons
 B.I. = beam ions
 B.E. = beam electrons
 Region I = region between V.A. and R.C.
 Region II = region between V.C. and V.A.

The calculation will be performed separately in Region I and II.

In Region I, the problem to be solved is the

solution of Eqs. (2.13) and (2.14) with appropriate boundary conditions. Specifically, we need to solve

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} V(r) = & -4\pi e \left[\frac{2 I_{bi}^I (m_i)^{1/2}}{4\pi r^2 e} \frac{1}{[2(-eV)]^{1/2}} \right. \\ & - \frac{2 I_{be} (m_e)^{1/2}}{4\pi r^2 e} \frac{1}{[2(-eV_0 - eV)]^{1/2}} \\ & \left. - \frac{2\pi \beta_e}{m_e^{5/2} r^2} (2e(|V_0| - |V|))^{1/2} \right] \end{aligned} \quad (3.7)$$

with the boundary conditions

$$V_{\text{cathode}} = -|V_0| = \text{Applied Voltage}$$

$$V(r_{\text{v.a.}}) = 0$$

$$\left. \frac{dV}{dr} \right|_{r_{\text{v.a.}}} = 0$$

Change variable

$$\xi \equiv \ln \frac{r}{r_{\text{v.a.}}}$$

and define

$$\phi \equiv \frac{V(r)}{V_0}$$

$$K_+^I \equiv \frac{I_{bi}^I}{|V_0|^{3/2}} \left(\frac{2m_i}{e} \right)^{1/2}$$

$$K_- \equiv \frac{I_{be}}{|V_0|^{3/2}} \left(\frac{2m_e}{e} \right)^{1/2}$$

$$\lambda_+^I \equiv \frac{K_-^I}{K_+^I} = \frac{I_{be}}{I_{bi}^I} \left(\frac{m_e}{m_i} \right)^{1/2}$$

$$C_- \equiv \frac{8\pi^2 e \beta_e}{m_e^2} \left(\frac{2e}{m_e |V_0|} \right)^{1/2}$$

(3.8)

Equation (3.7) then reduces to

$$\begin{aligned} \frac{d}{d\xi} (e^{2\xi} (\frac{d\phi}{d\xi})^2) &= 2e^{2\xi} \left[\frac{K_+^I}{\phi^{1/2}} - \frac{K_-}{(1-\phi)^{1/2}} - C_-(1-\phi)^{1/2} \right] \\ \phi(\xi = \ln \frac{r_{K.A.}}{r_{V.A.}}) &= 1 \\ \phi(\xi = 0) &= 0 \\ \frac{d\phi}{d\xi} \Big|_{\xi=0} &= 0 \end{aligned} \quad (3.9)$$

Equation (3.9) is the problem to be solved in Region I.

In Region II, trapped ions are introduced, so that

$$F_{ti}(E, J) = \frac{\beta_i^{\text{II}} \delta(J)}{J} [h(E + e|V|) - h(E)] \quad (3.10)$$

and

$$N_{ti}(r) = \frac{2\pi\beta_i^{\text{II}}}{m_i^{5/2} r^2} \int_{-e|V|}^0 \frac{dE}{[2(E - eV)]^{1/2}} = \frac{2\pi\beta_i^{\text{II}}}{m_i^{5/2} r^2} (2e|V|)^{1/2} \quad (3.11)$$

Defining quantities similar to those given in

Equation (3.8), we obtain, in Region II,

$$\frac{d}{d\xi} (e^{2\xi} (\frac{d\phi}{d\xi})^2) = 2e^{2\xi} \left[\frac{K_+^{\text{II}}}{\phi^{1/2}} + C_+^{\text{II}} \phi^{1/2} - \frac{K_-}{(1-\phi)^{1/2}} - C_-(1-\phi)^{1/2} \right] \quad (3.12)$$

where

$$C_+^{\text{II}} = \frac{8\pi^2 e}{m_i^2} \left(\frac{2e}{m_i |V_0|} \right)^{1/2} \beta_i^{\text{II}} \quad \xi \equiv \ln \frac{r}{r_{V.C.}}$$

The boundary conditions are:

$$\begin{aligned} \phi(\xi = 0) &= 0 & \frac{d\phi}{d\xi} \Big|_{\xi=0} &= 0 \\ \phi(\xi = \ln \frac{r_{V.A.}}{r_{V.C.}}) &= 0 & \frac{d\phi}{d\xi} \Big|_{\xi = \ln \frac{r_{V.A.}}{r_{V.C.}}} &= 0 \end{aligned}$$

Notice that K_- and C_- are the same in Regions I and II, so they are not superscripted.

3-3. PARAMETERS OF INTEREST

3-3-1. Parameters of Interest in Region I

In Equation (3.9) there are 3 independent parameters K_+^I , K_- and C_- which are related to the particle densities in Region I. The usual procedure for calculation is as follows: (a). fix K_+^I , (b). choose K_- (which is restricted to be smaller than $K_{-max} \equiv K_+^I \times \lambda_{+max}^I$, where λ_{+max}^I is given in Figure 5), (c). find the maximum C_- that still can satisfy the equation and all boundary conditions.

3-3-2. Parameters of Interest in Region II

In Region II, there are four parameters related to particle densities, viz, K_+^{II} , K_- , C_+^{II} , C_- . The usual calculational procedure is that, with the same K_- and C_- used in Region I, vary C_+^{II} to find the K_+^{II} that can satisfy Eq. (3.5) until some C_{+max} is reached. Beyond this C_{+max} no K_+^{II} exists which will satisfy the boundary conditions of Eq. (3.5).

3-3-3. Ion Current Amplification Factor

The inward trapped ion current density J_{-ti} is obtained by substituting Eq. (3.11) into Eq. (2.8).

We obtain

$$J_{-ti} = \frac{\pi e^2 |V_0|}{m_i^3 r^2} \beta_i \phi(r) \quad (3.13)$$

The inward trapped ion current I_{-ti} is then

$$I_{-ti} = 4\pi r^2 J_{-ti} = 4\pi^2 \beta_i \left(\frac{e^2 |V_0|}{m_i^3} \right) \phi \quad (3.14)$$

The ion current amplification factor is defined as

$$\alpha \equiv \frac{I_{ti}^{\pi} \max + I_{bi}^{\pi}}{I_{bi}^{\pi}} \quad (3.15)$$

By putting

$$K_+ = \frac{I_{bi}}{|V_0|^{\frac{1}{2}}} \left(\frac{2 m_i}{e} \right)^{\frac{1}{2}}$$

and using Equations (3.14) and (3.12) (for β_i in terms of C_+^{II})

$$\alpha = \frac{C_+^{\pi}}{2 K_+^{\pi}} \phi_{\max} + \frac{K_+^{\pi}}{K_+^{\pi}} \quad (3.16)$$

3-4. DERIVED QUANTITIES

In an electrostatic confinement device, the following quantities interest us:

3-4-1. Partial Particle Density

This is a fundamental quantity, and we are interested in how much faster it varies than the $1/r^2$ variation, since this is a good indication of current amplification.

The equations of interest are (2.14), (3.4), (3.11), which give

$$\begin{aligned} N_{bi} &= \frac{V_0}{4\pi r^2 e} K_+ \phi^{-\frac{1}{2}} \\ N_{ti} &= \frac{V_0}{4\pi r^2 e} C_+ \phi^{\frac{1}{2}} \\ N_{be} &= \frac{V_0}{4\pi r^2 e} K_- (1-\phi)^{-\frac{1}{2}} \\ N_{te} &= \frac{V_0}{4\pi r^2 e} C_- (1-\phi)^{\frac{1}{2}} \end{aligned} \quad (3.17)$$

Where the definitions of K_+ , K_- , C_+ , C_- are given by (3.8) and (3.12).

Also of interest is the ratio of trapped particle density to beam particle density. This is expressed by the quantities

$$\pi_i(r) = \frac{N_{ti}(r)}{N_{bi}(r)} = \frac{C_+}{K_+} \phi(r) \quad (3.18a)$$

and

$$\pi_e(r) = \frac{N_{te}(r)}{N_{be}(r)} = \frac{C_-}{K_-} (1 - \phi(r)) \quad (3.18b)$$

3-4-2. Partial Particle Pressure and Electrostatic Pressure:

Here we are interested in knowing how we obtain a pressure balance, what pressures exist in the system, and what components are producing the pressure.

The general form for the particle pressure of species s in the radial direction is given by

$$\begin{aligned} P_{rr}s &= m_s \int v_r v_r f_s(r, v) d^3v \\ &= 2\pi m_s \int v^4 \mu^2 f_s(r, v, \mu) dv d\mu \\ &= 2\pi m_s \int_{ev(r)}^{\infty} \int_0^{mrV(E)} v^3 \mu^2 \frac{J F_s(E, J)}{m^{5/2} r^2 (1 - \frac{J^2}{m^2 r^2 v^2})^{3/2}} \frac{dE dJ}{[2(E - qV)]^{1/2}} \end{aligned} \quad (3.19)$$

For B.I.,

$$F_{bi} = \frac{I_{bi} m_i^3 f(E) f(J)}{8\pi^2 e J}$$

Hence we obtain

$$P_{bi} = \frac{K_+ V_0^2}{2\pi r^2} \phi^{1/2} \quad (3.20)$$

Similarly

$$P_{be} = \frac{K - V_0^2}{2\pi r^2} (1 - \phi)^{1/2} \quad (3.21)$$

For T.I.,

$$F_{ti} = \frac{\beta_i \delta(\gamma)}{J} [h(E + e|V|) - h(E)]$$

Hence we obtain

$$\begin{aligned} P_{ti} &= \frac{2\pi m_i \beta_i}{3r^2} \left(\frac{2e|V|}{m_i} \right)^{3/2} \\ &= \frac{C + V_0^2}{6\pi r^2} \phi^{3/2} \end{aligned} \quad (3.22)$$

Similarly

$$P_{te} = \frac{C - V_0^2}{6\pi r^2} (1 - \phi)^{3/2} \quad (3.23)$$

Electrostatic Pressure:

$$\begin{aligned} P_{es} &= \frac{E^2}{8\pi} = \left(\frac{dV}{dr} \right)^2 / 8\pi \\ &= \frac{\left(\frac{d\phi}{dr} \right)^2 V_0^2}{8\pi r^2} \end{aligned} \quad (3.24)$$

The pressure point of view is also useful for establishing upper bound on amplification. Under some fairly general conditions that excluded trapped ions outside the virtual anode, Simon⁵⁾ showed that it would be difficult to obtain current amplification larger than 50. We have performed a more detailed analysis in Chapter VI, using the beam and trapped distributions already discussed, and

find an exact amplification upper bound of 1.5 if there are no trapped ions outside the virtual anode. However, when trapped ions are added outside the virtual anode, there is no upper bound. This is also discussed in Chapter VI.

IV. RESULTS ON NUMERICAL CALCULATIONS

Hirsch's¹⁾ and Porter's³⁾ pure beam cases predicted the existence of virtual electrodes. However, the value they predicted for the outermost V.A. was much smaller than that indicated by Hirsch's experiment¹⁾. This suggests that there must exist trapped particles which alter the location of the V.A.. Barnes' computer simulations indicated that only electrons were trapped in Region I due to an electron instability during start-up. Collisions and charge exchange were neglected in the simulations, these processes could lead to significant ion trapping in Region I. But, for simplicity, we will start with Barnes' result of only adding the T.E. in Region I. This will give us quite general knowledge on the qualitative and quantitative influence of trapped particles, and will be extended in Chapter VI to the case of including the T.I. in this region.

The systematic study will be:

In Region I:

- (1). For a given K_+^I , determine K_{-max} by Fig. 5.
- (2). Lower K_- to determine the amount of the T.E., i.e., the value of C_- , that can be used to replace the B.E., until C_{-max} is reached. .
Certainly C_{-max} is a function of K_+^I , and K_- .
Also, we examine the variation of the location of the V.A. with the addition of T.E..

In Region II:

(A). Assuming no T.I. in this region

(1). For $C_- = 0$ and given K_+^I , $K_- (\leq K_{-max})$, determine the K_+^{II} that can satisfy all the boundary conditions.

(2). For $C_- \neq 0$ and given K_+^I , K_- , find a new $K_+^{II'}$.
Certainly $K_+^{II'} \geq K_+^{II}$.

(B). Assuming T.I. exist in this region

(1). For a given set of K_- , C_- , examine the combinations of K_+^{II} , C_+^{II} that can satisfy all the boundary conditions.

(2). Examine the variation of the location of the V.C., and the influence of each species on the results.

For the above study, we choose $K_+^I = 0.7$, which corresponds to the following typical physical system: an ion beam current, which is 50% deuterium and 50% tritium, of 10.8 Amp, and an applied voltage of 100 KV. (This is equivalent to 7.74 Amp input current and 80 KV applied voltage.) The ion mass was taken to be 2.5 a.m.u.. The effect of varying K_+^I will also be discussed. The choice $K_+^I = 0.7$ is an intermediate current which exhibits all of the basic behavior.

4-1. Calculation of C_- in Region I for Different K_- ,

Assuming Fixed K_+^I and $C_+^I = 0$

For $K_+^I = 0.7$, Table 1 reports the results of calculating C_{-max} for different K_- , in which the parameter

λ_+ , instead of K_- is used, where λ_+ is defined by

$$\lambda_+ \equiv \frac{K_-}{K_+} = \frac{I_{be}}{I_{bi}} \left(\frac{m_e}{m_i} \right)^{1/2}$$

The third column of Table 1 shows the ratio of the virtual anode radius to virtual cathode radius when $C_- = C_{-max}$, while the fourth column shows the same ratio for $C_- = 0$. This variation is shown in greater detail on Fig. 8.

The most striking thing depicted by Table 1 and Fig. 8 is the ability of the T.E. to bring the V.A. inward. Qualitatively, this can be understood as follows:

In the pure beam case, the ion density goes to infinity at the V.A., while the electron density is a minimum. Hence the degree of neutralization by the electrons is low, and the counteracting force against the build-up of a potential well is relatively small. When the T.E. are added, their neutralizing ability is high, because their density is also a maximum when the B.I. density is a maximum, and decreases as the B.I. density decreases. Thus the ability of the B.I. to form a virtual anode is attenuated relatively heavily. So a longer distance is needed to achieve a V.A.. This effect is more significant when the amount of B.E. is less (hence more T.E. are allowed.)

4-2. Calculation of C_- in Region I for Different K_+^I ,

Assuming fixed λ_+ and $C_+^I = 0$

Figure 8 Effect on the Location of the Virtual Anode of Adding Trapped Electrons alone for Given Input Ion Current Parameter $K_+ = 0.7$ and Different Beam Electron Parameter λ_+

For a fixed λ_+ , Fig. 5 shows there is a K_{+min}^I . All K_+^I we pick should be greater than this minimum value. Porter 3) showed that in Region I $\lambda_+ \leq 1$. We will study three different cases.

1. Large λ_+

As a specific example we take $\lambda_+ = 0.8$ and $K_{+min} = 6.3$. This corresponds to an electron to ion input current ratio of 54.5, and the minimum ion current input will be 97.5 Amp for 100 KV applied voltage ($m_+ = 2.5$ amu.) The results are shown in Table 2.

2. Medium λ_+

For calculations we choose $\lambda_+ = 0.1$ and $K_{+min} = 0.05$. This corresponds to an electron to ion input current ratio 6.78, and the minimum input ion current will be 775 mA for 100 KV applied voltage. The results are shown in Table 3.

3. Small λ_+

For this case we choose $\lambda_+ = 0.01$. This corresponds to an electron to ion input current ratio 0.678. The results are presented in Table 4.

The influence of the parameters C_- , K_+ and λ_+ on the ratio $r_{V.A.} / r_{V.C.}$ is shown in Fig. 9.

Following Table 1, it was pointed out for a fixed K_+^I , the trapped electrons are much more effective than the beam electrons in bringing the V.A. inward, i.e. when C_{-max} / K_+^I gets larger, $r_{V.A.} / r_{R.C.}$ gets smaller. Tables 2, 3 and 4 show that for a fixed λ_+ , when K_+^I gets

λ_+	$C_{\text{-max}}$	$(\frac{r_{\text{V.A.}}}{r_{\text{R.C.}}})_{C_{\text{-}}=C_{\text{-max}}}$	$(\frac{r_{\text{V.A.}}}{r_{\text{R.C.}}})_{C_{\text{-}}=0}$	trap-to-beam ratio $\pi_e(r)$
0.45	0.000	0.256	0.256	0.00
0.40	0.140	0.242	0.274	$0.35 \times \phi(r)$
0.35	0.294	0.225	0.287	$0.84 \times \phi(r)$
0.30	0.455	0.2047	0.297	$1.52 \times \phi(r)$
0.25	0.623	0.1801	0.307	$2.54 \times \phi(r)$
0.20	0.800	0.1508	0.317	$4.00 \times \phi(r)$
0.15	0.994	0.1105	0.325	$6.50 \times \phi(r)$
0.10	1.232	0.045	0.332	$12.32 \times \phi(r)$
0.05	1.463	3×10^{-5}	0.340	$29.00 \times \phi(r)$
0.00	1.547	$\sim 10^{-4}$	0.346	

Table 1. Values of $C_{\text{-max}}$ for $K_+^I = 0.7$ and different λ_+ ($\lambda_+ \leq \lambda_{\text{+max}} = 0.454$). Also shown is $r_{\text{V.A.}} / r_{\text{R.C.}}$ with no trapping and with maximum trapping.

K_+^I	$C_{\text{-max}} / K_+^I$	$(\frac{r_{\text{V.A.}}}{r_{\text{R.C.}}})_{\text{no trapping}}$	$(\frac{r_{\text{V.A.}}}{r_{\text{R.C.}}})_{\text{max trapping}}$
7000	0.601	0.99005	0.98812
700	0.560	0.96851	0.93613
70	0.430	0.90213	0.88873
7	0.022	0.69350	0.69143

Table 2. Values of $C_{\text{-max}} / K_+^I$ for $\lambda_+ = 0.8$ and different K_+^I , $K_{\text{+min}}^I = 6.3$. Also shown is the ratio $r_{\text{V.A.}} / r_{\text{R.C.}}$ with no trapping and with maximum trapping.

K_+^I	C_{-max} / K_+^I	$\frac{r_{V.A.}}{r_{R.C.}}$) no trapping	$\frac{r_{V.A.}}{r_{R.C.}}$) max trapping
700	2.695	0.99999	0.91028
70	2.606	0.91940	0.75000
7	2.388	0.75580	0.39560
0.7	1.752	0.33230	0.04757

Table 3. Value of C_{-max} / K_+^I for $\lambda_+ = 0.1$ and different K_+^I $K_{+min}^I = 0.05$

K_+^I	C_{-max} / K_+^I	$\frac{r_{V.A.}}{r_{R.C.}}$) no trapping	$\frac{r_{V.A.}}{r_{R.C.}}$) max trapping
7000	2.914	0.99999	0.93137
700	2.892	0.99999	0.84360
70	2.830	0.92045	0.56744
7	2.646	0.75957	0.11754
0.7	2.190	0.34473	5.25700×10^{-4}
0.07		2.06440×10^{-4}	0.00000

Table 4. Values of C_{-max} / K_+^I for $\lambda_+ = 0.01$ and different K_+^I

Figure 9. the Ratio $\frac{V.A.}{R.C.}$ vs. Electron Trapping Parameter C_- for Different Combinations of K_+ and λ_+

larger, C_{-max} / K_+^I is also getting larger, i.e., relatively more T.E. are allowed for larger K_+^I . This is a reasonable result since λ_{+max} is larger for a larger K_+ , so that for a given λ_+ , $(\lambda_{+max} - \lambda_+)$ is larger, and consequently trapped electrons are replacing beam electrons. However it is also shown in Tables 2, 3 and 4, that for a fixed λ_+ , $(r_{V.A.} / r_{R.C.}) C_{-max}$ is larger for a larger K_+^I (or equivalently for a larger C_{-max} / K_+^I) rather than smaller. This indicates that even with relatively more T.E., the V.A. does not move in. Hence our general conclusion is that trapped electrons are more effective than beam electrons in moving the virtual anode inward; however, B.I. are even more powerful than T.E. in altering the location of the V.A.

4-3. Calculation of K_+^{II} in Region II for Varying K_- with Different C_- , Assuming K_+^I Fixed and $C_+^I = C_+^{II} = 0$

In these calculations, K_+^I was fixed at 0.7, and three different cases were investigated:

- (1). C_- was assigned the maximum value corresponding to the given K_+^I , and K_- (see Table 1).
- (2). C_- was assigned one half the maximum value.
- (3). C_- was set equal to zero. This case had been done by Porter³⁾, and the present results are in agreement with his calculations.

The results of the calculations for K_+^{II} vs. λ_+ are shown in Fig. 10. Also shown in Fig. 10 is

Figure 10

Transmitted Ion Beam vs λ_+ for Different Amounts of Trapped Electrons, $K_+^I = 0.7$

$r_{V.C.} / r_{V.A.}$ VS. λ_+ for C_{-max} . It is seen that for the beam electron parameter λ_+ approaching its maximum value, there is little variation in K_+^{II} . However, for small λ_+ and large electron trapping, the effect is dramatic. The value of K_+^{II} can increase by two orders of magnitude. Nonetheless, K_+^{II} / K_+^I is still less than one so that there is a decrease in the beam ion current across the virtual anode.

In the above calculations it was found that for a given K_- , if K_+^{II} was computed only to the third decimal place, the derivative at the V.C. is not very close to zero. To check if the K_+^{II} so found was correct, we calculated it to the fourth or the fifth decimal place. If it isn't correct, no matter how many decimal place we calculate, $\left| \frac{d\phi}{d\xi} \right|_{\phi=1}$ is fixed and greater than zero. If $\left| \frac{d\phi}{d\xi} \right|_{\phi=1}$ converges to zero when we calculate K_+^{II} to more decimal place, we need further to check $\left| \frac{d\phi}{d\xi} \right|_i$ at the last few points; if it is decreasing (i.e., converging to zero), then the K_+^{II} so found can be considered correct. Table 5 is an example to show the result of calculating K_+^{II} to more decimal place.

Referring to Table 5, for example, when $\lambda_+ = 0$, i.e., no B.E. in Region II, when we calculate K_+^{II} to the third decimal place $\left| \frac{d\phi}{d\xi} \right|_{\phi=1}$ equals 1.1. When we improve the accuracy of K_+^{II} to the fifth decimal place $\left| \frac{d\phi}{d\xi} \right|_{\phi=1}$ still equals 1.1. This means there is no indication that the requirement $\left| \frac{d\phi}{d\xi} \right|_{\phi=1} = 0$ can be satisfied, hence no

K_+^{II} exists for $\lambda_+ = 0$ for which a V.C. can be formed.

The conclusion is that a V.C. can never be formed by T.E.

alone. However, for the $\lambda_+ = 0.2$ case, when we calculate K_+^{II} to the third decimal place $\left| \frac{d\phi}{d\beta} \right|_{\phi=1}$ equals 0.280, but when we calculate K_+^{II} to fifth decimal place, it drastically decreases to 0.04. Thus there is a strong indication the requirement $\left| \frac{d\phi}{d\beta} \right|_{\phi=1} = 0$ will be satisfied with a sufficiently accurate K_+^{II} .

4-4. Calculation of K_+^{II} for Varying C_+^{II} , Assuming Fixed K_- , C_- , K_+^I and $C_+^I = 0$

The results of the calculation of C_+^{II} and K_+^{II} for $K_+^I = 0.7$, $K_- = 0.14$, $C_- = 0.7$ are shown in Table 6.

In Table 6, the data for $C_+^{II} = 0.670$ is invalid, because the derivative at the V.C. is so large and would not converge to zero as we calculate K_+^{II} to more decimal places. This indicates that the upper limit of C_+^{II} is around 0.588. We must notice the fact that it is very obscure when the C_{+max}^{II} is reached, because the only criterion for C_{+max}^{II} is that beyond C_{+max}^{II} the derivative at the V.C. will not converge to zero no matter how many decimal place we calculate for K_+^{II} ; or, even if it converges, the derivative near the V.C. is small but increasing.

The ion current amplification α is easily obtained, and is around 0.57. Although it is much larger than the pure beam case or the case of adding T.E. alone, it is

λ_+	third decimal place		fifty decimal place	
	K_+^{II}	$\left \frac{d\phi}{d\beta}\right _{\phi=1}$	K_+^{II}	$\left \frac{d\phi}{d\beta}\right _{\phi=1}$
0.00	0.001	1.1	0.00098	1.1
0.05	0.003	0.857	0.00209	0.6399
0.10	0.007	0.668	0.00607	0.1900
0.15	0.012	0.505	0.01169	0.0640
0.20	0.020	0.280	0.01932	0.0400

Table 5. Effect of calculating K_+^{II} to more decimal place

C_+^{II}	K_+^{II}	$\frac{r_{V,C_+}}{r_{V,A_+}}$	$\left \frac{d\phi}{d\beta}\right _{\phi=1}$
0.00	0.1930	0.0545	0.012
0.10	0.1740	0.0350	0.048
0.20	0.1590	0.032	0.04
0.30	0.1492	0.0182	0.050
0.40	0.1340	7.6×10^{-3}	0.074
0.50	0.1248	1.08×10^{-3}	0.08
0.588	0.1040	0	1.08
0.670	0.0360	0	0.355

Table 6. Values of K_+^{II} for varying C_+^{II} and $K_+^I = 0.7$,
 $K_- = 0.14$, $C_- = 0.7$ and $C_+^I = 0$

still far below the 10^4 value which is desired for fusion application.

The case $K_- = 0.0245$, $C_- = 1.480$ was also calculated. C_{+max} is less than 0.35. The amplification is poorer than the former case. This implies the beam particles are more effective for trapping the opposite species. We will discuss this closely later.

We have also obtained the value of C_{+max} for an unsaturated well (i.e., $\phi_{max} < 1$) (see Fig. 11). Table 7 gives some information of these unsaturated wells.

From Table 7, we see that C_+ can be large for an unsaturated well. However this does not imply a large amplification, since the ion current amplification factor is also determined by V_{max} (i.e., ϕ_{max}). The combination of C_+^{II} and ϕ_{max} is still small, and, moreover, in this case we have a lot of slow particles. Hence we will not give further attention here to the unsaturated well problem.

The calculations in this section were performed by starting at the V.A. and stopping at the V.C. As a check, the calculations have been performed from the V.C. to the V.A.. The results are consistent. The V.C. to V.A. approach has one benefit over the other in that if an assigned C_+^{II} is too big, it can be detected in a few steps during the calculation. This is because the V.C. is formed by some optimal combination of ions and electrons. Since the T.I. are a maximum at the V.C., the excess of C_+^{II} is very significant in the beginning and can destroy the

C_+^{II}	K_+^{II}	$\frac{V_{\max}}{V_0}$	$\frac{r_{V.C.}}{r_{V.A.}}$
0.6	0.115	0.581	4.7×10^{-3}
0.8	0.100	0.422	6.9×10^{-3}
1.3	0.075	0.203	0.034
1.8	0.059	0.115	0.059
2.3	0.048	0.052	0.322

Table 7. Values of C_+^{II} and K_+^{II} for unsaturated well
 $(V_{\max} / V_0 < 1)$ for $K_+^I = 0.7$, $K_- = 0.14$ and
 $C_- = 0.7$

Figure 11. Illustration of an Unsaturated well (marked as I), and a full well (marked as II)

formation of the V.C. immediately; hence, they can be detected very quickly. In the V.A. to V.C. approach, the T.I. are a minimum in the beginning, so that, even if C_+ is too big, the excess of ions is unperceptible in the beginning, and when the excess is detected, many calculations have passed.

4-5 Derived Quantities

Equations (3.17) to (3.24) show that all the derived quantities are of the form

$$\frac{\text{constant}}{r^2} \left\{ \begin{array}{l} \phi^{1/2} \\ (1-\phi)^{1/2} \\ \phi^{3/2} \\ (1-\phi)^{3/2} \end{array} \right\}$$

Hence, for convenience, in the figures for the derived quantities, normalized quantities in the brackets are used. Figures 12 and 13 report calculations of these quantities.

The Parameters chosen in Figs. 12 and 13 correspond to a physical system with input ion current of 10.8 Amp, input electron current of 146 Amp, trapped ion current of 0.77 Amp (in Region II), and trapped electron current of 362 Amp with 100 KV applied voltage. If we have a fusion reactor of $r_{R.C.} = 50$ cm, and $|V_0| = 100$ KV, then $V_0 / (4\pi e) = 5.4 \times 10^{10}$. Now we calculate all the derived quantities at the location

Figure 12 Normalized Partial Particle Pressure and Total Particle Pressure vs normalized radius for a System with $K_+^I = 0.7$, $K_+^{II} = 0.179$, $K_- = 0.14$, $C_+^I = 0$, $C_+^{II} = 0.100$, $C_- = 0.70$. (Notice inside the virtual anode (V.A.) the scale of the radius changes).

$$r/r_{R.C.} = 0.023$$

$$\phi^{1/2} = 0.837$$

$$\phi^{-1/2} = 1.194$$

$$(1-\phi)^{1/2} = 0.567$$

$$(1-\phi)^{-1/2} = 1.830$$

At the above r , we obtain from (3.17) to (3.24):

$$N_{bi} = 8.7 \times 10^{10} / \text{cm}^3$$

$$N_{ti} = 3.4 \times 10^9 / \text{cm}^3$$

$$N_{be} = 1.04 \times 10^{10} / \text{cm}^3$$

$$N_{te} = 1.62 \times 10^{10} / \text{cm}^3$$

$$P_{bi} = 2.0 \times 10^3 \text{ dynes} / \text{cm}^2 = 7.2 \times 10^{-2} \text{ psi}$$

$$P_{ti} = 2.54 \times 10^2 \text{ dynes} / \text{cm}^2 = 9.5 \times 10^{-3} \text{ psi}$$

$$P_{be} = 1.04 \times 10^3 \text{ dynes} / \text{cm}^2 = 3.74 \times 10^{-2} \text{ psi}$$

$$P_{te} = 1.90 \times 10^2 \text{ dynes} / \text{cm}^2 = 6.84 \times 10^{-3} \text{ psi}$$

$$P_{\text{electrostatic}} = 3.53 \times 10^2 \text{ dynes} / \text{cm}^2 = 1.27 \times 10^{-2} \text{ psi}$$

These calculations are performed at a place quite close to the virtual cathode where the trapped ion density reaches the maximum. However the calculated trapped ion density appears to be too small ($3.4 \times 10^9 / \text{cm}^3$) for the fusion purpose. So are all the calculated pressure terms. This is a result of our incomplete model as discussed in Chapter V.

Figures 12 and 13 depict very clearly the deviation of all quantities from the $1/r^2$ variation which should be a flat line in these figures. Also interesting is the variation of the total pressure in Fig. 12. The quantity

$r^2 P_{\text{total}}$ decreases greatly from the R.C. to the V.A.. This is because when we approach the V.A., the B.I. lose their kinetic energies, and the T.I. decrease both in quantity and kinetic energy, hence the variation of the total pressure is weaker than $1 / r^2$.

Figure 12 shows that, neglecting the $1 / r^2$ effect, the fractional total pressure reduction, defined by $(r^2 P_{\text{total}})_1 - (r^2 P_{\text{total}})_2 / (r^2 P_{\text{total}})_1$, from the V.A. (labeled as 1) to the V.C. (labeled as 2) is around 0.38, while that from the R.C. (1) to the V.A. (2) is around 0.5. At the V.C. P_{total} loses all the electron (B.E. and T.E.) contribution, while at the V.A. P_{total} loses all the ion (B.I. and T.I.) contribution. Thus the loss of ion contribution results in almost the same (at least to the same order) fractional total pressure decrease as that for the electron contribution. Consequently, electrons and ions have the same effectiveness in contributing to the total pressure.

V. DISCUSSION

All calculations under the model we set up indicate no significant ion current amplification. The reason can be understood by closely examining the finite difference equation for the squared derivative $(\frac{d\phi}{d\xi})^2$.

The finite difference equations for solving Eq. (3.13) from V.C. to V.A. read:

$$\begin{aligned}\phi_{i+1} &= \phi_i - \Delta\xi \times \left| \frac{d\phi}{d\xi} \right|_i \\ \left(\frac{d\phi}{d\xi} \right)_{i+1}^2 &= \left(\frac{d\phi}{d\xi} \right)_i^2 e^{-2\Delta\xi} + 4 \times \left[K_+ (\phi_{i+1}^{1/2} - \phi_i^{1/2}) + K_- (\sqrt{1-\phi_{i+1}} - \sqrt{1-\phi_i}) \right. \\ &\quad \left. + \frac{C_+}{3} (\phi_{i+1}^{3/2} - \phi_i^{3/2}) + \frac{C_-}{3} ((1-\phi_{i+1})^{3/2} - (1-\phi_i)^{3/2}) \right] \\ &\hspace{15em} (5.1)\end{aligned}$$

If we start from the V.C., i.e., $\phi=1$, $\frac{d\phi}{d\xi}=0$, the most favorable situation for forming a full space-charged-limited potential well (see Fig. 14a) is that, in the beginning, $(\frac{d\phi}{d\xi})_{i+1}^2$ increases as i increases. After a certain point the contribution of the quantity in brackets becomes zero and then negative, so that $(\frac{d\phi}{d\xi})_{i+1}^2$ starts to decrease, i.e., ϕ_{i+1} is decreasing, but now at a slower rate. Ideally, we hope $(\frac{d\phi}{d\xi})_{i+1}^2$ decreases to zero when ϕ_{i+1} just reaches 0, i.e., at the V.A.. But, in practice two things might happen if incorrect parameters are chosen: either the negative contribution is too big so that $(\frac{d\phi}{d\xi})_{i+1}^2$ becomes zero when the V.A.

has not been reached yet (see Fig. 14b), or the negative contribution is too small to bend the potential profile flat, so that when $\phi_{i+1}=0$, $(\frac{d\phi}{d\xi})_{i+1}^2 > 0$ (see Fig. 14c)

For the pure beam case, for a given K_- (or K_+), we can always find a K_+ (or K_-) to satisfy the conditions for a virtual anode. This is because, if we start from the V.C., in the beginning

$$\phi_i = 1 - \delta_i$$

$$\phi_{i+1} = 1 - \delta_2$$

$$\delta_2 > \delta_1 > 0 \quad \text{but very small}$$

$$\text{B.E. contribution } A = (1 - \phi_{i+1})^{1/2} - (1 - \phi_i)^{1/2} = \delta_2^{1/2} - \delta_1^{1/2}$$

$$\text{B.I. contribution } B = \phi_{i+1}^{1/2} - \phi_i^{1/2} \cong -\frac{1}{2}(\delta_2^{1/2} + \delta_1^{1/2})(\delta_2^{1/2} - \delta_1^{1/2})$$

$$\text{Because } \delta_1, \delta_2 \sim 0, \text{ this implies } |A| \gg |B|.$$

Hence there is a big net positive contribution to $(\frac{d\phi}{d\xi})_{i+1}^2$.

When the V.A. is being approached, the roles of ions and electrons are reversed, so there will be a big negative contribution to bring $(\frac{d\phi}{d\xi})_{i+1}^2$ to zero; hence there always exists an optimal K_+ .

However, when the B.I., B.E., T.I. and T.E. are all added, the situation is different. If we start from the V.C. to the V.A., in the beginning

$$\phi_i = 1 - \delta_1$$

$$\delta_2 > \delta_1 > 0 \quad \text{but very small}$$

$$\phi_{i+1} = 1 - \delta_2$$

Figure 14a a Full Potential Well, $\left|\frac{d\phi}{dr}\right|=0$ at $\phi=0$

Figure 14b Unacceptable Well, Case i, $\left|\frac{d\phi}{dr}\right|=0$ at $\phi \neq 0$

Figure 14c Unacceptable Well, Case ii, $\left|\frac{d\phi}{dr}\right| \neq 0$ at $\phi=0$

Figure 14 Illustrations of Different Potential Wells, "→" means direction of calculations

Hence the contributions of each species in Eq. (5.1) are

$$\text{B.I.} \quad K_+^{\text{II}} (\phi_{i+1})^{1/2} - (\phi_i)^{1/2} \simeq -K_+^{\text{II}} \frac{1}{2} (\delta_2 - \delta_1) < 0 \quad (5.2)$$

$$\text{B.E.} \quad K_- ((1-\phi_{i+1})^{1/2} - (1-\phi_i)^{1/2}) \simeq K_- (\delta_2^{1/2} - \delta_1^{1/2}) > 0 \quad (5.3)$$

$$\text{T.I.} \quad C_+^{\text{II}} (\phi_{i+1})^{3/2} - (\phi_i)^{3/2} \simeq -C_+^{\text{II}} \frac{1}{2} (\delta_2 - \delta_1) < 0 \quad (5.4)$$

$$\text{T.E.} \quad C_- ((1-\phi_{i+1})^{3/2} - (1-\phi_i)^{3/2}) \simeq \frac{C_-}{3} (\delta_2^{3/2} - \delta_1^{3/2}) > 0 \quad (5.5)$$

If we want big ion current amplification we need

$$C_+^{\text{II}} \gg K_+^{\text{I}}$$

However, calculations in Chapter V have shown that when

$$C_+^{\text{I}} = 0, K_+^{\text{I}} \geq K_- \quad \text{and} \quad K_- \gg K_+^{\text{II}}. \quad \text{Hence, overall}$$

$$C_+^{\text{II}} \gg K_- \gg K_+^{\text{II}}$$

Equations (5.4) and (5.5) indicate that in the beginning the contribution of the T.I. is big and negative, and, in comparison, that of B.E. and B.I. are negligible. To prevent $(\frac{d\phi}{ds})_{i+1}^2$ from becoming zero or negative before reaching the V.A., a big positive contribution is needed to compensate the big negative T.I. contribution, and this can only be achieved by the positive contribution of the undetermined T.E.. However, our previous calculations indicated that with no T.I. in Region I, K_+^{I} , and K_- can trap only a relatively small amount of electrons, i.e., C_- is of the same order of K_+^{I} or K_- . Hence the T.E. cannot achieve this purpose either.

Physically, this can be understood as follows:

If we desire to achieve a large ion current amplification by radially trapped ions, the amount of T.I. required by

the amplification will be so enormous that the V.C. will be swept out, because the radial T.I. density is maximum around the V.C.

We stated above, because no T.I. are present in Region I, C_- will be restricted to be comparable to K_+^2 , or K_- . But even if, by some mechanism, C_- were allowed to be very big to compensate the negative T.I. contribution near the V.C., this C_- must be much bigger than C_+ , because (5.4) shows that the T.I. contribution is of order δ while the T.E. contribution is of order $\delta^{3/2}$. But at the other end, i.e., near the V.A., the roles of the T.I. and the T.E. are reversed, hence this big C_- will sweep out the V.A. even more strongly!

All the above arguments explain why no big ion current amplification is found under our model if no T.I. are in Region I. Our conclusion is that with no ion trapping in Region I, big ion current amplification can never be achieved by trapping a large amount of ions with a large amount of trapped electrons.

VI. THE EFFECTS OF TRAPPED IONS BETWEEN THE REAL CATHODE AND VIRTUAL ANODE

In this chapter, we add trapped ions in Region I and explore the consequences. It will be demonstrated first, by a pressure analysis similar to that used by Simon⁵⁾, that for our choice of distribution functions, large amplification is theoretically possible with trapped ions in Region I and impossible without trapped ions in Region I. Specific calculations appropriate to the experiments of Hirsch are then performed which show that trapped ions in Region I pull the virtual anode out toward the real cathode and do, in fact, enable us to achieve substantial amplifications.

6-1. Amplification Analysis

Simon⁵⁾ showed from the general relation $\nabla \cdot (\bar{P} + \bar{T}) = 0$, where $\bar{P}$ is the plasma stress tensor, and $\bar{T}$ is the electromagnetic stress tensor, that, for the case of radially moving particles, and assuming $\vec{B} = 0$, the total particle pressure P and the electrostatic pressure $\frac{E^2}{8\pi}$ satisfy the relation

$$r \frac{d}{dr}(r^2 p) = \frac{1}{r} \frac{d}{dr}(r^4 \frac{E^2}{8\pi}) \quad (6.1)$$

Using the identity

$$\frac{d}{dr}(r^3 p) = r \frac{d}{dr}(r^2 p) + r^2 p$$

Equation (6.1) becomes

$$\frac{d}{dr} \left[r^3 p - r^3 \frac{E^2}{8\pi} \right] = r^2 p + r^2 \frac{E^2}{8\pi} \quad (6.2)$$

Defining

$$\psi \equiv r^3 \left(p - \frac{E^2}{8\pi} \right)$$

Equation (6.2) becomes

$$\frac{d}{dr} \psi = \frac{\psi}{r} + \frac{r^2 E^2}{4\pi} \quad (6.3)$$

Defining

$$\chi \equiv \psi/r$$

Equation (6.3) becomes

$$\frac{d}{dr} \chi = r \frac{E^2}{4\pi} \quad (6.4)$$

Integrate Eq. (6.4), we obtain

$$\chi - \chi_a = \int_{r_a}^r \frac{E^2}{4\pi} r dr \quad (6.5)$$

or

$$r^2 \left(p(r) - \frac{E^2(r)}{8\pi} \right) - r_a^2 \left(p(r_a) - \frac{E^2(r_a)}{8\pi} \right) = \int_{r_a}^r \frac{E^2}{4\pi} r dr \quad (6.6)$$

When $r = r_{V.A.}$ (radius of the V.A.), and $r_a = r_{R.C.}$ (radius of the R.C.), we know $\vec{E}(r_{V.A.}) = 0$, so Eq. (6.6) becomes

$$r_{V.A.}^2 p_{V.A.} + r_{R.C.}^2 \frac{E_{R.C.}^2}{8\pi} + \int_{r_{V.A.}}^{r_{R.C.}} \frac{E^2}{4\pi} r dr = r_{R.C.}^2 p_{R.C.} \quad (6.7)$$

Since $\gamma_{R.C.}^2 \frac{E_{R.C.}^2}{8\pi} + \int_{r_{V.A.}}^{r_{R.C.}} \frac{E^2}{4\pi} r dr$ is positive definite,

we obtain the relation

$$\gamma_{V.A.}^2 P_{V.A.} \leq \gamma_{R.C.}^2 P_{R.C.} \quad (6.8)$$

At the R.C. the electron kinetic energy is zero, so only the ions will contribute to the particle pressure. Using Eq. (3.20) to (3.23) for partial particle pressure, we obtain

$$P_{R.C.} = \frac{V_0^2}{6\pi \gamma_{R.C.}^2} C_+^I + \frac{V_0^2}{2\pi \gamma_{R.C.}^2} K_+^I \quad (6.9)$$

Similarly, at the V.A., only the electrons will contribute to $P_{V.A.}$, so we have

$$P_{V.A.} = \frac{V_0^2}{6\pi \gamma_{V.A.}^2} C_-^I + \frac{V_0^2}{2\pi \gamma_{V.A.}^2} K_-^I \quad (6.10)$$

With Eqs. (6.9) and (6.10), Eq. (6.8) becomes

$$K_- + \frac{C_-}{3} \leq K_+^I + \frac{C_+^I}{3} \quad (6.11)$$

By the same arguments, if we put $r = r_{V.C.}$, and $r_a = r_{V.A.}$ in Eq. (6.6), we obtain

$$K_+^{II} + \frac{C_+^{II}}{3} \leq K_- + \frac{C_-}{3} \quad (6.12)$$

Equations (6.11) and (6.12) together give

$$K_+^{II} + \frac{C_+^{II}}{3} \leq K_- + \frac{C_-}{3} \leq K_+^I + \frac{C_+^I}{3} \quad (6.13)$$

Equation (6.13) is a very important expression, for example, in the pure beam case, $C_+^I = C_- = 0$, which implies

$$K_- \leq K_+^I$$

This result was obtained previously by Porter³). Next, if following Barnes computer simulation, put $C_+^I = 0$, Equation (6.13) implies

$$\frac{1}{2} \frac{K_+^I}{K_+^I} + \alpha \leq 1.5, \text{ or } \alpha \leq 1.5 \quad (6.14)$$

where α is the ion current amplification factor defined by Equation (3.16). Our detailed calculations (Table 6) are consistent with this result.

If $C_+^I \neq 0$, Eq. (6.13) implies

$$\alpha \leq 1.5 + \frac{C_+^I}{2K_+^I}$$

Mathematically, there is no upper limit for C_+^I , so that, in contrast to Eq. (6.14), there is no definite upper bound imposed on α . In the next section, an example is presented where α is approximately 150.

6-2. Application to the Interpretation of Hirsch's Experiment

In this section, we add trapped ions in Region I and apply the results to the interpretation of Hirsch's experiment. His experimental parameters are: input ion current $I_+ = 20$ MA, applied voltage $V_0 = 90$ KV, $M_+ = 2.5$

a.m.u.. Converting to C.G.S. units, by Eq. (3.8), the normalized beam ion current parameter $K_+^I = 1.52 \times 10^{-3}$.

With such a small K_+^I , if we have only ion beams and electrons in Region I, but not trapped ions, the V.A. would be very close to the center. However, from X ray and neutron flux measurements, it appears that a V.A. exists at approximately 1.25 cm for a cathode radius of 5.7 cm. This indicates that there must be a substantial trapped ion current in Region I to bring the V.A. all the way out to 1.25 cm. With these T.I., the corresponding K_- , C_- will be much bigger than that without T.I., which will in turn give much bigger K_+^{II} , C_+^{II} than before. A big ion current amplification is thus made possible.

It has been found in Region I that C_+ needs to be around 10 to bring the V.A. out to 1.25 cm. Hence we choose $C_+^I = 10$, and the following combinations of K_- , C_- , shown in Table 8, can give the required V.A.:

K_-	0.29	0.25	0.20	0.15	0.10
C_-	0.00	0.037	0.087	0.137	0.187

Table 8. Combinations of K_- , C_- to give $\frac{r_{V.A.}}{r_{V.C.}} = 0.22$
for $K_+^I = 1.52 \times 10^{-3}$, and $C_+^I = 10$

From our previous study, we know the B.E. are more effective in trapping the ions (c.f. Chapter V). However, intuitively, from the stability point of view, a certain combination of the T.E. and the B.E. will be a more stable

configuration, so we choose $K_- = 0.2$, and $C_- = 0.087$ to find the combinations of K_+^{II} and C_+^{II} in Region II. These results are presented in Table 9.

C_+^{II}	0.0	0.2	0.3	0.4	0.5
K_+^{II}	0.0610	0.0367	0.0295	0.025	no good
$\frac{r_{V.C.}}{r_{V.A.}}$		1.08×10^{-2}	1.88×10^{-3}		

Table 9. Results for Simulating Hirsch's Experiment

Table 9 indicates that

$$0.4 < C_{+max}^{II} < 0.5$$

so the ion current amplification factor is of the order:

$$\alpha = \frac{C_{+max}^{II}}{2K_+^{II}} + \frac{K_+^{II}}{K_+^{II}} \simeq 15^{\circ}$$

While this provides a theoretical explanation of Hirsch's experimentally detected V.A., and makes a significant ion current amplification possible, it should be noted that the present model contained no angular momentum. Angular momentum on the ions trapped in the center is expected to reduce the trapping and amplification. On the other hand, we might be able to further increase the ion trapping by allowing for circulating ion current through the cathode and allowing even more ion trapping in Region I. These calculations are left for the future.

VII. SUMMARY AND SUGGESTION FOR FURTHER WORK

SUMMARY

The essential consequences of adding trapped particles, in addition to the beam particles, to an electrostatic confinement device are summarized as follows:

A. When only trapped electrons are added in Region I, i.e., between the virtual anode and the real cathode

- (1). For a given input current K_+^I , there is a maximum beam electron current K_{-max} from the real cathode, at which no trapped electrons are allowed if a virtual anode is desired.
- (2). If we lower the beam electron current, i.e., let the beam electron current parameter $K_- < K_{-max}$, there is a maximum amount of trapped electrons $C_{-max}(K_+, K_-)$ which can be put into Region I in place of the beam electrons.
- (3). When trapped electrons increase to a certain amount, the space-charge-limited condition, i.e., $\frac{dV}{dr} = 0$, at the real cathode can no longer be satisfied.
- (4). When beam electrons are reduced and are replaced by trapped electrons, the virtual anode moves toward the center, so trapped electrons are more effective than beam electrons in moving the virtual anode toward the center.

- (5). For a fixed beam electron to beam ion current ratio $\lambda_+ \equiv \frac{K_-}{K_+^I}$, relatively more trapped electrons are allowed for larger K_+^I , i.e., C_{-max} / K_+^I is larger for larger K_+^I . The calculations showed that the larger C_{-max} / K_+^I case is less effective in bringing the virtual anode inward. Keeping in mind that trapped electrons move the virtual anode inward, the above stated effect implies beam ions are more effective than trapped electrons in altering the location of the virtual anode.
- (6). For high ion input current, say, $K_+^I > 100$, the virtual anode is very close to the real cathode. In the model of particles having only radial motion, for large K_+^I , because of consequences (4) and (5), there is no way to move the virtual anode inward by any significant amount.
- (7). With the trapped electrons, which exist in both Region I and Region II (the region between the virtual cathode and the virtual anode), some range of combinations of beams ions (K_+^I) and trapped ions (C_+^{II}) exist for forming a virtual cathode. The amount of ions in this case is much more than that without trapped electrons; i.e., $(C_+^{II} + K_+^{II})_{\text{with T.E.}} \gg (K_+^{II})_{\text{without T.E.}}$. However, in the case $C_+^I = 0$,

i.e., no trapped ions in Region I, the ion current amplification can never be greater than 1.5. Both numerical calculations and pressure analysis show this.

- (8). Trapped ions in Region II move the virtual cathode toward the center. In some cases they can move a virtual cathode which is far from the center very close to the center.
- (9). Without beam ions a virtual anode cannot be formed; without beam electrons, a virtual cathode cannot be formed. In fact, beam particles are completely responsible for the formation of virtual electrodes, because trapped particles at the corresponding virtual electrodes are zero.

E. When both trapped ions and trapped electrons are added in Region I:

- (1). Pressure analysis shows mathematically there is no upper limit for the amount of trapped ions, i.e., C_+^I in Region I has no upper limit. For a given K_+^I , with large C_+^I , the allowed K_- and C_- can get much larger than that without C_+^I . With the new K_- and C_- , K_+^{II} can consequently be much larger than K_+^I , which is in contrast to Porter's pure beam result³⁾:

$K_+^I > K_+^{II}$. The new K_+^{II} and C_+^{II} indicate very

high ion current amplification is now at least mathematically possible. In fact, the pressure analysis shows mathematically there is no upper limit for the amplification.

- (2). Trapped ions in Region I can move the virtual anode outward.
- (3). Theoretical work with trapped ions in Region I provides an explanation for the possible virtual anode detected by Hirsch¹⁾.

SUGGESTIONS FOR FURTHER WORK

- (1). Calculate the neutron flux and X-ray intensity due to the interactions of the particle distributions proposed in this study and compare with Hirsch's experimental data.
- (2). Investigate a more realistic model which includes the angular distribution for trapped particles. Intuitively, we expect the angular distribution would lower the ion current amplification because the angular distribution will greatly weaken the focusing property of a spherical device.
- (3). Analyze the stability of solutions obtained in this study which allow large amplification to see if they can exist in the device.

APPENDIX I METHOD FOR NUMERICAL CALCULATIONS

FORMULATION OF THE FINITE DIFFERENCE EQUATION

Integrating over Eq.(3.12), with the boundary condition $\left. \frac{d\phi}{d\xi} \right|_{\xi=0} = 0 \left(\xi = \ln \frac{r_{i4}}{r} \right)$ we obtain

$$\left(\frac{d\phi}{d\xi} \right)^2 = 2e^{-2\xi} \left[\int_{\xi=0}^{\xi} e^{2\xi'} \left(\frac{K_+}{\sqrt{\phi}} + C_+ \phi^{1/2} - \frac{K_-}{\sqrt{1-\phi}} - C_- (1-\phi)^{1/2} \right) \frac{d\phi}{d\xi'} d\xi' \right] \quad (1)$$

Integrating by parts, and using the first order expansion

$$e^{2\Delta\xi} \simeq 1 + 2\Delta\xi$$

$$\int_{\phi_i}^{\phi_{i+1}} f(\phi') d\phi' \simeq f(\phi_i) (\phi_{i+1} - \phi_i)$$

Equation (1) can be written into the finite difference equation:

$$\begin{aligned} \left(\frac{d\phi}{d\xi} \right)_{i+1}^2 = e^{2\Delta\xi} \left\{ \left(\frac{d\phi}{d\xi} \right)_i^2 + 4 \left[K_+ (\sqrt{\phi_{i+1}} - \sqrt{\phi_i}) + K_- (\sqrt{1-\phi_{i+1}} - \sqrt{1-\phi_i}) \right. \right. \\ \left. \left. + \frac{C_+}{3} (\phi_{i+1}^{3/2} - \phi_i^{3/2}) + \frac{C_-}{3} ((1-\phi_{i+1})^{3/2} - (1-\phi_i)^{3/2}) \right] \right\} \quad (2) \end{aligned}$$

Equation (2) is used together with

$$\phi_{i+1} = \phi_i + \Delta\xi \times \left| \left(\frac{d\phi}{d\xi} \right)_i \right|$$

to perform calculations.

METHOD FOR FINDING THE OPTIMAL K_+^{II}

(A). Calculating from the V.A. to the V.C.

Equation (2) is the finite difference equation to be solved. For a given set of K_- , C_- and C_+ , if C_+ is in the allowed range, there will exist an optimal K_+ to satisfy all the boundary conditions. K_+ is determined as follows:

To determine the first significant digit, we make an estimation of the order of K_+ , say $K_+ = 10^m$, where m is some estimated integer (positive or negative), and start the iteration.

A-1.

If the K_+ so chosen is too big, because its contribution is positive, $(\frac{d\phi}{dz})_{i+1}^2$ will be appreciably greater than zero when ϕ_{i+1} reaches the supposed V.C.. Hence K_+ is adjusted to 10^{m-1} and so forth, until, when K_+ is adjusted to $10^{m'-1}$, $(\frac{d\phi}{dz})_{i+1}^2$ becomes zero before $\phi_{i+1} = 1$ during the iteration. This means the adjusted K_+ starts to be a little bit smaller than the optimal K_+^{II} ; hence K_+^{II} must lie between $10^{m'-1}$ and $10^{m'}$.

Then start with $K_+ = 2 \times 10^{m'-1}$; if it's small, $(\frac{d\phi}{dz})_{i+1}^2$ becomes zero before $\phi_{i+1} = 1$ and K_+ is adjusted by adding $10^{m'-1}$ to it, and so forth until, when $K_+ = N_1 \times 10^{m'-1}$, ϕ_{i+1} can reach 1 before $(\frac{d\phi}{dz})_{i+1}^2$ becomes zero. This means K_+ starts to be a little bit larger than the optimal K_+^{II} . Hence the optimal K_+^{II} must lie between $(N_1 - 1) \times 10^{m'-1}$ and $N_1 \times 10^{m'-1}$. The first significant digit is thus definitely determined to be $N_1 - 1$. The second significant digit is determined by starting with

$K_+ = (N_1 - 1) \times 10^{m-1} + 0.1 \times 10^{m-1}$ and follow the same procedures for determining N_1 . Similarly, we can determine as many significant digits as we want.

A-2.

If the starting estimation of K_+ is too small, $(\frac{d\phi}{dz})_{i+1}^2$ will become zero before $\phi_{i+1} = 1$; K_+ is then adjusted to 10^{m+1} and continue. The range of K_+ can be determined similarly as in A-1.

The significant things are: if the C_+ is in the allowed range, as we improve the accuracy of K_+ , the derivative at the V.C. will converge to zero, and the location of the V.C. so obtained will be more and more accurate. If C_+ is beyond the allowed range, a "cheating" K_+ will be obtained by this method, and the way out is to calculate K_+ to more decimal place. If the derivative at the V.C. doesn't converge to zero, the C_+ so chosen must be beyond the allowed range, or even if it converges, when we print out the last few points near the V.C., we find that $(\frac{d\phi}{dz})_{i+1}$ is tending to increase. In the latter case, the C_+ so chosen is still beyond the allowed range, because the tendency of $(\frac{d\phi}{dz})_{i+1}$ near the V.C. should be to decrease to zero.

(B). Calculation from the V.C. to the V.A.

The finite difference equations will be written as

$$\begin{aligned} \left(\frac{d\phi}{d\xi}\right)_{i+1}^2 &= \left(\frac{d\phi}{d\xi}\right)_i^2 e^{-2\Delta\xi} + 4 \left[K_+ (\sqrt{\phi_{i+1}} - \sqrt{\phi_i}) + K_- (\sqrt{1-\phi_{i+1}} - \sqrt{1-\phi_i}) \right. \\ &\quad \left. + \frac{C_+}{3} (\phi_{i+1}^{3/2} - \phi_i^{3/2}) + \frac{C_-}{3} ((1-\phi_{i+1})^{3/2} - (1-\phi_i)^{3/2}) \right] \end{aligned}$$

$$\phi_{i+1} = \phi_i - \Delta\xi \left| \left(\frac{d\phi}{d\xi}\right)_i \right|$$

Notice now the contribution of B.I. to $\left(\frac{d\phi}{d\xi}\right)_{i+1}^2$ is negative. Thus, for a chosen K_+ , if $\left(\frac{d\phi}{d\xi}\right)_{i+1}^2$ becomes zero before reaching the V.A., i.e., $\phi_{i+1} = 0$, this means K_+ so chosen is too large, i.e., the negative contribution will be so big that $\left(\frac{d\phi}{d\xi}\right)_{i+1}^2$ is brought from positive to zero too soon. If K_+ chosen is too small, the positive contribution of B.E. and T.E. dominates throughout the potential well, hence $\left(\frac{d\phi}{d\xi}\right)_{i+1}^2$ will not become zero when the V.A. is reached. With these points in mind, we can calculate as many decimal places for K_+ as we want following the same procedures used in case (A).

APPENDIX II. RESULTS FOR CUTOFF MAXWELLIAN TRAPPING

Assume the trapped electrons are of the cut-off Maxwellian form:

$$F_{\pm}(E, J) = \epsilon \frac{\delta(J)}{J} e^{-E/kT} [h(E - e|V|) - h(E - e|V_0|)] \quad (1)$$

The density expression, Eq. (2.14d), becomes

$$N_{te}(r) = \frac{4\pi}{m_e^{3/2} r^2} \left(\frac{\pi kT}{2}\right)^{1/2} \epsilon e^{-\eta\phi} \text{erf}(\sqrt{\eta(1-\phi)}) \quad (2)$$

where η is defined as

$$\eta = \frac{e|V_0|}{kT}$$

and $\text{erf}(x)$ is the error function defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

Poisson's equation then reads (see Eqs. (3.8) and (3.9))

$$\frac{d}{dz} \left(e^{2\phi} \left(\frac{d\phi}{dz} \right)^2 \right) = 2e^{2\phi} \left[\frac{K_+}{N\phi} - \frac{K_-}{N(1-\phi)} + M_- e^{-\eta\phi} \text{erf}(\sqrt{\eta(1-\phi)}) \right] \quad (3)$$

where

$$\begin{aligned}
 \xi &\equiv \ln \frac{r}{r_{V.A.}} \\
 K_+ &\equiv \frac{I_{bi}}{V_0^{3/2}} \left(\frac{2m_i}{e} \right)^{1/2} \\
 K_- &\equiv \frac{I_{be}}{V_0^{3/2}} \left(\frac{2m_e}{e} \right)^{1/2} \\
 M_- &\equiv \frac{(4\pi)^2 e}{V_0 m_e^{5/2}} \left(\frac{\pi kT}{2} \right)^{1/2} e
 \end{aligned} \tag{4}$$

Equation (3) can be written into finite difference equation

$$\begin{aligned}
 \left(\frac{d\phi}{d\xi} \right)_{i+1}^2 &= \left(\frac{d\phi}{d\xi} \right)_i^2 e^{-2\Delta\xi} + 4 \left\{ K_+ (\phi_{i+1}^{1/2} - \phi_i^{1/2}) + K_- ((1-\phi_{i+1})^{1/2} - \right. \\
 &\quad \left. (1-\phi_i)^{1/2}) + M_- \left[\frac{1}{\eta} (e^{-\eta\phi_{i+1}} \operatorname{erf}(\sqrt{\eta(1-\phi)}) - e^{-\eta\phi_i} \operatorname{erf}(\sqrt{\eta(1-\phi_i)})) - \right. \right. \\
 &\quad \left. \left. \frac{1}{\sqrt{\eta}} e^{-\eta} (\sqrt{1-\phi_{i+1}} - \sqrt{1-\phi_i}) \right] \right\}
 \end{aligned} \tag{5}$$

Equation (5) was solved with the equation

$$\phi_{i+1} = \phi_i + \Delta\xi \left| \left(\frac{d\phi}{d\xi} \right)_i \right|$$

and the boundary conditions

$$\begin{aligned}
 \phi(\xi=0) &= 0 \\
 \phi\left(\xi = \ln \frac{r_{R.C.}}{r_{V.A.}}\right) &= 1 \\
 \frac{d\phi}{d\xi}(\xi=0) &= 0
 \end{aligned}$$

The results are shown in Figures 15 to 18.

Comparison of Fig. 15 (or Fig. 16) with Fig. 9 shows that the shape of the $r_{V.A.} / r_{R.C.}$ vs T.E. curve

for given K_+^I and λ_+ is unchanged when the trapped electron distribution is changed from uniform in speed space to cut-off Maxwellian. This reveals an important piece of information: the properties we obtained for electrons uniformly trapped in speed space are quite general for different trapped distributions, because the curve we just mentioned is the most illustrative one for showing the influence of different amounts of trapped electrons (up to a maximum amount) on the location of the virtual anode. The only major difference is that the maximum allowed trapping differs greatly for different trapped distributions.

Next examine the trapped-to-beam ratio $\pi_e(r)$:

$$\pi_e(r) \equiv \frac{N_{te}(r)}{N_{be}(r)} = \frac{M_-}{K_-} e^{-\eta\phi} \operatorname{erf}(\sqrt{\eta(1-\phi)})$$

Referring to Fig. 17 and Fig. 18 for M_- and η , we find this factor at every r is much smaller than that for particles uniformly trapped in speed space (see Table 1 or Fig. 9), which means that for the cut-off Maxwellian distribution, the amount of trapped particles will be much smaller than that for uniform trapping. The current amplification will, in turn, be much poorer for the cut-off Maxwellian case than the uniform trapping case. Our calculations have also shown this. Briefly, this can be understood as follows;

Referring to Eq. (3) and (5), it is seen that when

we start from the V.A., the contribution of the T.E. (i.e., the quantity in the squared brackets in Eq. (5)) is negative and small, but because the density of the T.E. in this case consists of the product of the exponential function and error function, i.e., $e^{-\eta\phi} \text{erf}(\sqrt{\eta}(1-\phi))$, the variation of this density will be much more rapid than that for uniform T.E. (i.e., $(1-\phi)^{1/2}$). Hence after a few steps, the T.E. term becomes relatively large and negative, hence a relatively smaller M_- is allowed in front of that squared bracket to allow ϕ_{i+1} to increase to 1 (i.e., the real cathode) before $(\frac{d\phi}{dz})_{i+1}$ becomes zero. Since the amount of trapping is directly related to M_- , the trapping is limited to a very small amount, as indicated in the last paragraph.

Inside the V.A., the case of cut-off Maxwellian T.I. instead of T.E. was explored. In this case, one more condition is imposed on M_+ (the T.I. coefficient; the counterpart of M_- in (5)), namely, for the rapidly varying T.I. density $N_{ti}(r) = \frac{V_0 M_+}{4\pi r^2 e} e^{\eta\phi} \text{erf}(\sqrt{\eta}\phi)$, M_+ needs not only to be such that ϕ_{i+1} will reach 1 before $(\frac{d\phi}{dz})_{i+1}$ becomes negative, but also M_+ needs to be such that $(\frac{d\phi}{dz})_{i+1} = 0$ when $\phi_{i+1} = 1$. M_+ is thus limited to a very small value. Numerical calculations indicate that M_+ is so small that the ion current amplification factor was smaller than 10^{-4} . We can conclude that if the cut-off Maxwellian is the appropriate distribution, the amplification will be very small indeed.

Figure 15. The Ratio $r_{V.A.}/r_{R.C.}$ vs Electron Trapping Parameter M_-/K_+ (for Cut-off Maxwellian Distribution) for Different Combinations of K_+ and λ_+ with $\eta = 1.25$

Figure 16 The Ratio $\gamma_{V.A.}/\gamma_{R.C.}$ VS Electron Trapping Parameter $M-/K_+$ (for Cut-off Maxwellian Distribution) for Different Combinations of K_+ and λ_+ with $\eta=10$

Figure 17 M_{max} vs λ_+ for Different η

Figure 18 $\frac{r_{V.A.}}{r_{R.C.}}$ vs λ_+ for Different η , with $K_+ = 0.7$ and Maximum Cut-off Maxwellian Trapped Electrons Present in Region I

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