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ACCELERATOR DEPARTMENT
Informal Report

TRANSVERSE COLLECTIVE INSTABILITY EXCITED
BY A NONUNIFORM v-SHIFT IN INTENSE BEAMS

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MASTER

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ABSTRACT

A model which describes the effect of a nonuniform space charge ν -shift on the resistive wall instability associated with transverse collective oscillations of a beam of particles is proposed. It successfully predicts the overall characteristics of the instability as observed at the CERN ISR: (1) the observation of a current limitation in spite of the proportionate increase in ν -spread as charge is stacked; and (2) the fact that at the onset of instability, beam losses occur at a particular edge of the beam. The suggestion is simply that the original model of the resistive wall instability is applicable, but the effect on the betatron tune of the space-charge forces is directly on the ν -value density distribution. It is shown that under certain circumstances, this distortion of the density function can excite the instability. An important result is that the stable region in W -space (W is a complex quantity depending on the beam characteristics and its environment) is not diminished by the distortion, but rather shifted. The implications of this result is the notion of an optimization of design such that stability remains even as the beam intensifies and the distortion increases.

1. Introduction

Observations at the CERN ISR have demonstrated the presence of a current limiting transverse collective instability, termed the "brickwall" effect.¹ The classical transverse resistive wall instability² does not in the first instance seem adequate to explain the phenomenon. It has, in fact, been shown^{2,3} that the threshold for the instability is related to the parameter I/Δ , where I is the circulating current and Δ is a measure of the width of the ν -value (betatron tune) distribution. Now, since in the ISR, current accumulation is performed with I/Δ kept constant, or in any case below the design threshold for the transverse instability, it is necessary to seek another physical factor to explain the observation.

The suggestion has been made that space-charge effects can cause a change in the beam distribution of ν -values^{1,4} and that this distortion of the ν -distribution will produce a current limiting instability threshold, even though the parameter I/Δ remains constant.^{1,3,5} (Δ is assumed to be a nominal value, independent of space-charge considerations.) A qualitative explanation is given, as follows: The low current ν -value is a linear function of momentum, p or radius, R . As the current is increased the function is altered into a curved, approximately parabolic shape. At some current value, the curve $\nu(R)$ will become flat, $\nu' \sim 0$, at one edge of the beam. Thus the ν -value density at this edge of the beam tends to infinity and, the idea is, the beam becomes locally unstable. This is a very appealing explanation in view of the observation that beam loss does indeed occur at the edge of a stacked coasting beam.^{1,5}

However, this explanation, attractive as it is, is not entirely satisfying in that one has not reconciled the effect with a "collective growth of the beam."

In other words, one must remember that the instability is, after all, excited by the dipole moment of the entire beam.

In order to make the connection between the qualitative explanation given above and the dispersion relation theory of the collective instability,² we propose a simple model that takes account of the nonuniform space charge ν -shift. What is suggested here is that the space-charge effect be included by altering the distribution of ν -values in a manner consistent with the perturbed ν versus R curve. With the perturbed ν -distribution function, one can then solve the proper dispersion relation, finding the stability threshold and growth rate when unstable conditions apply.

This model has the basic property that the current limitation, i.e. the onset of the transverse instability, is a function only of the space-charge distortion. Thus, quantitative knowledge of this latter effect as a function of the charge distribution in the beam will allow the prediction of the current limitation. Here we will assume that the space-charge effect is a parabolic distortion and we will show how the instability threshold changes as a function of a single distortion parameter. We will also show how the model predicts initial beam losses at the beam edge when the instability threshold is crossed.

2. Effect of Space Charge ν -Shift

ν -Value Density Distribution

According to our model, the actual density distribution, that is, the one to be used in the dispersion relation, can be found from an unperturbed density distribution, an unperturbed ν value and a function describing the nonuniform space charge ν -shift. We work in terms of normalized ν -values, x , defined by

$$x = \frac{\nu - \nu_c}{d}, \quad (2.1)$$

where v_c is the center of the range of v -values,
and d is the range, a finite quantity.

Thus, x is limited by

$$-1 \leq x \leq 1 \quad . \quad (2.2)$$

If we let x_0 and x represent unperturbed (v_0) and perturbed v -values, then since x_0 is a linear function of radius, R we can relate the two by

$$x = x_0 + f(x_0) \quad , \quad (2.3a)$$

where, for a parabolic distorting function,

$$f(x_0) = r(1 - x_0^2) \quad , \quad (2.3b)$$

and r is the single distortion parameter. The perturbed density function, $n(x)$ can then be obtained from the unperturbed density function $n_0(x_0)$ through the relation

$$n(x) = n_0(x_0) \left. \frac{dx_0}{dx} \right|_{\text{evaluated at } x} \quad , \quad (2.4)$$

or,

$$n(x) = \frac{n_0(x_0)}{1 + f'(x_0)} = \frac{n_0(x_0)}{1 - 2rx_0} \quad . \quad (2.5)$$

For purposes of numerical work which we will describe later, we introduce a finite range distribution function, $n_0(x_0)$, which has a flat top and $\sin^2$ tails. Specifically, we have,

$$n_0(x_0) = \begin{cases} \xi \sin^2 \frac{\pi}{2} \left(\frac{1 + x_0}{1 - \alpha_L} \right) , & -1 \leq x_0 \leq -\alpha_L \\ \xi , & -\alpha_L \leq x_0 \leq \alpha_R \\ \xi \sin^2 \frac{\pi}{2} \left(\frac{1 - x_0}{1 - \alpha_R} \right) , & \alpha_R \leq x_0 \leq 1 \end{cases} \quad (2.6)$$

where,

$$\xi = \frac{1}{1 + \frac{1}{2}(\alpha_R + \alpha_L)} \quad . \quad (2.7)$$

For simplicity, we will assume $\alpha_L = \alpha_R$, and consider two types of unperturbed distributions. For rectangular-like distributions, we will use $\alpha_L = \alpha_R = 0.8$, while for gaussian-like distributions, we will take the above $n_0(x_0)$, with $\alpha_L = \alpha_R = 0.2$.

To define completely the perturbed density function, we need to specify the distortion parameter, r . From Eq. (2.5), we see immediately

$$|r| < \frac{1}{2} \quad . \quad (2.8)$$

We will thus describe the onset of instability as $|r| \rightarrow \frac{1}{2}$. We will use, for numerical work, representative values of r in the range 0 to $-\frac{1}{2}$. Note that negative r signifies a space-charge depression of the v -value.^{1,4,5} It should also be pointed out that the parameter, r , is a function of the slope, dv_0/dR . If sextupoles are used to increase the slope, the magnitude of r diminishes. This is, in fact, the technique used at the CERN ISR to reduce the effect and increase the stacked current.¹

We note that we are limiting ourselves to physical situations where v is a function of momentum or radius, but not betatron amplitude. This allows a simpler analysis, and the situation does in fact apply to the CERN ISR where the energy stacking scheme does produce a beam where the dominant v -variation is due to momentum spread. This is also the case for the ISA design where longitudinal phase-space stacking has also been proposed.⁶

The Dispersion Relation

It can be shown that the large t behavior of the dipole moment for transverse motion of an intense beam of particles will, in general, be of the form^{2,3,7}

$$P(\theta, t) \sim e^{i(k\theta - \omega_c t)} \quad (2.9)$$

where k is the mode number related to the azimuth, θ , and ω_c is a complex frequency,

$$\omega_c = \omega + i/\tau \quad (2.10)$$

Actually the collective solution is a good approximation only if $\tau \geq 0$ and only after a sufficiently long time such that other transients have damped to negligible size. For $\tau > 0$, τ represents the e-folding growth time for the unstable mode, k .

To compute the complex frequency, ω_c , for the oscillating system, we must know the particle distribution of betatron tunes, ν , and we must also be given certain beam and environment characteristics. First, we define a function $Z(x_c)$ by the relation,

$$\frac{1}{Z^*(x_c)} = \int \frac{n(x)}{x - x_c} dx \quad (2.11)$$

where the dependence on the mode number, k , is implicit,

$$x_c = x_R + i x_I \quad (2.12)$$

the real oscillating frequency ω is related to x_R by

$$\omega = \Omega(x_R d + k - \nu_c) \quad (2.13)$$

with³ $k, \omega > 0$, and $k > \nu_c$, while τ is connected to x_I by

$$\frac{1}{\tau} = \Omega x_I d \quad (2.14)$$

Ω is the particle angular frequency, and x is defined in Eq. (2.1). Next we introduce a complex quantity related to beam and environment characteristics:

$$Z_c = \alpha(U' + V' + iV') \quad (2.15)$$

with

$$\alpha = \left(\frac{I}{d} \right) \frac{Rr_p}{ev_c \Omega \beta^2 \gamma^3} \quad (2.16)$$

while U' and V' are functions of beam and chamber characteristics, and for a circular geometry, are given by

$$U' = - \left(\frac{1}{a^2} - \frac{1}{b^2} \right) , \quad (2.17)$$

and

$$V' = \frac{e \beta^2 \gamma^2}{b^3} \sqrt{\frac{2\epsilon_0 \rho_e}{|\omega_c|}} . \quad (2.18)$$

The various parameters are as follows:

I is the beam current in amps,

R is the radius of the ring,

r_p is the classical proton radius $= e^2/4\pi\epsilon_0 mc^2 = 1.54 \times 10^{-18}$ m,

ϵ_0 is the free space dielectric constant, $\epsilon_0 = 1/36\pi \times 10^{-9}$ sec/ohm-m,

ρ_e is the resistivity of the vacuum chamber wall, in ohm-m,

a is the beam radius,

b is the chamber radius,

γ is the energy in units of proton rest mass, m ,

β is the particle velocity in units of c , the velocity of light,

and e is the electric charge, $e = 1.602 \times 10^{-19}$ coulombs.

The angular frequency, ω_c , that appears in the expression for V' , Eq. (2.18), is a function of the mode number, k , and can generally be replaced by its approximate value

$$|\omega_c| \approx \Omega(k - \nu_c) . \quad (2.19)$$

We can now write the dispersion relation determining the complex frequency, ω_c , in the form

$$Z(x_c) = Z_c = \alpha(U' + V' + iV') , \quad (2.20)$$

With $Z(x_c)$ defined in Eq. (2.11). The density function to be used here is the space charge distorted one discussed earlier. Thus we are interested in

solutions of Eq. (2.20) for x_c as a function of the distortion parameter, r . The current enters implicitly since r is a function of I . Explicitly, I enters through its dependence on α . From Eq. (2.16) this dependence is of the form I/d . This means, as we have pointed out before, that if r is neglected, and if stacking is such that d is proportional to I , there should be no current limitation. The current limitation in our model is thus through the distortion of the density function, which is an increasing function of I .

In particular, we will map the function $Z(x_c)$ onto the Z -plane. If we consider the case $\text{Im}(x_c) = 0$, then we have the stability boundary for Z . If the physical value of Z , Z_c , lies within this boundary, stable conditions prevail. If, on the other hand, Z_c lies outside this boundary, the beam is unstable. We will show how a stable beam for $r = 0$ will become unstable as r approaches $-\frac{1}{2}$ by plotting stability boundaries for different values of r and relating these plots to the value of Z_c for the CERN ISR⁸ and for some possible values for the ISA.⁶ Under unstable conditions, we will compute growth times, τ .

Beam Oscillation Amplitude

The dipole moment of the beam is a measure of the beam oscillation amplitude. As previously mentioned, the large time behavior is of the form $e^{[i(k\theta - \omega t) + t/\tau]}$. However, the behavior of separate portions of the beam, in terms of ν -value, will be different. By defining a dipole moment density per unit ν , $P_\nu(\theta, t)$, that is,

$$P(\theta, t) = \int P_\nu(\theta, t) d\nu, \quad (2.21)$$

then it can be shown^{3,7} that in terms of the variables x , defined in Eq. (2.1), and x_c , defined in Eqs. (2.12, 2.13, and 2.14), we can express the ν -dependence

of the beam oscillation amplitude through the function $A(x)$, given by

$$A(x) = \frac{n(x)}{\sqrt{\left(\frac{x - x_R}{x_I}\right)^2 + 1}} \quad (2.22)$$

This expression applies, of course, when unstable conditions are present, i.e. $x_I > 0$. Thus the time dependence, $e^{t/\tau}$, means the amplitude of the entire beam is growing. The expression (2.22) simply gives the relative amplitude for different "v-bites" of the beam at a given time near the onset of growth. In general terms, loss occurs at the v-bites with the largest amplitude. As the current is reduced, the distortion parameter is reduced and stable conditions reappear. Thus, the model predicts beam loss at the largest amplitudes. As stable conditions develop, the oscillations damp and the beam reestablishes itself at a lower current. This, in qualitative terms, accurately represents the CERN observations.⁵ We can obtain a measure of the amplitude per "radius-bite" if we recall that $n(x)dx = n_0(x_0)dx_0$ and that x_0 is proportional to ΔR . Thus, the amplitude per unit R is related to the quantity

$$\bar{A}(x_0) = \frac{n_0(x_0)}{\sqrt{\left(\frac{x(x_0) - x_R}{x_I}\right)^2 + 1}} \quad (2.23)$$

We will later consider in detail the function $\bar{A}(x_0)$ under conditions of instability induced by space-charge distortion.

3. Results of Computations

We will describe the impact on beam stability of variation of two parameters. On the one hand, we consider the effect of the shape of the unperturbed distribution. This, in our terms, means varying the flat top of the distribution. In the second place, we will show the influence of the distortion parameter. We plot three types of curves:

(a) Stability Boundaries in the $W = (\alpha U', \alpha V')$ Plane. This consists of a map of Eq. (2.11), with $\text{Im}(x_c) = 0$, using Eq. (2.15) to obtain

$$W = Z_R - Z_I + iZ_I. \quad (3.1)$$

In Figs. 1(a) and (b) are shown such maps. The (a) curves have the flat top parameter $\alpha_L (= \alpha_R) = 0.2$, the (b) curves have $\alpha_L = 0.8$. Three different r values are used - ($r = 0, -0.2, -0.49$).

(b) Solution of the Dispersion Relation under Unstable Conditions. This is done diagrammatically by mapping Eq. (2.11) in the W -plane for different $x_I \geq 0$.

Figures 2(a) and (b) are examples of such maps, with the (a) curves corresponding to $\alpha_L = 0.2$, $r = -0.49$ and the (b) curves having $\alpha_L = 0.8$, $r = -0.49$.

(c) ν -Dependence of Beam Oscillation Amplitude under Unstable Conditions. The function $\bar{A}(x_c)$, Eq. (2.23), is plotted in Fig. 3. The curve corresponds to $\alpha_L = \alpha_R = 0.8$. The values of x_R, x_I are obtained from Fig. 2(b). The value of the distortion parameter is $r = -0.49$.

Let us consider Fig. 1. The region of stability in the W -plane is the boundary curve together with the real ($\alpha U'$) axis. The most striking feature of these curves is the fact that the stable area is not diminished, just shifted. This has an implication for design considerations which we just mention. From the curves we can see that there is a region of stability common to all the curves. Thus, in the design of the ISA, for example, the physical values for W_R, W_I might be chosen to lie in this "stable" region. If, however, the physical values of W_R, W_I are such that $|U'| \ll V'$ as is the case at the CERN ISR, this represents a point in Fig. (1) somewhere along the vertical axis, i.e. $W_R = 0$. If, therefore, a stable condition exists with $r = 0$, then as $|r|$ is increased, the stability boundary approaches this point, ultimately crossing it, and thus making the beam unstable.

In order to see this reasoning process more clearly, let us consider a concrete example. If we assume $R = 193$ m, $v_c = 8.75$, $\gamma = 30$, $\Omega = 2\pi(247.7) \times 10^3$, $d = 0.03$, $I = 10$ amps, $V' = 55,000$ m⁻², (presumably for the lowest mode, $k = 9$) then $W_I = 0.09$. These values correspond roughly to the CERN ISR.^{1,5,8} As can be seen from Fig. 1(b), i.e. for a wide stack, and $r \rightarrow -\frac{1}{2}$, this point is crossed by the stability boundary and instability is predicted.

We can pursue this example by first determining the growth rate from Fig. 2(b) and then, using the particular values of x_R and x_I corresponding to the unstable state, plot the beam oscillation amplitude function, $\tilde{A}(x_0)$, as a function of x_0 , i.e. R . This is done in Fig. 3. For the lowest mode, we obtain from x_R and x_I , through Eqs. (2.13) and (2.14), the oscillation frequency, ω , and growth time, τ , for the instability: $\omega = 0.22 \Omega$ (frequency = 54.5 kHz), $\tau = 2.1$ msec. (It is interesting to compare this actual growth time to the zero- v -width growth time,² τ_0 , which is given by $\tau_0 = ev_c^2 \beta^2 \gamma^3 / IR r_p V' = 0.23$ msec. As is evident, this latter number bears little relationship to the actual growth time in this case.) Observation of the beam oscillation amplitude function, $\tilde{A}(x_0)$ for this case (Fig. 3), allows us to see how initial beam loss is expected to occur in our model. Qualitatively it is evident that beam loss occurs initially at the lower edge of the v -distribution which is related directly to the fact that $r < 0$. In order to translate this to beam loss in physical space, we require v_0 as a function of radius. With a linear function, beam loss will occur at a "small" radius if $dv_0/dR > 0$, and at a "large" radius if $dv_0/dR < 0$. This is essentially consistent with the observations at the CERN ISR.

4. Conclusions

We have proposed a model which describes the effect of a nonuniform space charge v -shift on the resistive wall instability associated with transverse collective oscillations of a beam of particles. It successfully predicts the overall characteristics of the instability as observed at the CERN ISR: that is, (1) the observation of a current limitation in spite of the proportionate increase in v -spread during the stacking of charge; and (2) the fact that at the onset of instability beam losses occur at a particular edge of the beam. The suggestion is simply that the original model of the resistive wall instability² is applicable, but the effect on the betatron tune of the space-charge forces is directly on the v -value density distribution. Under certain circumstances, as we have indicated, this distortion can excite the instability. An important result of the computations is that the stable region in $W = (\alpha U', \alpha V')$ space (these quantities are functions of the beam and environment characteristics) is not diminished by the distortion, but rather shifted. Thus, one may conceive of an optimization of design such that stability remains even as the beam intensifies and the distortion increases.

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Figure Captions

Fig. 1 Stability Boundaries in W-plane.

(a) $\alpha_L = \alpha_R = 0.2$

(b) $\alpha_L = \alpha_R = 0.8$

x in (b) corresponds to the physical point used in the example in the text: $W_R = 0$, $W_I = 0.09$. Boundaries are drawn for three values of the distortion parameter, r .

Fig. 2 W-plane map for unstable conditions.

(a) $\alpha_L = \alpha_R = 0.2$

(b) $\alpha_L = \alpha_R = 0.8$

The distortion parameter, $r = -0.49$. The boundaries correspond to different growth rates. x in (b) corresponds to the physical point used in the example in the text: $W_R = 0$, $W_I = 0.09$. The values of (x_R, x_I) corresponding to the solution of the dispersion relation, Eq. (2.20), can be estimated from the graphs: $x_R = -0.987$, $x_I = 0.01$.

Fig. 3 v-dependence of beam oscillation amplitude under unstable conditions.

$\alpha_L = \alpha_R = 0.8$, $r = -0.49$. The values of the x_R, x_I are obtained from Fig. 2(b). The variable x_0 is linearly related to the radius, R .

Fig. 1

Fig. 2

Fig. 3