326
12-9-64

UCRL-7784

MASTER

University of California

Ernest O. Lawrence
Radiation Laboratory

**NUMERICAL CALCULATION OF STATIC
MAGNETIC FIELDS IN AN IRREGULAR TRIANGLE MESH**

Livermore, California

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

NUMERICAL CALCULATION OF STATIC MAGNETIC FIELDS
IN AN IRREGULAR TRIANGLE MESH

Alan M. Winslow

August 4, 1964

NUMERICAL CALCULATION OF STATIC MAGNETIC FIELDS IN AN IRREGULAR TRIANGLE MESH

Alan M. Winslow

Lawrence Radiation Laboratory, University of California
Livermore, California

August 4, 1964

ABSTRACT

A general two-dimensional magnetostatic code has been developed which calculates the vector potential and magnetic induction in any plane arrangement of iron, air, and conductors. It uses a mesh composed of irregular triangles in which the mesh lines can be distorted to conform to irregular interfaces and boundaries; thus eliminating the need for special equations at these places. The speed and accuracy of the calculation appear to be comparable to that obtainable using regular meshes.

I. INTRODUCTION

When solving partial differential equations by finite difference methods it is customary to use a mesh composed of rectangular (or sometimes equilateral triangular) zones. These zones are uniform in size except in regions where more (or less) accuracy in the solution is desired, in which case the zones may be correspondingly smaller (or larger). The mesh spacing may be varied by inserting or deleting mesh points, a procedure usually leading to special equations which complicate the calculation and may cause loss of accuracy.

A more serious problem connected with meshes of this type arises from the presence of interfaces or boundaries which do not conform to the mesh lines. These must either be approximated by fitting them to the nearest mesh lines, or be permitted to cut across zones, again leading to extra mesh points and special equations.

These difficulties can be avoided if one is willing to give up the use of regularly shaped zones and permit mesh lines to be distorted to conform to interfaces and boundaries. No special mesh points are then needed, and whole zones can be used to fit any line that can be approximated by straight segments. Furthermore, variations in mesh spacing may be spread out over several zones and thus take place smoothly.

The first calculation in which an irregular mesh was used to solve partial differential equations was apparently that of R. H. MacNeal in 1953,¹ who used an electrical-circuit analog computer in which the mesh lines were represented by resistors to solve the equation $\vec{\nabla} \cdot (\sigma \vec{\nabla} \phi) + \tau = 0$ in an irregular triangle mesh. His method shows striking similarities to that described in this report.

In 1958, C. E. Leith of this Laboratory, who was unaware of MacNeal's work, derived a similar finite difference approximation to the Laplacian in an irregular triangle mesh² and wrote a digital computer code to solve diffusion problems in such a mesh using an adaptation of the alternating direction method. In Part II we give derivations of this finite difference approximation, and in Part III we describe a recent application of the technique by the writer to the calculation of static two-dimensional magnetic fields, using the method of point successive overrelaxation.

For a given problem, after the zoning of interfaces and boundaries has been prescribed, there is of course no unique way to construct an irregular mesh. Since it seems desirable for the mesh lines to be smooth, a useful method is to make use of the averaging properties of solutions to the Laplace equation and interpret the zoning as a pseudo-potential problem with the mesh lines as the equipotentials. A zoning code employing an approximate form of such a method was also developed by Leith² and improved by W. P. Crowley and the writer in 1962.³ This code has been used to produce all the meshes described in this report.

Meshes composed of irregular quadrilaterals or other geometric figures can also be used. Triangle meshes are easier to adapt to irregular boundaries, and also have the advantage of making it particularly simple to approximate functions within the mesh by linear expressions when deriving the finite difference equations.

II. DERIVATION OF THE DIFFERENCE EQUATIONS

We use a Cartesian coordinate system, with coordinates x, y . For simplicity we have chosen a mesh which is topologically regular and which can be represented by the mesh of which a portion is shown in Fig. 1, with six triangles meeting at every interior mesh point. This restriction, however, is not essential (see Ref. 1). In order to conform to boundaries and interfaces, the triangles are usually irregular in size and shape, the irregularity being more or less smoothly distributed by the zoning code referred to above (see Figs. 6-21 below).

Fig. 1. Structure of the triangle mesh.

The relation between the topology and the geometry of the mesh can best be understood by studying an actual problem. Figure 7a shows the topological connections of the mesh shown in Figs. 7 and 9. The circled points in Fig. 7a were specified in the input to the problem, the remaining points being placed by the zoning code.

Let us consider the solution of the nonlinear diffusion equation

$$c(\phi) \frac{\partial \phi}{\partial t} = \vec{\nabla} \cdot [\lambda(\phi) \vec{\nabla} \phi] \quad (1)$$

in such a mesh, with prescribed boundary conditions, where the positive coefficients $c(\phi)$ and $\lambda(\phi)$ may also be functions of position. We define $\phi(x, y, t)$ at triangle vertices. Within each triangle the quantity ϕ is assumed to be the linear function of position determined by the three vertex values,

and consequently each triangle has a $\text{grad } \phi$ vector associated with it. We consider the coefficients c and λ to be constant within a given triangle.

Associated with the primary triangle mesh we define a secondary mesh of 12-sided figures whose vertices are alternately the centroids of the six adjacent triangles and the midpoints of the six adjacent sides. This is shown in Fig. 2, in which a single such figure is shaded. The secondary mesh element surrounding a given vertex includes one-third of the area of each of the six primary mesh triangles sharing that vertex.

Fig. 2. Relation of secondary to primary mesh.

In Fig. 3 is shown a typical triangle divided into the three pieces which it contributes to the three secondary mesh figures surrounding the three vertices.

Fig. 3. Contributions of a triangle to the secondary mesh.

For such a triangle we have:

The area of the triangle:

$$A = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \frac{1}{2} [y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)]; \text{ and}$$

The area of the quadrilateral associated with a given vertex:

$$a = \frac{A}{3}.$$

We shall give three different derivations of the same finite difference approximation (the first being Leith's original derivation) and then describe MacNeal's derivation of his own approximation.

1. Derivation by Flux Conservation

Consider the triangle $i + 1/2$ defined by the two side vectors $\vec{s}_i, \vec{s}_{i+1}$, with values ϕ_i, ϕ_{i+1}, ϕ at the respective vertices as shown in Fig. 4.

Fig. 4. Vectors used for flux calculation.

The gradient vector $\vec{\nabla}\phi_{i+1/2}$ satisfies the equation

$$\phi_j = \phi + \vec{s}_j \cdot \vec{\nabla}\phi_{i+1/2} \quad j = i, i+1 \quad (2)$$

and is given by

$$\vec{\nabla}\phi_{i+1/2} = \frac{(\phi_i - \phi) \vec{s}_{i+1}^\perp - (\phi_{i+1} - \phi) \vec{s}_i^\perp}{\vec{s}_i \cdot \vec{s}_{i+1}^\perp} \quad (3)$$

where the vector $\vec{s}^\perp$ represents the vector $\vec{s}$ rotated clockwise by an angle $\pi/2$.

Within each triangle the flux of the diffusing quantity is given by

$$\vec{F}_{i+1/2} = -\lambda_{i+1/2} \vec{\nabla}\phi_{i+1/2}.$$

The conservation law can be expressed through Gauss' theorem by equating the surface integral of the left side of Eq. (1) over the secondary mesh element to the integral of the normal component of $\vec{F}$ over the boundary of the secondary mesh element.

The contribution $R_{i+1/2}$ from the triangle $i+1/2$ shown in Fig. 4 to the rate of change R in the secondary mesh element is

$$\begin{aligned} R_{i+1/2} &= \vec{F}_{i+1/2} \cdot (\vec{m}_1 + \vec{m}_2) \\ &= \vec{F}_{i+1/2} \cdot \left(\frac{\vec{s}_{i+1}^\perp}{2} - \frac{\vec{s}_i^\perp}{2} \right) \\ &= \frac{1}{2} \vec{F}_{i+1/2} \cdot (\vec{s}_{i+1}^\perp - \vec{s}_i^\perp). \end{aligned}$$

Summing around the vertex and using the central vertex value ϕ as the average value over the dodecagon, we obtain

$$\sum_{i=1}^6 c_{i+1/2} \frac{\Delta\phi_{i+1/2}}{\Delta t} a_{i+1/2} \doteq \frac{\Delta\phi}{\Delta t} \sum_{i=1}^6 c_{i+1/2} a_{i+1/2} = \sum_{i=1}^6 \vec{F}_{i+1/2} \cdot \frac{1}{2}(\vec{s}_{i+1} - \vec{s}_i)$$

or

$$\Delta\phi = \frac{\Delta t \sum_{i=1}^6 \vec{F}_{i+1/2} \cdot \frac{1}{2}(\vec{s}_{i+1} - \vec{s}_i)}{G}, \quad (4)$$

where

$$G = \sum_{i=1}^6 c_{i+1/2} a_{i+1/2}. \quad (5)$$

Using (3), and letting $\psi_i = \phi_i - \phi$, we can express the sum in Eq. (4) as

$$\begin{aligned} S &= \frac{1}{2} \sum_i \frac{\lambda_{i+1/2} (\psi_i \vec{s}_{i+1} - \psi_{i+1} \vec{s}_i) \cdot (\vec{s}_{i+1} - \vec{s}_i)}{\vec{s}_i \cdot \vec{s}_{i+1}} \\ &= \frac{1}{2} \sum_i \frac{\lambda_{i+1/2} [\psi_i \vec{s}_{i+1} \cdot (\vec{s}_{i+1} - \vec{s}_i) + \psi_{i+1} \vec{s}_i \cdot (\vec{s}_i - \vec{s}_{i+1})]}{\vec{s}_i \cdot \vec{s}_{i+1}} \end{aligned}$$

where we have made use of the relations

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{u} \cdot \vec{v} \\ \vec{u} \cdot \vec{v} &= -\vec{v} \cdot \vec{u}. \end{aligned}$$

Since the sum is cyclic, we can reduce the index by one in the second term, obtaining

$$\begin{aligned} S &= \frac{1}{2} \sum_i \left[\frac{\lambda_{i+1/2}}{\vec{s}_i \cdot \vec{s}_{i+1}} \vec{s}_{i+1} \cdot (\vec{s}_{i+1} - \vec{s}_i) + \frac{\lambda_{i-1/2}}{\vec{s}_{i-1} \cdot \vec{s}_i} \vec{s}_{i-1} \cdot (\vec{s}_{i-1} - \vec{s}_i) \right] \psi_i \\ &= \sum_i w_i (\phi_i - \phi). \quad (6) \end{aligned}$$

The coefficient of ψ_i in (6) is called the coupling coefficient for the line joining the vertex i and the center. It depends only on the nature of the two triangles having this as a common side, and can be written

$$w_i = \frac{1}{2} \left(\lambda_{i+1/2} \cot \theta_{i+1/2} + \lambda_{i-1/2} \cot \theta_{i-1/2} \right). \quad (7)$$

The angles $\theta_{i\pm 1/2}$ are shown in Fig. 5a.

Fig. 5a. Angles associated with the coupling coefficient w_i .

We note that the coupling between two points x_1, y_1 and x_2, y_2 is symmetric, so that

$$w_{12} = w_{21}. \quad (8)$$

Moreover, w_i can be positive or negative depending on the relative magnitudes of $\lambda_{i\pm 1/2}$ and whether the angles are acute or obtuse. However, the sum of the couplings around a given point is

$$\sum_{i=1}^6 w_i = \frac{1}{4} \sum_i \frac{\lambda_{i+1/2}}{A_{i+1/2}} (\vec{s}_{i+1} - \vec{s}_i)^2, \quad (9)$$

(since $\vec{s}_i \cdot \vec{s}_{i+1} = 2A_{i+1/2}$) which is positive.

Thus, Eq. (1) can be written in finite difference approximation

$$\frac{\Delta \phi}{\Delta t} = \frac{1}{G} \sum_i^6 w_i (\phi_i - \phi). \quad (10)$$

The Laplacian itself is approximated by

$$\vec{\nabla} \cdot (\lambda \vec{\nabla} \phi) = \frac{1}{A_D} \sum_{i=1}^6 w_i (\phi_i - \phi), \quad (11)$$

where

$$A_D = \sum_{i=1}^6 a_{i+1/2}$$

is the area of the secondary mesh dodecagon. Thus we have replaced the fluxes by currents $w_i(\phi_i - \phi)$ flowing along mesh lines.

Lines joining boundary points are treated in the same manner as lines joining interior points except that the coefficient λ of material outside the boundary is set equal to zero. If the outward normal gradient $\partial\phi/\partial n$ is prescribed at a boundary, we add an external current I_j at each surface point j given by

$$I_j = \left(\frac{\partial\phi}{\partial n} \right)_j \frac{1}{2} (\lambda_{j-1/2} s_{j-1/2} + \lambda_{j+1/2} s_{j+1/2})$$

where $\frac{1}{2} (s_{j-1/2} + s_{j+1/2})$ is the length of boundary associated with the boundary point and $\lambda_{j\pm 1/2}$ are the coefficients of the associated triangles (see Fig. 5b).

Fig. 5b. Segments used for boundary currents.

I_j is equal to the total flux crossing the boundary over the segments a to b shown in Fig. 5b.

2. Derivation by Line Integration⁴

An alternative derivation can be obtained by approximating derivatives by area averages and using Gauss' theorem in two dimensions⁵ which states that over a region R

$$\iint_R [f_x(x, y) + g_y(x, y)] dx dy = \oint [f(x, y) dy - g(x, y) dx],$$

where the integral on the right is a line integral around the boundary of the region R. Thus we write

$$f_x = \frac{\iint f_x dx dy}{\iint dx dy} = \frac{\oint f dy}{\oint x dy}$$

$$g_y = \frac{\iint g_y dx dy}{\iint dx dy} = \frac{-\oint g dx}{\oint x dy}.$$

To evaluate the second derivatives in Eq. (1) we use the dodecagon shown in Fig. 2 as our contour, obtaining

$$\left(\lambda \phi_x \right)_x = \frac{\oint \lambda \phi_x dy}{\oint x dy}$$

$$\left(\lambda \phi_y \right)_y = \frac{-\oint \lambda \phi_y dx}{\oint x dy}.$$

Within each triangle, ϕ_x and ϕ_y are assumed to be constant and given by

$$\phi_x = \frac{\oint \phi dy}{\oint x dy}$$

$$\phi_y = \frac{-\oint \phi dx}{\oint x dy},$$

where the contour is now the boundary of the given triangle. Replacing the integrals by finite sums, we obtain

$$(\lambda \phi_x)_x = \frac{1}{A_D} \sum_i X_i (\phi_i - \phi) \quad (12)$$

$$(\lambda \phi_y)_y = \frac{1}{A_D} \sum_i Y_i (\phi_i - \phi),$$

where

$$X_i = \frac{1}{2} \left[\frac{\lambda_{i+1/2} (y_{i+1} - y)(y_{i+1} - y_i)}{\vec{s}_i \cdot \vec{s}_{i+1}} + \frac{\lambda_{i-1/2} (y_{i-1} - y)(y_{i-1} - y_i)}{\vec{s}_{i-1} \cdot \vec{s}_i} \right]$$

$$Y_i = \frac{1}{2} \left[\frac{\lambda_{i+1/2} (x_{i+1} - x)(x_{i+1} - x_i)}{\vec{s}_i \cdot \vec{s}_{i+1}} + \frac{\lambda_{i-1/2} (x_{i-1} - x)(x_{i-1} - x_i)}{\vec{s}_{i-1} \cdot \vec{s}_i} \right]$$

and

$$X_i + Y_i = w_i.$$

Thus the x and y components of the scalar products in the numerators of Eq. (6) are just the y and x components, respectively, of our finite difference approximation to the Laplacian.

3. Derivation by Variational Method⁶

The generalized Laplace equation

$$\vec{\nabla} \cdot (\lambda \vec{\nabla} \phi) = 0$$

can be obtained by minimizing the integral

$$I = \iint g \, dx dy$$

with respect to ϕ , where g is a function of $(\vec{\nabla} \phi)^2$ such that $g' = \lambda$. We can obtain a finite difference approximation by substituting for $\vec{\nabla} \phi$, in the equation above, the expression (3) and again minimizing with respect to ϕ at each mesh point.

Summing over the six triangles around the vertex at which ϕ is defined, we have

$$I = \sum_{i=1}^6 g_{i+1/2} A_{i+1/2}. \quad (13)$$

Taking the variation of (13) with respect to ϕ we get

$$\delta I = \sum_{i=1}^6 \delta g_{i+1/2} A_{i+1/2} = \sum_{i=1}^6 \lambda_{i+1/2} \delta (\vec{\nabla} \phi)_{i+1/2}^2 A_{i+1/2}.$$

From (3) we have

$$\vec{\nabla} \phi_{i+1/2} = \vec{a}_{i+1/2} + \vec{\beta}_{i+1/2} \phi,$$

where

$$\vec{a}_{i+1/2} = \frac{\phi_i \vec{s}_{i+1} - \phi_{i+1} \vec{s}_i}{\vec{s}_i \cdot \vec{s}_{i+1}}$$

$$\vec{\beta}_{i+1/2} = \frac{\vec{s}_i - \vec{s}_{i+1}}{\vec{s}_i \cdot \vec{s}_{i+1}},$$

so that

$$(\vec{\nabla} \phi)_{i+1/2}^2 = (\alpha^2 + \beta^2 \phi^2 + 2 \vec{a} \cdot \vec{\beta} \phi)_{i+1/2}$$

and

$$\delta I = 2\delta \phi \sum_i \lambda_{i+1/2} (\beta^2 \phi + \vec{a} \cdot \vec{\beta})_{i+1/2} A_{i+1/2} = 0.$$

Since

$$A_{i+1/2} = \frac{1}{2} (\vec{s}_i \cdot \vec{s}_{i+1}),$$

we have

$$-\phi \sum_i \lambda_{i+1/2} \frac{(\vec{s}_{i+1} - \vec{s}_i)^2}{\vec{s}_i \cdot \vec{s}_{i+1}} + \sum_i \lambda_{i+1/2} \left[\frac{\phi_i (\vec{s}_{i+1}^2 - \vec{s}_{i+1} \cdot \vec{s}_i) + \phi_{i+1} (\vec{s}_i^2 - \vec{s}_i \cdot \vec{s}_{i+1})}{\vec{s}_i \cdot \vec{s}_{i+1}} \right] = 0.$$

Shifting the index by one in the last term and using (9), we obtain

$$\sum_{i=1}^6 w_i (\phi_i - \phi) = 0,$$

where the w_i are the same as in (7). Except for the factor $1/A_D$, this is the same result as (11).

4. Derivation of MacNeal's Approximation¹

In solving the equation $\vec{\nabla} \cdot (\sigma \vec{\nabla} \phi) + \tau = 0$, MacNeal interprets $\phi(x, y)$ as the electrical potential in a plane region of conductivity $\sigma(x, y)$ and $\tau(x, y)$ as a prescribed current density. Then $-\sigma \vec{\nabla} \phi$ represents the current density flowing in the material.

After triangulating the region, he proceeds as follows¹:

"Consider a portion of this network shown in Fig. 4. The perpendicular bisectors of the sides of the triangles divide the region into polygons surrounding each point. A network of resistors is now constructed connecting the vertices of the triangles. The voltage across each resistor shall be interpreted as the line integral of the gradient of the potential between the two points it connects. For example, for points A and B of Fig. 4:

$$V_B - V_A = \int_A^B \nabla \phi \cdot d\mathbf{l}. \quad (5)'$$

[Fig. 4. Portion of the asymmetrical network of triangles. (From R. H. Mac Neal, Quarterly of Applied Mathematics, Vol. 11, p. 295, 1953.)]

The current in the resistor shall be interpreted as the total normal flux crossing the common boundary of the dotted polygons surrounding the two points. Since the current density is $-\sigma \vec{\nabla} \phi$, we have:

$$I_{AB} = - \int_1^2 \sigma (\nabla \phi \cdot \mathbf{n}) dr, \quad (6)'$$

where n is a unit vector normal to dr . If $\nabla\phi$ and $\sigma\nabla\phi$ are now expanded in Taylor's series about the point O , the midpoint of the segment $A-B$, and all terms except the first are neglected, $\nabla\phi \approx (\nabla\phi)_0$ and $\sigma\nabla\phi \approx \sigma_0(\nabla\phi)_0$. Since the segment AB is normal to the segment $1-2$, the projection of $\nabla\phi$ on dl is the same as the projection of $\nabla\phi$ on n . Hence

$$V_B - V_A \approx \int_{AB} |\nabla\phi_0| \cos \alpha, \quad (7)'$$

$$I_{AB} \approx -\sigma_0 r_{12} |\nabla\phi_0| \cos \alpha. \quad (8)'$$

The value of the resistor connecting A and B is

$$R_{AB} = \frac{V_A - V_B}{I_{AB}} = \frac{\int_{AB}}{\sigma_0 r_{12}} \quad (9)'$$

Hence R_{AB} depends only on the physical properties of the material and the manner in which the region is subdivided. If the segment $1-2$ were not perpendicular to AB , the value of the resistor would depend on the orientation of the field. It can also be shown by a simple geometrical argument that

$$Y_{AB} = \frac{1}{R_{AB}} = \frac{\sigma_0}{2} (\cot \beta_1 + \cot \beta_2), \quad (10)'$$

where β_1 and β_2 are the interior angles of the triangles subtended by the segment AB . If both β_1 and β_2 are acute angles R_{AB} will be physically realizable."

Applying Gauss' theorem to the polygon he obtains:

$$\sum_P I_{pB} + \tau_B A_B = 0 \quad (14)'$$

where I_{pB} is the current flowing into node B from the p -th adjacent node. Equation (14) is Kirchhoff's law for the sum of the currents entering a node. It shows that the appropriate area for calculating the current to be inserted into node B is the area interior to the dotted polygon surrounding B .

By substituting Eq. (9) into Eq. (14) the generalized difference equation for node B is obtained:

$$\sum_P \sigma_0 \frac{r_{Bp}}{l_{Bp}} (V_p - V_B) + \tau_B A_B = 0, \quad (15)'$$

where l_{Bp} is the distance between node B and node p and r_{Bp} is the length of the segment that is common to the polygons surrounding node B and node p ."

MacNeal's conductances Y_{AB} are exactly Leith's coupling coefficients w_i for the case of uniform conductivity σ , and his derivation permits a simple geometrical interpretation of the couplings in terms of the ratio r/ℓ as long as the angles β_1 and β_2 are not obtuse.

It will be noted that MacNeal does not assume that ϕ varies linearly with position, but instead assumes that $\vec{\nabla}\phi$ is constant over each side of the polygon. His statement concerning the necessity for segment 1-2 to be perpendicular to AB is not true if one assumes linear variation.

Since Leith's derivation depends only on the fact that the polygon bisects the triangle sides, and not on the actual shape of the polygon, the two derivations give the same couplings and differ only in the shape, and consequently area, of the polygon. MacNeal later discusses changes in the shape of this polygon for improved accuracy.

III. APPLICATION TO THE CALCULATION OF STATIC MAGNETIC FIELDS

1. Derivation of the Difference Equations

Consider an arbitrary distribution of infinite straight parallel conductors carrying constant currents parallel to the z axis. The magnetic field $\vec{H}(x, y)$ and the magnetic induction $\vec{B}(x, y)$ have components only in the x, y plane while the current density $\vec{j}$ and magnetic vector potential $\vec{A}$ have only z components which we label simply $j(x, y)$ and $A(x, y)$.

From the relations

$$\vec{B} = \mu \vec{H} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{H} = 4\pi \vec{j},$$

where $\mu(x, y, |\vec{B}|)$ is the magnetic permeability, we obtain

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = 4\pi \vec{j}. \quad (14)$$

Because of the single-component nature of $\vec{A}$ and $\vec{j}$, Eq. (14) reduces to

$$\vec{\nabla} \cdot (\gamma \vec{\nabla} A) = -4\pi j, \quad (15)$$

where $\gamma \equiv \mu^{-1}$.

Equation (15) is equivalent to the nonlinear diffusion Eq. (1) with the currents playing the role of sources constant in time and with a "flux" given by $\vec{F} = -\gamma \vec{\nabla} A$. Its finite difference approximation in a triangle mesh is therefore

$$\sum_{i=1}^6 w_i (A_i - A) = -4\pi I, \quad (16)$$

where

$$I = \sum_{i=1}^6 j_{i+1/2} a_{i+1/2} \quad (17)$$

is the total current through the secondary mesh dodecagon surrounding the vertex x, y at which A is defined. The coefficients w_i in (16) are calculated by (7) using γ in place of λ .

The boundary conditions require that at an interface the normal component of $\vec{B}$, $\vec{B}_n$ and the tangential component of $\vec{H}$, $\vec{H}_t$ be continuous. Our derivation of (10) by Gauss' theorem insures the continuity of the normal component of $\vec{F}$, so that

$$\vec{F}_n = -(\gamma \vec{\nabla} A)_n$$

is continuous. But because $\vec{A}$ has only a z component, the gradient and curl are equal in magnitude and orthogonal to each other, so that

$$(\gamma \vec{\nabla} A)_n = (\gamma \vec{\nabla} \times \vec{A})_t = \vec{H}_t,$$

and therefore $\vec{H}_t$ is continuous.

The requirement that $\vec{B}_n$ be continuous means that

$$\vec{B}_n = (\vec{\nabla} \times \vec{A})_n = (\vec{\nabla} A)_t$$

be continuous. This requirement is satisfied by our definition (3) of the gradient.

The external boundary conditions for these problems may be mixed. On all external boundaries we assume a condition of no leakage of magnetic flux, so that $A = 0$. If we consider the magnet to be symmetric about its median plane, we need to calculate only one half of the magnet and set the normal derivative of A equal to zero on the median plane. We can easily accomplish this by setting equal to zero all external coupling coefficients on the median plane.

2. Solution of the Difference Equations¹⁰

We use the method of successive overrelaxation to solve the difference Eqs. (16). Solving (16) for A gives

$$A = \frac{\sum_i w_i A_i + 4\pi I}{\sum_i w_i} \quad (18)$$

The denominator is positive by (9). Introducing the overrelaxation factor $\rho (0 < \rho < 2)$ we have, for the $(n+1)$ st cycle,

$$A^{n+1} = \rho \left(\frac{\sum_i w_i^n A_i^{n,n+1} + 4\pi I}{\sum_i w_i^n} - A^n \right) + A^n$$

or

$$A^{n+1} = A^n + \frac{\rho}{\sum_i w_i^n} \left[\sum_i w_i^n (A_i^{n,n+1} - A^n) + 4\pi I \right]. \quad (19)$$

The nearest-neighbor values $A_i^{n,n+1}$ represent either A_i^n or A_i^{n+1} , depending on the sequence of the calculation in each cycle, new values when obtained being substituted for old.

When there are no ferromagnetic materials present, the permeability μ is constant and equal to one. The coupling coefficients are then constant and the equations are linear. In iron, however, μ is a function of $|\vec{B}|$ - that is, of $|\vec{\nabla} A|$ - and varies over a wide range, so that we recalculate the coupling coefficients for iron regions each cycle. Because the coupling coefficients are functions of the gradient of the dependent variable A , the nonlinearity is more severe than in the nonlinear diffusion equation (1) where they are functions of the dependent variable ϕ . Experience has shown that the equations are unstable unless the coupling coefficients in iron are prevented from changing too rapidly in successive cycles. We have found that if w_i is a newly-calculated value, the linear weighting scheme⁷

$$w_i^{n+1} = \rho_w w_i + (1 - \rho_w) w_i^n \quad (20)$$

stabilizes the equations, where ρ_w is a positive fraction of the order of 0.1, depending on the rate of change of μ with $|\vec{B}|$.

For stability it is also necessary that the overconvergence factor ρ in Eq. (19) be not much greater than one in iron regions or on iron-air interfaces, and in fact we use one. In air, however, we try to choose ρ to optimize the convergence rate η defined as

$$\eta^n = \frac{\sqrt{\sum_i \left[(A_i^{n+1})^2 - (A_i^n)^2 \right]}}{\sqrt{\sum_i \left[(A_i^n)^2 - (A_i^{n-1})^2 \right]}} \quad (21)$$

where the summation is taken over the whole mesh. According to theory⁸ the optimum value ρ_{opt} is given by

$$\rho_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \lambda^2}}, \quad (22)$$

where, for a given ρ and the resulting η , we can obtain λ from the equation

$$\lambda = \frac{\rho + \eta - 1}{\rho \eta^{1/2}}. \quad (23)$$

We have found that for linear problems (constant μ) Eq. (23) with a constant λ fits the results extremely well. For problems with nonlinear (iron) regions as well as for linear problems we have obtained a satisfactory automatic calculation of ρ by using the following scheme:

$$\begin{aligned} \lambda &= \frac{\rho^n + \eta^n - 1}{\rho^n (\eta^n)^{1/2}} \\ \rho_{\text{opt}} &= \frac{2}{1 + \sqrt{1 - \lambda^2}} - \rho_0 \\ \rho^{n+1} &= \beta \rho_{\text{opt}} + (1 - \beta) \rho^n, \end{aligned} \quad (24)$$

with $\rho_0 = 0.01$ and $\beta = 0.05$. We require that η be less than one for fifteen successive cycles before ρ is changed.

Typical optimized values of η in 20×50 meshes are: 0.94 for iron (fixed boundary potential) problems; 0.97 for air (infinite μ); and 0.97-0.985 for a general problem with air and with iron of finite μ . ρ_{opt} may be as large as 1.95.

The criterion used for convergence is the smallness of the ratio

$$\epsilon = \frac{\sqrt{\sum_i (A_i^{n+1} - A_i^n)^2}}{\sqrt{\sum_i (A_i^n)^2}}$$

summed over the entire mesh.

3. Results for Plane Problems

In Figs. 6 to 21 are shown some results for three magnets being studied by the accelerator design group at LRL, Berkeley. Equipotentials have been drawn in by linear interpolation between vertex values. Because of the special relationship between the gradient and the curl of A in plane geometry, equipotentials are also lines of flux B .

A graph of the median plane field for the H magnet is given in Fig. 22, obtained by taking $(A_{i+1} - A_i)/(x_{i+1} - x_i)$ along the median plane. Also shown for comparison are results obtained by the use of R. Christian's magneto-static code at LRL, Berkeley, using a square mesh with nearly the same number of zones. The difference between results of the two codes is less than 2%.⁹

4. Some Features of the Code

The principal features of the magnet code described here are simplicity and generality. Any triangle may be made to represent air, iron, copper, etc., carrying any prescribed current, and interfaces and boundaries may occur anywhere in the mesh. With a $32678 (=2^{15})$ word computer the calculation is memory-contained for meshes up to 1600 points (3200 triangles).

The calculating time on an IBM 7094 is 0.6 millisecond per mesh point per cycle for points not in iron and 3.0 milliseconds for points in iron, giving 2.2 milliseconds, for example, as the average time for a mesh composed two-thirds of iron. For a 20×50 mesh the number of cycles required for convergence to $\epsilon = 10^{-6}$ is 400 to 800, or 15 to 30 minutes of machine time. Infinite- μ problems run five to seven times faster.

5. Extension to Problems with Cylindrical Symmetry

The diffusion equation (1) and the magnetic field equation (14) with cylindrical symmetry can be treated in cylindrical coordinates r, z, θ by transforming to Cartesian coordinates.

Fig. 6. C magnet showing material interfaces.

○ Points specified in problem input.

Fig. 7a. Logical map of C magnet mesh.

Fig. 7b. C magnet showing triangle zones.

Fig. 8. C magnet showing equipotentials.

Fig. 9. Enlarged view of C magnet.

Fig. 10. Figure 9 without zone lines.

Fig. 11. H magnet showing material interfaces.

Fig. 12. H magnet showing triangle zones.

Fig. 13. H magnet showing equipotentials.

Fig. 14. Enlarged view of H magnet.

Fig. 15. Figure 14 without zone lines.

Fig. 16. Conformal magnet showing material interfaces.

Fig. 17. Conformal magnet showing triangle zones.

Fig. 18. Conformal magnet showing equipotentials.

Fig. 19. Enlarged view of conformal magnet, left coil region.

Fig. 20. Enlarged view of conformal magnet, central region.

Fig. 21. Enlarged view of conformal magnet, right coil region.

Fig. 22. Median plane field vs position for H magnet.

Fig. 23. n vs r for CERN proton synchrotron magnet (open sector).

(a) Diffusion Equation

Let $\phi = \phi(r, z)$, independent of θ , and let $\vec{\nabla}$, $\vec{\nabla}_c$ represent the del operator in Cartesian coordinates and cylindrical coordinates respectively.

Then for any vector function $\vec{f}$ which is independent of θ we have

$$\vec{\nabla}_c \cdot \vec{f} = \frac{1}{r} \vec{\nabla} \cdot (r \vec{f}).$$

Hence Eq. (1) becomes

$$\frac{1}{r} \vec{\nabla} \cdot (r \lambda \vec{\nabla} \phi) = c \frac{\partial \phi}{\partial t}$$

or

$$\vec{\nabla} \cdot (r \lambda \vec{\nabla} \phi) = rc \frac{\partial \phi}{\partial t}.$$

Thus we see that by replacing λ by $r\lambda$, and c by rc , we can treat the cylindrical coordinates z, r as if they were Cartesian coordinates x, y respectively.

In finite difference form we have in place of (5) and (7) respectively

$$G = \sum_i c_{i+1/2} \bar{r}_{i+1/2} a_{i+1/2}$$

$$w_i = \frac{1}{2} (\lambda_{i+1/2} \bar{r}_{i+1/2} \cot \theta_{i+1} + \lambda_{i-1/2} \bar{r}_{i-1/2} \cot \theta_{i-1}),$$

where $\bar{r}_{i+1/2}$, the average radius of triangle $i+1/2$, is given by

$$\bar{r}_{i+1/2} = \frac{1}{3} (r + r_i + r_{i+1})$$

and $\bar{r}_{i+1/2}$ is the average radius of a quadrilateral at vertex r given by

$$\bar{r}_{i+1/2} = \frac{7}{12} \bar{r}_{i+1/2} + \frac{5}{12} r.$$

(b) Magnetic Field Equation

For a vector function $\vec{f}(r, z)$ which has only a θ component we have

$$\vec{\nabla}_c \times \vec{f} = \frac{1}{r} \vec{\nabla} \times (r \vec{f}).$$

For a vector function $\vec{g}(r, z)$ which has only r and z components

$$\vec{\nabla}_c \times \vec{g} = \vec{\nabla} \times \vec{g}.$$

Since $\vec{A}(r, z)$ and $\vec{j}(r, z)$ have only θ components,

$$\vec{B} = \vec{\nabla}_c \times \vec{A} = \frac{1}{r} \vec{\nabla} \times (r \vec{A}).$$

Since $\vec{B}$ has only r and z components,

$$\vec{\nabla}_c \times (\gamma \vec{B}) = \vec{\nabla} \times (\gamma \vec{B}) = \vec{\nabla} \times \left[\frac{\gamma}{r} \vec{\nabla} \times (r \vec{A}) \right] = 4\pi \vec{j}.$$

Thus (15) becomes

$$\vec{\nabla} \cdot \left[\frac{\gamma}{r} \vec{\nabla} (rA) \right] = -4\pi j. \quad (25)$$

We see that γ/r replaces γ , $r\vec{A}$ replaces $\vec{A}$, and $r\vec{B}$ replaces $\vec{B}$.

For a current loop, $A \sim r$ for small r , $A \sim 1/r^2$ for large r . Therefore the boundary conditions on rA are $\lim_{r \rightarrow 0} (rA) = \lim_{r \rightarrow \infty} (rA) = 0$.

For computational purposes (25) may be written

$$\vec{\nabla} \cdot (\gamma \vec{\nabla} A) = -4\pi j - \frac{\partial}{\partial r} \left(\frac{\gamma A}{r} \right). \quad (26)$$

A finite difference approximation to the second term on the right may be obtained by the line integral method.

IV. CONCLUSION

The irregular triangle technique has been successfully applied to the calculation of static magnetic fields by means of a digital computer, and a code for the IBM 7094 has been written which has sufficient flexibility and speed to be of practical interest. The accuracy of the calculation appears to be comparable to that obtained using regular meshes.

V. ACKNOWLEDGMENTS

I wish to thank C. F. Andrews, of LRL Berkeley, for calling my attention to the possibility of using the irregular mesh technique to calculate magnetic fields, and for helpful suggestions.

REFERENCES

- ¹R. H. MacNeal, "An Asymmetrical Finite Difference Network," Quarterly of Applied Mathematics 11, 295 (1953).
- ²C. E. Leith, Lawrence Radiation Laboratory (Livermore, California) unpublished (1958).
- ³A. M. Winslow, "'Equipotential' Zoning of Two-Dimensional Meshes," Lawrence Radiation Laboratory (Livermore, California) Rept. UCRL-7312 (1962).
- ⁴A. M. Winslow, "Description of An Irregular Triangle Mesh Generator," Lawrence Radiation Laboratory (Livermore, California) Rept. UCRL-7880 (1964).
- ⁵R. Courant, Differential and Integral Calculus, Vol. II (Interscience Publishers, Inc., New York, 1936) p. 360.
- ⁶Based on a derivation by Milton E. Rose.
- ⁷Taken from the magnetostatic code of R. Christian (MURA), in use at LRL, Berkeley.
- ⁸G. E. Forsythe and W. R. Wasow, Finite Difference Methods for Partial Differential Equations (John Wiley & Sons, Inc., New York, 1960) Sec. 22.
- ⁹Figure 23 gives triangle mesh results for the index n , defined to be $n = |(R_0/B_0) \partial B/\partial r|$ where $R_0 = 70.079$ m, for the CERN proton synchrotron magnet. With 1 cm zones on the median plane the error is less than 1.5% over the vacuum chamber region. The straight line fit to the pole face profile was not particularly accurate, and it is believed that a better fit would improve the accuracy of the results.

Numerical results for the median plane field and for the stored magnetic energy also show about the same accuracy.
- ¹⁰Note added in proof: A more rapidly converging method for the solution of equations (16), based on Newton's method, is described in UCRL-12132-T.

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.