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Equilibria of Collisionless Plasma
Part II

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Equilibria of Collisionless Plasma
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by

Richard L. Morse

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EQUILIBRIA OF COLLISIONLESS PLASMA - PART II

by

Richard L. Morse

ABSTRACT

Part II of this report applies the Vlasov equilibrium methods of Part I to straight z-pinches and multipoles and then to toroidal equilibria. The last section contains a conjecture about the role of the rigid drift equilibria in real plasma containment experiments.

V. Z-PINCH EQUILIBRIA

We now consider a class of z-independent z-pinch equilibria which have much in common with the rigid rotor equilibria of Eq. (21). The rigid rotor distributions are so named because the mean rotational particle velocity is proportional to radius, i.e., the rotation, when viewed as a flow, has no shear. For a z-pinch in which the current flow and therefore the mean particle motions are in the z-direction, the magnetic field has only a B_θ component which can be described by the vector potential component $A_z(r)$. By symmetry, the canonical $\hat{z}$ momentum, $P_z = m_s(V_z + q_s A_z/c)$, is a constant of particle motion and can be employed much as P_θ was in Eq. (21) to give

$$f_s = f_s \left[-(\epsilon - V_s P_z)/T_s \right],$$

$$= f_s \left[-\frac{m_s}{2T_s} \{V_r^2 + V_\theta^2 + (V_z - V_s)^2\} + \frac{m_s}{2T_s} \left\{ -\frac{2q_s \phi}{m_s} + \frac{2V_s}{m_s} \left(\frac{q_s A_z}{c} \right) + V_s^2 \right\} \right]. \quad (60)$$

Here V_s is the constant (i.e., r independent) mean $\hat{z}$ velocity of species s.

Following the same procedure as in Eqs. (22) through (28) for the quasi-Maxwellian form of f_s gives

$$f_s = \frac{n_o}{(2\pi T_s/m_s)^{3/2}} \exp \left[\frac{m_s}{2T_s} \left\{ -\frac{2q_s \phi}{m_s} + \frac{2V_s}{m_s} \left(\frac{q_s A_z}{c} \right) \right\} \right] \times \exp \left[-\frac{m_s}{2T_s} \{V_r^2 + V_\theta^2 + (V_z - V_s)^2\} \right], \quad (61)$$

$$n_s = n_o \exp \left[\frac{m_s}{2T_s} \left\{ -\frac{2q_s \phi}{m_s} + \frac{2V_s}{m_s} \left(\frac{q_s A_z}{c} \right) \right\} \right],$$

$$J_{zs} = q_s n_s V_s, \quad (62)$$

and

$$\rho = \sum_s q_s n_s \text{ and } J_z = \sum_s J_{zs} \quad (63)$$

(This distribution was first written down and integrated as below by Ed Harris.)¹

The charge neutrality condition which parallels Eq. (29) is now

$$\frac{m_i}{2T_i} \left\{ -\frac{2e\phi}{m_i} + \frac{2V_i}{m_i} \left(\frac{eA_z}{c} \right) \right\} = \frac{m_e}{2T_e} \left\{ +\frac{2e\phi}{m_e} - \frac{2V_e}{m_e} \left(\frac{eA_z}{c} \right) \right\} \quad (64)$$

or

$$e\phi(r) = \frac{(T_e V_i + T_i V_e)}{T_e + T_i} \left(\frac{eA_z}{c} \right). \quad (65)$$

When Eq. (65) is substituted into Eq. (62) and terms are regrouped, the resulting charge neutral density

profile is

$$n = n_e = n_i = n_0 \exp \left[\frac{(V_i - V_e)}{(T_i + T_e)} \left(\frac{eA_z}{c} \right) \right]. \quad (66)$$

The total current is now

$$J_z = n_0 e (V_i - V_e) \exp \left[\frac{(V_i - V_e)}{(T_i + T_e)} \left(\frac{eA_z}{c} \right) \right]. \quad (67)$$

Though it could have been seen at the beginning, it is now clear that the magnetic field will be unaffected by the axial component of the rest frame of the plasma because J_z Eq. (67) depends only on the relative drift, $(V_i - V_e)$. In this light, Eq. (65) for $\varphi(r)$ can be thought of as effecting a Lorentz transformation of the $E_r(r)$ in going from one frame moving in the $\hat{z}$ direction to another.

Similar arguments do not apply to the rigid rotor distributions since a change in mean rate of rotation of the plasma cannot be transformed away.

To obtain $A_\theta(r)$ from $J_z(A_\theta)$ we need the field equations: they are

$$B_\theta = \vec{\nabla} \times \vec{A} \Big|_\theta = - \frac{dA_z}{dr} \quad (68)$$

and

$$\vec{\nabla} \times \vec{B} \Big|_z = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \Big|_z = - \frac{1}{r} \frac{d}{dr} \left(\frac{rdA_z}{dr} \right) = \frac{4\pi J_z}{c}. \quad (69)$$

Combining Eqs. (67) and (69) gives

$$\frac{1}{r} \frac{d}{dr} \left(\frac{rdA_z}{dr} \right) = - \frac{4\pi n_0 e (V_i - V_e)}{c} \exp \left[\frac{(V_i - V_e)}{(T_i + T_e)} \left(\frac{eA_z}{c} \right) \right]. \quad (70)$$

Again we define a Ψ as

$$\Psi = - \frac{(V_i - V_e)}{(T_i + T_e)} \left(\frac{eA_z}{c} \right), \quad (71)$$

and Eq. (70) becomes

$$\frac{d}{d(r^2)} \left(r^2 \frac{d\Psi}{d(r^2)} \right) = - \frac{\pi n_0 e^2 (V_i - V_e)^2}{c^2 (T_i + T_e)} \exp^{-\Psi}, \quad (72)$$

$$\Psi(r^2 = 0) = 0.$$

Equation (72) has the solution

$$\Psi = \ln(1 + r^2/r_0^2)^{+2}, \quad (73)$$

where

$$r_0^2 = \frac{2c^2 (T_i + T_e)}{\pi n_0 e^2 (V_i - V_e)^2}. \quad (74)$$

From Eq. (71)

$$A_z = \frac{c(T_i + T_e)}{e(V_i - V_e)} \ln(1 + r^2/r_0^2)^{-2}. \quad (75)$$

From Eqs. (68), (73) and (74)

$$B_\theta = \frac{4c(T_i + T_e)}{e(V_i - V_e)} \frac{r/r_0^2}{(1 + r^2/r_0^2)} \\ = \text{sgn}(V_i - V_e) \sqrt{8\pi n_0 (T_i + T_e)} \times \frac{r/r_0}{1 + r^2/r_0^2} \quad (76)$$

and from Eq. (66)

$$n = \frac{n_0}{(1 + r^2/r_0^2)^2} \quad (77)$$

The radial E field is obtained from Eq. (64) for $\varphi(r)$ and

$$E_r = - \frac{d\varphi}{dr}, \quad (78)$$

and is

$$E_r = - \left\{ \frac{(T_i V_e + T_e V_i)}{c(T_i + T_e)} \right\} \frac{dA_z}{dr} = \{ \} B_\theta. \quad (79)$$

Clearly, if

$$T_i V_e + T_e V_i = 0, \quad (80)$$

then $E_r = 0$ everywhere. On the other hand, if no mechanism exists by which the plasma can acquire a net $\hat{z}$ momentum, $m_i \gg m_e$ implies that in the laboratory frame $V_i < V_e$ and in this approximation Eq. (79) becomes

$$E_r \approx - \frac{T_i V_e}{c(T_i + T_e)}. \quad (81)$$

In this case there is no net ion current and if, in addition, $T_e = 0$ so that all electron current is due to $E \times B$ drift, then we recover the MHD result,

$$E_r = - V_e B_\theta / c. \quad (82)$$

Again it is useful to obtain the line density, N ,

from Eq. (77) for n ,

$$N \equiv \int_0^\infty dr 2\pi r n = n_0 \int_0^\infty \frac{dr 2\pi r}{1-r^2/r_0^2} = \pi n_0 r_0^2 \quad (83)$$

so that from Eq. (74) for r_0^2 ,

$$N = \frac{2c^2 (T_i + T_e)}{e^2 (V_i - V_e)^2} \quad (84)$$

Also, since the total current is

$$I = e(V_i - V_e)N, \quad (85)$$

we have the relationship between measurable macroscopic quantities

$$N(T_i + T_e) = \frac{I^2}{2c^2} \quad (86a)$$

What has been obtained here, as pointed out by Harris,¹ is a microscopic derivation of the well-known Bennett pinch. It should be pointed out, however, that here as in the rigid rotor distributions, particles of both species possess isotropic Maxwellian distributions and are therefore immune to ambipolar diffusion as well as any ills that accrue to velocity space anisotropy of one species by itself. In view of these properties it is a little surprising that there exists no microscopic theory of the stability of this class of z-pinch equilibria.

Finally, a word is in order about combined θ -pinch and z-pinch equilibria, usually called a screw pinch because of the helical shape of the combined B_z and B_θ magnetic field lines. Such equilibria can be obtained in the θ - and z-independent geometry from rigid drift distributions which are a hybrid of the z-pinch distributions of this section and the θ -pinch distribution of Section IV:

$$f_s = f_s [-(\epsilon - V_s P_z - \omega_s P_\theta)/T_s] \quad (86b)$$

or specifically

$$f_s \sim \exp - (\epsilon - V_s P_z - \omega_s P_\theta)/T_s$$

Although no general analytic solution is known for the latter form, solutions have been generated numerically for parameters in ranges of interest and it can be said that there are no surprises. The

results clearly combine the features of rigid drift θ - and z-pinches.

VI. LOW- β MULTIPOLE EQUILIBRIA

It is convenient to consider the straight multipole confinement geometry now because it is possible to find equilibria for this scheme which, like those discussed for the z-pinch, rely on the conservation of canonical P_z . The straight quadrupole will now be explicitly discussed but obvious changes give similar results and conclusions for other multipoles, including octupoles. Quadrupoles consist of two parallel bar-like conductors carrying about equal currents in the same direction, surrounded by a closed return conductor.

1. Magnetic Field Configuration

The magnetic field is almost entirely determined by the currents in these conductors because the ratio of plasma pressure to magnetic field pressure, β , in multipoles is usually so low that currents flowing in the confined plasma are too weak to have any significant effect. Hence it is not necessary to solve the $\vec{\nabla} \times \vec{B} = 4\pi\vec{J}/c$ equation with self-consistent plasma currents, as was done above for the θ - and z-pinches. It is sufficient to represent the bar conductor currents by two parallel line currents, I , flowing in the z-direction, separated by a distance $2a$ in the x-direction and to remove the return conductor to infinity. The resulting magnetic field lies in the x, y plane everywhere and can be completely described by the one vector potential component, $A_z(x, y)$. When the potentials for the two-line currents are superimposed, the result is (using a convenient choice of the arbitrariness in $\vec{A}$)

$$A_z(x, y) = \frac{2I}{c} \ln \frac{a^2}{r_1 r_2}, \quad (87)$$

where r_1 and r_2 are the distances from the two line currents to the field point, or

$$A_z(x, y) = \frac{2I}{c} \ln \left[\frac{a^2}{\sqrt{y^2 + (x-a)^2} \sqrt{y^2 + (x+a)^2}} \right] \quad (88)$$

The $\vec{B}$ field is then given by

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x} \quad (89)$$

As before, surfaces of constant A_z are flux surfaces and therefore contain field lines. The pattern of the field can be visualized by locating and tracing the $A_z = 0$ surface. Starting at $x, y = 0$ and by starting at Eq. (88), we see that from there the field separately loops out around both line currents and returns, thus forming a figure 8 with a current line in the middle of each loop. This singular flux surface or field line, which can be seen from Eq. (89) to have a $\bar{B}$ field null at $x, y = 0$, is the separatrix. All field lines inside the loops encircle their respective line currents; all field lines outside encircle both line currents. It is the general idea of multipole plasma containment that the plasma density should be largest on the separatrix and adjacent flux surfaces and fall to zero at the inner bar conductors and at large distances outside the separatrix where material surfaces are encountered.

2. Plasma Distribution

In the quadrupole geometry the only ignorable coordinate, i.e., symmetry axis, is z with its conserved conjugate $P_z = m_s V_z + q_s A_z / c$ for each particle species. It is tempting to try to use P_z in the rigid drift distribution, Eq. (60), as was done for the z -pinch, but the density behaves badly. Equation (66) for n would then apply, but the $A_z(x, y)$ given in Eq. (88) goes to $+\infty$ at both of the current lines and to $-\infty$ at infinity for $I > 0$ or the other way around for $I < 0$. In either event the density given by Eq. (66) is not maximum near the separatrix but diverges either at the current lines or at infinity, where the walls would really be, depending on the signs of I and $(V_i - V_e)$. The field configuration used here is general enough to permit the following conclusion: a rigid drift distribution will not fit into a quadrupole or any other multipole.

Clearly, any distribution that does fit must be an even function of P_z , and the closest such to a Maxwellian would seem to be

$$f_s \sim \exp \left[- (e + \alpha_s P_z^2 / 2m_s) / T_s \right] \quad (90)$$

for each species, s , where α_s is a dimensionless anisotropy parameter. Making the substitutions for e and P_z and normalizing gives

$$f_s = \frac{n_0 (1 + \alpha_s)^{1/2}}{(2\pi T_s / m_s)^{3/2}} \times \exp \left[- \frac{m_s}{2T_s} \left\{ v_x^2 + v_y^2 + (1 + \alpha_s) (v_z - v_{ds})^2 \right\} \right] \times \exp \left[- \frac{m_s}{2T_s} \left\{ \frac{2q_s \phi}{m_s} + \frac{\alpha_s}{1 + \alpha_s} \left(\frac{q_s A_z}{m_s c} \right)^2 \right\} \right] \quad (91)$$

where

$$v_{ds} = - \frac{\alpha_s}{1 + \alpha_s} \left(\frac{q_s A_z}{m_s c} \right).$$

Hence

$$n_s = n_0 \exp \left[- \frac{m_s}{2T_s} \left\{ \frac{2q_s \phi}{m_s} + \frac{\alpha_s}{1 + \alpha_s} \left(\frac{q_s A_z}{m_s c} \right)^2 \right\} \right]. \quad (92)$$

Charge neutrality now requires that

$$\frac{m_i}{2T_i} \left\{ \frac{2e\phi}{m_i} + \left(\frac{eA_z}{m_i c} \right)^2 \right\} = \frac{m_e}{2T_e} \left\{ \frac{2e\phi}{m_e} + \frac{\alpha_e}{1 + \alpha_e} \left(\frac{eA_z}{m_e c} \right)^2 \right\} \quad (93)$$

or

$$2e\phi = \frac{T_i T_e}{(T_i + T_e)} \left(\frac{eA_z}{c} \right)^2 \left[\frac{\alpha_e}{m_e T_e (1 + \alpha_e)} - \frac{\alpha_i}{m_i T_i (1 + \alpha_i)} \right]. \quad (94)$$

Substituting this charge neutrality relationship between ϕ and A_z into Eq. (88) for $n_s(\phi, A_z)$ gives

$$n = n_e = n_i = n_0 \exp \left\{ - \frac{1}{s(T_i + T_e)} \left(\frac{eA_z}{c} \right)^2 \left[\frac{\alpha_e}{m_e (1 + \alpha_e)} + \frac{\alpha_i}{m_i (1 + \alpha_i)} \right] \right\}. \quad (95)$$

Equation (95) satisfies all the requirements on density if $\alpha_e, \alpha_i > 0$ and $A_z(x, y)$ is given by Eq. (88). It is largest on the separatrix where $A_z = 0$ and goes to zero at infinity and on the current lines where A_z is $\pm \infty$.

Many observations can now be made about the forms of Eqs. (88 through 95). It can be seen from Eq. (94) that it is possible to choose α_i and α_e so that $\phi = 0$, which would be required if some sort of line shorting occurred. From Eq. (95) we see that

as the α 's increase the plasma density becomes more localized to the separatrix, $A_z = 0$. This may be understood in a mathematical sense by noting that as $\alpha_s \rightarrow \infty$, Eq. (90) for f_s becomes $\sim \delta(P_z) e^{-\epsilon/T_s}$. Since P_z is proportional to the value of A_z on the flux surface to which the particles' guiding center is constrained (to the extent that the gyro radius is small enough to permit the guiding center concept), then f_s represents only particles with guiding centers on the separatrix and the thickness of the plasma will be only one gyro diameter.

The meaning of the α 's as anisotropy parameters can be seen from Eq. (91) for f_s in terms of the velocities. If $\alpha_s > 0$, then the thermal spread in V_z is smaller than the thermal spread in V_x and V_y . In effect, two different temperatures have emerged:

$$\begin{aligned} T_x &= T_y = T_s \\ T_z &= \frac{T_s}{1+\alpha_s} \end{aligned} \quad (96)$$

which is related to the fact (Eq. 91) that the drift of each species, V_{ds} , is now spatially dependent. Though the derivation here has been specific, the result is general; such transverse anisotropies are a necessary consequence of departure from rigid drift distributions. Now then, the plasma stability literature is stiff with examples of instabilities provoked by such velocity space anisotropies. In particular, the author has recently derived a stability criterion for a sort of transverse mirror instability in a low but finite β sheet plasma with the same distribution for the ions as the f_s of Eq. (90). However, it is not known how dangerous these instabilities are in this usually very low- β application. But it is possible that the inability of multipole containment devices to accommodate rigid drift particle distributions is a weakness.

However, of the various possible multipole equilibria, the class discussed here, Eq. (90), seem to be about the most isotropic possible and are therefore a state toward which other less isotropic equilibria might decay.

VII. TOROIDAL EQUILIBRIA

The subject of toroidal equilibrium may be

very important to controlled thermonuclear reactions because straight containment systems have open ends and it appears that such open-ended systems must either be very long or very lossy.

1. Symmetry

By now the role of constants of particle motion in determining collisionless plasma equilibria should be clear. Some symmetry of the system causes one or more of the components of the canonical momentum, $\vec{P}$, to be constant (in addition to the energy, ϵ , which is always conserved in a stationary system). When a distribution function is taken to depend on these components of $\vec{P}$, it acquires a spatial dependence through the corresponding components of the vector potential, $\vec{A}$, which is contained in $\vec{P}$. In the θ -pinch treatment, the necessary confining spatial dependence came from $A_\theta(r)$, in the z -pinch, from $A_z(r)$, and in the multipole, from $A_z(x,y)$. Although the electrostatic potential, ϕ , in the energy, ϵ , is spatially dependent, this alone is not sufficient; at least one component of $\vec{P}$ must be conserved, i.e., one coordinate must be ignorable.

In discussing tori, we will use the r, θ, z cylindrical system of coordinates with the z axis being the major axis. Since the idea of toroidal confinement is that the extent of the plasma should be restricted in z as well as r , the only possible ignorable coordinate is θ . Hence, if an exact collisionless toroidal equilibrium is to be found, the system must be symmetric about the z (major) axis, i.e., be θ -independent, and the desired spatial dependence of the particle distribution functions enters through $A_\theta(r, z)$. This is the same field geometry discussed in connection with θ -pinches (Eq. (4), Sec. I). $A_\theta(r, z)$ describes general B_r and B_z , but not B_θ fields. Remember that θ is the long way around the torus, the coordinate that varies along the minor axis; B_θ is the toroidal field; B_r and B_z constitute the so-called poloidal field. The fact that the equilibrium particle distributions are influenced by the magnetic fields only through $A_\theta(r, z)$ means that the form of the possible plasma equilibria is not affected by any symmetric B_θ which might be present and vice versa; the B_θ must be a vacuum field. An integral part of this picture is that the presence of $A_\theta(r, z)$

in the particle distribution functions can give rise to only J_θ electric currents. These last statements of the consequences of depending only on θ -symmetry, and hence A_θ alone, can be proved rigorously but should be reasonably clear from the implied physical arguments.

Before developing specific cases of toroidal equilibria, we briefly discuss the many well-known toroidal containment schemes which we have excluded by our insistence on θ -symmetry. This group includes mostly minimum B and Stellarator-like low- β devices which are not θ -symmetric by virtue of various bumps and wiggles in the magnetic field, even when the external shape of the machine is a symmetric torus. The reader may recall that the predictions of equilibria in such devices are based on the adiabatic invariance of the magnetic moment of particle motion, μ , and such things as the closure of drift surfaces or higher adiabatic invariants. The invariance of μ , through which particles are tied to field lines, seems hard to assail except by invoking high-frequency instabilities and these are alleged not to exist in tori. On the other hand, the expectation of drift surface closure is often based on such assumptions as vanishing gyro radii, precision in the production of the bumps and wiggles in the magnetic field, and sometimes the specification of special plasma profiles.* Hence these nonsymmetric toroidal equilibria, in addition to being outside the spirit of the rest of this report, would seem slightly less reliable than the symmetric equilibria for which less stringent assumptions are required.

This preference for axisymmetric toroidal equilibria accepts the fairly conventional view stated above that such equilibria are subject only to low frequency instabilities. In contrast with this, the last section of this report offers a more radical conjecture which implicitly assumes the failure of μ under some circumstances and which, if correct, would also demand that toroidal equilibria be symmetric but for different or additional reasons.

*Note that the application of the invariance of μ to confinement in an axisymmetric mirror machine is a very different matter. Here this invariance serves only to prevent end loss. The rest is taken care of by conserved P_θ , as in a θ -pinch, and the concept of drift surfaces need never be introduced.

2. Toroidal Multipole

When a straight z -independent multipole is bent into a symmetric toroidal multipole, the original z -axis becomes the minor axis of the torus, i.e., a circular loop of constant r and z . All physical quantities are then independent of θ , and the conservation of canonical P_θ plays the same role as P_z in the straight case. In a quadrupole the two wires become coaxial loops straddling the minor axis and may be in the same constant z -plane, in the same constant r surface, or have some other less convenient mechanical arrangement. The same sort of options apply to higher multipoles. It is easy to see that in the limit of large aspect ratio, i.e., when the radius of the minor axis is much larger than the separation of the wires, equilibria in the vicinity of the conductors in this configuration pass continuously into the straight equilibria of Sec. VI if distribution functions of form

$$f_s \sim \exp \left[- \left(\epsilon + P_\theta^2 / 2m_s r_s^2 \right) / T_s \right] \quad (97)$$

are assumed. Naturally, A_θ is chosen to be zero on the separatrix and the dimensionless constant α_s of Eq. (90) has had to be replaced by $1/r_s^2$, where r_s is a length constant. Formulae for $A_\theta(r, z)$ from coaxial loops are available in many E and M books. Although these formulae contain the inconvenient elliptic functions, for large aspect ratios the fields in the vicinity of the conductors differ only by small toroidal corrections from the fields given by equations such as Eqs. (87) and (88) for the straight quadrupole. Altogether, then, there is no difficulty in finding toroidal multipole equilibria.

Multipoles suffer, however, from the need for internal conductors which must somehow be supported, either by mechanical hangers which pass through the containment region or by magnetic field levitation (or in fast pulsed experiments by merely being dropped). It is thought that hangers can be much improved by passing auxiliary currents through them to divert the plasma-bearing flux surfaces or tubes from contact with the material structure. Such shielded hangers appear to make some sense from the closed drift containment point of view discussed at the beginning of this section. But if there is any merit to the preference expressed for symmetry con-

tainment, then levitation is better because no matter how cleverly they are shielded, hangers break the θ -symmetry and force toroidal multipoles to rely on closed drift containment.

3. Toroidal Pinches

In the transformation of a multipole from straight to toroidal form, no symmetry was lost; all exact, collisionless, straight equilibria survived the bending process without need of asking how large the aspect ratio must be before a toroidal system thinks it is straight and that sort of thing.

Not so for the pinches. The straight z -pinch equilibria, which depend on P_z conservation, transform smoothly as the straight P_z becomes the toroidal P_θ . For large aspect ratio, the small toroidal corrections are harmless as discussed below. But the straight θ -pinch relies on symmetry in the angle variable which goes around the plasma column, not along its length, and when the plasma column bends in any way this coordinate is no longer ignorable. This is the obverse of the observation at the beginning of this section that the toroidal field, B_θ , which the B_z field of a θ -pinch turns into in the transition to a toroidal configuration, cannot actively participate in exact toroidal equilibria.

A review of the possibilities shows that only the toroidal z -pinch* can possess a toroidal rigid drift equilibrium and hence is a subject of some interest. Because, in practice, most of these devices have fairly large aspect ratios and circular cross sections (with the exception of the Doublets), we settle here for calculating the toroidal corrections to a straight, axisymmetric, rigid drift z -pinch equilibrium.

The appropriate starting point for development of the toroidal rigid drift equilibrium is the rigid rotor formalism of Sec. IV without the restriction to z -independence. The arguments between Eqs. (29) and (30) for dropping centrifugal force terms on the grounds that a system which has always been axisymmetric should have no net angular momentum are equally valid here. These terms could easily be added as in the rotating θ -pinch treatment if needed. As seen in the laboratory frame, the

electrons carry the diamagnetic current and the ion cloud sits at rest in an electrostatic potential well. In short, the entire procedure of Sec. IV for obtaining the self-consistent field equation applies with the addition of the z -dependence in $\vec{v} \times (\vec{v} \times \vec{A})$, giving in place of Eq. (33)

$$\frac{\partial^2 A_\theta}{\partial z^2} + \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial (r A_\theta)}{\partial r} = \frac{4\pi n_o e \omega_e r}{c} \exp - \left[\frac{\omega_e r}{(T_e + T_i)} \left(\frac{e A_\theta}{c} \right) \right]. \quad (98)$$

As before, we substitute the flux function

$$r A_\theta = \frac{(T_e + T_i) c \psi}{e \omega_e} \quad (99)$$

into Eq. (98) and obtain

$$\frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{4\pi n_o e^2 \omega_e^2 r^2}{c^2 (T_e + T_i)} \exp - \psi. \quad (100)$$

Now, in the spirit of large aspect ratio toroidal geometry, we also make a change of coordinates:

$$r = r_T + \rho \cos \varphi, \quad z = \rho \sin \varphi. \quad (101)$$

For boundary conditions we specify that the plasma-confining region is bounded by a flux-confining conducting wall which is a toroidal shell with a circular cross section of radius ρ_T centered on the mirror axis $r = r_T, z = 0$.

The idea is that in the limit $\rho_T/r_T \rightarrow 0$ the equilibrium we find will be that of a cylindrical (φ -independent) z -pinch with its center at $r = r_T, z = 0$. In fact, it will be the same rigid drift equilibrium described in Sec. V, with ρ replacing r in the formulae. With this anticipated result in mind we define

$$V_e \equiv \omega_e r_T \quad (102)$$

which together with the change of coordinates, Eq. (101), gives Eq. (100) the new form

$$\begin{aligned} & \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{1}{r_T (1 + \rho/r_T \cos \varphi)} \\ & \left(- \cos \varphi \frac{\partial \psi}{\partial \rho} + \frac{\sin \varphi}{\rho} \frac{\partial \psi}{\partial \varphi} \right) \\ & = \frac{4\pi n_o e^2 V_e^2}{c^2 (T_e + T_i)} \left(1 + \frac{\rho}{r_T} \cos \varphi \right)^2 \exp - \psi. \end{aligned} \quad (103)$$

* The Perhapsatron configuration.

Up to here the formulae are exact. Equation (103), or perhaps more conveniently Eq. (100) could be solved numerically on a two-dimensional grid of ρ and φ or r and z with the $\psi = \text{const.}$ boundary condition on the flux at the conducting wall. However, the first toroidal corrections to the straight equilibrium can be obtained analytically from an expansion procedure.

We assume that

$$\psi(\rho, \varphi) = \psi^{(0)}(\rho) + \psi^{(1)}(\rho) \cos\varphi + \text{negligible higher order terms,} \quad (104)$$

where $\psi^{(1)}/\psi^{(0)}$ is of the order of ρ_T/r_T , i.e., small. A more complete expansion to higher order in ρ_T/r_T would contain terms with factors of $\sin m\varphi$ and $\cos m\varphi$, $m > 1$. The reader who is interested in such a formal procedure together with more detailed rationale for the form of Eq. (104) should read LAMS-2282² where this problem is treated generally in the context of hydromagnetic equilibria. However, the reader will see from the following that the correction term in Eq. (104) does have the proper form. When Eq. (104) for $\psi(\rho, \varphi)$ is substituted into the field Eq. (103), and the result is expanded so that terms of different orders in ρ_T/r_T can be grouped separately, then the zero-order terms give

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi^{(0)}}{\partial \rho} \right) = \left[\frac{4\pi n_0 e^2 v_e^2}{c^2 (T_e + T_i)} \right] \exp - \psi^{(0)} \quad (105)$$

and, after cancellation of a common factor of $\cos\varphi$, the first-order terms give the correction field equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi^{(1)}}{\partial \rho} \right) - \frac{\psi^{(1)}}{\rho^2} - \frac{1}{r_T} \frac{\partial \psi^{(0)}}{\partial \rho} = \left[\frac{4\pi n_0 e^2 v_e^2}{c^2 (T_e + T_i)} \right] \left(\frac{2\rho}{r_T} \psi^{(1)} \right) \exp - \psi^{(0)}. \quad (106)$$

The mathematically meticulous reader may have noted that by now n_0 is not n at $r = 0$ and ψ does not go to zero at $r = 0$ as in the θ -pinch treatment from which we started. Instead, n_0 is the maximum density and ψ is appropriately large at $r = 0$. The switch, however, is legitimate.

Equation (105) is the Bennett-Harris¹ equation

(Eqs. 70-71), with ρ replacing r as we anticipated, and $V_i = 0$. Thus

$$\psi^{(0)}(\rho) = \ln (1 + \rho^2/\rho_0^2)^2, \quad (107)$$

$$\text{where } \rho_0^2 = \frac{2c^2 (T_e + T_i)}{\pi n_0 e^2 v_e^2}. \quad (108)$$

When Eq. (106) is regrouped with terms in $\psi^{(1)}$ on the left and terms in the now known $\psi^{(0)}$ on the right, it is seen to be a second-order, linear, inhomogeneous equation for $\psi^{(1)}$; the general solution of such an equation has the form

$$\psi^{(1)} = \psi_H + \psi_I, \quad (109)$$

where ψ_H is the general solution of the homogeneous equation and ψ_I is any solution of the complete inhomogeneous equation. Since Eq. (106) reduces to the homogeneous equation when $r_T \rightarrow \infty$, it follows that a solution to the homogeneous equation is a $\psi^{(1)}$ which would exactly describe an r displacement of the $\psi^{(0)}$ solution; this is easily shown to be $-\frac{\partial \psi^{(0)}}{\partial \rho} \Delta r$, where Δr is that displacement. In fact, the other independent solution required in linear combination with the first to form the general ψ_H has an unphysical singularity, and the ψ_I which we will find describes a distortion of ψ from $\psi^{(0)}$ but no r displacement. Hence $\psi^{(1)}$ has the form

$$\psi^{(1)} = -\Delta r \frac{d\psi^{(0)}}{d\rho} + \psi_I, \quad (110)$$

where Δr is the r displacement from $r = r_T$ of the density maximum of the resulting equilibrium. A standard formal procedure for obtaining a solution to the inhomogeneous Eq. (106) from any solution of the homogeneous equation gives

$$\psi_I = \frac{\rho_0^2}{2r_T} \frac{d\psi^{(0)}}{d\rho} \left\{ \left(\frac{\rho}{\rho_0} \right)^2 + \frac{1}{2} \int_0^{(\rho/\rho_0)^2} dx \left[-\left(\frac{x+1}{x} \right) + \left(\frac{x+1}{x} \right)^2 \ln(x+1) \right] \right\} \quad (111)$$

so that altogether

$$\psi(\rho, \varphi) = \psi^{(0)} + \frac{d\psi^{(0)}}{d\rho} \cos\varphi \times$$

$$\left(-\Delta r + \frac{\rho_o^2}{2r_T} \left\{ \left(\frac{\rho}{\rho_o} \right)^2 + \frac{1}{2} \int_0^{\rho/\rho_o} dx \right. \right. \\ \left. \left. \times \left[-\left(\frac{x+1}{x} \right) + \left(\frac{x+1}{x} \right)^2 \ln(x+1) \right] \right\} \right) \quad (112)$$

The integral inside the { } brackets cannot be expressed in terms of common tabulated functions but it is convergent and positive for all ρ/ρ_o , and for small values the first term is proportional to $(\rho/\rho_o)^2$. Hence the whole $\psi^{(1)}$ goes like $(\rho/\rho_o)^2$ for small (ρ/ρ_o) , which confirms the statement above that this ψ_I gives no contribution to the r displacement.

The flux, ψ , can only be constant on the conducting boundary $\rho = \rho_T$ if the second term with its $\cos\varphi$ factor is zero there. The toroidal displacement of the density maximum is therefore

$$\Delta r = \frac{\rho_o^2}{2r_T} \left\{ \left(\frac{\rho_T}{\rho_o} \right)^2 + \frac{1}{2} \int_0^{\rho_T/\rho_o} dx \left[-\frac{x+1}{x} + \left(\frac{x+1}{x} \right)^2 \ln(x+1) \right] \right\}, \quad (113)$$

which is always positive because the integral is always positive. To display the results of numerical integration compactly, we write Eq. (113) in the form

$$\frac{r_T \Delta r}{\rho_o^2} = \frac{1}{2} + \frac{1}{4} \left(\frac{\rho_o}{\rho_T} \right)^2 \int_0^{\rho_T/\rho_o} dx \left[\dots \right]. \quad (114)$$

In this form $r_T \Delta r / \rho_o^2$ is a function of $(\rho_T/\rho_o)^2$ and this function is plotted in the figure. (Note that as $(\rho_T/\rho_o)^2 \rightarrow 0$ the right-hand side of Eq. (114) is a finite number, 5/8, in spite of possible appearances to the contrary).

Now a few observations. In initiating this calculation of the first small toroidal corrections to z-pinch equilibria, we took the position that $\rho_T < r_T$ was the criterion for validity of the expansion. A better criterion might be $\Delta r < \rho_T$, i.e., the displacement of the plasma much smaller than the radius of the confining boundary. The latter condition can be seen to be a little stronger than the former for cases in which the radius of the confining region, ρ_T , is much larger than the mean plasma radius, ρ_o , and these are probably the

physically interesting cases. We observe for now that in such cases and for typical laboratory values of the aspect ratio, r_T/ρ_T , the Δr given by the figure can be quite noticeable. Otherwise, there is little to say about the departures from straight z-pinch equilibrium represented by $\psi(\rho, \varphi)$, Eq. (112). To this order, the displacement of the plasma from the center of the conducting shell causes only the sort of smooth distortions of field and density profiles that would be expected, with perhaps more noticeable effects near the center than near the wall. The toroidal z-pinch equilibrium discussed above is a major improvement over the straight z-pinch for the reason that in contrast to the straight θ -pinch the ends of a straight z-pinch break the essential symmetry, in addition to allowing end loss.

We have shown that toroidal rigid rotor equilibria exist, that they are z-pinch--the Perhasatron configuration--and that for reasonably large aspect ratios they differ only slightly from the Bennett-Harris straight z-pinch discussed in Sec. V. This concludes the formal treatment of equilibria.

VIII. CONJECTURE ABOUT THE ROLE OF RIGID DRIFT EQUILIBRIA IN CONTAINMENT

There is a growing body of experimental results which indicate that high-temperature plasmas do not obey the infinite conductivity MHD model in ways that Sherwood theory has predicted they should. These failures can be described as a selective occurrence of what appears to be large anomalous resistivity. The following is a short list of what the author thinks are the most persuasive examples. They are necessarily drawn from experiments which are relatively easy to diagnose because of some feature such as high density.

A. High Energy θ -Pinches

θ -pinches like Scylla IV or ISAR I develop ion temperatures of a few keV, densities of a few $\times 10^{18} \text{ cm}^{-3}$ and are long enough, ~ 1 meter, to allow good density resolution from optical interferometry parallel to the plasma axis, in addition to Faraday rotation and flux loop measurements of internal magnetic field. Analysis of these data shows that by the time the straight plasma column is formed in the middle of the tube, the plasma

and B_z field have interpenetrated to the extent that the density profile is bell-shaped and the B_z has a complementary bell-shaped diamagnetic depression. After this time, no further penetration seems to occur. This penetration which takes place so rapidly at first is too fast to be explained by classical diffusion and seems inconsistent with simple models of anomalous resistivity because of the way it stops with large diamagnetic currents remaining in the plasma. No adequate theory exists to explain why the B_z field does not remain outside one of the various skin depths, which are all smaller than the observed penetration depth, or why, given that there is considerable penetration, that it stops where it does.

B. Z-Pinch and Turbulent Heating Experiments

What limited data there are on field penetration into high-temperature z-pinches and the similar field penetration in turbulent heating experiments (notably the recent work by Jensen and Scott at General Atomic)³ show again that there is very rapid field penetration beyond what various skin depths would predict. In this case, the B_θ field, which could not be finite on axis, assumes what looks like a Bennett-Harris profile.

C. Tokamaks

As this report is being written, recent results from the Tokamak devices are gaining acceptance.⁴ These symmetric tori have a large toroidal field, B_θ , and an induced θ current that produces a smaller poloidal field, B_ϕ . The induced toroidal current which generates the B_ϕ also heats the plasma to ion temperatures of ~ 500 eV, starting from a uniform neutral gas filling of $\sim 3 \times 10^{13} \text{ cm}^{-3}$. Although the standard MHD equilibrium theory applied to these devices imagines that most of the plasma pressure should be balanced by gradients of B_θ with B_ϕ serving only to balance the toroidal drift, experimental results are clearly consistent with the z-pinch equilibrium conditions of Sec. V, Eq. (86a), with insignificant toroidal corrections of the kind described in Sec. VII-3. Experimental efforts to increase the plasma pressure, for a given induced poloidal current, significantly beyond the z-pinch values given by Eq. (86a) have failed. Another interesting Tokamak result is that the minor radius of the plasma, which of course starts at the tube (or limiter) radius, quickly falls to a much smaller

value and then stays relatively constant for the remainder of the plasma life. This behavior is similar in a way to the early rapid partial field penetration in θ -pinch plasmas.

In general, it should be noted that of all the reasonably high-temperature CTR experiments, two stand out in regard to containment with values of $n\tau \gtrsim 5 \times 10^{11}$, the large θ -pinches and the Tokamaks (here n is the maximum particle density in cm^{-3} and τ is the containment time in sec).

These experimental results and the unique mathematical properties of the rigid drift equilibria suggest that for sufficiently high temperatures these equilibria may be physically favored by being states of the plasma in which an otherwise large anomalous resistivity vanishes. Furthermore, it is suggested that when the confinement geometry permits, initial plasma fillings which aren't in such a state may relax by diffusion into a rigid drift equilibrium. In some cases this relaxation would necessarily cause a loss of some of the plasma to material walls. In this view, the θ -pinch plasma and B field diffuse into one another until a rigid rotor equilibrium develops with its smooth bell-shaped density profile; this situation also obtains for the B_θ penetration in z-pinch and turbulent heating experiments. The initial decrease in Tokamak plasma radius could be interpreted as diffusive or resistive loss of plasma to the walls (or limiters) until a toroidal rigid rotor equilibrium remains, with a radius which is small enough so that the spatial tails of the bell-shaped density profile are virtually zero at the walls. This of course would also be consistent with the failure to exceed the z-pinch equilibrium conditions, since any other equilibrium would not have the rigid drift property.

The foregoing implicitly assumes that the functional form of the rigid drift distributions, $f(e+P_\theta)$ or $f(e+V_d P_z)$, is either the Maxwellian form discussed above or something close to it. So far we have failed to note the now relevant fact that there is a fairly direct correspondence between the form taken for the rigid drift f 's and the resulting spatial density profile. If the f 's are bell-shaped functions, the density profile will be also. If the f 's have tails that go out to $\pm \infty$, then so will the density profile, and, correspondingly, if the tails of the f 's go to zero at some point, then so will the

density profile. Any turbulent or shock heating process, together with a slight binary collisional effect, should produce reasonably smooth bell-shaped f 's out to about twice the mean thermal velocity. The tails, however, are a little different. As is known from Fokker-Planck theory, collisions cause the tails of a truncated Maxwellian to fill out in velocity at about the rate of one thermal unit per collision period, with less and less plasma being involved as the procedure progresses to lower density parts of the tail. If the postulated anomalous resistivity kept pace in modifying the density profile to match the development of the bell shape and then the slower collisional development of the tails of the f 's, the loss of plasma to walls would be expected to decrease considerably if the thickness of the nearly vacuum magnetic buffer between plasma and walls were to become of the order of, or large compared to, the mean thickness of the plasma.

The large lateral compression which accompanies the heating of a θ -pinch plasma automatically imposes a thick buffer. Hence the negligible sideways loss of θ -pinch plasma, which is also consistent with MHD theory, proves little or nothing here. But the fact mentioned above that the Tokamaks quickly develop such a buffer by loss of plasma and then become relatively quiescent is certainly consistent.

In this connection it may be significant that the tails of the Maxwellian θ -pinch density profiles fall off like $\exp[-(\frac{r}{r_0})^2]$, while the Maxwellian z -pinch (and therefore Tokamak) profiles go like $[1 + (\frac{r}{r_0})^2]^{-2}$ (r_0 is the mean plasma radius). These longer z -pinch tails could compensate for the greater economy of providing the magnetic buffer with the $1/r$ z -pinch field.

In summary, the implications of this conjecture would be (i) symmetric containment geometries are necessary, (ii) plasma pressure (β) in a symmetric torus should not be expected to exceed the z -pinch equilibrium condition, Eq. (86), and (iii) fairly thick magnetic buffers are required for long time containment. This says nothing about the gross instabilities such as the Kruskal-Shafranov $m = 1$ for a toroidal or screw pinch or the $m = 0$ for a z -pinch, which according to MHD theory should in some cases disassemble a confined plasma in unacceptably

short times. However, the appearance of a large anomalous resistivity in response to any gross asymmetry of the plasma could have an important effect on these instabilities.

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Fig. 1. This curve shows the relationship between the toroidal shift, Δr , of the plasma major radius from the containment tube major radius, r_T , the plasma minor radius, ρ , and the tube minor radius, ρ_T . From this curve it can be seen that for parameters of interest $\Delta r/\rho_T \approx \rho_T/r_T$.