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NBC - AN IBM 7090 COMPUTER PROGRAM FOR
SOLVING THERMOELASTIC PROBLEMS

F. J. Witt

OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION

for the

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JANUARY 1965

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NOMENCLATURE

Symbols

D	Physical constant defined by Eqs. (27) or (28)
E	Modulus of elasticity
G	Shear modulus
T	Temperature or radiation-exposure function
t	Normalized temperature function
u	Displacement in the x direction
v	Displacement in the y direction
α	Coefficient of thermal expansion or irradiation-induced dimensional change coefficient
σ	Unit normal stress
τ	Unit shear stress
ϵ	Unit normal strain
γ	Unit shear strain
μ	Poisson's ratio
ϕ	Airy stress function

Subscripts

x	In the direction of the x coordinate
y	In the direction of the y coordinate
z	In the direction of the z coordinate
xy	When used with shear stress or strain, the first subscript indicates the direction normal to the plane and the second, the direction of the component of stress or strain
zx	When used with Poisson's ratio, it means the ratio of strain in x direction to strain in z direction when stressed in the z direction

DEFINITIONS

biharmonic mesh point number	The number associated with a particular mesh point obtained by numbering the mesh points that enter into the analysis sequentially, starting with 1, from the upper left location to the right
boundary point	A point arbitrarily chosen to lie on a boundary but not necessarily falling on a mesh point
exterior point	A mesh point that is a member of a biharmonic array about some interior point but does not lie on the cross section
external boundary	A boundary not interior to the cross section; in short, any boundary that does not describe a hole
interior point	A mesh point that lies on the cross section of the configuration being analyzed
internal boundary	A boundary that is interior to the cross section; it is also an interior boundary if it is interior by symmetry; in short, a hole has an internal boundary
interpolation point	A mesh point from which a boundary condition is interpolated to a boundary point
mesh unit	A constant distance between two consecutive mesh points of rectangular array
rectangular mesh point number	The number associated with a particular mesh point obtained by numbering the mesh points sequentially, starting with 1, from the upper left corner of the rectangular array to the right (such points may not enter into the analysis; those that do are listed in the ITABCØ table)

NBC — AN IBM 7090 COMPUTER PROGRAM FOR SOLVING
THERMOELASTIC PROBLEMS

F. J. Witt

Abstract

The work of various individuals and groups has been combined to provide a numerical means for calculating stresses in bodies with the use of either a plane-strain or a plane-stress model. The particular problem considered is the evaluation of thermal or irradiation-induced stresses for arbitrary temperature or irradiation-exposure distributions. A computer program, NBC, is described that may be used to solve the two-dimensional nonhomogeneous biharmonic equation pertinent to the analysis of simply connected and multiply connected regions. The material may be isotropic or it may only exhibit isotropy in the plane of the cross section. The biharmonic equation and related boundary conditions are derived and an example is presented. Complete instructions for using the program are given.

1. INTRODUCTION

The material presented here represents a compilation of the work done by various individuals and groups to provide a numerical means for calculating stresses in bodies through the use of the Airy stress function. A computer program is described for solving the nonhomogeneous biharmonic equation applicable to plane-strain or plane-stress thermoelastic analyses of simply connected and multiply connected regions. The material may be isotropic or it may be isotropic only in the plane of the cross section; that is, the material properties are independent of direction in the plane of the cross section but vary in the normal (axial) direction. Such materials are said to be transversely isotropic.

Battelle Memorial Institute (BMI) was placed under contract by the Oak Ridge National Laboratory to write an IBM-7090 computer program¹ for analyzing thermal stresses and strains in bodies for which either plane-strain or plane-stress assumptions are adequate. The problem was generalized so that the right-hand side of the nonhomogeneous biharmonic equation can be either a constant or a function of the coordinates in the plane.

The personnel of the EDPF* group at the Oak Ridge Gaseous Diffusion Plant later assisted in revising the program and wrote the matrix transpose and simultaneous system solver routines that are now used. The program is called the Nonhomogeneous Biharmonic Code (NBC). While NBC may be used to calculate thermal stresses and strains, it may also be used for calculating stresses and strains resulting from irradiation-induced dimensional changes in bodies used in nuclear reactors. Dimensional changes are induced in graphite, for example, that are proportional to the doses associated with the integrated fast-neutron flux above a given energy, and the resulting strains are analogous to thermal strains.

In this report, the differential equation and the boundary conditions, based on the Airy stress function, are derived for both isotropic and transversely isotropic materials. A numerical procedure for solving the differential equations is given, and NBC is described in detail. Guidelines for successful use of the program are discussed in Appendix A and an example is presented in Appendix B.

2. THERMOELASTIC ANALYSIS

Plane-strain and plane-stress analyses of isotropic, homogeneous bodies have been considered in detail,^{2,3} and an extension of such analyses to transversely isotropic materials is presented here. A cross section under consideration is assumed to be subjected to neither external forces nor body force potentials. The isotropic and transversely isotropic analyses differ basically only in the expression for Hooke's law, and this difference is discussed.

The equilibrium, strain-displacement, and compatibility equations are given below:

1. Equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0, \quad (1)$$

where $\tau_{xy} = \tau_{yx}$.

*Electronic Data Processing Facility.

2. Strain-displacement equations:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \quad (2)$$

3. Compatibility equation:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \quad (3)$$

where x and y are the rectangular coordinates of the cross section.

For materials that are transversely isotropic, the following relationships exist between the material properties:⁴

$$\begin{aligned} \mu_x &= \mu_{xy} = \mu_{yx}, & \mu_{xz} &= \mu_{yz}, \\ E_x &= E_y, & \frac{\mu_{xz}}{E_x} &= \frac{\mu_{zx}}{E_z}. \end{aligned} \quad (4)$$

As noted by Nightingale,⁵ some materials, in particular, graphite, undergo dimensional changes upon fast-neutron irradiation that lead to induced strains in the material. Such strains are analogous to thermally induced strains; that is, that part of the total strain due to the shrinkage or growth is proportional to the irradiation. Only normal strains, and no shear strains, arise in this manner. The remaining part of the total strain is that required to maintain the continuity of the body. Thus no distinction is made in this report between thermally induced and irradiation-induced strains. For transversely isotropic materials, the generalized Hooke's law,⁶ with the induced strains added, reduces to the following form:

$$\epsilon_x = \frac{1}{E_x} \left(\sigma_x - \mu_x \sigma_y \right) - \frac{\mu_{zx}}{E_z} \sigma_z + \alpha_x^T, \quad (5)$$

$$\epsilon_y = \frac{1}{E_x} \left(\sigma_y - \mu_x \sigma_x \right) - \frac{\mu_{zx}}{E_z} \sigma_z + \alpha_x^T, \quad (6)$$

$$\epsilon_z = \frac{\sigma_z}{E_z} - \frac{\mu_{zx}}{E_z} (\sigma_x + \sigma_y) + \alpha_z T , \quad (7)$$

$$\gamma_{xy} = \frac{2(1 + \mu_x)}{E_x} \tau_{xy} , \quad (8)$$

$$\gamma_{yz} = \frac{1}{G_z} \tau_{yz} , \quad (9)$$

$$\gamma_{zx} = \frac{1}{G_z} \tau_{zx} , \quad (10)$$

where $\alpha_x = \alpha_y$. There are five independent material constants, but only three enter into the calculation of stresses. These five constants reduce to two for the familiar isotropic case, and the relationships between the constants in Eqs. (5) through (10) are

$$\mu_{zx} = \mu_x , \quad E_z = E_x , \quad G_z = \frac{E_x}{2(1 + \mu_x)} , \quad \alpha_z = \alpha_x . \quad (11)$$

A stress function is defined as follows:^{2,3}

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} , \quad (12)$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} , \quad (13)$$

$$\tau_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} , \quad (14)$$

where ϕ is the Airy stress function. This definition of a stress function requires the following constraint conditions (called Michell's conditions^{1,7}):

$$\int_{C_i} \frac{\partial}{\partial n} (\nabla^2 \phi + t) \, d\bar{s} = 0 , \quad (15)$$

$$\int_{C_i} (y \frac{\partial}{\partial n} - n_y)(\nabla^2 \phi + t) d\underline{s} = 0 , \quad (16)$$

$$\int_{C_i} (x \frac{\partial}{\partial n} - n_x)(\nabla^2 \phi + t) d\underline{s} = 0 , \quad (17)$$

where $\int_{C_i}$ indicates the line integral about and along closed boundary C_i , n and $\underline{s}$ indicate the normal and tangential directions on the boundary as defined in Fig. 1, and n_x and n_y are the x and y components, respectively, of the outward unit normal vector at the boundary. Also

$$t = DT , \quad (18)$$

where D is a constant defined by Eqs. (27) or (28) below. Equation (15) guarantees the single valuedness of the rotation about a closed boundary,

Fig. 1. Biharmonic Array of Mesh Points Near a Boundary.

while Eqs. (16) and (17) guarantee single valuedness of the displacements. These equations must be satisfied about every boundary of the cross section. In particular, they are identically satisfied on external boundaries. The equations given in ref. 7 may be applied to any closed path enclosing (and only enclosing) a single closed boundary, while Eqs. (15), (16), and (17) may be applied only to the exact boundary.

Under the condition of plane strain, ϵ_z is a constant (K, say),* and, from Eq. (7),

$$\frac{\sigma_z}{E_z} = \frac{\mu_{zx}}{E_z} (\sigma_x + \sigma_y) + K - \alpha_z T . \quad (19)$$

Substituting for σ_z , Eqs. (5) and (6) become

$$\begin{aligned} \epsilon_x = \frac{1}{E_x} \left[\left(1 - \frac{E_x \mu_{zx}^2}{E_z} \right) \sigma_x - \left(\mu_x + \frac{E_x \mu_{zx}^2}{E_z} \right) \sigma_y \right] \\ + (\alpha_x + \mu_{zx} \alpha_z) T - \mu_{zx} K \end{aligned} \quad (20)$$

and

$$\begin{aligned} \epsilon_y = \frac{1}{E_x} \left[\left(1 - \frac{E_x \mu_{zx}^2}{E_z} \right) \sigma_y - \left(\mu_x + \frac{E_x \mu_{zx}^2}{E_z} \right) \sigma_x \right] \\ + (\alpha_x + \mu_{zx} \alpha_z) T - \mu_{zx} K . \end{aligned} \quad (21)$$

The equilibrium equations, Eqs. (1), are satisfied identically by the stress function, and it remains only for ϕ to satisfy compatibility Eq. (3). Substituting Eqs. (12), (13), and (14) into Eqs. (8), (20), and (21) and taking the derivatives, as indicated, yield

*The actual expression for ϵ_z depends on the end constraint conditions; see refs. 8 and 9.

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{1}{E_x} \left[\left(1 - \frac{E_x \mu_{zx}^2}{E_z} \right) \frac{\partial^4 \phi}{\partial y^4} - \left(\mu_x + \frac{E_x \mu_{zx}^2}{E_z} \right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \right] + (\alpha_x + \mu_{zx} \alpha_z) \frac{\partial^2 T}{\partial y^2}, \quad (22)$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{1}{E_x} \left[\left(1 - \frac{E_x \mu_{zx}^2}{E_z} \right) \frac{\partial^4 \phi}{\partial x^4} - \left(\mu_x + \frac{E_x \mu_{zx}^2}{E_z} \right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \right] + (\alpha_x + \mu_{zx} \alpha_z) \frac{\partial^2 T}{\partial x^2}, \quad (23)$$

and

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = - \frac{2}{E_x} (1 + \mu_x) \frac{\partial^4 \phi}{\partial x^2 \partial y^2}. \quad (24)$$

Hence, the compatibility equation, Eq. (3), becomes

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2 \partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -D \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (25)$$

or

$$\nabla^4 \phi = -D \nabla^2 T, \quad (26)$$

where

$$D = \frac{E_x E_z (\alpha_x + \mu_{zx} \alpha_z)}{(E_z - E_x \mu_{zx}^2)}. \quad (27)$$

Similar equations may be derived by assuming conditions of plane stress for which $\sigma_z = 0$. For this case,

$$D = E_x \alpha_x. \quad (28)$$

In the numerical procedure described later, the constant D may also be factored out of the right-hand side of the integral (Michell) equations. Thus the system of difference equations may be normalized by D . In irradiation shrinkage problems, $\nabla^2 T$ is proportional to T , where T is the exposure. The constant of proportionality is B_2 , as described in the Input Summary. This constant may not, in general, be factored out of the system of equations.

In the following, all internal and external boundaries are assumed to be free from applied forces. If n_x and n_y are the components of the outward unit normal vector at a boundary (see Fig. 1), then, as shown by Holm,¹⁰

$$n_x \sigma_x + n_y \tau_{xy} = 0 , \quad (29)$$

$$n_x \tau_{xy} + n_y \sigma_y = 0 . \quad (30)$$

If $\underline{s}$ is the unit tangent vector oriented as shown in Fig. 1, then

$$n_x = \frac{dy}{ds} , \quad (31)$$

$$-n_y = \frac{dx}{ds} . \quad (32)$$

Combining Eqs. (29) through (32) yields

$$\frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) = 0 , \quad (33)$$

$$\frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) = 0 . \quad (34)$$

It follows that $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial x}$ are constants on a boundary. Thus on each boundary, C_i ,

$$\phi = a_i x + b_i y + c_i , \quad (35)$$

and the constants must be determined. However, these constants are not

arbitrary but are uniquely determined, since Eqs. (15), (16), and (17) are satisfied on an internal boundary. Hence, it remains only to satisfy the constants for the external boundary. This may be done easily by taking

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial y} = 0 \quad (36)$$

(i.e., $a = b = 0$ on the external boundary). This makes the stress function unique to within a constant. Thus, if ϕ is arbitrarily chosen at some point, the function is completely determined. It should be noted that the stresses and displacements are independent of Eq. (35).

If there is a line of symmetry on the cross section, then, along the line,

$$\frac{\partial\phi}{\partial n} = 0, \quad (37)$$

and

$$\frac{\partial \nabla^2 \phi}{\partial n} = 0, \quad (38)$$

where $\frac{\partial}{\partial n}$ denotes the partial derivative normal to the line of symmetry. These equations follow directly from the symmetry exhibited by the stress function and the first stress invariant.

3. THE NUMERICAL METHOD OF SOLUTION

NBC is a modified version of the BMI program¹ used for calculating the stresses in the graphite columns of the Experimental Gas-Cooled Reactor being built at Oak Ridge, Tennessee.¹¹ Four programs were added and two deleted from the BMI version. Also the Fortran chain feature was incorporated to produce a composite program, and various aspects of data input were altered. A description of the numerical techniques used in the analysis is given below.

The usual finite difference method of selecting a rectangular array of equally spaced points on the cross section of the body and expressing

the differential equation and boundary conditions in terms of these points is used. A linear algebraic system of equations involving the values of ϕ at every point is found. From the solution of the system, the stresses and displacements for a cross section are determined. Complete biharmonic arrays, as illustrated by the numbered points in Fig. 1, are written at every mesh point on the cross section. This gives external points; for example, if the boundary line in Fig. 1 is a right edge, then points 4, 8, 9, and 12 are external points. These external points may be considered as an analytical continuation of the stress function. Actually, such points are very useful because an equation must exist in the system for each point, and some boundary condition may be satisfied. For a particular mesh spacing, boundary points are chosen on each boundary consistent with the total number of external points. Usually two boundary conditions are associated with each boundary point.

A consistent method must be used in forming the difference equations from the differential equation and boundary conditions discussed in Section 2. The function chosen for forming the equations results from a double application of a truncation of Stirling's polynomial interpolation formula.¹² With reference to Fig. 1, the function is

$$\begin{aligned}
 \phi(r,s) = & \phi(7) + \frac{r}{2} [\phi(8) - \phi(6)] + \frac{s}{2} [\phi(3) - \phi(11)] + \frac{r^2}{2} [\phi(8) + \phi(6) \\
 & - 2\phi(7)] + \frac{s^2}{2} [\phi(3) + \phi(11) - 2\phi(7)] + \frac{rs}{4} [\phi(4) + \phi(10) \\
 & - \phi(2) - \phi(12)] + \frac{r(r^2 - 1)}{12} [\phi(9) - 2\phi(8) - \phi(5) + 2\phi(6)] \\
 & + \frac{s(s^2 - 1)}{12} [\phi(1) - 2\phi(3) - \phi(13) + 2\phi(11)] + \frac{rs^2}{4} [\phi(4) \\
 & + \phi(12) - 2\phi(8) - \phi(10) - \phi(2) + 2\phi(6)] + \frac{r^2 s}{4} [\phi(4) + \phi(2) \\
 & - 2\phi(3) - \phi(12) - \phi(10) + 2\phi(11)] + \frac{r^2(r^2 - 1)}{24} [6\phi(7) - 4\phi(8) \\
 & - 4\phi(6) + \phi(9) + \phi(5)] + \frac{s^2(s^2 - 1)}{24} [6\phi(7) - 4\phi(3) - 4\phi(11) \\
 & + \phi(1) + \phi(13)] + \frac{r^2 s^2}{4} [\phi(4) + \phi(2) + \phi(12) + \phi(10) - 2\phi(3) \\
 & - 2\phi(8) - 2\phi(6) - 2\phi(11) + 4\phi(7)] .
 \end{aligned} \tag{39}$$

This formula assumes the correct value of the function at each of the 13 points of the array. It may be differentiated and evaluated at any point, $\phi(r,s)$. In particular,

$$\begin{aligned} \nabla^4 \phi(r,s) = & 20\phi(7) - 8[\phi(3) + \phi(6) + \phi(8) + \phi(11)] + 2[\phi(2) + \phi(4) \\ & + \phi(10) + \phi(12)] + [\phi(1) + \phi(5) + \phi(9) + \phi(13)] , \end{aligned} \quad (40)$$

which does not involve r and s . Also,

$$\frac{\partial \phi}{\partial r}(0,0) = \frac{2}{3} \phi(8) - \frac{2}{3} \phi(6) - \frac{1}{12} \phi(9) + \frac{1}{12} \phi(5) , \quad (41)$$

which is the same as the result obtained by accurately applying Taylor series techniques to a fourth-order correction factor.¹³ This also holds true for the second derivatives.

The condition of plane strain assumed in the program is that there is no resultant force acting in the axial direction. This defines the axial strain to be a constant, K [see Eq. (19)], which is determined with

$$\int_A \sigma_z dA = 0 . \quad (42)$$

The resultant moments acting about the x and y axes, respectively, are given by

$$M_x = \int_A (\sigma_z - \bar{\sigma}) y dA , \quad (43)$$

$$M_y = \int_A (\sigma_z - \bar{\sigma}) x dA , \quad (44)$$

where $\bar{\sigma}$ is the average axial stress. Further discussions on this subject are available in refs. 8 and 9. Trapezoidal integration is used by the program in defining the Michell conditions.

4. NBC - THE NONHOMOGENEOUS BIHARMONIC CODE

NBC is an IBM 7090 computer program* for analyzing a body of arbitrary cross section for which the analyses presented in Sections 2 and 3 are applicable. The finite difference forms are all taken from the basic interpolation formula given in Eq. (39). The stress function is usually obtained in normalized form and differs from the stress function of interest by the factor D given in Eqs. (27) or (28). Thus the physical constants enter only into the calculation of stresses and displacements. In particular, the stress function is independent of the physical properties of the material being considered. Also, the program calculates stresses for a plane-strain analysis for constant strain. If D as defined by Eq. (28) is given as part of the input, the stresses for a plane stress analysis are computed, and references to axial stresses should be ignored.

Since the program is quite general, the utmost care must be taken in preparing the input. The many different aspects of input preparation to be considered are discussed below.

The first step in preparing a problem for analysis is to cover the cross section with a rectangular array of equally spaced mesh points. The only points that enter into the calculations are those which form part of the biharmonic array of some point that lies on the cross section. Symmetry properties should be considered in deciding whether a whole cross section must be analyzed or just a symmetrical portion. Consideration should also be given to selecting a mesh spacing such that mesh points fall at the more pertinent locations on the cross section; for example, on the boundaries. The points on the rectangular array are numbered consecutively, row after row, from the upper left corner. The horizontal direction to the right is designated by N; the vertical direction downward, by M. The sequentially numbered points of the rectangular array that enter into the analysis are listed consecutively in the table of input designated by ITABCØ. A correspondence may then be set up between the ITABCØ table and the tables designating the temperature or exposure (TEMP) and the heat

*Portions of the System Solver Program and associated Matrix Transpose Program are written in FAP. The remaining programs are written in FORTRAN. A 32K machine is required.

generation function (QTEMP). The QTEMP table actually gives the right-hand side of the type 1 equation listed in the Input Summary and is necessary only if IQC is 3. For exterior points the QTEMP entries may be taken as zero; however, if the integral equations are written for a boundary or the displacements are to be calculated along the boundary, entries in the TEMP table must be given for all exterior points along this boundary that enter into the Laplacian array about mesh points around which boundary points are expanded (this is discussed below). This means that the set of exterior points closest to the boundary in question should have a corresponding entry in the TEMP table. The TEMP values so taken must be analytical continuations of the TEMP values of the points on the cross section. Zero entries may be made to correspond to the set of points for which an entry is not required. If the boundary stress routine is to be run, all the TEMP values at exterior points must be given. The largest number of points that may enter into an analysis is 1000.

Boundaries are described by choosing boundary points on the boundary in question and giving both the locations of such a point with respect to the nearest point on the cross section (i.e., interpolation point) and the components of the outward normal vector of the boundary point. The boundary conditions are specified at the boundary points by using the interpolation formula of Eq. (39). On a given boundary the boundary points should be equally spaced, especially if displacements are to be computed. The r and s components (see Fig. 1) are listed in sequence (corresponding to the numbering of the boundary points) in tables designated by RA and SA, respectively. The x and y components of the outward normal vector are similarly listed in the XNA and YNA tables. Likewise the N and M coordinates of the interpolation point are listed in tables designated by NA and MA, respectively.

The number of boundary points chosen for a particular boundary is arbitrary but should be such that the boundary is well defined. As indicated later, a boundary condition is associated with each exterior point and two boundary conditions may usually be written at each boundary point. It may then be concluded that the total number of boundary points is approximately one-half the number of exterior points. A rule of thumb is to

place the boundary points at intervals approximating a mesh unit. Frequently, boundary conditions may not be specified at the first and last point of a boundary; this situation is discussed later. With one exception, boundary points are numbered consecutively with respect to the total number of boundary points and those on a particular boundary. The exception is that a point which is common to two boundaries (for example, each edge of an outer rectangular boundary is usually considered a boundary) is given two different boundary point numbers because it has two different normal vectors. For complete channels or holes in a cross section the first and last boundary points must coincide, and there are two designations for this point. The sign convention is described in the Input Summary.

The consideration of boundary conditions is equally important, since failure to specify these conditions properly may result in a singular matrix of rank considerably less than that required.* Care must be taken not to over-specify conditions at the beginning or end of a boundary that may also be a point of another boundary. In order to specify the boundary conditions, it is first necessary to renumber the points pertinent to the analysis. The points are numbered in the same sequence as in the ITABCØ table except that they are numbered consecutively starting with 1. The numbers associated with the points in this fashion are called biharmonic mesh point numbers. This numbering is the sequence in which the system of finite difference equations is arranged. Four tables with corresponding entries must then be set up. The first table is designated by IPMSN and contains, in sequence, the list of exterior (using the biharmonic mesh point numbers) points of the array. The ICN table contains the type number (see Input Summary, below) of the difference equation to be written about a boundary point listed in the IBDSN table. The IEXPNS table contains the mesh point number about which the boundary point is expanded (i.e., interpolation point number). Certain other information may also be given in this last table. If a point number does not occur in the IPMSN table, it is

* If there are n points, the rank of the matrix should be n ; however, a satisfactory solution is often obtained if the rank is $n - 1$. This situation is discussed later.

assumed to lie on the cross section, and the biharmonic difference equation [Eq. (1) of the equation types given in the Input Summary] is written about each such point.

General rules concerning the choice of boundary conditions (that is, entries in the ICN table) are difficult to give, especially when certain assumptions of symmetry are made. However, a discussion of equations of types 1 through 12 of the Input Summary is given below. All the types, except 1, 2, and 12, are written about or with regard to boundary points. The entries in the IEXPNS table are not referred to except when equations of type 2 or 12 are specified in the ICN table. With this noted exception, the entries in this table are arbitrary; however, entries must be made at least through the last IEXPNS number used.

The type 1 equation is never specified in the ICN table and is written about every point not referred to in the IPMSN table. The right-hand side of the difference equation is determined by the input value of IQC. If IQC is 3, the QTEMP table must be given, and as noted previously, it must contain the numerical value of the right-hand side of the biharmonic equation. If IQC is 4, the value of B2 is given.

In the type 2 equation the value of the stress function at the mesh point given in IEXPNS that corresponds to a 2 in the ICN table is set equal to zero. A singular system of equations results unless this equation is used at least once. Also, this designation may be used to set the stress function equal to zero along a boundary on which mesh points fall. When this is done, the stress function at the points lying on the extension of the boundary should also be set equal to zero. It may be observed that the normal stress is then identically zero. This type of equation may be used simultaneously with a type 3 equation to specify a free boundary. Since a type 2 equation applies only to a mesh point, this procedure is applicable only to a boundary on which mesh points fall.

A type 3 equation may be used simultaneously with a type 2 equation, as indicated above, or with a type 4 equation to specify that the cross section is symmetrical with respect to the boundary in question. Since

$$\frac{\partial \phi}{\partial n} = \cos \theta \frac{\partial \phi}{\partial x} + \sin \theta \frac{\partial \phi}{\partial y} ,$$

where θ is the angle between the horizontal and normal derivative vectors, no more than two of the three quantities $\frac{\partial\phi}{\partial n}$, $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$ may be specified at a given boundary point.

As noted in the discussion of the type 3 equation, a type 4 equation is used in specifying symmetry about a boundary. If symmetry exists in mesh points also, a type 12 equation should be used in place of equations of types 3 and 4.

Equations of types 5 and 6 imply a free edge condition for an external boundary and may be applied to a boundary of any shape. If symmetry conditions exist, care must be used not to over-specify the end points of a boundary (see App. B). Again symmetry conditions may imply that these types may be used on internal boundaries instead of types 7 and 8.

Equations of types 7 and 8 are applicable to internal boundaries. The derivative at the boundary point in question is set equal to the derivative at the first boundary point of the boundary in question. This implies that the derivative along the boundary is a constant. Since the last and first points of a complete internal boundary coincide, these conditions must not be written about such points. Often an assumption of symmetry gives only a part of a complete internal boundary, and in this case, symmetry conditions may imply that only one of these conditions may be specified at the last boundary point on the symmetrical portion of the complete boundary (see App. B). Application of these conditions to external boundaries may also be made (see App. A).

Equations of types 9, 10, and 11 are the integral equations (Michell's conditions)^{1,7,14} that must be satisfied about every internal boundary and must guarantee single valuedness in rotation and displacements. For each boundary specified in the problem there corresponds a table called THET in which the entries are closely related to equations of types 9, 10, and 11. If, for a particular internal boundary, no symmetry exists, then equations of types 9, 10, and 11 must be specified, and the entry into the THET table for the boundary in question is zero. If the boundary is cut by a single line of symmetry, then equations of types 9 and 11 must be specified, and a type 10 equation must not be specified. The corresponding entry in the THET table is the angle in radians required to rotate the symmetry line in

a clockwise direction until it is parallel to the M axis. If two or more symmetry lines cut the boundary, only a type 9 equation is specified, and the entry in the THET table is again zero. The boundary point number specified with each of these types (i.e., the IBDSN entry) is always the first boundary point of the boundary in question. Trapezoidal integration is used by the program.

The use of a type 12 equation specifies that the value of the stress function for the mesh point specified in IPMSN is equal to the value of the stress function specified in IEXPNS. This type of equation is used to imply symmetry. An additional application is suggested in Appendix A.

Certain other information must be supplied to the program, but the definitions are straightforward and are given only in the Input Summary.

Input Summary

With the exception of the first card, all the input must be given in DIP format. DIP is an input subroutine that enables the input to be read in with variable field width. This proves to be a most useful property when several tables of some length are a part of the input. A description of the data format, as applicable to NBC, is given below.*

DIP input contains a control punch in column 1 of the card that indicates the type of input numbers to be read in and whether the card under consideration is the last card of the input. The first entry of a particular set of data on a card must also contain the title of the data being read in. After the title, the numbers of the table are listed in sequence on the card in question through column 72** and on the following cards until all that particular set of data is given. (The title is given only on the first card.) Columns 73 to 80 are not defined and should be used to identify each card. A field is terminated with either one comma or any number of blank spaces. If the control punch in column 1 for two different sets of data is the same, such data (including the name) may be listed on the same card. The control punch in the first column, pertinent to NBC, is:

* A complete description of DIP may be found in ref. 15.

** A decimal point must not occur in column 72.

- 3 Floating-point data (contains a decimal point)
- 4 Fixed-point data (integers)
- 3 and 8 Floating-point data -- end of input
- 4 and 8 Fixed-point data -- end of input

Exponents for floating points are indicated by a slash followed by the exponent. A negative value is indicated by a minus; for example, -10,000 may be given by -1./4 and 0.0001 by .1/-3.

The first card of a complete set of data is for identification, and the first 78 columns may be used. This card is not read in by the DIP subroutine. With this exception the individual sets of data may be grouped in any convenient order.

NBC consists of five routines set up in chain sequence. The Initializing Program, which is described later, arranges the data pertinent to each routine. If some routine is not to be run, the data peculiar to that program may not be included in the input. A specification number is associated with each program that must be entered sequentially in the ISPEC table. The numbers are:

- 1 Coefficient Matrix Routine
- 2 Transpose and System Solver Routine
- 3 Stress Calculation Routine
- 4 Displacement Routine
- 5 Boundary Stress Routine

It is not necessary to run the program in its entirety; however, should this not be done, Logical Tape 7 (defined as A6 in present usage), which has been generated by the programs designated by smaller numbers, must be available, with one exception. To run the displacement routine satisfactorily, Logical Tape 7 generated by the first three programs must be available; however, the boundary stress routine does not depend on the displacement routine having been run previously.

The titles and descriptions of the input data are given below. The type of input numbers and the programs, numbered as above, for which the input must be given are indicated in the parentheses. The dimensions of the input data formats are given in Table 1.

Table 1. Dimension Table for Input

Dimension	Input Title
5	RØ, NSIGZ, HTMESH, EZ, EX, ALPHAX, ALPHAZ, PRXZ, DDD, EPS, ISPEC
(5,5)	V
10	DELS, IBEG, IEND, ZM, ZN, THET, AMDA
1000	ITABCØ, RA, SA, XNA, YNA, MA, NA, IPMSN, ICN, IEXPNS, IBDSN, TEMP, QTEMP

Input Terminology

The first 78 columns of the first card are for identification.

Individual Terms

ISPEC	The data contain, in sequence, the routine numbers described above for the routines to be run (Fixed points: 1, 2, 3, 4, 5)
IE	Total number of mesh points (Fixed points: 1, 3, 4, 5)
IB	Number of exterior mesh points (Fixed points: 1, 3)
IC	Number of boundary points (coincident points counted twice) (Fixed points: 1, 4, 5)
ICUR	Total number of specified boundary curves of configuration (Fixed points: 1, 4)
IQC	Number indicating the form of the right-hand side of the nonhomogeneous biharmonic equation; see type 1 equation given below (Fixed point: 1)
MMAX	Number of rows of mesh points in rectangular array (Fixed points: 1, 3, 4)
NMAX	Number of columns of mesh points in rectangular array (Fixed points: 1, 3, 4, 5)
IPN*	Identification number in Stress Calculation Routine (Fixed point: 3)

*This number does not enter into the calculations and may be omitted.

B2 Constant used only when IQC is 4. The units of B2 must be (mesh units)⁻² (Floating point: 1)

Tables

1. Boundary Curve Tables. The input for various boundaries is listed in the same sequence in each table. Any boundary on which the integral equations are applied or on which displacements or boundary stresses are to be computed must be defined by boundary points. Each table has ICUR entries.

IBEG	A table containing the beginning boundary point number for each curve (Fixed points: 1, 4)
IEND	A table containing the last boundary point number for each curve (Fixed points: 1, 4)
THET	A table containing the angle (radians) of rotation for lines of symmetry of each curve. The values to use for internal boundaries are discussed under equation types 9, 10, and 11. For external boundaries, zeros are used (Floating point: 1)
ZM	A table containing the M coordinate of the center of each curve in rectangular mesh units; a value of zero may be taken if the boundary is not closed; i.e., if no integral equations are written about the boundary (Floating points: 1, 4)
ZN	A table containing the N coordinate of the center of each curve in rectangular mesh units, a value of zero may be taken if the boundary is not closed; i.e., if no integral equations are written about the boundary (Floating points: 1, 4)
DELS	A table containing the distance between boundary points in mesh units for each boundary (Floating point: 4)

2. Geometry Tables. The input for boundary points, beginning with the first, is listed sequentially. Each table has IC entries.

MA	M coordinate of interpolation point (Fixed points: 1, 4, 5)
NA	N coordinate of interpolation point (Fixed points: 1, 4, 5)

RA	Horizontal distance (r value) from interpolation point to boundary point, positive to right (Floating points: 1, 4, 5)
SA	Vertical distance (s value) from interpolation point to boundary point, positive upward (Floating points: 1, 4, 5)
XNA	x component of unit outward normal vector of boundary point, positive to the right (Floating points: 1, 4, 5)
YNA	y component of unit outward normal vector of boundary point, positive upward (Floating points: 1, 4, 5)

3. Equation Tables. The equation tables are used to describe the boundary conditions. Each table has IB entries.

IPMSN	Biharmonic mesh point numbers of exterior point in sequence (Fixed points: 1, 3)
IBDSN	Boundary point numbers to which the corresponding ICN numbers relate certain conditions (Fixed point: 1)
IEXPNS	Interpolation point numbers that correspond to the boundary points listed in the IBDSN table. A different definition is given for an ICN of 2 or 12 (see below) (Fixed point: 1)
ICN	Equation type numbers that specify, in general, the conditions on corresponding boundary points referred to in the IBDSN table. In some instances they are related to the IEXPNS number (Fixed point: 1)

The equation type numbers that may be entries in the ICN table are:

Type	Equation
1	$\nabla^4 \phi = \begin{cases} 0, & \text{IQC} = 1 \\ 1, & \text{IQC} = 2 \\ f(x,y), & \text{IQC} = 3 \\ -(B2)T, & \text{IQC} = 4 \end{cases}$
2	$\phi = 0$ at corresponding mesh point listed in IEXPNS table
3	$\frac{\partial \phi}{\partial n} = 0$ at corresponding boundary point listed in IBDSN table (n indicates the normal direction)
4	$\frac{\partial^2 \phi}{\partial n^2} = 0$ at corresponding boundary point listed in IBDSN table (n indicates the normal direction)
5	$\frac{\partial \phi}{\partial x} = 0$ at corresponding boundary point listed in IBDSN table

6 $\frac{\partial \phi}{\partial y} = 0$ at corresponding boundary point listed in IBDSN table

7 $\frac{\partial \phi}{\partial x}$ at corresponding boundary point listed in IBDSN table
 $= \frac{\partial \phi}{\partial x}$ at first boundary point on boundary

8 $\frac{\partial \phi}{\partial y}$ at corresponding boundary point listed in IBDSN table
 $= \frac{\partial \phi}{\partial y}$ at first boundary point on boundary

9 $\int \frac{\partial}{\partial n} \nabla^2 \phi \, d\underline{s} = - \int \frac{\partial t}{\partial n} \, d\underline{s}$

10 $\int \left(x \frac{\partial}{\partial n} - n_x \right) \nabla^2 \phi \, d\underline{s} = - \int \left(x \frac{\partial}{\partial n} - n_x \right) t \, d\underline{s}$

11 $\int \left(y \frac{\partial}{\partial n} - n_y \right) \nabla^2 \phi \, d\underline{s} = - \int \left(y \frac{\partial}{\partial n} - n_y \right) t \, d\underline{s}$

12 ϕ at mesh point entry in IPMSN table = ϕ at corresponding mesh point in IEXPNS table

An equation of type 1 is set up at each mesh point lying on the body and, in general, is not specified in the ICN table.

4. Tables for Specifying Rectangular Mesh-Point Numbers and Thermo-elastic Loadings. Each table has IE entries.

ITABCØ Table of rectangular mesh point numbers taken in a sequence that coincides with the biharmonic mesh points (Fixed points: 1, 3, 4, 5)

TEMP Table of temperature or exposure values associated with each mesh point; there exists a one-to-one correspondence between TEMP and ITABCØ (Floating points: 1, 3, 4, 5)

QTEMP Table that specifies the value of $f(x,y)$ when IQC is 3. There is a one-to-one correspondence between the entries in this table and those in ITABCØ. Zero entries may be made for points not lying on the body (Floating point: 1)

5. Material Property Tables. All the input described below is required only in the stress, displacement, and boundary stress routines. The constants used directly in the routines may be a part of the input or only mechanical properties may be given and the constants computed.

The stress, displacement, and boundary stress routines are set up to run parameter studies, if such is desired. Up to five different sets of constants used in the calculation of in-plane stresses and displacements may be entered, and up to five entries of the constant V used in the axial stress calculation can be made for each set of in-plane data. The listing of the data in the various tables must be in the same sequence, except in the table for V , in which the data are listed in an unbroken sequence. In the option check, if $DDD = 0$, all data pertinent to the case at hand are computed for plane-strain conditions; if $DDD \neq 0$, the data required have been read in. For each case that DDD is read in directly, the computed axial stresses must be multiplied by $E_z \alpha_z / D$ to produce the correct axial stresses. If DDD is calculated, the axial stresses are correctly calculated. For the Boundary Stress Routine, if E_z and α_z are not given, the axial stress is computed as zero. This program does not use V . The input for the options is listed under options A and B. The input for a given case is the following:

HTMESH	Length of a mesh unit in terms of the set of dimensions used. Usually this value is the length of a mesh unit in inches (Floating point: 3)
NSIGZ	The number of V 's associated with a given set of physical properties. A table of axial stresses is computed for each V (Fixed points: 3, 5)
$R\phi$	Optional scaling factor. In computing axial stresses each TEMP value is divided by this quantity. If $R\phi$ is not included as input, it is taken as 1.0 (Floating points: 3, 5)

Option A: (ALPHAZ and EZ are necessary for boundary stress routine)

PRXZ	Poisson's ratio; ratio of strain in x direction to strain in z direction when stressed in the z direction (Floating points: 3, 4, 5)
ALPHAX	Coefficient of thermal expansion (or shrinkage coefficient) in the isotropic plane (Floating points: 3, 4, 5)
ALPHAZ	Coefficient of thermal expansion (or shrinkage coefficient) in the axial or z direction (Floating points: 3, 4, 5)

EX Young's modulus of elasticity in the isotropic plane (Floating points: 3, 4, 5)

EZ Young's modulus of elasticity in the axial or z direction (Floating points: 3, 4, 5)

Option B:*

$$\begin{aligned}
 \text{DDD} \quad \text{DDD} &= \begin{cases} \frac{E_X E_Z (\alpha_X + \mu_{ZX} \alpha_Z)}{E_Z - \mu_{ZX}^2 E_X} \\ \text{or} \\ E_X \alpha_X \end{cases} \quad [\text{Floating points: 3, 5, see Eqs. (27) or (28)}] \\
 V \quad V &= \frac{\mu_{ZX} E_X (\alpha_X + \mu_{ZX} \alpha_Z)}{\alpha_Z (E_Z - \mu_{ZX}^2 E_X)} \quad (\text{Floating point: 3}) \\
 \text{EPS} \quad \text{EPS} &= \alpha_X + \mu_{ZX} \alpha_Z \quad (\text{Floating point: 4}) \\
 \text{AMDA} \quad \text{AMDA} &= \frac{\mu_{ZX} \alpha_Z}{\alpha_X + \mu_{ZX} \alpha_Z} \quad (\text{Floating point: 4})
 \end{aligned}$$

Input Associated with Each Routine

A t in parentheses following a designation indicates a table; o indicates optional input. The first card (identification) and ISPEC are common to all routines.

1. Coefficient Matrix Routine.

IE	IB
IC	ICUR
IQC	IBEG (t)
IEND (t)	MMA
NMAX	IPMSN (t)
ICN (t)	IBDSN (t)
IEXPNS (t)	ITABCØ (t)
MA (t)	NA (t)
ZM (t)	ZN (t)

* Terms entering into the formulas below are defined in the nomenclature.

B2 (o)	TEMP (t)
QTEMP (t,o)	THET (t)
RA (t)	SA (t)
XNA (t)	YNA (t)

2. Transpose and System Solver Routine. With the exception of the identification and ISPEC cards no external input is required.

3. Stress Calculation Routine.

IE	NMAX
MMAX	IPN (o)
IB	IPMSN (t)
ITABCØ (t)	V (t,o)
HTMESH	DDD (t,o)
RØ (o)	TEMP (t)
NSIGZ (t)	PRXZ (t,o)
ALPHAX (t,o)	ALPHAZ (t,o)
EX (t,o)	EZ (t,o)

4. Displacement Routine.

IE	IC
ICUR	NMAX
MMAX	ITABCØ (t)
TEMP (t)	NA (t)
MA (t)	IBEG (t)
IEND (t)	RA (t)
SA (t)	XNA (t)
YNA (t)	ZN (t)
ZM (t)	EPS (t,o)
DELS (t)	AMDA (t,o)
PRXZ (t,o)	ALPHAX (t,o)
ALPHAZ (t,o)	EX (t,o)
EZ (t,o)	

5. Boundary Stress Routine.

IE	NMAX
MA (t)	NA (t)
RA (t)	SA (t)
XNA (t)	YNA (t)

ITABCØ (t)	TEMP (t)
NSIGZ	RØ (o)
PRXZ (t,o)	ALPHAX (t,o)
ALPHAZ (t)	EX (t,o)
EZ (t)	DDD (t,o)
IC	

Output Summary

The identification is read out prior to the output of each routine.

Coefficient Matrix Routine

All the input to the coefficient matrix routine is read out and is followed by a listing of the system of equations. Only nonzero entries are listed for the left-hand side, and the biharmonic mesh point numbers are given corresponding to the coefficients. The right-hand side is designated by the number IE + 1. Each equation is normalized by the square root of the sum of squares of its coefficients (that is, by the length of the vector).

Transpose and System Solver Routine

Three columnal tables are listed first. The first table gives the number of the pivot;* the second, the equation in which the pivot was taken; the third, the value of the pivot. The solution of the system of equations is given next. The solution gives the values of the stress function at the biharmonic mesh points designated. Next, checking data are listed, and the pertinent output data are defined as follows (assuming the system of equations is given in matrix form by $Ax = y$):

CØS	Cosine of the angle between Ax and y ; it should be approximately 1
RSQ	Sum of the square of the residuals
SXY	Scalar product $(Ax)y^T$; it should approximate SY for a good solution

* Gaussian elimination is used to achieve a solution. A pivot is the largest coefficient of the unknown being eliminated.

SY	Scalar product, yy^T (length of Y squared); it should approximate SAX
SAX	Scalar product $(Ax)(Ax)^T$; length of Ax squared
Ax(I)	Product of A times solution vector x
Y(I)	y vector; for any given I, it should approximate the corresponding entry in Ax(I)
RI	Absolute value of residual vector, $ AX(I) - Y(I) $, for a given I. These entries should be small compared with the corresponding entry in Y(I). If the numbers involved in calculating the residuals are six orders of magnitude greater than the residual, then the residual is taken as zero and AX(I) and Y(I) are made equal for the particular I in question.

Stress Calculation Routine

All the input to the stress calculation routine is read out first. A number designated by TBAR is read out that is the average of the TEMP values. The axial strain, ϵ_z (or K), is calculated as α_z (TBAR). This approximation assumes that the Poisson's ratio effect is negligible. The stresses are then read out and are followed by a summary of maximum and minimum stresses. The moments about the x and y axes ($N = 1$, $M = 1$, respectively) are given. These values may have little significance except when a complete body is analyzed. SBAR is the average axial stress. If there is no axial force, SBAR should be approximately zero. Tensile stresses are positive; compressive stresses, negative.

Displacement Calculation Routine

The input, including TBAR, is read out followed by a listing of the rotation and the U and V displacements. (In Section 2 the u and v displacements correspond to U and V, respectively.) These values are calculated using numerical integration with respect to the first boundary point on the curve [see Eq. (2)]. The U displacement of a point with respect to the first point on a curve is negative if the point has moved to the right; the V displacement is positive if the point has moved downward. The

rotation is positive counterclockwise. The rotations and displacements are calculated by integrating around the boundary from the initial point.

Boundary Stress Routine

The input, including TBAR, is listed. A number designated as EPIZ is also given that is the axial strain calculated as α_z (TBAR). The stresses on the boundaries are then read out and a summary of maximum and minimum stresses is obtained. Three tables in the listing need to be defined. One table, designated BOUNDARY TEMP, contains the temperature (or fast-neutron exposure) as calculated using the interpolation formula, Eq. (39), at each boundary point. Another table designated NUM ANGLE contains the counterclockwise angle between the N coordinate and the direction of principal stress. This angle, θ , is restricted to the range $-90^\circ < \theta < 90^\circ$ so that, for example, an angle calculated to be 120° would appear as -60° . Since the transverse shear at a boundary is zero, the principal stress direction is defined by the tangent and normal at the boundary. The table designated NORMAL ANGLE is computed from XNA and YNA and is the angle the outward normal vector makes with the N-axis in the positive counterclockwise direction. The range for this angle, θ , is $-90^\circ \leq \theta \leq 90^\circ$. The corresponding values in these last two tables should be approximately the same or the sum of their absolute values should be approximately 90° , with one exception. If $\sigma_x = \sigma_y$ and $\tau_{xy} \neq 0$, $\tan 2\theta$ is infinite and 2θ is $\pm 90^\circ$. However, the arc tangent subroutine of the program cannot handle this, and it computes a zero for the angle (the NUM ANGLE table entry). The NORMAL ANGLE table entry is correct. This situation exists at boundary point 4 of Fig. B.1 of the example (see App. B).

Error Statements

There are several error statements in the programs and most are self-explanatory. In some instances the statement is not entirely correct but indicates the region in which some input is in error. For example, if a reference is made to an unspecified boundary point (it would be interpreted as being at $N = M = 0$), the error statement indicates an error in ITABCØ,

since a biharmonic array would not exist about $N = M = 0$. If the computer stops on HPR 6,6, a bad tape is indicated.

Program Machine Time

The actual machine time used in running a particular case varies with the number of mesh points and the complexity of the configuration, but an accurate estimate may be made from Fig. 2. Actual times are plotted in this figure. For configurations with many mesh points, most of the time is used by the transpose and system solver routine.

Operating Instructions for NBC

The five programs of NBC were combined into a one-job run by using the Fortran chain feature. To accomplish this an Initializing Program was

Fig. 2. Computer Time for Analyzing Various Problems with NBC.

written that processes the input data and directs the flow of the problem. The Initializing Program is always executed, but the other programs are executed only as specified by the ISPEC entries.

The program chain tape is prepared by the CDP monitor by submitting the programs in specified order with proper chain designation. The deck setup is given below:

```

CDP control cards (*ID, *CHARGE, *TAPE, etc.)
*CHAIN (6,3)
*TYPE (BINARY)
INITIALIZING PROGRAM
*CHAIN (1,3)
*TYPE (BINARY)
COEFFICIENT MATRIX PROGRAM
*CHAIN (2,3)
*TYPE (BINARY)
MATRIX TRANSPOSE PROGRAM
*CHAIN (20,3)
*TYPE (BINARY)
SYSTEM SOLVER PROGRAM
*CHAIN (3,3)
*TYPE (BINARY)
STRESS CALCULATION PROGRAM
*CHAIN (4,3)
*TYPE (BINARY)
DISPLACEMENT PROGRAM
*CHAIN (5,3)
*TYPE (BINARY)
BOUNDARY STRESS ROUTINE
*DATA

```

When the program chain tape is prepared the Initializing Program assumes control if one of the CDP control cards is an *EXECUTE card.

Preparation of the program chain tape takes approximately 3 min, and therefore the tape should be saved for later use. A one-statement program has been written to call the Initializing Program from the tape. The set-up for use of a previously prepared program chain tape is as follows:

```

CDP control cards
*TYPE (BINARY)
  Program to call chain (6,3)
*DATA
  INPUT data

```

The tapes used by NBC are:

<u>CHANNEL A</u>		<u>CHANNEL B</u>	
<u>Logical Number</u>	<u>Assignment</u>	<u>Logical Number</u>	<u>Assignment</u>
2	Any	11	3, program chain
7	Any (6 in present		tape
	usage)	5	Any
1	Any	4	Any
		6	Any
		8	Any

Logical Tape 7 on Channel A contains sufficient information from the preceding programs to run the stress calculation, displacement, and boundary stress programs subject to the restrictions discussed in the above section on "Input Summary." If a nondimensional analysis is run, the Logical Tape 7 may be used to run a parameter study on physical properties at some future time. It is therefore recommended that this tape be saved until such time as it can no longer be of use.

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APPENDICES

Appendix A

EXPERIENCE WITH THE USE OF NBC

NBC has been used to analyze multiply connected bodies in the form of cylinders and prisms with rectangular and hexagonal cross sections. Also, simply connected configurations have been analyzed. Because of the complexity of the program, experience is needed in diagnosing difficulties that may arise. Therefore, a general discussion of difficulties encountered in using NBC and the methods used to eliminate these difficulties is given below.

A major concern in setting up a problem is the total number of mesh points to be used in the analysis. As indicated in Section 3 the interpolation formula yields finite difference forms that are correct in the first and second derivatives at mesh points to order h^4 , where h is the mesh unit length. From this it may be implied that a relatively coarse mesh may be sufficient in the analysis of most configurations. (This is shown to be true for the example presented in App. B.) The best approach is to analyze the coarser mesh problem and then use a finer mesh to check the convergence of the solution. This technique can produce considerable savings in machine time, especially when several runs are to be considered for the same configuration.

As previously mentioned, when a configuration possesses symmetry about a given line or lines, the symmetry should be included in the analysis of the configuration. This will allow for a relatively fine mesh spacing as compared with that which may be used for the whole configuration.

In the analysis of almost any configuration, there is considerable leeway as to the choice of boundary conditions. A judicious choice can mean the difference between obtaining a correct solution and a completely erroneous one. This does not mean that using one set of boundary conditions is better theoretically than using another; but it does mean that numerically the choice is important. The finite difference equations inherently lead to ill-conditioned matrices,¹⁶ and the conditioning can be improved by specifying certain types of boundary conditions. A general procedure cannot be set forth because by actual calculations the techniques

that yield good results for one configuration may fail for another somewhat similar configuration. There are certain guide lines which, when followed, appear to result in good solutions. These guides, together with certain checking techniques and cautions, are given below.

1. On a closed boundary, if there is a choice of using boundary conditions 5 and 6 or 7 and 8, conditions 7 and 8 will lead to success more often. By the use of conditions 7 and 8, more mesh points are related more strongly. The accuracy is the same if both procedures are successful.

2. On an exterior boundary, conditions 5 and 6 are usually specified. In some analyses, however, these conditions may produce erroneous results. Should this occur, certain steps may be taken that usually produce acceptable results. One technique is to use conditions 7 and 8 at all the boundary points except the first point on the boundary. At this point, conditions 5 and 6 are specified. In this manner distant points are interrelated, and the conditioning of the system of equations is apparently improved. Another technique, which has proved useful if a boundary lies along a row or column of mesh points, is to make each point along the boundary zero. The points that lie on the extension of the boundary should also be made zero. This automatically forces the normal stress to be zero. For specifying the other conditions on such a boundary the procedure that has proven most successful is to force symmetry about the boundary with regard to the mesh points one mesh unit from the boundary and its extension. The transverse shear in this case is forced to be zero. This procedure is equivalent to using a first-order approximation in the expression for the normal derivative. Ignoring a boundary condition on a boundary so specified often results in unrealistic results in the region where a condition is not specified. Thus the specification should be complete, with one exception. The exception is a point lying on the boundary extension for which the boundary is specified as zero. In this case, if the point lies in only one biharmonic array of the cross section, it may not be necessary to set the point in question equal to zero, as such points enter only weakly in the stress calculation. However, extreme caution must be used in this procedure. If two such points that are not specified as being zero fall in the same biharmonic array and if they occur in only the related equation, a singular system of equations may result in a manner peculiar to the method of solution

used. In the Gaussian elimination, when one of the unknowns is eliminated the other is also. Thus, there is no equation by which the second unknown may be found. Each time this occurs a zero is listed as the equation number, and the equation corresponding to the missing number is not satisfied. The situation described above usually occurs in an exterior rectangular corner. Also a zero pivot number sometimes occurs in a manner that cannot be foreseen. However, a respecification of the boundary conditions in the area at which the equation is not satisfied will usually eliminate these difficulties. A common error is to specify the same boundary condition at the same point; this also leads to a zero pivot.

3. For general cross sections it is difficult to choose boundary points in such a manner that the total number of exterior points is exactly equal to the number of required boundary conditions. It is sometimes even not desirable to have this situation, especially if the same cross section is to be analyzed under various symmetry conditions. A logical question is whether some boundary conditions are to be ignored, and if so, which ones should they be. Again only guide lines can be indicated in answering this question. If possible, boundary conditions should be well specified in regions of the cross section at which the boundary curvature undergoes fast or abrupt changes. If a boundary is constrained, as in the case of a hole, the boundary condition may not need to be fully specified.

4. The major factor in evaluating the acceptance of a solution is how well the known boundary conditions are satisfied. It is not uncommon for a solution to satisfy the system of equations fairly well, from a residual standpoint, but give stress results that are entirely erroneous. A significant point is that on every boundary the normal in-plane stress and transverse in-plane shear should be zero. Such precise conditions are seldom attainable in numerical procedures, unless, of course, the boundaries are forced to satisfy the boundary conditions, as discussed above. However, such nonzero conditions on the boundaries should be only in nominal error when compared with maximum stress values. For instance, in the example of Appendix B, the normal in-plane stress at mesh point 87 should have been zero but was computed as -101.

5. The effect and propagation of input error is also a significant factor in an analysis. The effect of input error depends on the type and

magnitude of the error. For example, if an RA entry is incorrectly given but is 90% accurate, the effect on the results would probably be negligible. If the sign of an RA entry that is not close to zero is in error, a large region of the cross section may be incorrectly analyzed.

6. Often a solution appears to be fairly acceptable in a given region of a cross section but entirely erroneous in other regions. The acceptable results should not be assumed to be the correct ones without careful investigation, as examples have shown that in some cases such an assumption may be in serious error.

7. Careful inspection of the pivots should be made. The system of equations is scaled so that the pivots should usually not be less than one by more than one or two orders of magnitude. Should the pivots be extremely small, round-off error may dominate the solution and erroneous results be produced. If only the last few pivots are very small the system is nearly singular; however, good results are often obtained.

8. The residuals should be carefully inspected to see whether they are small compared with the equation constant. A good solution usually results if the residuals are four or five orders of magnitude less than the right-hand sides of the related equations. The manner in which the residuals are computed has been described previously, and zero values for residuals should not be misinterpreted.

9. The system of equations is singular if the type 2 equation is not specified at least once. If this is not done, an adequate solution of very large numbers is often obtained anyway, since it is the gradient of the stress function that is important. In this case the residuals are usually all calculated as zero, as previously explained. Again, the numbers may be too large to produce acceptable results.

10. As previously stated, the distance between boundary points should be about one mesh unit ($DELS \approx 1.0$). This rule should be rather closely adhered to. For the example given in Appendix B, less accurate results are obtained when the boundary points are placed every $11 \frac{1}{4}^\circ$ ($DELS = 0.79$) on the internal boundary. In this case erroneous answers would result from completely specifying the inner boundary. This discussion applies mainly to a configuration covered by a coarse mesh, since for analyses involving

a large number of mesh points, the distances between boundary points are usually forced to be near one mesh unit.

Appendix B

EXAMPLE: THICK-WALLED CYLINDER SUBJECTED TO
IRRADIATION-INDUCED DIMENSIONAL CHANGES

Analytical calculations of the stresses in thick-walled cylinders subjected to radially dependent temperature-induced strains are available and have been extended to the case of irradiation-induced dimensional changes in a transversely isotropic material.¹⁷ In this section results obtained from NBC for a thick-walled cylinder of transversely isotropic material are compared with those obtained analytically.

The cylinder to be analyzed has an inside diameter of 30 in. and an outside diameter of 90 in. The exposure for an integrated fast-neutron flux at energies above 0.18 Mev is given by

$$T = 5.25 \times 10^5 e^{-0.26r} ,$$

where r is the distance from the center in inches and T is in megawatt days per adjacent ton (Mwd/AT), a term used to describe irradiation exposure.¹⁸ The material is assumed to undergo dimensional changes that are proportional to the exposure. The material properties are

$$E_x = 1.11 \times 10^6 \text{ psi} ,$$

$$E_z = 1.5 \times 10^6 \text{ psi} ,$$

$$\alpha_x = -1.3 \times 10^{-7} (\text{in./in.})/(\text{Mwd/AT}) ,$$

$$\alpha_z = -3.4 \times 10^{-7} (\text{in./in.})/(\text{Mwd/AT}) ,$$

$$\mu_{zx} = 0.35 .$$

The configuration analyzed with the use of NBC was the 45° sector shown in Fig. B.1. Exterior, interior, and boundary points are indicated. Symmetry is assumed about the radial boundaries to produce an analysis for a complete cylinder. A relatively coarse mesh spacing was chosen so that

Fig. B.1. 45° Sector of a Thick-Walled Cylinder.

not only a complete set of input data could be included but also the degree of accuracy of such a spacing could be noted. The data for the example contain only one set of material properties, and all five programs were run. The tabular input data are given in Tables B.1 through B.4. The individual input data are given below:

IE:	11	NMAX:	15	EZ:	1.5/6
IB:	60	NSIGZ:	1	ALPHAX:	-1.3/-7
IC:	15	IPN:	117	ALPHAZ:	-3.4/-7
ICUR:	2	B2:	0.758604	HTMESH:	3.75
IQC:	4	PRXZ:	0.35	MMAX:	13
EX:	1.11/6				

The stresses along the horizontal radial edge for the numerical and analytical solution are shown in Fig. B.2. Excellent agreement exists between the two sets of results, and this is especially significant in the light of the rather coarse mesh spacing of the grid. The calculated values for the stress function are tabulated in Table B.5.

Table B.1. Boundary Curve Data for Example

Curve No. ^a	IBEG	IEND	THET	ZM	ZN	DELS
1	1	4	0	11.0	1.0	1.047
2	5	15	0	11.0	1.0	0.942

^aNot part of input.

Fig. B.2. Comparison of Numerical and Analytical Stresses Along Horizontal Radial Edge in Example Problem.

Table B.2. Geometry Data for Example

Boundary Point No. ^a	NA	MA	RA	SA	XNA	YNA
1	5	11	0	0	-1.0000	0
2	5	10	-0.1363	0.0353	-0.9659	-0.2588
3	5	9	-0.5359	0	-0.8660	-0.5000
4	4	8	-0.1716	-0.1716	-0.7071	-0.7071
5	13	11	0	0	1.0000	0
6	13	11	-0.0370	0.9415	0.9969	0.0785
7	12	9	0.8523	-0.1228	0.9877	0.1564
8	12	8	0.6684	-0.1986	0.9724	0.2335
9	12	7	0.4127	-0.2918	0.9511	0.3090
10	12	7	0.0866	0.5922	0.9239	0.3827
11	11	6	0.6921	0.4479	0.8910	0.4540
12	11	5	0.2317	0.2700	0.8526	0.5225
13	10	4	0.7082	0.0535	0.8090	0.5878
14	10	4	0.1249	0.7934	0.7604	0.6495
15	9	3	0.4853	0.4853	0.7071	0.7071

^aNot part of input.

Table B.3. Equation Data

IPMSN	ICN	IBDSN	LEXPNS	IPMSN	ICN	IBDSN	LEXPNS
1	12	0	9	71	6	7	70
2	12	0	14	72	5	7	70
3	12	0	8	73	7	3	63
4	6	15	7	74	8	3	63
5	12	0	21	83	6	8	58
6	12	0	13	84	5	6	95
8	5	14	14	85	9	1	87
9	6	14	14	86	8	2	75
10	2	0	117	96	6	6	95
11	12	0	20	97	5	5	95
15	7	2	75	98	12	0	74
16	6	13	14	99	12	0	75
17	12	0	40	100	12	0	76
18	12	0	29	101	12	0	77
24	5	13	14	102	12	0	78
25	6	12	23	103	12	0	79
26	12	0	52	104	12	0	80
27	12	0	39	105	12	0	81
34	5	12	23	106	12	0	82
35	6	11	45	107	12	0	83
36	12	0	63	108	12	0	84
37	12	0	51	109	12	0	63
46	5	11	45	110	12	0	64
47	6	9	45	111	12	0	65
48	12	0	74	112	12	0	66
49	12	0	62	113	12	0	67
59	5	9	45	114	12	0	68
60	5	8	58	115	12	0	69
61	6	10	45	116	12	0	70
62	7	4	50	117	12	0	71

Table B.4. Thermoelastic Loadings and Rectangular Mesh Points

Biharmonic Mesh Point No. ^a	ITABCØ	TEMP	Biharmonic Mesh Point No. ^a	ITABCØ	TEMP
1	9	2.3	60	119	1.4
2	23	9.0	61	123	34,508.6
3	24	4.9	62	124	16,339.7
4	25	2.5	63	125	7,098.8
5	37	34.8	64	126	2,949.3
6	38	19.0	65	127	1,194.6
7	39	9.8	66	128	476.4
8	40	4.9	67	129	188.1
9	41	2.3	68	130	73.7
10	51	133.5	69	131	28.8
11	52	73.7	70	132	11.2
12	53	38.3	71	133	4.3
13	54	19.0	72	134	1.7
14	55	9.0	73	138	61,012.2
15	56	4.2	74	139	25,029.0
16	57	1.9	75	140	9,931.3
17	65	509.1	76	141	3,883.9
18	66	286.1	77	142	1,507.5
19	67	149.4	78	143	582.6
20	68	73.7	79	144	224.5
21	69	34.8	80	145	86.4
22	70	15.9	81	146	33.2
23	71	7.0	82	147	12.7
24	72	3.1	83	148	4.9
25	73	1.3	84	149	1.9
26	79	1,920.7	85	153	76,567.8
27	80	1,107.7	86	154	29,258.0
28	81	582.6	87	155	11,180.0
29	82	286.1	88	156	4,272.1
30	83	133.5	89	157	1,632.4
31	84	60.0	90	158	623.8
32	85	26.2	91	159	238.4
33	86	11.2	92	160	91.1
34	87	4.7	93	161	34.8
35	88	1.9	94	162	13.3
36	93	7,098.8	95	163	5.1
37	94	4,272.1	96	164	1.9
38	95	2,270.9	97	165	0.7
39	96	1,107.7	98	169	25,029.0
40	97	509.1	99	170	9,931.3
41	98	224.5	100	171	3,883.9
42	99	96.1	101	172	1,507.5
43	100	40.3	102	173	582.6
44	101	16.6	103	174	224.5
45	102	6.8	104	175	86.4
46	103	2.7	105	176	33.2
47	104	1.1	106	177	12.7
48	107	25,029.0	107	178	4.9
49	108	16,339.7	108	179	1.9
50	109	8,852.5	109	185	7,098.8
51	110	4,272.1	110	186	2,949.3
52	111	1,920.7	111	187	1,194.6
53	112	825.9	112	188	476.4
54	113	345.0	113	189	188.1
55	114	141.2	114	190	73.7
56	115	57.0	115	191	28.8
57	116	22.8	116	192	11.2
58	117	9.0	117	193	4.3
59	118	3.6			

^aNot part of input data.

Table B.5. Stress Function for the Example

Biharmonic Mesh Point No.	Stress Function	Biharmonic Mesh Point No.	Stress Function
1	237.84	60	756.13
2	130.92	61	10,625
3	-8.0506	62	22,065
4	193.99	63	22,763
5	1,603.0	64	19,226
6	737.69	65	14,565
7	176.01	66	10,105
8	-8.0506	67	6,347.1
9	237.84	68	3,446
10	4,678.6	69	1,429.3
11	3,157.8	70	285.21
12	1,776.6	71	-21,644
13	737.69	72	575.09
14	130.92	73	-8,719.3
15	7.4668	74	18,437
16	393.85	75	23,806
17	9,860.0	76	21,070
18	7,395.9	77	16,161
19	5,130.1	78	11,279
20	3,157.8	79	7,145.1
21	1,603.0	80	3,949.9
22	545.99	81	171.0
23	35.544	82	397.54
24	97.768	83	-12.935
25	732.15	84	473.22
26	16,570	85	-42,252
27	13,492	86	13,338
28	10,352	87	23,743
29	7,395.9	88	21,725
30	4,801.9	89	16,757
31	2,699.9	90	11,712
32	1,170.4	91	7,435.8
33	259.31	92	4,132.5
34	-7.0515	93	1,812.3
35	399.92	94	440.36
36	22,763	95	-13.628
37	20,485	96	441.25
38	17,197	97	1,819.3
39	13,492	98	18,437
40	9,860.0	99	23,806
41	6,617.8	100	21,070
42	3,944.3	101	16,161
43	1,930.7	102	11,279
44	618.83	103	7,145.1
45	27.065	104	3,949.9
46	161.75	105	1,710.0
47	990.59	106	397.54
48	18,437	107	-12.935
49	22,065	108	473.22
50	22,869	109	22,763
51	20,485	110	19,226
52	16,570	111	14,565
53	12,343	112	10,105
54	8,458.9	113	6,347.1
55	5,215.3	114	3,446.1
56	2,729.4	115	1,429.3
57	1,037.2	116	285.21
58	143.98	117	0
59	48.300		

In this example there are three boundary conditions that are not satisfied. Specifically, there is only one condition specified at boundary points 10 and 4, and the stress function value at point 10 should be equal to the value at 30. The missing conditions do not noticeably affect the analysis.

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