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ENGINEERING APPLICATION OF WEIBULL STATISTICS

Livermore, California

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ENGINEERING APPLICATION OF WEIBULL STATISTICS*

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Livermore, California

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ABSTRACT

The Weibull theory is an attempt to account for the variability of measured strength in "brittle" material. As might be expected, it is not of general applicability although there are many cases where the theory is satisfactorily confirmed by experimental evidence. Of greater importance perhaps is the fact that its qualitative aspects are very frequently confirmed, and that even in its simplest form the theory may be used to approximate fairly complex data and is amenable to the development of conservative but meaningful design rules. This paper reviews some methods of analyzing data in terms of the Weibull distribution and gives a particularly convenient normalized expression. Formulas are given for the statistical behavior of the weakest specimens in a given sample and the application is demonstrated using data on beryllia and alumina. Brief consideration is given to the subject of proof-test (truncation) and bias (pre-stress or axis-shift) and the effects on a "primitive" (unbiased, untruncated) sample.

Some qualitative confirmation is presented from material published in the literature.

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The limitations of the theory are discussed and a number of design rules are evolved which, though approximate, rest on a simple and meaningful analytical basis.

The design rules recommend using a "primitive" distribution, if at all possible. Such a distribution, if a compromise is necessary, is made to approximate the least (weakest) values observed. The conservatism can be increased by reducing the Weibull modulus. The "extreme safety factor" is defined as the ratio of most probable least strength to the operational stress and is used in the light of the statistical behavior of the least value. Design must be based on a required survival probability.

In summary then, the major aspects of Weibull's theory are touched upon; some new modifications are presented; the application to data analysis is demonstrated; and a few design rules are proposed.

Introduction

The strength of a brittle material is unpredictable; that is, within the requirements of engineering precision. But, if in the past, the strength of any given ceramic was not very well known, the operational stress was so low and failure so inconsequential that the attrition suffered from failure of the weakest samples did not constitute a problem. Today, however, the problem is immense. Refractory materials, with the exception of a few exotic metals and alloys, are brittle, a property which is most typified by a great variability in strength. The same is true of metals operating below the ductile transition temperature. Hypersonic leading edges, uncooled rocket nozzles, high temperature nuclear reactor fuel elements, high temperature gas turbines, high temperature heat exchangers, and a host of

structural applications promise remarkable technological advantages if only the strength of the current crop of refractory materials were more reliable.

The measurement of strength of brittle material even under closely controlled conditions is difficult. The absence of ductility can lead to spurious results in the laboratory testing of brittle specimens owing to various eccentricities and geometric irregularities. As a result, a number of refined test techniques have been evolved specifically for brittle specimens [1, 2, 3].¹ Nevertheless, most of the variability in strength remains and it is evident that a statistical interpretation of strength for purpose of design is unavoidable. The most popular statistical theory currently is that of Weibull, first proposed in 1939 [4]. This acceptance is owed to the theory's ability to predict failure under any stress distribution from information of a given stress distribution, its relatively simple mathematical form, and satisfactory agreement with many data. Forms of the Weibull distribution have been applied with encouraging results in several fields of endeavor, such as fatigue, corrosion, as well as the natural sciences [5, 6, 7].

This paper is concerned with a review of the application of Weibull statistics to data, and the development of some design principles. In keeping with the original exposition by Weibull, stress will be the variable under consideration, although other applications are recognized.

The Weibull Distribution

Without going into the essentially heuristic derivation, we give the formula first proposed by Weibull for the probability that a given specimen

¹Numbers in brackets refer to similarly numbered referenced in bibliography at end of paper.

will survive an applied tensile stress of unspecified shape:

$$S = e^{-\int \left(\frac{\sigma}{\sigma_0}\right)^m dV} \quad (1)$$

where

S = probability of survival

σ = applied stress

m = material constant (Weibull modulus)

σ_0 = scaling factor

V = volume under tension.

If a normalized stress variable is used, obtained from dividing each observed value of strength by the mean value, then equation (1) can be changed to (the reasoning is given in Appendix A):

$$S = e^{-\beta^m [\Gamma(1+1/m)]^m} \quad (2)$$

where

$\beta = \sigma/\sigma_{\text{mean}}$

m = Weibull modulus

Γ = the gamma function.

Here, the specification of m and the mean value completely define the distribution.

The premises which underlie these formulas are (a) that failure results only from tensile stress and (b) failure is described by "weakest-link" behavior in which a localized failure precipitates system or specimen failure.

We are now prepared to test a set of data and see if it may be fitted by a distribution of the form of equation (2). The strength data on n specimens is arranged in decreasing order of strength and each specimen numbered.

(ranked) serially from 1 to n. The strength of each specimen is normalized by dividing by the mean strength, giving the variable β . The survival probability of each specimen is computed in accordance with the recommendation of Gumbel [8] by

$$S = \frac{r}{n+1} . \quad (3)$$

This equation is convenient but may not be precisely accurate. More complicated methods of assigning probability are available [9] if desired.

It remains to be seen if a single value of m exists which, by the form of equation (2) will give a satisfactory approximation to the data. The usual way is to take logs of equation (2) to get

$$\log \ln (1/S) = m \log \beta + \log [\Gamma(1 + 1/m)]^m . \quad (4)$$

If $\ln (1/S)$ is plotted against β on log-log paper ("ln" denotes natural logarithmic and "log" denotes the base 10) and a straight line is satisfactory, then its slope will give m and equation (2) is confirmed. For this purpose the graph of Fig. 1 may be used. This graph is simply log-log paper with failure probability (1-S) as the labeled ordinate. A more sensitive graphical approach is afforded by use of Fig. 2, which is a plot of equation (2) where the coordinates are S and m with parameter β . If the data are approximated here by a vertical line equation (2) applies. Table 1 and Figs. 3 and 4 demonstrate these applications to a set of strength data on BeO tubes which were pressurized to failure [10].

Effect of Size and Stress Distribution

Although equation (2) is convenient for analyzing a set of data and gives a complete formula for the survival probability, the prediction of the effect

of stress distributions and/or the effect of size is best carried out with equation (1). These effects will be illustrated by the two common examples of uniform tension and uniform bending.

In the case of uniform tension of a beam with volume V_T , equation (1) gives

$$S = e^{-(\sigma/\sigma_0)^m} V_T \quad (5)$$

The strength in uniform tension of specimens differing in size from the test sample may be predicted if m and σ_0 remain unchanged. Since then the survival probability at the mean value is the same, we get

$$e^{-(\sigma_1/\sigma_0)^m} V_1 = e^{-(\sigma_2/\sigma_0)^m} V_2 \quad (6)$$

and, therefore

$$\sigma_1/\sigma_2 = (V_2/V_1)^{1/m} \quad (7)$$

The larger specimen is weaker and this effect of size increases as m decreases.

The case of uniform bending in a rectangular beam requires the substitution of the stress distribution $\sigma_{\max} y/c$ in equation (1) and, to compare with uniform tension, reasoning similar to the above leads to (using σ_B for bend strength and σ_T for tensile strength) a form similar to equation (6):

$$e^{-\left(\frac{\sigma_B}{\sigma_0}\right)^m \frac{V_B}{2(m+1)}} = e^{-\left(\frac{\sigma_T}{\sigma_0}\right)^m V_T},$$

and, therefore

$$\frac{\sigma_B}{\sigma_T} = \left[\frac{V_T}{V_B} 2(m+1) \right]^{1/m} \quad (8)$$

Here we see the combined effect of volume and stress distribution on the relative strengths of tensile and bending specimens (note that V_B is the total beam volume in bending).

Generally speaking, large volumes and low m -values will produce low strength and high variability, respectively. As an example of some typical values of m , steel is characterized by m -values of 40 to 100 in the ductile state and 20 to 30 in the brittle state. BeO is described by $m = 7$ to 11 while some graphites may be as low as 3 or 4.

Strength of Weakest Specimens

It is evident, looking at Fig. 2, that though the mean value may be comfortably high, when m -values are low there may be an uncomfortably high probability of observing relatively weak specimens. The incidence of weak specimen increases as the sample size increases and as m decreases. It is possible to predict the most likely least value in a set of n specimens and, in fact, to describe the distribution of the least values. Following Epstein, [11] we see that the probability of n specimens simultaneously surviving the stress σ is given by

$$S_n = [S(\sigma)]^n, \quad (9)$$

where S gives the survival of any one specimen. The probability density is found by differentiating once:

$$\frac{dS_n}{d\beta} = nS^{n-1} S' . \quad (10)$$

The most probable least value corresponds to the maximum of the probability density which is obtained by another differentiation and equating to zero, giving the formula

$$(n - 1)S_1^2 = - S''S. \quad (11)$$

The value of β satisfying this equation will be the most likely least value. Substitution of equation (2) (Appendix B) gives for the least value β^*

$$\beta^* = \left(\frac{m - 1}{mn} \right)^{1/m} \frac{1}{\Gamma\left(1 + \frac{1}{m}\right)}. \quad (12)$$

Equation (12) is plotted on Fig. 5.

This equation and graph are extremely useful for design purposes and may also be used to make a first estimate of the Weibull modulus m from the observed extreme value. The application of equation (12) is demonstrated by the summary of BeO strength tests in Table 2. Table 2 shows the m -value estimated from the observed least value, and from the graphical analysis. The least value is predicted according to equation (12), by the preferred m -value. The agreement for engineering purposes is quite satisfactory. As a further confirmation, some bend-strength data on BeO from the literature [13] are also included.

Truncation and Bias

Although some quantitative aspects of the least (weakest) values are now understood, the presence of these weak specimens is a major design problem. There are two basic ways of avoiding the "least-value problem." If, in a set of specimens, the weakest ones could be removed by some suitable examination, the resulting group would exhibit a threshold stress below which no failure would occur. This threshold, or truncation level, would be predicated on operational stress requirements and the economics of rejecting the "weak" fraction from the starting sample. It must be noted that a selection process such as a proof test may degrade the statistics of the survivors [12].

The second method is to reduce the net operational tensile stress by the judicious use of residual and pre-stress. In this case the form of the survival distribution may remain essentially unchanged but will be biased [shifted along the stress (β) axis] an amount corresponding to the compressive residual or pre-stress.

The effect of truncation and bias on a distribution of the form of equation (2) will be examined. The unbiased and untruncated distribution, equation (2), will be referred to as the "primitive" distribution.

Figure 6 shows a typical primitive, an arbitrary truncation, ψ (normalized to the mean value) and a representation of the distribution of the truncated population.

The survival probability of the truncated distribution is given by

$$S_{\psi} = \frac{S(\beta)}{S(\psi)}, \quad (13)$$

where $S(\psi)$ is the primitive evaluated at ψ and β is arbitrary, provided, of course, that $\beta \geq \psi$ since survival is assured, in principle, when $\beta < \psi$.

Putting in the function of $S(\psi)$ by equation (2), we have

$$S_{\psi} = e^{-(\beta^m - \psi^m)} [\Gamma(1+1/m)]^m \quad (14)$$

for the distribution of a primitive truncated at ψ . We can now inquire into the behavior of the least values in such a truncated population.

It is clear that the least value can never be less than ψ . Application of the general formula of equation (11) produces a result identical with equation (12). The interpretation is that the least value of the truncated distribution is identical with the least value of the corresponding primitive so long as this least value exceeds ψ ; otherwise the least value is ψ itself. The reason

is that the least value depends on the shape of the probability density [the derivative of the distribution function, equation (10)]. If this graph has a maximum, the position at this maximum (mode) will not change as truncation occurs, until the truncation exceeds the mode and the most probable value becomes ψ itself. This is depicted graphically in Fig. 7.

Although the least value remains essentially unaffected, the mean value of the truncated distribution will exceed the primitive mean value. The improvement of the mean value by truncation is shown on Fig. 9. The analysis for this mean value involves the incomplete gamma function and is given in Appendix C.

The other side of the truncation "coin" is the situation in which an unknown population is being examined and it is desired to know whether a truncation exists. One approach is to test several samples of increasing size and observe the extreme values. If, as the sample size increases, the least value approaches an asymptote, a truncation or bias is indicated (see Fig. 8). This method can simultaneously be used to estimate the Weibull modulus m . It appears, judging from Fig. 9, that for most values of m a relatively high truncation level is required to cause significant changes in the mean value.

Experience shows that some data plotted on Fig. 1 or 2 may not give straight lines. The presence of non-linearity in such data is usually attributed to a threshold in the variable, i. e., a value below which the failure probability vanishes. There may, however, be several reasons for the non-linearity of data analyzed by a primitive distribution:

- a) The sample has been subject to truncation as discussed above.
- b) The sample is non-homogeneous. For example, there may be

more than one value of Weibull modulus, or, even if a single m -value applies, there may be more than one mean value, i. e., the mean value is not unimodal.

c) There exists residual stress of either sign.

d) Some form of bias exists in the distribution, as discussed below.

The introduction of bias into equation (2) follows more from the necessity to account for data which could not be approximated by equation (2) than for the prediction of an effect of bias on a primitive sample. In his original paper Weibull proposed a more general form than equation (1) to account for some of his data.

This form was

$$S = e^{-\int_V \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m dV} \quad (15)$$

Here σ_u is a threshold or bias which mathematically is simply a shift along the stress axis to a new origin at σ_u . Physically, σ_u can be regarded as a residual stress which, when it is algebraically positive in equation (15), is analogous to a compressive stress and retards the failure probability at the applied stress σ . It could conceivably be tensile as well and hence cause a truncation at the level of zero nominal stress. The distribution of σ_u in space is arbitrary, consistent with the absence of a net moment. This form, equation (15), is more complicated analytically and in most cases is not integrable in closed form. It is, however, consistent with the premises of the theory and leads, in a natural way, to predictions of size effect and stress distribution effect when a bias σ_u is present. Some of Weibull's later publications indicate that he is using the following simplified distribution to

approximate experimental data:

$$S = e^{-\left(\frac{x-a}{b}\right)^m} \quad (16)$$

In doing so all possibility of computing in general a stress distribution effect and a size effect are abandoned since this equation does not follow generally from equation (6) except for uniform tension and unit volume. It is, however, a simple analytical form which can be made to approximate a wide range of experimentally observed data.

It is possible to analyze bias in a general way if the assumption is made that the applied stress σ and the threshold stress σ_u are linearly related. This would mean that σ_u must be distributed in space and be geometrically similar to the applied stress. Physically this seems to be very artificial, but it can be justified to some extent. A residual stress distribution may be present which is approximately geometrically similar to the applied stress, at least in the tensile region, and even if not similar, the fact that most failures will occur in a small region near the maximum stress, the deviation may not be very great. This assumption will lead to a simple and general form containing a bias and if it leads to reasonable engineering approximations it will be justified.

By methods identical with Appendix A the distribution of a primitive with bias is obtained:

$$S = e^{-(\beta-\alpha)^m} [\Gamma(1+1/m)]^m, \quad (17)$$

where α is the normalized bias (divided by the primitive mean). The relative mean value of the biased population will be $1 + \alpha$. In the case of an unknown sample containing a bias the normalizing factor will implicitly be $1 + \alpha$, the

sample mean. Transforming equation (17) to parameters which are normalized to the unknown sample mean gives:

$$S = e^{-\left(\frac{\beta-\alpha}{1-\alpha}\right)^m \left[\Gamma(1+1/m)\right]^m} \quad (18)$$

Note that $\beta \geq \alpha$ and $1 > \alpha$. When $\beta = 1$ (the mean value) this equation is identical with equation (2) and the survival probability at the mean value can (by Fig. 2) be used to estimate m , though this is not precise because of the high uncertainty in m at $\beta \approx 1$. The behavior of the extreme values of a biased primitive distribution follows directly from application of formula (10) and yields, in terms of the primitive mean:

$$\beta^* = \left(\frac{m-1}{mn}\right)^{1/m} \frac{1}{\Gamma\left(1 + \frac{1}{m}\right)} + \alpha \quad (19)$$

and in terms of the biased sample mean:

$$\beta^* = (1 - \alpha) \left(\frac{m-1}{mn}\right)^{1/m} \frac{1}{\Gamma\left(1 + \frac{1}{m}\right)} + \alpha \quad (20)$$

Appendix D describes methods of estimating m and α in an unknown sample. For purpose of illustration Fig. 10 shows for $m = 7$, the primitive with unit mean, and the result of truncation at 0.7, and of a bias of 0.7 (based on sample mean).

As a demonstration of application, some data on Al_2O_3 [13] which exhibits a threshold will be analyzed.

The data (taken from data sheets at IITRI) are given in Table 3 and were analyzed by the graph of Fig. 2. It is seen in Fig. 11 that a satisfactory engineering approximation of the extreme values is obtained with a primitive when $m = 15$ and it is not necessary to invoke a threshold. However, using

the method of Appendix D, we estimate $m = 2.5$ and therefore $\alpha = 0.78$. This agrees quite well with the published conclusions, and although there are slight differences in the predicted extreme values, both give quite satisfactory agreement, but the primitive approximation, not having a lower bound, is more conservative.

Summary and Recommendations

The above discussion of truncation and bias implicitly assumed that the truncation resulted from a stress distributed similarly to the operational stress while the bias is restricted to a fairly special condition. The general expressions for truncation and bias where no assumptions are made other than equation (15), can become absolutely formidable and are likely to be of very little practical value. It is clear, however, that whenever possible the most satisfying analysis results from the primitive distribution and it will be desirable to attempt to approximate, in a conservative manner, more complex data by a primitive form, i. e., without bias or truncation. If such an approximation cannot satisfactorily be made, then the idealized biased distribution may be tried.

Experience in design with brittle material usually demonstrates the great difficulty of avoiding cracking. In large systems this may be accounted for simply by equation (12) or, in essence, the size effect. Hence design techniques which allow cracking to occur without subsequent collapse (fail-safe or "crack-tolerant" design) are of primary importance. It is, indeed, possible to separate initial fracture from collapse by the use of redundant supports [14]. The penalty is loss of detailed information on stress magnitude and distribution; however, this may not be an unreasonable price to pay. If

compressive stability can be induced, for example by arching, then the compressive strength of the material can be made the limiting property. Since in most ceramics the compressive strength exceeds the tensile strength by factors of 4 to 10, the load carrying capability of the system may substantially increase before ultimate collapse (14). The possibility of profitably employing redundant supports should be carefully examined in any design with brittle material.

An interesting new definition of safety factor can be made using, as reference, the most probable least value in the sample. If the extreme safety factor K_X is defined as the ratio of least strength to operational stress and the classical safety factor K_C is the ratio of mean strength to operational stress, the two are related by equation (12) and Fig. 5:

$$\frac{K_X}{K_C} = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma(1+1/m)} . \quad (21)$$

If a certain value of failure probability F is desired, the extreme safety factor will be given by ($F = 1 - S$):

$$K_X = \left(\frac{m-1}{m} \right)^{1/m} \left(\ln \frac{1}{S} \right)^{1/m} \underset{\text{high } S}{\approx} \left(\frac{m-1}{mF} \right)^{1/m} . \quad (22)$$

Assume, as an example, $m = 10$ (like BeO) and $F = 10^{-7}$ (for a man-rated system). This gives $K_X \approx 6$. If the number of specimens, or alternatively the volume, is $n = 100$, the classical safety factor will be about 9 from Fig. 5.

As a general approach to design a number of design rules should be considered:

- 1) Well characterized component or simulation data with (usually)

more emphasis on the shape of the survival distribution function than on localized studies of probability.

2) **An initial analysis or approximation of data by a "primitive"**

Weibull function. In this case the precision of the approximation at the least values should be a dominant criterion, since the variance of the least values is less than the variance about the mean, as shown in Appendix E.

- 3) The extreme value and its distribution can be studied analytically by equations (9) and (12) while the variance is given in Appendix E.
- 4) The degree of conservatism in design may be conveniently expressed by the extreme safety factor, equation (22).
- 5) If proof-testing or screening is to be applied, a careful study should be made of the possible physical degradation of the surviving specimens.
- 6) Some rules of thumb:
 - a) Estimate m from the observed extreme and Fig. 5.
 - b) Estimate low failure probabilities by $F = n\beta^m$
 - c) Estimate the extreme safety factor by $K_X = F^{-1/m}$
 - d) Estimate the least value in n by $\beta^* = (1/n)^{1/m}$
 - e) Employ redundant support and think in terms of "crack-tolerant" design.

Appendix A. Derivation of the Normalized Weibull Distribution

We have

$$S = e^{-\int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV}$$

Assume

$$\sigma = \sigma f(\xi),$$

where $f(\xi)$ is a normalized geometry function describing the stress distribution and σ is a reference stress.

$$S = e^{-\left(\frac{\sigma}{\sigma_0}\right)^m V \int_{\xi} f(\xi)^m d\xi}$$

At the mean value the probability is S_a :

$$S_a = e^{-\left(\frac{\sigma_a}{\sigma_0}\right)^m V \int_{\xi} f(\xi)^m d\xi}$$

Eliminating

$$\frac{V}{\sigma_0^m} \int_{\xi} f(\xi)^m d\xi \quad \text{gives}$$

$$S = e^{-\left(\frac{\sigma}{\sigma_a}\right)^m \ln \left(\frac{1}{S_a}\right)} \quad (A-1)$$

Using the definition of the mean value and $\sigma/\sigma_a = \beta$, i. e.,

$$\sigma_a = \int_0^{\infty} S d\sigma = \sigma_a \int_0^{\infty} S d\beta$$

and putting for S

$$\sigma_a = \sigma_a \int_0^{\infty} e^{-\beta^m \ln \frac{1}{S_a}} d\beta .$$

$$\text{If } w = \beta^m \ln \frac{1}{S_a} ,$$

$$d\beta = \frac{1}{\left(\ln \frac{1}{S_a}\right)^{1/m}} \frac{1}{m} w^{1/m-1} dw .$$

Putting back,

$$1 = \frac{1}{m \left(\ln \frac{1}{S_a}\right)^{1/m}} \int_0^{\infty} e^{-w} w^{1/m-1} dw ,$$

Since the integral is the Gamma Function,

$$m \left(\ln \frac{1}{S_a}\right)^{1/m} = \Gamma\left(\frac{1}{m}\right) ;$$

$$\ln \frac{1}{S_a} = \left[\Gamma\left(\frac{1}{m} + 1\right) \right]^m . \quad (\text{A-2})$$

Putting in (A-1), we get equation (2):

$$S = e^{-\beta^m \left[\Gamma\left(1 + \frac{1}{m}\right) \right]^m} . \quad (\text{A-3})$$

Appendix B. The Mode of Least Values

From equation (11),

$$(n - 1)S'^2 = - S''S, \quad (B-1)$$

we have

$$S = e^{-\beta^m \Gamma^m},$$

where $\Gamma = \Gamma(1 + 1/m)$

$$S' = - m\beta^{m-1} \Gamma^m e^{-\beta^m \Gamma^m} = - m\beta^{m-1} \Gamma^m S$$

$$S'^2 = m^2 \beta^{2(m-1)} \Gamma^{2m} S^2$$

$$S'' = \left[m^2 \beta^{2(m-1)} \Gamma^{2m} - m(m-1) \beta^{m-2} \Gamma^m \right] S.$$

Putting in (B-1) we get

$$(n - 1)m^2 \beta^{2(m-1)} \Gamma^{2m} S^2 = \left[m(m-1) \beta^{m-2} \Gamma^m - m^2 \beta^{2(m-1)} \Gamma^{2m} \right] S^2,$$

$$nm\beta^m \Gamma^m = (m-1)$$

$$\beta^* = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma(1 + 1/m)}. \quad (B-2)$$

Appendix C. The Mean Value of a Truncated Primitive

This discussion refers to the graph of probability density in Fig. C-1. It is desired to find $\hat{\beta}$, the mean of the distribution truncated at ψ . Application of the mean-value theorem gives

$$\hat{\beta} = \frac{\int_{\psi}^{\infty} \beta f d\beta}{\int_{\psi}^{\infty} f d\beta} , \quad (C-1)$$

where $\psi = -dS/d\beta$. The denominator immediately gives $S(\psi) = e^{-\psi^m} [\Gamma(1+1/m)]^m$ and the numerator may be integrated by parts to give

$$\psi S(\psi) + \int_{\psi}^{\infty} S d\beta .$$

Combining, then, and using $\Gamma = \Gamma(1 + 1/m)$,

$$\hat{\beta} = \psi + e^{\psi^m \Gamma^m} \int_{\psi}^{\infty} e^{-\beta^m \Gamma^m} d\beta . \quad (C-2)$$

Integration tables [15] are available for the function

$$e^x E = e^x \int_1^{\infty} e^{-xu} u^{-n} du .$$

By letting $x = \psi^m \Gamma^m$, $u = \beta^m / \psi^m$ and $d\beta = (\psi^m / m) u^{1/m-1} du$, we have, as required,

$$\hat{\beta} = \psi + \frac{\psi^m}{m} \left[e^x E_{(1/m-1)} \right] . \quad (C-3)$$

Selected values of m and ψ specify x and the function $E_{(1/m-1)}$. The results are plotted in Fig. 6.

Appendix D. Estimation of Parameters in a Biased Primitive Distribution

We have, using $\Gamma = \Gamma(1 + 1/m)$,

$$S = e^{-\left(\frac{\beta-\alpha}{1-\alpha}\right)^m} \Gamma^{-m}$$

and

$$\frac{\beta - \alpha}{1 - \alpha} = \frac{1}{\Gamma} \left(\ln \frac{1}{S} \right)^{1/m}.$$

Solving for α :

$$\alpha = \frac{\beta - \frac{1}{\Gamma} \left(\ln \frac{1}{S} \right)^{1/m}}{1 - \frac{1}{\Gamma} \left(\ln \frac{1}{S} \right)^{1/m}}. \quad (D-1)$$

Before (D-1) can be used, an estimate of m is needed.

There are several possibilities for estimating m . Initial estimates can be based on the survival probability at the mean value. Several computations of α can be made, selecting β values which are not too far from the least value, and if they give satisfactory agreement, a representative value of α can be selected.

Alternatively, various values of α can be selected and $(\beta - \alpha)/(1 - \alpha)$ computed. The resulting values can be plotted on Fig. 1 and, if a straight line results, the m -value is given. Figure 2 may similarly be used with various assumptions on α .

Appendix E. Variance of Extreme Values

Before computing the variances, it will be found useful to obtain the integral of the following function:

$$H(p, n) = \int_0^{\infty} \beta^p e^{-\beta^m n \Gamma^m} d\beta . \quad (E-1)$$

Here $\Gamma = \Gamma(1 + 1/m)$ and the result will be in terms of the parameters p and n . Parameter p will depend on the order of the moment of the distribution which is being computed, and n is the number of specimens in the sample for which the statistics of the least values are desired. Note that the behavior of the mean value is obtained when $n = 1$, that of the extreme value when $n =$ the sample size. In this sense the number of specimens n corresponds to relative volume, since the least value in n specimens should represent the strength of the n -sized specimens.

If we let

$$\begin{aligned} \omega &= \beta^m \Gamma^m; \quad \beta = \frac{1}{\Gamma} \left(\frac{\omega}{n} \right)^{1/n} , \\ d\beta &= \frac{1}{mn^{1/m} \Gamma} \omega^{1/n-1} d\omega , \\ \beta^p &= \frac{1}{\Gamma^p n^{p/m}} \omega^{p/m} . \end{aligned}$$

Putting back,

$$H(p, n) = \int_0^{\infty} \frac{1}{\Gamma^p n^{p/m}} \omega^{p/m} e^{-\omega} \frac{1}{m \Gamma n^{1/m}} \omega^{1/n-1} d\omega .$$

Combining,

$$H(p, n) = \frac{1}{mn^{p+1/m} \Gamma^{p+1}} \int_0^{\infty} e^{-\omega} \omega^{p+1/m-1} d\omega .$$

The integral is $\Gamma\left(\frac{p+1}{m}\right)$, giving

$$H(p, n) = \frac{1}{mn^{p+1/m}} \frac{\Gamma\left(\frac{p+1}{m}\right)}{\left[\Gamma(1 + 1/m)\right]^{p+1}} . \quad (E-2)$$

Now the variance of the unbiased, untruncated Weibull distribution is given by

$$a^2 = \int_0^\infty (\beta - \beta_{\text{mean}})^2 f d\beta \quad (E-3)$$

where the probability density f is given by $-\frac{dS}{d\beta}$.

Using

$$S_n = e^{-n\beta^m \Gamma^m},$$

which is the distribution of least values in a sample of size n , gives for f ,

$$f = -\frac{dS}{d\beta} = n\Gamma^m m\beta^{m-1} e^{-\beta^m \Gamma^m n}.$$

The mean of this density function is at $(1/n)^{1/m}$ and, putting in (E-3),

$$a^2 = mn\Gamma^m \int_0^\infty (\beta - 1/n^{1/m})^2 e^{-n\Gamma^m \beta^m} d\beta.$$

We can expand this into three integrals each of which may be immediately solved by the formula (E-2):

$$a^2 = mn\Gamma^m \left\{ \underbrace{\int_0^\infty \beta^{m+1} e^{-\beta^m \Gamma^m n} d\beta}_I - \frac{2}{n^{1/m}} \underbrace{\int_0^\infty \beta^m e^{-\beta^m \Gamma^m n} d\beta}_II + \frac{1}{n^{2/m}} \underbrace{\int_0^\infty \beta^{m-1} e^{-\beta^m \Gamma^m n} d\beta}_III \right\},$$

$$H_I(m+1, n) = \frac{1}{mn^{1+2/m}} \frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^{m+2}},$$

$$\frac{2}{n^{1/m}} [H_{II}(m, n)] = \frac{2}{mn^{1+2/m}} \frac{1}{[\Gamma(1+1/m)]^m},$$

$$\frac{1}{n^{2/m}} [H_{III}(m-1, n)] = \frac{1}{mn^{1+2/m}} \frac{1}{[\Gamma(1+1/m)]^m}.$$

Multiplying by $mn[\Gamma(1+1/m)]^m$ gives,

$$a^2 = \frac{1}{n^{2/m}} \left\{ \frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^2} - 1 \right\}. \quad (E-4)$$

This formula gives the variance of the least value in a sample of size n and demonstrates that the least values are more reliable than the mean values. Its square root, the standard deviation, is plotted in Fig. E-1 for various n , that is,

$$\text{Std. dev.} = \frac{1}{n^{1/m}} \sqrt{\frac{\Gamma(1+2/m)}{[\Gamma(1+1/m)]^2} - 1}. \quad (E-5)$$

Appendix F. Miscellaneous Confirmation

Figure F-1 shows the normalized strength of potassium chloride and sodium chloride whiskers as a function of their diameters. These data are given in Ref. [16] and have been replotted here on log-log coordinates. If the trends predicted by a primitive Weibull function are followed, the data should be approximated by a straight line. The slope should be $\propto V^{1/m} \propto r^{2/m}$.

Figure F-2 shows the strength of iron whiskers coated with nickel (Ref.[17]). Again a plot on log-log coordinates seems to imply the type of exponential variation predicted by the primitive Weibull function.

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Table 1. Burst test of round tubes I. D. = 0.230, O. D. = 0.310; $\xi_i = 0.74$; $\sigma/p = 3.4$; Stressed length = 4.5 in.

Run No.	Observed data	Ordered data	Max. stress (ksi)	Norm. stress	Cumulative failure probability	Cumulative survival probability
	Burst pressure (psi)	Burst pressure (psi)				
1	6200	4600	15.64	0.743	0.032	0.968
2	5600	5000	17.0	0.808	0.064	0.936
3	6000	5200	17.68	0.840	0.097	0.903
4	6500	5300	18.04	0.857	0.129	0.871
5	5200	5500	18.7	0.888	0.161	0.839
6	5600	5600	19.04	0.904		
7	6900	5600	19.04	0.904		
8	6600	5600	19.04	0.904	0.250	0.742
9	6500	5700	19.38	0.921	0.290	0.710
10	5500	5900	20.06	0.953	0.322	0.678
11	6400	6000	20.4	0.969	0.355	0.645
12	5300	6100	20.74	0.985	0.387	0.613
13	6500	6200	21.08	1.00	0.419	0.581
14	7000	6300	21.42	1.02	0.452	0.548
15	6900	6400	21.76	1.03		
16	6800	6400	21.76	1.03	0.516	0.484
17	6900	6500	22.10	1.05		
18	6700	6500	22.10	1.05		
19	5000	6500	22.10	1.05		
20	4600	6500	22.10	1.05	0.645	0.355
21	5600	6600	22.44	1.07		
22	5900	6600	22.44	1.07	0.709	0.291
23	6300	6700	22.78	1.08	0.742	0.258
24	5100	6800	23.12	1.10	0.774	0.226
25	6100	6900	23.46	1.11		
26	6900	6900	23.46	1.11		
27	6500	6900	23.46	1.11		
28	7400	6900	23.46	1.11	0.903	0.097
29	6400	7000	23.80	1.13	0.935	0.065
30	6600	7400	25.16	1.19	0.968	0.032
	Mean	6200	21.08			

Table 2. Strength of BeO tubes; Weibull modulus and least value prediction.

Sample	No. of specimens	Mean strength (ksi)	Value of m from			Observed least value	Predicted least value
			least value Fig. 3	Fig. 1	Fig. 2 (preferred)		
1	30	21.1	11	9	9.5	15.64	15.4
2	37	21.8	8	7.5	9.5	14.28	15.3
3	106	20.1	9	7	7	12.2	11.1
Ref.[13]	140	15.6	8	7.25	Not used	8.87	8.3

Table 3. Experimental results of bending tests on alumina (Wesgo AL-995) at room temperature tested by IITRI.

Specimen B $\square \frac{1}{3}$ 4-in. span-uniform bending
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Rank	Failure stress	Norm. stress	Survival fraction	Rank	Failure stress	Norm. stress	Survival fraction
1	23544	0.790	0.0164	31	29700	0.997	0.525
2	23976	0.805	0.033	32	29700	0.997	0.525
3	25380	0.852	0.049	33	29808	1.000	0.558
4	26136	0.877	0.066	34	29808	1.000	0.558
5	26244	0.881	0.082	35	30024	1.008	0.574
6	26676	0.896	0.098	36	31032	1.012	0.607
7	27324	0.917	0.115	37	30132	1.012	0.607
8	27546	0.925	0.131	38	30456	1.022	0.623
9	27972	0.939	0.164	39	30672	1.030	0.640
10	27972	0.939	0.164	40	30780	1.033	0.672
11	28080	0.943	0.180	41	30780	1.033	0.672
12	28188	0.946	0.213	42	30888	1.037	0.705
13	28188	0.946	0.213	43	30888	1.037	0.705
14	28404	0.953	0.246	44	30996	1.041	0.722
15	28404	0.953	0.246	45	31104	1.044	0.738
16	28512	0.957	0.262	46	31752	1.066	0.754
17	28620	0.961	0.279	47	31860	1.070	0.771
18	28836	0.968	0.295	48	32076	1.077	0.804
19	28944	0.972	0.344	49	32076	1.077	0.804
20	28944	0.972	0.344	50	32184	1.080	0.853
21	28944	0.972	0.344	51	32184	1.080	0.853
22	29052	0.975	0.361	52	32184	1.080	0.853
23	29160	0.979	0.394	53	32292	1.084	0.867
24	29160	0.979	0.394	54	32724	1.099	0.886
25	29484	0.990	0.476	55	32832	1.102	0.902
26	29484	0.990	0.476	56	32940	1.106	0.918
27	29484	0.990	0.476	57	33156	1.113	0.935
28	29484	0.990	0.476	58	33480	1.124	0.951
29	29484	0.990	0.476	59	33588	1.128	0.968
30	29592	0.993	0.492	60	33804	1.135	0.984

Avg. 29782

Table 3. (Continued)

	<u>Results of Analysis</u>						Observed least value (ksi)
	<u>Primitive Approximation</u>		<u>Biased Distribution</u>				
		Predicted least value				Predicted least value	
	m	(ksi)	m	α	σ_u (ksi)	(ksi)	
This study	15	23.5	2.5	0.78	23.2	24.4	23.5
Data from reference 13	none	none	2.5	0.82	24.4	25.2*	

* Not given in reference 13.

Fig. 1. Weibull probability paper.

Fig. 2. Weibull probability paper.

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Fig. 2. Weibull probability paper. (Continued)

Fig. 3. Burst strength of BeO.

Fig. 4. Burst strength of BeO.

Fig. 5. Extreme values (minima) of the reduced Weibull distribution (normalized).

Fig. 6. Effect of truncation on a cumulative distribution.

(a)

(b)

(c)

Fig. 7. Effect of truncation on the probability density.

Fig. 8. The least values of the truncated distribution.

Fig. 9. Strength improvement due to truncation.

Fig. 10. Bias and truncation on a distribution with $m = 7$.

Fig. 11. Alumina bending data from Ref. [13]; type B specimen at room temperature.

Fig. C-1. Probability density of primitive distribution.

Fig. E-1. Standard deviation of various sample sizes as a function of the Weibull modulus.

Fig. F-1. The strength of potassium chloride and sodium chloride whiskers.

Fig. F-2. The strength of nickel-coated iron whiskers.

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