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Carnegie Institute of Technology

Department of Physics

APPLICATION OF THE BETHE-SALPETER EQUATION
TO THE STATIC BOOTSTRAP MODEL

by

Keh-Ying Lin

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

May 1966

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Carnegie Institute of Technology

Pittsburgh, Pennsylvania

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ABSTRACT

The bootstrap equations derived from the Bethe-Salpeter equations with ladder approximation are applied to the static models of baryons. An approximation similar to the effective range approximation is made in the static model, and then we get a set of algebraic equations which are convenient for numerical calculation. Among the models considered, one is the well-known π - N - N^* static model, for which we obtain an unique self-consistent solution which fits the experimental data quite well by choosing a suitable cutoff. A similar result is obtained if $SU(3)$ symmetry is assumed, and the so-called F-D mixing angle of the coupling constants turns out to be approximately 38° . We have considered a static model which includes π , K , N , N^* , Y_1^* , Σ and Λ in which no self-consistent solution is found. The bootstrap equations in this method are similar to those got from the N/D method except that the normalization equations are somewhat different.

CHAPTER I

INTRODUCTION

The discovery of the resonance (the $(3,3)$ resonance or N^*) in the elastic and charge-exchange pion-nucleon scattering amplitudes with the quantum numbers I (isotopic spin) = J (spin) = $3/2$ in the early fifties was a milestone in strong-interaction physics. As the particle accelerators became more and more powerful, the discoveries of new resonances turned out to be routine works for the experimentalists. From the theoretical point of view, the only difference between a resonance and a stable particle is the difference of their lifetimes, which can be explained by the simple fact that a particle is forbidden to decay strongly by some selection rules (e.g., conservation of strangeness) while a resonance is not, and we shall call both of them strongly interacting particles or hadrons. It is natural to ask whether some of the hadrons are really composite particles. The $(3,3)$ resonance has been explained by the famous Chew-Low theory⁽¹⁾ using a static model (a brief review of their works is given in Chap. III). The essence of their work is that the N^* is a composite state of N and π , held together mainly by the exchange of a nucleon. It has been suggested^(2,3) that no particle is more elementary than others. The bootstrap hypothesis that each hadron helps to generate other hadrons was proposed by Chew and Frautchi⁽⁴⁾. The basic idea of the bootstrap theory is that each hadron is a bound state of those channels with which it communicates, and the forces among the virtual particles of the channels are associated with the exchange of the hadrons that communicate with the crossed channels.

In this kind of bootstrap model, the much weaker electromagnetic and weak interactions are treated as the perturbations since the photon and the leptons seem not to bootstrap themselves. In fact, Chew⁽⁵⁾ has argued that the photon may play a unique and essential role in the connection between the concepts of energy-momentum and macroscopic space-time.

The bootstrap hypothesis has been applied to many models^(6-17, 43-50); in the vector meson model of Cutkosky⁽⁶⁾, symmetry under a Lie algebra appears automatically from the dynamics, and the concept of parity has been investigated from this point of view⁽⁷⁾. A general discussion of symmetry, according to the bootstrap mechanism, has been given by Belinfante, Cutkosky and Renninger⁽¹⁸⁾.

Chew⁽⁸⁾ has pointed out that an argument similar to that used in the Chew-Low model of N^* works for the nucleon also (that is, the nucleon can be considered as a composite state of N and π , held together mainly by the exchange of a N^*) and suggested a static bootstrap model of N , N^* and π . He used the approximated N/D method to get a relation between two coupling constants which is compatible with experiment. Chew's work has caused a great excitement among the bootstrap physicists^(9-12, 43, 44, 51). Unfortunately, the approximate N/D method has many objections which can be avoided by using the Bethe-Salpeter formalism^(11,19,20). A comparison of these two methods is given in Chap. 4.

The theory that the strong interaction satisfies approximately the $SU(3)$ symmetry was suggested by Gell-Mann and Ne'eman^(52,53) as a generalization of the isotopic spin symmetry. The famous Gell-Mann-Okubo (GMO) mass formula^(52,54) works surprisingly well except in the case of vector mesons which can be explained as due to an accidental degeneracy of a singlet and the eighth member of an octet⁽⁵⁵⁻⁵⁷⁾. The discovery of Ω^- ⁽⁵⁸⁾ which was predicted by $SU(3)$ ⁽⁵⁹⁾ is a great triumph

for the theorists. According to this theory,⁽⁵⁹⁾ the baryons N, Σ, Λ , and Ξ are assigned to an octet representation, N^*, Ξ^*, Y_1^* and Ω^- a decuplet, and the mesons $\pi, K, \bar{K}$, and η an octet. However, not all physicists are happy about the $SU(3)$ theory^(61,62). What is the origin of this symmetry? Why the mass formula which was derived by using a first order perturbation theory works so well? These questions have been answered by bootstrap physicist by using several different models^(6,17,49,50,63).

The purpose of our investigation is to explain qualitatively the low-lying baryonic states from the bootstrap point of view using the static model⁽¹⁶⁻¹⁸⁾. We use the Bethe-Salpeter formalism^(11,19,20) instead of the N/D method⁽²¹⁻²³⁾ to write down the bootstrap equations. The general properties of the bootstrap equations and the criteria of self-consistency are given in Chs. 2 and 3. Chew's model is reinvestigated by using the Bethe-Salpeter equations (see Chap. 5). The cutoff (1.24 BeV) is chosen in such a way so that the calculated coupling constant $g_{N N \pi}$ agrees with experiment; then we find the mass difference between the N and N^* to be 0.3 in units of the average energy of the meson cloud, or 235 MeV, which is slightly smaller than the experimental value. If $SU(3)$ symmetry is assumed, then the calculated mass difference agrees very well with the averaged (over the experimental masses of the states within the same irreducible representation) mass difference. In both models, the ratio $g_{N^* N \pi} / g_{N N \pi}$ is quite close to the experimental value.

We have considered a simple bootstrap model which includes π, Σ, Λ and Y_1^* as a second example, analogous to the π, N, N^* model; of a very strongly broken $SU(3)$ symmetry. We find a solution with the following properties: $m(\Sigma) = m(Y_1^*)$, $g(Y_1^* Y_1^* \pi) : g(Y_1^* \Sigma \pi) :$

$g(\Sigma\Sigma\pi) = \sqrt{10} : \sqrt{2} : 2$ and $g(Y_1^* \Lambda \pi) : g(\Sigma \Lambda \pi) = -\sqrt{2} : 1$ (the normalization of the coupling constants is defined in sec. d of Chap. 4). This solution actually satisfies a symmetry higher than the isotopic spin symmetry. The ratios of the coupling constants turn out to be quite close to the corresponding ones in the SU(3) model, (see Chap. 6). However, there are strong attractions in the $T = 2$, $J = 3/2$, and $T = 2$, $J = 5/2$ channels. These attractions may give rise to new resonant states at a higher energy, which would violate the criteria of the self-consistency (see sec. e of Chap. 7).

A static model with non-degenerate mesons has been considered in Chap. 7. This model includes π , K , Y_1^* , Σ , Λ , N^* and N ; in other words, we ignore the relatively heavy particles (Ξ , Ξ^* , Ω and η) in the SU(3) multiplets. We find a solution whose mass spectrum does not agree with the one found in the nature. The coupling constants involving π , Y_1^* , Σ , and Λ turn out to be close to those of the previous model, and there also exists a strong attraction in the ($Y = \text{hypercharge} = 0$, $T = 2$, $J = 3/2$) channel. Besides, there are strong attractions in the channels $(0, 2, 3/2)$, $(1, 1/2, 3/2)$, and $(-1, 1/2, 3/2)$. The four channels are contained in the SU(3) channels (35, $J = 5/2$) and (27, $J = 3/2$) which give rise to attractions in the SU(2) $\times$ SU(3) model.

Part of the above results (concerning the (3,3) resonance) has been published separately (27) .

CHAPTER II

General Properties of
Bootstrap Equations in the Ladder Approximation

In this chapter we will review the general properties of the bootstrap equations derived by using the Bethe-Salpeter equations in the ladder approximation^(14,22).

- a. Bethe-Salpeter equation⁽²³⁾ and the
normalization condition^(22, 28-30)

The two-particle Green's function G satisfies the following integral equation:

$$G = G_1 G_2 + G_1 G_2 I G$$

where $G_1 G_2 = K^{-1}$ is the product of two one-particle Green's functions, and I is the interaction kernel (sum over all irreducible graphs). (All integration variables and irrelevant indices are suppressed, all exchange terms are ignored, and we usually drop the over-all energy-momentum delta function without changing notation.) We can write G near the bound-state pole as

$$G = i \phi_a \bar{\phi}_a / (P^2 - m_a^2) + \text{terms which are regular}$$

when $P^2 = (P_1 + P_2)^2 = m_a^2$. (2-1)

The Bethe-Salpeter equation can be written in the form:

$$\phi_a = K^{-1} I \phi_a . \quad (2-2)$$

Let us consider the Bethe-Salpeter equation off the bound-state mass shell,

$$\phi_a = \chi_a (P^2) K^{-1} I \phi_a, \text{ with } \chi_a (m_a^2) = 1;$$

we also have

$$\bar{\phi}_a = \chi_a \bar{\phi}_a I K^{-1}.$$

With the aid of the orthogonality relation

$$\begin{aligned} \bar{\phi}_a K \phi_b & \left[\equiv \int d^4q \bar{\phi}_a (P, q) K (P, q) \phi_b (P, q) \right] \\ & = 0 \text{ for } \chi_a (P^2) \neq \chi_b (P^2), \end{aligned}$$

we obtain

$$G = \sum_n \left[\phi_n \bar{\phi}_n / (1 - \chi_n^{-1}) \bar{\phi}_n K \phi_n \right].$$

(To prove this, set $G = \sum \phi_n \psi_n$ and apply the operator $\bar{\phi}_a (K - I)$ to both sides.) The comparison of this equation with Eq. (2-1) near the bound-state pole $P^2 = m_a^2$ would lead directly to the normalization condition

$$1 = -i \bar{\phi}_a K \phi_a \left[\partial \chi_a^{-1} (m_a^2) / \partial p^2 \right],$$

with
$$\chi_a^{-1} (p^2) = \bar{\phi}_a I \phi_a / \bar{\phi}_a K \phi_a.$$

Since
$$\delta [\bar{\phi}_a (K - I) \phi_a] / \delta \phi_a = 0 \text{ at } p^2 = m_a^2,$$

we have

$$\begin{aligned} \bar{\phi}_a K \phi_a \frac{\partial \chi_a^{-1} (m_a^2)}{\partial p^2} &= \frac{\partial [\bar{\phi}_a I \phi_a]}{\partial p^2} - \frac{\bar{\phi}_a I \phi_a}{\bar{\phi}_a K \phi_a} \frac{\partial [\bar{\phi}_a K \phi_a]}{\partial p^2} \\ &= \partial [\bar{\phi}_a (I - K) \phi_a] / \partial p^2 \\ &= \bar{\phi}_a \left[\partial (I - K) / \partial p^2 \right] \phi_a; \end{aligned}$$

then the normalization condition can be written in the form

$$1 = i \bar{\phi}_a \left[\partial(K - I) / \partial p^2 \right]_{p^2 = m_a^2} \phi_a \quad (2-3)$$

In order to make the subsequent discussion clearer, we shall rewrite the Eqs. (2-2) and (2-3) with all indices for the multichannel case:

$$\phi_a^{bc} = \sum_{d,e} (K^{-1})_{bc} I_{bc, de}^a \phi_a^{de} \quad (2-2')$$

$$1 = \sum_{\substack{b,c \\ d,e}} i \bar{\phi}_a^{bc} \left[\frac{\partial}{\partial p^2} (K_{bc} \delta_{de}^{bc} - I_{bc, de}^a) \right]_{p^2 = m_a^2} \phi_a^{de} \quad (2-3')$$

b Bootstrap equations

Bootstrap equations have been derived from the Bethe-Salpeter equations plus the normalization condition in the ladder approximation by Cutkosky and Leon⁽²²⁾; we will follow their notation and argument in this section.

Let us define the coupling constant g_a^{bc} from the residue of ϕ_a^{bc} at

$$p_b^2 = m_b^2, p_c^2 = m_c^2 :$$

$$\begin{aligned} \phi_a &\equiv \phi_a(p, q) \equiv \phi_a^{bc}(p_a, p_b, p_c) \\ &= g_a^{bc} \phi_a'^{bc}(p_a, p_b, p_c) , \end{aligned}$$

with

$$\left[U_{abc} K \phi_a'^{bc} \right]_{p_b^2 = m_b^2, p_c^2 = m_c^2} = Y_{abc} ,$$

where U_{abc} is any convenient momentum factor appearing at the vertex $a \rightarrow b + c$ to contract the spinor, vector, etc., and Y_{abc} is an angular factor normalized in some way. From the equation

$$\delta [\bar{\phi} (K - I) \phi] / \delta \phi = 0$$

and $\delta \phi = \phi' \delta g$, We obtain

$$\delta \left[\sum_{d,e} (g_a^{de})^2 \bar{\phi}_a^{de} K_{de} \phi_a^{de} - \sum_{d,e,f,h} g_a^{de} g_a^{fh} \bar{\phi}_a^{de} I_{de,fh}^a \phi_a^{fh} \right] / \delta g_a^{bc} = 0 ;$$

after rearrangement we have the coupling constant equation

$$g_a^{bc} = \sum_{e,f} i g_a^{ef} \bar{\phi}_a^{bc} I_{bc,ef}^a \phi_a^{ef} / \widetilde{K}_{bc}^a \quad (2-4)$$

where

$$\widetilde{K}_{bc}^a = i \bar{\phi}_a^{bc} K_{bc} \phi_a^{bc}$$

Now we introduce a generalized ladder approximation,

$$I_{bc,ef}^a = \sum_i g_i^{be} g_i^{cf} I_{bc,ef}^{ai},$$

in which the vertices are supposed to be so calculated that "vertex symmetry" is obtained (i.e. the same vertex equation is obtained no matter which particle is considered as the bound-state of the other two particles).

Then one can identify

$$g_a^{bc} = g_{abc} = g_{bac} = \dots$$

In this approximation, the vertex equation can be written in the form:

$$g_{abc} = \sum_{e,f,i} D_{abc}^{ife} g_{aef} g_{bie} g_{cif}, \quad (2-5)$$

where $D_{abc}^{ife} = \widetilde{I}_{bc,ef}^{ai} / \widetilde{K}_{bc}^a,$

and $\widetilde{I} = i \oint I' \oint'$

In this formulation, we have (because of "vertex symmetry")

$$D_{abc}^{ife} = D_{bac}^{fie} = D_{bca}^{fei} = \dots$$

From the identity $\widetilde{I}_{bc,ef}^{ai} = \widetilde{I}_{ef,bc}^{ai}$ (since they correspond to the same Feynman graph), we have the following identity:

$$\widetilde{K}_{bc}^a D_{abc}^{ife} = \widetilde{I}_{bc,ef}^{ai} = \widetilde{K}_{ef}^a D_{aef}^{icb}. \quad (2-6)$$

From the normalization condition, we have

$$1 = \sum_{b,c} (g_{abc})^2 \partial \widetilde{K}_{bc}^a / \partial m_a^2 - \sum_{b,c,e,f,i} g_{abc} g_{bie} g_{cif} g_{aef} \partial \widetilde{I}_{bc,ef}^{ai} / \partial m_a^2;$$

let us replace $(g_{abc})^2$ and $\widetilde{I}$ by

$$(g_{abc})^2 = g_{abc} \left[\sum_{e,f,i} D_{abc}^{ife} g_{aef} g_{bie} g_{cif} \right]$$

and $\widetilde{I} = \widetilde{K} D$ respectively, then the above equation becomes

$$1 = \sum_{b,c,e,f,i} \varepsilon_{abc} \varepsilon_{bie} \varepsilon_{cif} \varepsilon_{aef} \widetilde{K}_{bc}^a \frac{\partial D_{abc}^{ife}}{\partial m_a^2}.$$

Finally, the mass equation can be rewritten in the form

$$1 = \sum_{b,c,e,f,i} \varepsilon_{abc} \varepsilon_{bie} \varepsilon_{cif} \varepsilon_{aef} W_{bc,ef}^{ai} \quad (2-7)$$

where

$$\begin{aligned} W_{bc,ef}^{ai} &= W_{ef,bc}^{ai} \\ &= \frac{1}{2} \left[\widetilde{K}_{bc}^a \frac{\partial D_{abc}^{ife}}{\partial m_a^2} + \widetilde{K}_{ef}^a \frac{\partial D_{aef}^{icb}}{\partial m_a^2} \right]. \end{aligned}$$

The Eqs. (2-5) and (2-7) correspond to Figs. 1 (a) and (b). In fact one can derive the vertex equation directly from the Bethe-Salpeter equation (*). Therefore the D's are just the Feynman integrals corresponding to the triangle diagram. At each vertex the actual Bethe-Salpeter wave function is introduced, which provides form factors leading to an automatic cutoff. However, the previous way of calculating D's is more accurate, since the vertex equation satisfies $\delta(\dots)/\delta\phi = 0$ (i.e., a first order error in ϕ would lead to a second order error in D).

(*) Starting from the Bethe-Salpeter equation $\phi_a = K^{-1} I \phi_a$, we have

$$\begin{aligned} [U K \phi_a]_{p_b^2 = m_b^2, p_c^2 = m_c^2} &= Y_{abc} \varepsilon_{abc} \\ &= U I \phi_a = [U I' \phi'_a]_{p_b^2 = m_b^2, p_c^2 = m_c^2} \varepsilon_{aef} \varepsilon_{bie} \varepsilon_{cif}; \end{aligned}$$

so

$$YD = U I' \phi'$$

Crossing symmetry is satisfied exactly for the single particle poles but only approximately for the two particle and other continuum states.

Note that although $\widetilde{I}$ and $\widetilde{K}$ are exactly the Feynman integrals of the graphs appeared in Fig. 1 (b) and (c) respectively, the W 's are not the Feynman integrals of the corresponding diagrams; however it is very convenient to use the graphs as symbols of W 's.

c The conditions of self-consistency

A self-consistent solution must satisfy the bootstrap equations derived above. Besides it must also satisfy two additional conditions. The first condition is that the input particles should be the lowest states in the corresponding channels. The second condition is that our model should not introduce strong attractive forces for channels without bound states. We will now discuss the restrictions due to these two conditions.

Let us consider the first condition. We define a vector $\vec{g}_a$ whose components are $g_a^{bc} \sqrt{\widetilde{K}_{bc}^a}$. From Eq. (2-4), it is clear that the vector $\vec{g}_a$ is the eigenvector of a hermitian matrix (calling it potential matrix V). The matrix elements of V ,

$$V_{bc,ef}^a = i \bar{\phi}_a^{bc} L_{bc,ef}^a \phi_a^{ef} / \sqrt{\widetilde{K}_{bc}^a \widetilde{K}_{ef}^a},$$

may be explained as the average value (with respect to the Bethe-Salpeter wave functions) of the interaction kernel. The first condition requires that our solution should correspond to the strongest attraction in channel (a), in other words, the eigenvalue corresponding to our solution (which is normalized to one) should be much bigger than any other

eigenvalues. This assumes that if the attraction between the virtual particles becomes stronger, the mass of the bound state would be smaller.

Now we turn to the second condition whose meaning is somewhat imprecise. We have to choose a set of suitable test functions (say ψ) to calculate the matrix element of V ,

$$V_{bc,ef}^a(P^2) = \frac{i \bar{\psi}_a^{bc} I_{bc,ef}^a \psi_a^{ef}}{\sqrt{(\bar{\psi}_a^{bc} iK_{bc} \psi_a^{bc}) (\bar{\psi}_a^{ef} iK_{ef} \psi_a^{ef})}},$$

then it is reasonable to expect that the eigenvalues of V are not very sensitive to P^2 and the set of test functions. Then the restriction of the second condition means that all eigenvalues of V should be much smaller than one.

The above discussion is independent of any kind of approximation; if we use the ladder approximation, then in the first self-consistency condition, the hermitian matrix V is equivalent (i.e., up to a similarity transformation) to another matrix (in general not hermitian) whose elements are $(\neq)$

$$\sum_i D_{abc}^{ife} g_{bie} g_{cif}. \text{ This can be also used for the second condition}$$

($\neq$) Using Eq. (2-6), it is easy to show that

$$V_{bc,ef}^a = \frac{\sum_i g_i^{be} g_i^{cf} \widetilde{I}_{bc,ef}^{ai}}{\sqrt{\widetilde{K}_{bc}^a \widetilde{K}_{ef}^a}} = \sqrt{\frac{\widetilde{K}_{bc}^a}{\widetilde{K}_{ef}^a}} \sum_i D_{abc}^{ife} g_i^{be} g_i^{cf}.$$

if we define

$$D_{abc}^{ife} \text{ by } D_{abc}^{ife} \equiv \frac{\overline{\psi}_a^{bc} I'_{bc,ef} \psi_a^{ef}}{\overline{\psi}_a^{bc} K_{bc} \psi_a^{bc}}$$

CHAPTER III

Bootstrap Equation in the Generalized Static Modela Review of the classical static model⁽¹⁹⁻²¹⁾

The static model has been used⁽¹⁾ to explain satisfactorily the scattering of pions by nucleons at kinetic laboratory energies of pions not much larger than 200 MeV. It is well-known that the phase shifts for the P-states are large compared to that of s-states. Furthermore, the phase shift for the P-state with total angular momentum $j = 3/2$ and total isotopic spin $T = 3/2$ is particularly big.

In this model, we neglect the recoil of the nucleons and treat them as infinitely heavy^(#), but the mesons must be treated relativistically. The total Hamiltonian H is a scalar in ordinary space. Since pions have odd intrinsic parity and H is linear in the meson field operators, the coefficients of the meson field operators must be a pseudoscalar. Besides, the Hamiltonian must behave like a scalar in isospin space. Starting from the spin operator σ of the nucleons (a pseudovector) and the momentum vector K of the pion, the form of the most general pseudoscalar which can be constructed from these two vectors is the three-dimensional scalar product of these two vectors multiplied by an arbitrary function of the magnitude p of the momentum vector.

^(#) It is easy to convince oneself that the effect of recoil of nucleon is indeed very small. For example, when the kinetic energy of the incoming pion is 200 MeV, the maximum kinetic energy of the recoiled nucleon (happening at backward scattering) is only about 12.5% of the mass of the nucleon.

Expanding the meson field $\vec{\phi}$ (an isovector) in terms of plane waves with annihilation operators $\vec{a}(K)$,

$$\vec{\phi} = \frac{1}{\sqrt{V}} \sum_K \frac{1}{\sqrt{2\omega}} \left[e^{iK \cdot X} \vec{a}(K) + \text{c.c.} \right],$$

(the normalization factor $1/\sqrt{2\omega}$ is so chosen that we have $[a_\gamma(K), a_{\gamma'}^*(K')] = \delta_{\gamma, \gamma'} \delta_{K, K'}$), we can write the Hamiltonian in the form:

$$H = \frac{f}{m} \sum_K \frac{v(p)}{\sqrt{2\omega V}} i \sigma \cdot K \vec{T} \cdot \vec{a}(K) + \text{c.c.}, \quad (3-1)$$

where f is a real dimensionless constant (the coupling constant), m is the mass of the pion, ω is the energy of pion with momentum K , V is the volume of a large box where all particles are supposed to be enclosed, $\vec{T}$ is the isospin operator of the nucleons, and the two letters "c.c." means that the complex conjugate of the first term has to be added to make H hermitian. The cut-off function $v(p)$ can be thought of as some kind of Fourier transform of the particle density inside the nucleon, and we shall consider it as a phenomenological function to be fitted by experimental data. However, it is more convenient to use the expansion of the meson field in terms of states with definite angular momenta,

$$\vec{\phi} = \frac{1}{\sqrt{R}} \sum_{p, l, m} \frac{P}{\sqrt{\omega}} j_l(pr) \left[Y_l^m(\Omega_x) \vec{a}_l^m(p) e^{-i\omega X_0} \right] + \text{c.c.},$$

where our normalization volume is a sphere with radius R . The symbol $j_l(pr)$ is the conventional spherical Bessel function which satisfies the asymptotic condition

$$j_l(pr) \approx \frac{1}{pr} \sin \left(pr - \frac{l}{2} \pi \right) \text{ for } pr \gg l.$$

The magnitude of the momentum is determined by the equation $j_L(p_n R) = 0$ or $p_n R \approx (\frac{l}{2} + n) \pi$.

In the limit when the radius R becomes very large, a summation over p_n can be replaced by an integration:

$$\sum_n f(p_n) \rightarrow \frac{R}{\pi} \int_0^\infty dp f(p)$$

The operators $\vec{a}_\ell^m(p)$ and $\vec{a}_\ell^{m*}(p)$ are the annihilation and creation operators for mesons and they fulfill the commutation relations (equal times)

$$[\vec{a}_\ell^m(p), \vec{a}_{\ell'}^{m'}(p')] = \delta_{\ell\ell'} \delta_{mm'} \delta_{pp'},$$

(all other commutators vanish) where τ denotes different components in isospin space. The operators $\vec{a}$ and $\vec{a}^\dagger$, are independent of time in the free field but depend on time in interacting field. It can be proved that the plane wave operators $\vec{a}(K)$ are related to the spherical wave operators $\vec{a}$ by $(\dagger)$

$$\vec{a}(K) = \frac{1}{p} \sqrt{\frac{8(\pi)^3}{V}} \frac{R}{\pi} \sum_{m,l} (-i)^l Y_\ell^m(\Omega_p) \vec{a}_\ell^m(p)$$

Since the spherical harmonics $Y_l^m(\Omega)$ is given by

$$Y_1^0 = \sqrt{3/4\pi} \cos \theta$$

and $Y_1^\pm = \mp \sqrt{3/8\pi} \sin \theta e^{\pm i\varphi},$

we can rewrite $\sigma \cdot K$ in the form:

$$\sigma \cdot K = p \sqrt{4\pi/3} \sum_m \sigma^{(m)} [Y_1^m(\Omega_p)]^\dagger,$$

$(\dagger)$ Ref. 19, Eq. (5-39).

where

$$\sigma^{(+1)} = \frac{1}{\sqrt{2}} (\sigma_1 - i \sigma_2)$$

and

$$\sigma^{(0)} = \sigma_3.$$

It follows from the orthogonal condition of Y_1^m that ⁽⁷⁾

$$H = \frac{f}{m} \frac{1}{\sqrt{12\pi}} \frac{1}{\sqrt{R}} \sum_p \frac{p^2 v(p)}{\sqrt{\omega}} \sum_{m, T_3} \sigma^{(m)} \tau^{(T_3)}_{\alpha_1} \alpha_1^{(m) (T_3)}(p) + \text{c.c.} \quad (3-2)$$

Now it is obvious that the interaction only involves mesons in P-states. Actually, we could immediately write down this expression as the interaction Hamiltonian since the mesons have to be in P-states according to the principle of conservation of parity. However we prefer to derive it from Eq.(3-1).

We shall summarize some well-known results without going into details ⁽²¹⁾. Let us define a function $h_\alpha(\omega)$ by the relation

$$h_\alpha(\omega + i\epsilon) = \frac{e^{i\delta_\alpha} \sin \delta_\alpha(p)}{v^2(p) \cdot p^3}$$

where α can assume the values 1, 2, and 3. The convention is that $\alpha = 1$ refers to the $T = j = 1/2$ state, $\alpha = 2$ refers to $T = 1/2$, $j = 3/2$ or $T = 3/2$, $j = 1/2$, and $\alpha = 3$ refers to $T = J = 3/2$. The functions $\delta_\alpha(p)$ are real and denote the phase shifts of the elastic

⁽⁷⁾ Ref. (19), Eq. 5-48.

meson-nucleon scattering. Using the one-meson approximation (this means we neglect contributions from states with two or more mesons), it can be shown that $h_\alpha(z)$ satisfy the following relation:

$$\operatorname{Re} \left[\frac{\lambda_\alpha}{\omega} \frac{1}{h_\alpha(\omega)} \right] = \frac{\lambda_\alpha p^3 v^2(p) \cot \delta_\alpha}{\omega} = 1 - \omega r_\alpha(\omega)$$

where

$$\lambda_\alpha = \frac{2}{3} \frac{f^2}{4\pi m^2} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$$

$$r_\alpha(\omega') = \frac{\lambda_\alpha}{\pi} p \int_m^\infty \frac{p^3 v^2 d\omega}{\omega^2 (\omega - \omega')} + \frac{1}{\pi} \int \frac{p^3 v^2 B_\alpha(\omega) d\omega}{\omega^2 (\omega + \omega')}$$

and $B_\alpha(\omega)$ are functions determined from the crossing relations and cannot be expressed simply in terms of the fundamental quantities of the model. The functions $r_\alpha(\omega)$ are almost constants for small ω , and it can be shown that

$$r_3 = \frac{1}{27 \pi^2 \lambda_3} \int_m^\infty \frac{(5/2 \sigma_3 + \sigma_1 + \sigma_2) d\omega}{\omega p v^2} > 0,$$

where σ_α is the total cross section for the state α . Let us assume the existence of a cut-off energy $\omega_{\max}$, i.e., $v^2(p) = 1$ if $\omega < \omega_{\max}$ and $v^2(p) = 0$ if $\omega > \omega_{\max}$. The approximation which neglects the energy dependence of $r_\alpha(\omega)$ for $\omega \ll \omega_{\max}$ is called the effective range approximation. The resonance will occur when $\omega = 1/r_3$. A comparison with experimental data gives the following numbers:

$$\frac{f^2}{4\pi} = 0.088,$$

$$r_3^{-1} = 2.18 \text{ m},$$

and $\omega_{\max} \approx 6m$. It is reasonable to expect that for $\alpha = 1$ and 2 , the sign of the function r_α is determined by the first term which gives a negative sign in both cases, therefore we expect that the phases δ_1 and δ_2 do not pass through a resonance at low energy; this result agrees with experiment.

It is interesting that the most important result of the Chew-Low theory -- the existence of a resonance in $T = j = 3/2$ channel (the so-called $(3,3)$ resonance in the literatures) and the non-existence of resonances in other channels -- depends only on three numbers λ_α which are related to the corresponding crossing matrix elements. The essence of this theory is that the resonance can be considered in first approximation as a pion-nucleon composite, held together mainly by exchange of a nucleon. Chew⁽⁸⁾ argued by using the N/D method and the effective range approximation that the nucleon is approximately a composite of a pion and a nucleon, bound together mainly by exchange of a $(3,3)$ resonance. We shall compare the N/D method with the Bethe-Salpeter approach in next chapter and point out some inconsistencies in Chew's argument at the beginning of Ch. 5. However, Chew's bootstrap model is very interesting and will be the subject of Chapter 5.

b. The generalized static model

We are going to discuss a generalized static model of the meson-baryon interaction. In this model, we assume that only mesons in P-states are involved in any interaction (in the case of the pion-nucleon interaction, the principle of parity conservation rules out any other possibility).

Besides, we shall assume all mesons are degenerate in this chapter, but isotopic spin invariance is not assumed here.

Under these assumptions, the effective interaction Hamiltonian H can be written as:

$$\begin{aligned}
 H = H_0 \sum_P P^2 / \sqrt{\omega} \left[\sum_{\substack{a,b,\gamma \\ (J_a \geq J_b)}} f_{ab\gamma} v_{ab\gamma}(p) \right. \\
 \times \sum_{\substack{T_a, T_b, T_\gamma \\ m_a, m_b, m_\gamma}} \delta_{T_a, T_b + T_\gamma} \frac{\langle J_a, m_a | J_b, m_b, 1, m_\gamma \rangle}{\sqrt{2J_a + 1}} \begin{matrix} m_a, T_a \\ a \\ J_a \end{matrix} \begin{matrix} m_b, T_b \\ b \\ J_b \end{matrix} \\
 \times \left(\gamma_1^{m_\gamma, T_\gamma}(p) + (-1)^{m_\gamma} \gamma_1^{*-m_\gamma, -T_\gamma}(p) \right) \left. \right\} + \text{c.c.} \quad (3-3)
 \end{aligned}$$

$$H_0 = (1/m \sqrt{12\pi R}) ,$$

where $a_{J_a}^{m_a, T_a}$ and $a_{J_a}^{*m_a, T_a}$ denote respectively the annihilation and creation operator of a baryon with total spin J_a , charge T_a , and the third component of total spin m_a , (γ refer to P-state mesons) and

$\langle J_a, m_a | J_b, m_b, 1, m \rangle$ is the familiar Clebsch-Gordan coefficient (we use the Condon-Shortley phase convention), and $f_{ab\gamma}$ is the (real) coupling constant. Conservation of angular momentum is implied by the definition of Clebsch-Gordan coefficients.

Let us define $f_{ab\gamma}$ to be zero if charge conservation is violated, then the matrix elements of the Hamiltonian can be written as:

$$\langle A | H | B, M \rangle = \langle B, M | H | A \rangle^* =$$

$$H_0 \sum_p p^2 f_{ab\gamma} v_{ab\gamma}(p) \frac{\epsilon_{ab} \langle J_a, m_a | J_b, m_b, 1, m \rangle}{\sqrt{\omega} \sqrt{2J_a + 1}}$$

where

$$A = a_{J_a}^{* m_a, T_a} | 0 \rangle ,$$

$$M = \gamma_{J_b}^{* m_\gamma, T_\gamma} | 0 \rangle ,$$

$\epsilon_{ab} = 1$ if $J_a \geq J_b$, $\epsilon_{ab} = -1$ if $J_a < J_b$, $f_{ab\gamma} \equiv f_{ba\bar{\gamma}}$
 and $v_{ab\gamma}(p) \equiv v_{ba\bar{\gamma}}(p)$. The form factors are so defined that
 $|v_{ab\gamma}(0)| = 1$. The form factors so defined are too general for numerical
 calculation. However, we prefer to see how the bootstrap equations look
 like in general before we make approximations for form factors (in
 section e).

c. Bootstrap equations in the generalized static model

We shall discuss the general form of the bootstrap equations in the
 generalized static model. Let us assume a set of degenerate mesons and
 a certain set of baryons which bootstrap each other in the sense that
 each baryon is a bound state of a P-state meson and any baryon (including
 itself). However, we do not assume a Fermi-Yang type model for mesons
 (by this we mean any model which considers mesons as bound states of
 baryon-antibaryon system), that means the existence and the properties of
 mesons are taken for granted in this bootstrap model.

Now we can write down the vertex equation which corresponds to Fig. 2(a) directly from the Bethe-Salpeter equation:

$$f_{aba} \frac{\langle J_a m_a | J_b m_b J_a m_a \rangle}{\sqrt{2J_a + 1}} = \sum_{c,d,\gamma} f_{ac\gamma} f_{cda} f_{db\bar{\gamma}} D_{dc\gamma}^{aba} \times$$

$$\frac{(-1)^{J_c - J_b}}{\sqrt{2J_a + 1} \sqrt{2J_c + 1} \sqrt{2J_b + 1}} \left[\sum_{m_c, m_d, m_\gamma} \langle J_a m_a | J_c m_c J_\gamma m_\gamma \rangle \times \right.$$

$$\left. \langle J_c m_c | J_d m_d J_a m_a \rangle \langle J_b m_b | J_d m_d J_\gamma m_\gamma \rangle \right]$$

where $D_{dc\gamma}^{aba}$ is an invariant under rotation and is defined as real; besides, it should satisfy the relation $D_{dc\gamma}^{aba} = D_{cd\gamma}^{baa}$. We shall discuss in detail the properties of D in the static model in sec. e. Since the Clebsch-Gordan coefficients are normalized in such a way that

$\sum_{m_b, m_a} \langle J_a m_a | J_b m_b J_a m_a \rangle^2 = 1$, we shall multiply both sides of the above expression by $\langle J_a m_a | J_b m_b J_a m_a \rangle$ and sum over all m_b and m_a ; then we get ($J_a = J_\gamma = 1$)

$$f_{aba} = \sum_{c,d,\gamma} f_{ac\gamma} f_{cd\gamma} f_{db\bar{\gamma}} D_{dc\gamma}^{aba} C_{J_d J_c}^{J_a J_b},$$

where

$$C_{J_d J_c}^{J_a J_b} \equiv \left[(-1)^{J_a + J_b + J_c + J_d - 2J_c} \right] \begin{Bmatrix} J_a & J_b & 1 \\ J_d & J_c & 1 \end{Bmatrix},$$

$$(-1)^{2J_a} = (-1)^{2J_b} = (-1)^{2J_c} = (-1)^{2J_d},$$

and $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ is the famous 6-j symbol⁽³¹⁾. The 6-j symbol is related to the Racah coefficient by the relation

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{a+b+d+e} W(a, b, e, d; c, f),$$

and it is invariant against any permutation of the columns or interchange of the upper and lower arguments in each of any two columns.

There are six different type of $C_{c d}^a b$ which differ from zero, they are given by the following formulas:

$$F_1(J) \equiv C_{J J}^{J J} = -1/6 \quad \text{if } J = 1/2$$

$$= \frac{2J(2J+2) - 4}{2J(2J+1)(2J+2)} \quad \text{if } J \geq 1,$$

$$F_2(J) \equiv C_{J J-1}^{J J-1} = \frac{2}{2J(4J^2 - 1)} \quad J \geq 1,$$

$$F_3(J) \equiv C_{J-1 J-1}^{J J-1} = \frac{2}{2J(2J-1)} \quad J > 1,$$

$$F_4(J) \equiv C_{J J-1}^{J J} = \frac{-2}{2J(2J+1)} \quad J \geq 1,$$

$$F_5(J) \equiv C_{J-2 J-1}^{J J-1} = \frac{1}{2J-1} \quad J \geq 2,$$

$$F_6(J) \equiv C_{J-1 J-1}^{J J} = \frac{1}{2J} \sqrt{\frac{4J^2 - 4}{4J^2 - 1}} \quad J > 1.$$

Note $C_{c d}^a b = C_{d c}^b a = C_{a b}^c d = C_{b a}^d c$. We put the numerical values

of $F_i(J)$ for $J \leq 2$ in the Table I.

TABLE I
NUMERICAL VALUES OF $F_i(J)$ for $J \leq 2$.

$J \backslash i$	1	2	3	4	5	6
$1/2$	$-1/6$					
1	$1/6$	$1/3$		$-1/3$		
$3/2$	$11/60$	$1/12$	$1/3$	$-1/6$		$\sqrt{10}/12$
2	$1/6$	$1/30$	$1/6$	$-1/10$	$1/3$	$\sqrt{5}/10$

The normalization equation corresponding to Fig. 2(b) can be written down immediately, it is

$$1 = \frac{1}{(2J_a + 1)} \sum_{b,c,d,a,\gamma} f_{aba} f_{ac\gamma} f_{cda} f_{db\bar{\gamma}} W_{c\gamma,ba}^{ad} x$$

$$\text{where } W_{c\gamma,ba}^{ad} = \frac{1}{4M} \left[\frac{\partial^D}{\partial M_a} \frac{dc\gamma}{\partial M_a} K_a^{ba} + \frac{\partial^D}{\partial M_a} \frac{dba}{\partial M_a} K_a^{c\gamma} \right].$$

Here K_{ba}^a is the self-energy integral corresponding to Fig. 2 (c), and M denotes the average mass of baryons. We shall discuss the properties of K and W in detail in sec. e.

d. Higher symmetry and bootstrap equations.

The above discussion can be generalized to the case where we try to find a solution with a symmetry higher than rotational invariance. One may assume that the solution is invariant under a transformation group which either includes the group $SU(2)$ (corresponding to the rotational invariance of the space) as a subgroup (for example, $SU(4)$ and $SU(6)$, or is the direct product of $SU(2)$ and an internal symmetry group (e.g., isotopic spin invariance and $SU(3)$). What we need is a generalized 6-j symbol which will be defined later; in the case of internal symmetry, it is the product of two generalized 6-j symbols corresponding to $SU(2)$ and the internal symmetry group.

(7)

Let us assume that the solution satisfies a group G whose irreducible unitary representations ($a, b, c, \text{etc.}$) are all finite dimensional ($N_a, N_b, \text{etc.}$). Let γ_a be the eigenvalues of a certain set of commuting operators which specify uniquely the states within an irreducible representation (IR). The product of two IR's is, in general, reducible. Since an IR may appear several times in the product of two IR's, one may use the symmetry properties (T 's) to distinguish them. We shall use a^* to denote the IR which is conjugate to a . By analogy with the

(7) It would be helpful to compare the following statements with the particular case of $SU(2)$.

Clebsch-Gordan (CG) coefficient of SU(2), we denote the corresponding CG coefficient of the group G by $\langle \tau^a | \gamma_b \gamma_c \rangle$ which can be chosen as real.

Let $A_{\gamma_a}^a$ ($A_{\gamma_a}^{*a}$) be the operator which annihilates (creates) a baryon state which is assigned to the IR a with quantum numbers γ_a . The meson may or may not belong to a self-conjugate IR. Let us use π and K to denote these two types of mesons, with obvious meaning. The CG coefficients have the following property:

$$\frac{\langle \tau^a | \gamma_b \gamma_c \rangle}{\sqrt{N_a}} = \xi(\gamma, a, b, c) \zeta(c, \gamma_c) \frac{\langle \tau^b | \gamma_a \gamma_c^* \rangle}{\sqrt{N_b}}$$

where ξ depends on γ, a, b , and c but is independent of γ_a , etc., it takes the value 1 or -1; ζ is defined in a similar way. We have the following identity:

$$\xi(\gamma, a, b, c) \zeta(c, \gamma_c) = \xi(\gamma, b, a, c^*) \zeta(c^*, -\gamma_c).$$

We shall use the convention that $\zeta(c, \gamma_c) = \zeta(c^*, -\gamma_c)$, then it follows

$$\xi(\gamma, a, b, c) = \xi(\gamma, b, a, c^*).$$

In the case of non-self-conjugate meson, the corresponding terms in the effective Hamiltonian can be written as (we drop the common

factor):

$$g_{abk}^{\tau} v_{abk}^{\tau}(p) \lambda_{abk}^{\tau} \frac{\langle \tau \begin{smallmatrix} a & b & k \\ \gamma_a & \gamma_b & \gamma_k \end{smallmatrix} \rangle}{\sqrt{N_a}} \frac{*}{\gamma_a} \frac{B}{\gamma_b} (K^k_{\gamma_k} + \zeta(k, \gamma_k) \bar{K}^{*K} - \gamma_k) + \text{c.c.}$$

where $\bar{K}$ means the operator corresponding to the antiparticle of meson K , and λ_{abk}^{τ} equals ± 1 and can be defined by using some convention. Let us introduce a new function $\lambda_{ba\bar{K}}^{\tau}$ which, by definition, is related to λ_{abk}^{τ} by the relation

$$1 = \lambda_{abk}^{\tau} \lambda_{ba\bar{K}}^{\tau} \zeta(\tau, a, b, k).$$

Then the next term (which we denoted by c.c.) can be rewritten in the form:

$$g_{abk}^{\tau} v_{abk}^{\tau*}(p) \lambda_{ba\bar{K}}^{\tau} \frac{\langle \tau \begin{smallmatrix} b & a & k^* \\ \gamma_b & \gamma_a & -\gamma_k \end{smallmatrix} \rangle}{\sqrt{N_b}} \frac{*}{\gamma_b} \frac{B}{\gamma_b} \frac{A}{\gamma_a} (\bar{K}^{*K} - \gamma_k + \zeta(k^*, -\gamma_k) K^{*K} \gamma_k),$$

where we have used the relation

$$\zeta(k, \gamma_k) = \zeta(k^*, -\gamma_k).$$

One can treat the case of the selfconjugate meson in exactly the same way.

The since we have

$$\sum_{\gamma_b, \gamma_a} \left[\lambda_{abu}^{\tau} \langle \tau \begin{smallmatrix} a & b & a \\ \gamma_a & \gamma_b & \gamma_a \end{smallmatrix} \rangle \right]^2 = 1,$$

one can define a generalized 6-j symbol (G_{6js}) which corresponds to a term in Fig. 2(b) as

$$= G = \frac{\lambda_{ac\gamma}^{\gamma} \lambda_{cda}^{\rho} \lambda_{bd\gamma}^{\sigma} \lambda_{aba}^{\omega}}{\sqrt{N_b N_c}} \sum_{\gamma_b, \gamma_c, \gamma_d, \gamma_{\gamma}, \gamma_{\alpha}}$$

$$\langle \gamma_a^a | \gamma_c^c \gamma_{\gamma}^{\gamma} \rangle \langle \rho_c^c | \gamma_d^d \gamma_{\alpha}^{\alpha} \rangle \langle \sigma_b^b | \gamma_d^d \gamma_{\gamma}^{\gamma} \rangle \langle \omega_a^a | \gamma_b^b \gamma_{\alpha}^{\alpha} \rangle$$

$$= N_a^{-1} \sum_{\gamma_a} G. \quad (3-4)$$

The symmetry property of G_{6js} is obvious from the second form. We adopt the following convention:

$$\lambda_{abK}^{\gamma} = 1,$$

$$\lambda_{ab\pi}^{\gamma} = 1 \quad \text{if } N_a \geq N_b.$$

In the case of $SU(2)$, the ordinary 6-j symbol is given by

$$\left\{ \begin{matrix} J_a & J_b & J_{\alpha} \\ J_d & J_c & J_{\gamma} \end{matrix} \right\} = (-1)^{J_b + J_c + J_{\alpha} + J_{\gamma}} \frac{1}{\sqrt{N_b N_c}} \left[\sum_{m_b, m_c, m_d, m_{\alpha}, m_{\gamma}} \right]$$

$$\left\langle J_a m_a | J_b m_b J_{\alpha} m_{\alpha} \right\rangle \left\langle J_a m_a | J_c m_c J_{\gamma} m_{\gamma} \right\rangle \left\langle J_c m_c | J_d m_d J_{\alpha} m_{\alpha} \right\rangle \times$$

$$\left\langle J_b m_b | J_d m_d J_{\gamma} m_{\gamma} \right\rangle ;$$

then we have

$$\begin{array}{c} a \quad c \quad \alpha \\ \diagdown \quad \diagup \\ \gamma \quad d \quad b \end{array} = \lambda_{ac\gamma} \lambda_{cda} \lambda_{bd\gamma} \lambda_{aba} (-1)^{J_b + J_c + J_a + J_\gamma} \begin{Bmatrix} J_a & J_b & J_d \\ J_d & J_c & J_\gamma \end{Bmatrix}$$

In the case of SU(3) the CG coefficient can be written as a product of two factors⁽³²⁾ :

$$\left\langle \begin{array}{c} a \\ \gamma_a \end{array} \middle| \begin{array}{cc} b & c \\ \gamma_b & \gamma_c \end{array} \right\rangle = \left\langle \begin{array}{c} I_a \ m_a \\ I_b \ m_b \ I_c \ m_c \end{array} \right\rangle \left\langle \begin{array}{c} a \\ I_a \ Y_a \end{array} \middle| \begin{array}{cc} b & c \\ I_b \ Y_b \ I_c \ Y_c \end{array} \right\rangle,$$

where the first factor is the CG coefficient of SU(2), the next factor is the so called isoscalar factor, and I, m, Y are the total isotopic spin, the third component of isotopic spin and the hypercharge of the particle respectively. Then the G6js of SU(3) can be put in the form:

$$\begin{array}{c} a \quad c \quad \rho \\ \diagdown \quad \diagup \\ \gamma \quad d \quad \omega \\ \diagup \quad \diagdown \\ \sigma \quad b \end{array} = \frac{1}{\sqrt{N_b N_c}} \lambda_{ac\gamma} \lambda_{cda}^\rho \lambda_{bd\gamma}^\sigma \lambda_{aba}^\omega$$

$$\sum_{\substack{I_b, I_c, I_d, I_a, I_\gamma \\ Y_b, Y_c, Y_d, Y_a, Y_\gamma}} (-1)^{J_b + J_c + J_a + J_\gamma} \sqrt{2I_b + 1} \sqrt{2I_c + 1} \begin{Bmatrix} I_a & I_b & I_a \\ I_d & I_c & I_\gamma \end{Bmatrix}$$

$$\left\langle \begin{array}{c} a \\ I_a Y_a \end{array} \middle| \begin{array}{cc} c & \gamma \\ I_c Y_c & I_\gamma Y_\gamma \end{array} \right\rangle \left\langle \begin{array}{c} \rho \\ I_c Y_c \end{array} \middle| \begin{array}{cc} d & a \\ I_d Y_d & I_a Y_a \end{array} \right\rangle \left\langle \begin{array}{c} \sigma \\ I_b Y_b \end{array} \middle| \begin{array}{cc} d & \gamma \\ I_d Y_d & I_\gamma Y_\gamma \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} \omega \\ I_a Y_a \end{array} \middle| \begin{array}{cc} b & a \\ I_b Y_b & I_a Y_a \end{array} \right\rangle$$

When one actually calculates the G6js's, there are two tricks. The first one is that since the formula of the G6js is independent of γ_a ,

a clever choice of γ_a would reduce the number of terms in the sum considerably. The second one is that the G6jS is closely related to the corresponding crossing matrix element. We shall use the following definition^(*) for the element of a crossing matrix corresponding to the scattering amplitude $c + \gamma \rightarrow b + a$ exchanging a state "d" by

$$\langle a \gamma \omega | M(c \gamma a b) | d \rho \sigma \rangle \equiv \sum_{\gamma_c, \gamma_b, \gamma_a, \gamma_\gamma, \gamma_d} \mathcal{L}(\gamma, \gamma_\gamma) \mathcal{L}(a, \gamma_a)$$

$$\begin{aligned} & \langle \gamma \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} c \\ \gamma_c \end{smallmatrix} \gamma \rangle \langle \omega \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} \alpha \rangle \langle \rho \begin{smallmatrix} d \\ \gamma_d \end{smallmatrix} | \begin{smallmatrix} c \\ \gamma_c \end{smallmatrix} \alpha^* \rangle \langle \sigma \begin{smallmatrix} d \\ \gamma_d \end{smallmatrix} | \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} \gamma^* \rangle \\ &= \xi(\rho c d a) \xi(\sigma b d \gamma) \frac{N_d}{\sqrt{N_b N_c}} \sum_{\gamma_c, \gamma_b, \gamma_a, \gamma_\gamma, \gamma_d} \end{aligned}$$

$$\langle \gamma \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} c \\ \gamma_c \end{smallmatrix} \gamma \rangle \langle \omega \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} \alpha \rangle \langle \rho \begin{smallmatrix} c \\ \gamma_c \end{smallmatrix} | \begin{smallmatrix} d \\ \gamma_d \end{smallmatrix} \alpha \rangle \langle \sigma \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} | \begin{smallmatrix} d \\ \gamma_d \end{smallmatrix} \gamma \rangle$$

Comparison of this result with Eq. (3-4) gives

$$= \langle a \gamma \omega | M(c \gamma a b) | d \rho \sigma \rangle \times \lambda_{ac\gamma}^\gamma \lambda_{dc\bar{a}}^\rho \lambda_{db\bar{\gamma}}^\sigma \lambda_{aba}^\omega / N_d$$

(*) Some people (e.g., de Swart⁽³⁴⁾) use a different definition which differs by a phase factor

$$\mathcal{P}(\omega, a, b, \alpha) \mathcal{P}(\sigma, d, b, \gamma^*),$$

where $\mathcal{P}$ is defined by

$$\langle \gamma \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} \alpha \rangle = \langle \gamma \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} | \begin{smallmatrix} a \\ \gamma_a \end{smallmatrix} \begin{smallmatrix} b \\ \gamma_b \end{smallmatrix} \rangle \mathcal{P}(\gamma, a, b, \alpha).$$

Note that the crossing matrix elements are independent of the phase convention of λ 's; however, it does depend on the phase convention of the function ξ . We shall adopt the following convention for the case of SU(2)

$$\xi(\pi, m) = (-1)^{T_\pi + m + 1},$$

$$\xi(K, m) = (-1)^{T_K + m + 1} = \xi(\bar{K}, -m);$$

the value of ξ is fixed by the following relation:

$$\xi(a, b, c) \xi(c, m) = (-1)^{J_b - J_a + m}$$

In the case of SU(3), we use the following convention:

$$\xi(a, \nu_a) = (-1)^{I_a + Y_a/2}$$

(A) Some people use different convention, for example, Carruthers and Krisch⁽³⁴⁾ use the following convention

$$\xi(K, m) = \xi(\bar{K}, m) = (-1)^{T_K + m + 1},$$

which does not satisfy the relation $\xi(K, m) = \xi(\bar{K}, -m)$. Therefore they have to distinguish the cases $a + \pi \rightarrow b + \pi$, $a + K \rightarrow b + \pi$ and $a + K \rightarrow b + K$. Our convention has the advantage that the formula for the crossing matrix element can be applied in all these cases.

Then the value of ξ is, using de Swart's notation, $\xi(2) \cdot \xi(3)$.

For example, we have

$$\xi(D, \underline{8}, \underline{8}, \underline{8}) = 1,$$

$$\xi(F, \underline{8}, \underline{8}, \underline{8}) = 1,$$

$$\xi(\underline{10}, \underline{8}, \underline{8}) = -1,$$

$$\xi(\underline{10}, \underline{10}, \underline{8}) = 1.$$

Several useful crossing matrices are given in Appendix I.

In our definition, the relation between the overall coupling constant and the coupling constants among the individual states are

$$g_{\gamma_a \gamma_b \gamma} = \frac{\lambda_{ab\gamma}^{\tau} g_{ab\gamma}^{\tau}}{\sqrt{N_a}} \left(\begin{array}{c|c} a & b \\ \gamma_a & \gamma_b \end{array} \middle| \begin{array}{c} \gamma \\ \gamma \end{array} \right)$$

In particular, the SU(3) coupling constants and the SU(2) coupling constants are related by

$$\frac{\lambda_{ab\gamma}^{\tau} g_{ab\gamma}^{\tau}}{\sqrt{N_a}} \left(\begin{array}{c|c} a & b \\ I_a Y_a & I_b Y_b \end{array} \middle| \begin{array}{c} \gamma \\ I_Y Y_Y \end{array} \right) = \frac{\lambda_{I_a I_b I_Y}}{\sqrt{2 I_a + 1}} g_{I_a I_b I_Y}.$$

e. Linear Approximation

Now we want to discuss how to evaluate the factors D and W which appeared in the bootstrap equations. We shall exploit as much as possible the symmetry properties of these factors and make as little as possible the approximations about the form factors. In the static model, only mass differences between baryons appear in $D_{dc}^{ab\alpha\beta}$ and $W_{c\gamma,ba}^{ad}$. Let us choose them as $(M_a - M_c)$, $(M_b - M_d)$, and $(M_c - M_d)$. Since we assume that the average mass of the baryons is much bigger than any relative mass difference of baryons and mass of meson, it is reasonable to assume that the factors D and W are slowly varying functions of the mass differences; therefore we shall keep only the linear terms. This approximation is equivalent to the effective range approximation⁽⁴⁾. However, in order to get linear terms in W , we need an expansion of D which should be good up to second order.

(4) Let us define $\frac{e^{i\delta_3} \sin \delta_3}{\rho} = \frac{N}{D}$ and let N be the contributions coming from the exchanges of one-baryon states, N is real if $\omega \geq 0$ and it is proportional to p^2 if ω is small. Then $\text{Re} D = N \rho \cot \delta \approx \frac{2 \text{Re} D}{\partial \omega_3} (\omega - \omega_3)$ in the effective range approximation, where ω_3 is the position of the (3,3) resonance. Using the determinantal approximation (see Ch. 4), the relation between the $D^{\text{B.S.}}$ of the Bethe-Salpeter method and the $D^{\text{N/D}}$ of the N/D method is given by

$$D_{bc,df}^{\text{N/D}}(M_a^2) = \delta_{bc,df} - \sum_e g_e^{bd} g_e^{cf} D_{abc}^{\text{B.S.}}{}_{efd}.$$

Let us now write down the explicit formulas of D and K in the static model; they are

$$\begin{aligned}
 D_{dc\gamma}^{aba} &= \text{Re} \frac{1}{m^2 \cdot 12\pi R} \frac{v_{cda}(0)}{v_{aba}(0)} \frac{R}{\pi} \int \frac{v_{ac\gamma}(p) v_{bd\gamma}^*(p) p^4 dp}{\omega (M_a - M_c - \omega)(M_b - M_d - \omega)} \\
 &= \text{Re} \frac{v_{cda}(0) v_{aba}^*(0)}{m^2 \cdot 12\pi^2} \int_m^\infty \frac{v_{ac\gamma}(p) v_{bd\gamma}^*(p) p^3 d\omega}{(\omega - M_a + M_c)(\omega - M_b + M_d)}
 \end{aligned}$$

(in order to get a symmetrical expression, we put both a and b on their mass shells and let the three-momentum of the outgoing meson be zero; the relation $pdp = \omega d\omega$ is used in order to get the second expression.)

$$K_a^{ba} = \frac{1}{m^2 \cdot 12\pi^2} \int_m^\infty \frac{|v_{aba}(p)|^2 p^3 d\omega}{(M_a - M_b - \omega)}$$

Note that the K-integral is more divergent than the D-integral (just counting the powers of ω as ω goes to infinity), therefore K is more sensitive to the form of the cutoff function.

The simplest assumption is, of course, to use the same cutoff function for all interaction vertices. But this assumption would lead to serious difficulties. Since some of the mass differences between varyons might be bigger than m (this is just the case in the classical static model), some of the masses would have negative imaginary parts corresponding to the fact that they can decay strongly, and therefore their vertex structures should be more complicated. We shall assume that we can use

a narrow resonance approximation. Besides, it is consistent to assume that the form factors are independent of the spins. Then we can write

$$\begin{aligned} D_{dc\beta}^{aba} &= D(M_a - M_c, M_b - M_d, M_c - M_d) \\ &= D(M_b - M_d, M_a - M_c, M_d - M_c) \end{aligned}$$

From $D(x, x, z) = D(x, x, -z)$ and $D(x, y, 0) = D(y, x, 0)$, we have

$$(\partial/\partial x) D(x, 0, 0) = (\partial/\partial y) D(0, y, 0) \quad \text{at } x = y = 0,$$

$$(\partial^2/\partial x^2) D(x, 0, 0) = (\partial^2/\partial y^2) D(0, y, 0) \quad \text{at } x = y = 0,$$

$$\text{and} \quad (\partial/\partial z) D(0, 0, z) = 0 \quad \text{at } z = 0.$$

It follows that

$$\begin{aligned} D(x, y, z) \approx & \left[1 + (x+y) \partial/\partial x + (x^2 + y^2) \partial^2/\partial x^2 + z^2 \partial^2/\partial z^2 \right. \\ & \left. + 2xy \partial^2/\partial x \partial y \right] D(0, 0, 0). \end{aligned}$$

Therefore from these two assumptions (linear approximations of D and W plus spin independence of v), we can write

$$D_{dc\beta}^{aba} = D_0 \left[1 + D_1 (M_a + M_b - M_c - M_d) \right],$$

$$W_{c\gamma, ba}^{ad} = W_0 \left[1 + W_1 (M_a - M_c + M_a - M_b) + W_2 (M_a - M_d) \right] / 2M,$$

then the following two relations always hold for any solution (*)

$$\sum_{a,b,\alpha} f_{aba}^2 = D_0 \sum_{a,b,c,d,\alpha,\gamma} f_{aba} f_{ac\gamma} f_{cda} f_{db\bar{\gamma}} C_{J_a J_c}^{J_a J_b}$$

$$\text{and } 2M = \frac{W_0}{\sum_a (2J_a + 1)} \sum_{a,b,c,d,\alpha,\gamma} f_{aba} f_{ac\gamma} f_{cda} f_{db\bar{\gamma}} C_{J_d J_c}^{J_a J_b}$$

$$= (W_0/D_0) \left(\sum_{a,b,\alpha} f_{aba}^2 \right) / \sum_a (2J_a + 1)$$

(*)

To see this, let us rewrite the vertex equation in the following form

$$f_{aba}^2 = D_0 \sum_{c,d,\gamma} \left\{ f_{aba} f_{ac\gamma} f_{cda} f_{db\bar{\gamma}} C_{J_d J_c}^{J_a J_b} \right. \\ \left. [1 + D_1 (M_a + M_b - M_c - M_d)] \right\}.$$

The symmetry property of the coefficient C assures that the terms linear in masses cancel each other after we sum over all a , b and α .

Now it is reasonable to assume that if $M_a - M_c$, $M_b - M_d$ and $M_c - M_d$ are all very small compared with m , the dependence of mass differences of the form factors should be negligible compared with the explicit mass dependence of the propagators of baryons. Then we have

$$D_{dc\beta}^{aba} \approx \frac{1}{m^2 12\pi^2} \int_m^\infty \frac{|v|^2 p^3 d\omega}{(\omega - M_a + M_c)(\omega - M_b + M_d)}$$

$$\approx D_0 \left[1 + \frac{(M_a + M_b - M_c - M_d)}{\langle \omega \rangle} + \eta \frac{(M_a - M_c)^2 + (M_b - M_d)^2 + (M_a - M_c)(M_b - M_d)}{\langle \omega \rangle^2} \right]$$

for $|M_a - M_c| \ll m$ and $|M_b - M_d| \ll m$,

where $D_0 = I_2$,

$$\langle \omega \rangle = I_2 / I_3,$$

$$\eta = I_2 I_4 / I_3^2$$

$$\text{and } 12 (m\pi)^2 I_n = \int_m^\infty \frac{|v|^2 p^3 d\omega}{\omega^n}$$

In our approximation, η must be larger than one according to Schwarz's inequality, but it is always close to one for any reasonable cutoff function. If we assume that

$$|v(p)|^2 = \begin{cases} 1 & \text{if } \omega \leq \Lambda \\ 0 & \text{if } \omega > \Lambda \end{cases},$$

then η , D_0 and $\langle \omega \rangle / m$ can be expressed as functions of Λ / m (see Figs. 3,4,5). Since the cutoff Λ / m is of the order 10, we have $\langle \omega \rangle / m \approx 5$. Therefore the linear approximation is good if $\Delta M \ll \omega \approx 5m$.

We can do the same thing for K-integral, however we prefer to derive it from the identity Eq. (2-6), since it is too sensitive to the form of v . Then we have

$$K_a^{ba} \approx K_0 \left(1 + \frac{M_a - M_b}{\langle \omega \rangle} \right),$$

where K_0 is a phenomenological constant which can not be determined from the Eq. (2-6) alone. Then it is a straight-forward calculation to show that

$$W_{c\gamma,ba}^{ad} = \frac{D_0 K_0}{2M} \left[1 + \left(1 + \frac{\eta}{2} \right) \frac{M_a - M_c + M_a - M_b}{\langle \omega \rangle} + \frac{\eta(M_a - M_d)}{\langle \omega \rangle} \right]$$

In this approximation, D is always positive if $|M_i - M_j| < \langle \omega \rangle$ for all i and j .

Finally, the bootstrap equations can be rewritten in the form:

$$g_{aba} = \sum_{c,d,\gamma} g_{ac\gamma} g_{cd\gamma} g_{db\bar{\gamma}} C \begin{matrix} J_a & J_b \\ J_d & J_c \end{matrix} (1 + a + b - c - d)$$

$$\frac{D_0^2}{W_0} = \frac{1}{2J_a + 1} \sum_{b,c,d,a,\gamma} \left\{ g_{aba} g_{ac\gamma} g_{cd\gamma} g_{db\bar{\gamma}} C \begin{matrix} J_a & J_b \\ J_d & J_c \end{matrix} \right.$$

$$\left. \left[1 + \left(1 + \frac{\eta}{2} \right) (a - c + a - b) + \eta(a - d) \right] \right\} \quad (3-5)$$

where $g \equiv \sqrt{D_0} f$, $W_0 = D_0 K_0/2M$ and $a = M_a/\langle\omega\rangle$.

We want to emphasize that we only need to assume a form factor for the equal mass case in order to evaluate the constant D_0 . We do not make any specific assumption about v for non-equal mass cases except that they should behave in such a way that D 's are slowly varying functions of mass differences and that the imaginary part of the vertex equation is automatically satisfied once we find a solution of Eqs.(3-5). Since the form factors should depend on the solution and cannot be determined before we solve it, this assumption is actually a very natural one.

D_0 , in general, is very sensitive to the form factor v ; for example, in our previous assumption, we have

$$\frac{D_0(\Lambda = 8m)}{D_0(\Lambda = 6m)} = 1.9.$$

On the contrary, the parameter η is essentially cutoff independent, therefore the ratios of coupling constants and the ratios of mass differences are well-defined in the static model. (For example,

$\eta(\Lambda = 8m)/\eta(\Lambda = 6m) = 1.05$.) $\langle\omega\rangle$ has the dimension of mass and depends on v ; one could interpret $\langle\omega\rangle$ as the average energy of meson cloud. Using the linear approximation, the matrix element of the potential matrix $V_{ba, c\gamma}^a$ becomes

$$\sum_d \left[g_{db\bar{\gamma}} g_{cda} C \begin{pmatrix} J_a & J_b \\ J_d & J_c \end{pmatrix} (1 + a - d) \right].$$

How good is the linear approximation? It should be good if we find a solution with all $|a-b| \ll 1$. However, even if this condition is not very well satisfied, the solution would still be very interesting since it would exhibit approximately the general properties of a true solution.

f. The problem of searching for solutions

Let us consider a model with n mass differences (labeling them by x_i , $i=1, \dots, n$) and m coupling constants (g_i , $i = 1, \dots, m$); then we have $(n + m)$ unknowns. If we use the linear approximation, then it is always possible to reduce the number of unknowns from $(n + m)$ to $(m - 1)$. This can be done in the following way: We have n normalization equations which depend on the ratios of the coupling constants and are linear functions of x 's; therefore x 's can be expressed as single valued functions of these ratios. Let us define $t_i = g_i/g_m$ for $i = 1, \dots, m - 1$ and $t_m = 1$. The vertex equations have the form

$$g_i = t_i g_m = f_i(g, x(t)) = g_m^3 f_i(t, x(t)),$$

where we drop the indices in the argument of f_i . Then the bootstrap equations can be rewritten in the form

$$1/g_m^2 = f_i(t, x(t))/t_i \quad i = 1, \dots, m.$$

Now the whole problem reduces to a set of equations involving the ratios of the coupling constants only.

The method discussed in the previous paragraph is only one of many ways to simplify the problem. In some cases, part of the bootstrap equations are particularly simple and they can be used to reduce the number of unknowns. However, no general rules can be given.

Let us assume that after some steps we have r equations ($F_i(Y_1, \dots, Y_r) = 0, i = 1, \dots, r$) left which are very complicated. Now one can solve it with the help of a computer. We shall describe one way of doing it. Let us construct a new function $\Psi(Y_1, \dots, Y_r)$ which has the following property:

$$\Psi = 0 \quad \text{if } F_i = 0 \text{ for all } i,$$

$$\Psi > 0 \quad \text{otherwise}$$

We shall give two examples:

$$\Psi = \sum_{i=1}^r (F_i^2),$$

and

$$\Psi = \sum_{i=1}^r |F_i|.$$

Then we ask the computer to do the following job: To find the zeroes of Ψ in the neighborhood of a given point (in the r -dimensional space Y_i). Theoretically one can "scan" the whole space, that is, to try sufficiently large number of points to make sure that we find all solutions. In the case where r is very large, this "scanning" procedure becomes very impractical. However, in some problems (for example, the problem of broken symmetry), we are particularly interested in finding solutions in the neighborhood of a particular point.

CHAPTER IV

Comparison of the Bootstrap Equations gotten
from the Bethe-Salpeter equation and from the
N/D method^(#).

a. Review of N/D Method⁽²⁴⁻²⁶⁾

Let us use $T(s)$ to denote the scattering matrix for a partial wave of definite angular momentum, parity, etc., as a function of the square of invariant mass. According to the principle of maximal analyticity⁽³⁵⁾, $T(s)$ is an analytic function of s except for certain poles and branch cuts which are determined by the Landau rule⁽³⁶⁾. These branch cuts are usually divided into two classes: the right-hand cuts (RHC) associated with the unitarity condition, and the left-hand cuts (LHC)^(#). From the unitarity condition which is valid along the real s -axis approached from the upper half-plane, we have

$$\text{Im } T = -T^{\dagger} \rho T,$$

or

$$\text{Im } (T^{-1}) = -\rho,$$

where $\rho(s)$ is the phase space factor. The scattering matrix $T(s)$ for a given partial wave can be written as

$$T(s) = D(s)^{-1} N(s),$$

(T , D , and N are n by n matrices for a n -channel system.) The matrix elements of D and N are defined to be analytic functions of s in such a

^(#) We shall follow closely the lectures of Prof. R. E. Cutkosky at the Brandeis summer school (1965).

^(#) This terminology is misleading because what we mean by LHC are all the branch cuts which are not associated with the unitarity condition.

way that the singularities of D and N are exactly RHC and LHC respectively. The poles of $T(s)$ correspond to the zeros of $D(s)$. Then, on the RHC, we have

$$\text{Im } (T^{-1}) = -\rho = \text{Im } (N^{-1} D) = N^{-1} \text{Im } D.$$

so that

$$\text{Im } D = -N \rho.$$

In order to write down the dispersion relations of the matrix elements of D , we have to know what happens when s goes to infinity. Let us assume that we need one subtraction and that the matrix D is normalized to 1 (the unit matrix) at the subtracting point s_0 . Then we have

$$D(s) = 1 - \frac{s - s_0}{\pi} \int \frac{N(s') \rho(s') ds'}{(s' - s_0)(s' - s - i\epsilon)} \quad (4-1)$$

for complex s . The threshold of each integral depends on the corresponding element of ρ . We add an infinitesimal imaginary part to s in the integrand since the physical value of D is obtained by approaching the real s -axis from the upper half-plane. We shall not write down the term $i\epsilon$ explicitly later.

Let us define disc N as the discontinuity of N across the LHC. Since D does not have any LHC by definition, we have

$$\text{disc } N = \text{disc } DT = D \cdot \text{disc } T.$$

Finally we can write down the dispersion relation of N :

$$\begin{aligned}
N(s) &= \frac{1}{2\pi i} \int_{\text{LHC}} \frac{\text{disc } N(s') ds'}{s' - s - i\epsilon} \\
&= \frac{1}{2\pi i} \int_{\text{LHC}} \frac{D(s') \text{disc } T(s') ds'}{s' - s - i\epsilon} \quad (4-2)
\end{aligned}$$

A bound state with mass M corresponds to $\det D(M^2) = 0$ (det means the determinant), and a resonance state with mass M and width Γ corresponds to $\det D((M - i\Gamma)^2) \neq 0$. In order to avoid the complication due to spin and width, we shall neglect them and consider bound states only in the following discussion.

Let $\phi_{ai}(s)$ and $\psi_{aj}(s)$ ($i = 1, \dots, n$) be the eigenvectors of D and D^T (T means transpose) respectively with the eigenvalue $d_a(s)$. Under the assumption that all eigenvalues are different (then all eigenvectors of D are linear independent), we can write D in the form:

$$D_{ij}(s) = \sum_a \phi_{ai}(s) \psi_{aj}(s) d_a(s).$$

Then it follows that:

$$\det D = \prod_a d_a$$

and

$$D^{-1}_{ij} = \sum_a \phi_{ai} \psi_{aj} d_a^{-1}.$$

For a bound state (a), we require $d_a(M_a^2) = 0$ and

($\neq$) We may as well define $\text{Re } \det D(M'^2) = 0$ for a resonance state, note $M' \neq M$; however since the width is very small, we have $M' \approx M$.

($\neq$) We use the following normalizations:

$$\sum_j \psi_{aj} \phi_{a'j} = \delta_{aa'}$$

$$\lim_{s \rightarrow M_a^2} (s - M_a^2) T_{ij} = -g_{ai} g_{aj} = \frac{\phi_{ai}(M_a^2) \left(\sum_k \psi_{ak}(M_a^2) N_{kj}(M_a^2) \right)}{\frac{d d_a}{d M_a^2}}$$

where g_{ai} is the coupling constant corresponding to the state (a). From time reversal invariance, we have

$$\begin{aligned} \phi_{ai}(M_a^2) \left(\sum_k \psi_{ak}(M_a^2) N_{kj}(M_a^2) \right) = \\ \phi_{aj}(M_a^2) \left(\sum_k \psi_{ak}(M_a^2) N_{ki}(M_a^2) \right); \end{aligned}$$

in other words, we have

$$\left(\sum_k \psi_{ak} N_{ki} \right) / \phi_{ai} = K(M_a^2) = \frac{\sum_{k,i} \psi_{ak} N_{ki} \phi_{ai}}{\sum_i (\phi_{ai})^2} \quad (4-3)$$

where K is independent of i . Therefore we have

$$g_{ai} g_{aj} = -\frac{k}{d'} \phi_{ai} \phi_{aj}.$$

From the above equation, we conclude that $\phi_{ai} \propto g_{ai}$ and the bootstrap equations are

$$D_{ij}(M_a^2) g_{aj} = 0 \quad (4-4)$$

$$\sum_i (g_{ai})^2 = -\frac{k}{d'} \left[\sum_i (\phi_{ai})^2 \right] \quad (4-5)$$

So far, we have treated the N/D method exactly. However, in order to get the exact bootstrap equations, we have to put the exact form of disc T in the integral equation of D ; and nobody knows how to do it in a self-consistent way. Therefore people usually use approximations which are the subjects of the next two sections.

b. The determinantal approximation^(13,37)

In the determinantal approximation (DA), we put

$$N = B,$$

where B denotes the contributions from the Feynman graphs where a single particle is exchanged (Born approximation). Under DA, we have

$$D = 1 - \frac{s - s_0}{\pi} \int_{\text{RHC}} \frac{ds' B(s') \rho(s')}{(s' - s_0)(s' - s)}.$$

It follows that

$$T = D^{-1} B.$$

Although the unitarity condition is satisfied, time reversal invariance is not. The degree of violation can be thought as kind of measure of error.

We can write B in the form (see Fig. 6):

$$B_{bc,df} = \sum_e g_{ebd} g_{ecf} C_{efd}^{abc} A_e^{bc,df}(s),$$

where C is the corresponding crossing matrix element and A is the "potential" produced by exchanging particle e. Let us define a function $\mathcal{A}_e$ by

$$\mathcal{A}_e^{bc,df}(s) = \frac{s - s_0}{\pi} \int_{\text{RHC}} \frac{ds' A_e^{bc,df}(s') \rho_{df}(s')}{(s' - s_0)(s' - s)}$$

Then the condition

$$D_{ij}(M_a^2) g_{aj} = 0$$

can be written in the form: (see Fig. 7)

$$g_{abc} = \sum_{d,e,f} \mathcal{D}_e^{bc,df}(M_a^2) g_{ebd} g_{ecf} g_{adf} C_{efd}^{abc} \quad (*)$$

The vertex symmetry requires

$$\mathcal{D}_e^{bc,df}(M_a^2) = \mathcal{D}_f^{ac,de}(M_b^2) = \mathcal{D}_d^{ab,fe}(M_c^2).$$

In order to see whether these conditions are satisfied, let us consider the triangle Feynman diagram (Fig. 7). Let us put the particles b and c on their mass shells, and consider the integral as a function of s_a . We shall assume that there is no anomalous threshold, then the discontinuity across the branch cut is simply given by putting particles d and f on the mass shell⁽³⁸⁾ and is the same as the discontinuity of the function $\mathcal{D}$. If no subtraction is needed then we can identify the $\mathcal{D}$ and the triangle Feynman integral at $s_a = M_a^2$. However we always can define $\mathcal{D}$ as the regularized third order Feynman integral so that the vertex symmetry holds. Therefore the vertex equations are the same as that derived from the Bethe-Salpeter equation under ladder approximation. However, the normalization equation (Eq. 4-5) has a different form although the physical basis (residue of poles) is the same.

(*) The coupling constants defined in this chapter are different from that of last chapter. g_{abc} , g_{bac} , and g_{cab} are related by the symmetries of Clebsch-Gordan coefficients.

Further approximations are needed^(*) in order to preserve the time reversal invariance and to bring the normalization equation to a form similar to that of the Bethe-Salpeter approach.

c. Pole approximation and linear D approximation

In the "pole" approximation (PA), one replaces all LHC by several poles. Let us write

$$\text{disc } T(s) = - \sum_{ij} i \beta_{ij}^e \delta(s - s_e) 2\pi$$

where β^e is a symmetric matrix. Time reversal invariance holds in this approximation⁽²⁵⁾. Then Eq. (4-2) gives

$$N_{ij}(s) = \sum_e D(s_e) \beta_{ij}^e / (s - s_e)$$

and Eq. (4-1) gives

$$D_{ij}(s) = \delta_{ij} - (s - s_0) \sum_e D(s_e) \beta_{ij}^e R_e(s),$$

where

$$R_e(s) \equiv \frac{1}{\pi} \int_{\text{RHL}} \frac{\rho(s') ds'}{(s' - s_e)(s' - s_0)(s' - s)}.$$

It is clear that $R_e(s)$ is not sensitive to s for a sufficiently small region far from the RHC. Then $R_e(s)$ is not sensitive to s_e also and can be replaced by its average value $R(s)$. Therefore it is compatible to make

^(*) For example, if we assume $\beta_{ij}(s) = \beta_{ij}(s_0) b(s)/b(s_0)$.

in addition to PA, a linear D approximation (LDA), which means that we assume constant eigenvectors with eigenvalues linear in s .

The purpose of these approximations is to obtain some simple relations among the coupling constants which may turn out to be insensitive to the approximations used. Let us define a matrix B such that

$$B = \frac{-1}{2\pi i} \int_{\text{LHC}} \text{disc } T(s') ds'$$

Under the LDA alone, we have

$$\begin{aligned} \sum_k \psi_{ak} N_{kj} m_a^2 &= \sum_k \psi_{ak} \frac{1}{2\pi i} \int_{\text{LHC}} \sum_{n,l} \frac{\phi_{nk} \psi_{nl} d_l(s') \text{disc}_l T(s') ds'}{s' - m_a^2} \\ &= - \sum_l d'_a(m_a^2) \psi_{al} B_{lj} \\ &= K_a \phi_{aj}, \end{aligned}$$

or

$$\psi_{ai} = - \frac{K_a}{d'_a} \sum_j B_{ij}^{-1} \phi_{aj}.$$

Using the orthonormal relations of the eigenvectors, we get

$$1 = - \frac{K_a}{d'_a} \sum_{i,j} B_{ij}^{-1} \phi_{aj} \phi_{ai}$$

Comparison with Eq. (4-5) gives the following normalization condition:

$$1 = \sum_{i,j} g_{ai} B^{-1}_{ij} g_{aj}. \quad (4-6)$$

If we further assume PA, then we have

$$B = \sum \beta^e.$$

In the static model, the contribution from the Feynman graph corresponding to Fig. 6 (assuming that particles d and c are mesons) is

$$T = \frac{C_{efd}^{abc} g_e^{bd} g_e^{cf}}{s + m_e^2 - m_b^2 - m_f^2},$$

where C is the corresponding crossing matrix element. Therefore in the static model, we have some simple relations among coupling constants. However, since $D(s)$ is linear only for the value of s which is far from the RHC, the normalization conditions are actually derived under the extra assumption that the effects of the non-linear terms of D cancel each other.

In certain static models, a "reciprocal bootstrap" situation^(8,10,11) occurs in which the main contribution to the attractive "potential" comes from exchange of nearly degenerate particles and also the principal constituents of particle a are nearly degenerate. In this case $\rho(s)$ is nearly proportional to the unit matrix and therefore one would expect that D is nearly hermitian. Then one can assume $\psi_{ai}(m_a^2) \approx \phi_{ai}(m_a^2) \equiv g_{ai} / (\sum_i g_{ai}^2)^{1/2}$. The normalization condition is

$$(\sum_i g_{ai}^2)^2 = - (\sum_{i,j} g_{ai} N_{ij} g_{aj}) / d'.$$

Using the PA and assuming that $S_e - m_a^2$ is sufficiently small to guarantee the LPA in this small region, we obtain the relation

$$\sum_{i,j} g_{ai} N_{ij} g_{aj} = -d' \sum_{i,j} g_{ai} B_{ij} g_{aj} ,$$

so finally we get

$$\left(\sum_i g_{ai}^2 \right)^2 = \sum_{i,j} g_{ai} B_{ij} g_{aj} \quad (4-7)$$

d. Trouble with Approximate N/D Method.

The relations among coupling constants derived in section c are not reliable and should not be taken too seriously. The determinantal approximation satisfies vertex symmetry and unitarity, but violates time reversal symmetry. Although the vertex equations have the same form as that gotten from the Bethe-Salpeter approach, the physical meaning of them is not the same. In DA, we consider only the triangle diagram without any ladder at all while in the Bethe-Salpeter approach we take into account a set of graphs which are explicitly symmetrical. If we use the same Born approximation for the expression of disc T in Eq. (4-2), and solve the N/D equations exactly, then the vertex symmetry is lost. This is because we sum over ladders in the (a) channel but not in the (b) channel. (see Fig. 7)

CHAPTER V

BOOTSTRAP MODELS OF THE (3,3) RESONANCEa. Introduction

As mentioned before, Chew has suggested a static bootstrap model of the (3,3) resonance. In his model, the nucleon (N) and the (3,3) resonance (N^*) are supposed to "bootstrap" each other. He applied the approximate N/D method (that is, assuming the pole approximation and the linear D approximation) to the elastic (p-wave) $N\pi$ scattering amplitude and obtained two identical relations between the coupling constants which belong to the vertices: $N \rightarrow N + \pi$ and $N^* \rightarrow N + \pi$. Although the result is promising, the method he used does not agree entirely with the philosophy of "nuclear democracy" ($\neq$). Because if we want to treat both N and N^* in the same way, then one should consider simultaneously the other two p-wave scattering amplitudes: $N^* + \pi \rightarrow N + \pi$ and $N^* + \pi \rightarrow N^* + \pi$. Besides, there is no reason to neglect the vertex $N^* \rightarrow N^* + \pi$.

We shall adopt the same convention as the one used in sec. d of Chap. III for the effective Hamiltonian. Let us define g_0 , g_1 and g_2 by $g_0 = g_{(N N \pi)}/2$, $g_1 = g_{(N^* N \pi)}/4$ and $g_2 = g_{(N^* N^* \pi)}/4$. If we assume the linear D and pole approximation to this "reciprocal bootstrap" situation, then we would end up with two different equations among the three coupling constants (cf. Eq. (4-7)):

$$\left(4g_1^2 + g_0^2 \right)^2 = \begin{bmatrix} 2g_1 & g_0 \end{bmatrix} A(N) \begin{bmatrix} 2g_1 \\ g_0 \end{bmatrix} \quad (5-1)$$

($\neq$) This term is suggested by M. Gell-Mann (see ref. 5, p. 105.)

$$(g_2^2 + g_1^2)^2 = \begin{vmatrix} g_2 & g_1 \end{vmatrix} A(N^*) \begin{vmatrix} g_2 \\ g_1 \end{vmatrix} \quad (5-2)$$

$$\text{where } A(N) = \begin{vmatrix} (g_1^2 + 4g_2^2)/9 & g_1 (8g_0 + 10g_2)/9 \\ g_1 (8g_0 + 10g_2)/9 & (g_0^2 + 16g_1^2)/9 \end{vmatrix},$$

$$A(N^*) = \begin{vmatrix} (4g_1^2 + 121g_2^2/25)/9 & (5g_0 + 4g_2)g_1/9 \\ (5g_0 + 4g_2)g_1/9 & (4g_0^2 + g_1^2)/9 \end{vmatrix},$$

and the usual multiplication rule of matrices are assumed. From these two equations one can determine the ratios of the g 's. However, it is easy to show that they don't have any solution (see Appendix II). One can try to relax the "reciprocal bootstrap" assumption and then the corresponding relations between the coupling constants are (cf. Eq. (4-6)):

$$1 = \begin{vmatrix} 2g_1 & g_0 \end{vmatrix} A(N)^{-1} \begin{vmatrix} 2g_1 \\ g_0 \end{vmatrix} \quad (5-3)$$

$$1 = \begin{vmatrix} g_2 & g_1 \end{vmatrix} A(N^*)^{-1} \begin{vmatrix} g_2 \\ g_1 \end{vmatrix} \quad (5-4)$$

It is amusing that they don't have any solution either (see Appendix II). If we consider only the $N-\pi$ channel, then these relations reduce to Chew's identity in both cases. We have just demonstrated that the relations gotten from assuming the PA and the LDA are not reliable, and therefore another method, such as that based on the Bethe-Salpeter equation and used here, is necessary.

b. The SU(2) x SU(2) Model

We shall consider a simple bootstrap model which consists of three kinds of particles: pion, nucleon and the (3,3) resonance. The isotropic spin symmetry is explicitly assumed. That means the solution, if one exists, is invariant under the SU(2) x SU(2) transformation (in other words, we do not assume any relation between the isotopic spin symmetry and the rotational symmetry).

Using the crossing matrices appeared in the appendix I, one can write down easily the bootstrap equations: (each term of the equations corresponds to, in the same order, a graph in Fig. 8).

$$f_0 = \frac{1}{9} (f_0^3 + 2 \times 16 f_0 f_1^2 (1-x) + 20 f_1^2 f_2 (1-2x)) \quad (5-5)$$

$$f_1 = \frac{1}{9} f_1 (4 f_0^2 (1+x) + f_1^2 + 5 f_0 f_2 + 4 f_2^2 (1-x)), \quad (5-6)$$

$$f_2 = \frac{1}{9} (5 f_0 f_1^2 (1+2x) + 2 \times 4 f_1^2 f_2 (1+x) + \frac{121}{25} f_2^3), \quad (5-7)$$

$$\begin{aligned} D_0^2/W_0 = W(N) &\equiv \frac{1}{9} \left\{ f_0^4 + 16 f_0^2 f_1^2 (1-x) + 2 \times 16 f_0^2 f_1^2 (1-x - \frac{1}{2}x) \right. \\ &+ 4 f_1^4 (1-2x - \frac{1}{2}x) + 2 \times 20 f_0 f_1^2 f_2 (1-x - \frac{3}{2}x) + \\ &+ 16 f_1^2 f_2^2 (1-2x - \frac{1}{2}x) \left. \right\} \\ &= W(N^*) \equiv \frac{1}{9} \left\{ 4 f_0^2 f_1^2 (1+2x + \frac{1}{2}x) + f_1^4 (1+2x + \frac{1}{2}x) + \right. \\ &+ 2 \times 5 f_0 f_1^2 f_2 (1+x + \frac{3}{2}x) + 4 f_1^2 f_2^2 (1+\frac{1}{2}x) + \\ &+ 2 \times 4 f_1^2 f_2^2 (1+x + \frac{1}{2}x) + \frac{121}{225} f_2^4 \left. \right\} \quad (5-8) \end{aligned}$$

$$= \frac{W(N) + 4 W(N^*)}{5}$$

$$= (f_0^2 + 8f_1^2 + 4f_2^2)/5$$

$$\sqrt{D}_0 = f_0/g_0 = f_1/g_1 = f_2/g_2, \quad x = \frac{M(N^*) - M(N)}{\langle \omega \rangle}$$

Since the assumptions of the static model is satisfied only if the absolute value of x is small compared to one, we shall consider only the case that x satisfies the following restriction: $|x| < 1$. Then it can be shown that these equations have exactly three solutions. We leave the details to Appendix III(a). The results are summarized in Table II, where the properties of each solution are listed in the same column. In order to check whether the solutions are self-consistent or not, we calculate the eigenvalues of the potential matrices $V(a)$'s. As mentioned in Chp. II, the condition of self-consistency implies that in the channels corresponding to the existence of bound states all eigenvalues of the potential matrices except the special one normalized to unit should be much smaller than one. The result shows that two of the three solutions are not self-consistent.

In order to check whether the remaining solution is really self-consistent, we have to check the eigenvalues of the potential matrices which correspond to the channels without bound states. As mentioned in Chap. II, we need a test function. We choose the test function to be the same as the wave function of a bound state, and introduce a parameter z which corresponds to the mass of the bound state. We vary the parameter z of the testing function from -1 to $+1$ and calculate the

TABLE II

The properties of the solutions of the $SU(2) \times SU(2)$ $N - N^*$ Model. In the last two rows, $\lambda(a)$ is the alternate eigenvalue of $V(a)$. For self-consistency,

$$\lambda(a) \ll 1.$$

x	0.311	0.188	0.412
f_0	0.98	0	1.66
f_1	0.552	3	0.626
f_2	0.57	0	-1.36
η	1.25	1.2	1.2
$\lambda(N)$	-0.39	13	0.23
$\lambda(N^*)$	-0.05	4.75	1.95

biggest eigenvalue $\lambda(a)$ in each channel. For example, in the channel corresponding to $I = J = 5/2$, the eigenvalue is $\lambda = (1+z) f_1^2 + \frac{4}{25} (1+z-x) f_2^2 = 0.377 (1+z) - 0.0161$; then the biggest eigenvalue occurs at $z = +1$, i.e., 0.738. The other results are:

$$\begin{aligned} \lambda(I = 3/2, J = 5/2) &= \lambda(I = 5/2, J = 3/2) \\ &= -0.03 \quad \text{at } z = -1, \end{aligned}$$

$$\begin{aligned} \lambda(I = 1/2, J = 5/2) &= \lambda(I = 5/2, J = 1/2) \\ &= 0.057 \quad \text{at } z = +1, \end{aligned}$$

$$\begin{aligned} \lambda(I = 1/2, J = 3/2) &= \lambda(I = 3/2, J = 1/2) \\ &= 0.1 \quad \text{at } z = +1. \end{aligned}$$

The above calculations indicate that this solution is self-consistent. However, there exists a strong attractive potential at the channel $I = J = 5/2$. It is not clear whether this attractive potential is strong enough to form a bound state. It is interesting to mention that in the old "strong coupling" theory, there exists a resonance corresponding to each channel with $I = J = 1/2, 3/2$, etc. Besides, it has been shown^(*) that there does exist a self-consistent solution corresponding to a model with N , N^* and also a $(5,5)$ resonance.

The solution is not sensitive to the values of γ . In Table III, we give the data of the self-consistent solution for several different values of γ .

* Private communication by Dr. F. S. Chen-Cheung.

TABLE III.

The self-consistent solution of the
 $SU(2) \times SU(2)$ $N - N^*$ model as a
 function of η .

η	$f_{0/3}$	$f_{1/3}$	$f_{2/3}$	x
0	0.233	0.171	0.33	0.314
0.25	0.253	0.178	0.3	0.33
0.4	0.266	0.182	0.283	0.337
0.5	0.275	0.184	0.267	0.34
0.6	0.283	0.185	0.256	0.339
0.7	0.291	0.185	0.243	0.338
0.9	0.306	0.186	0.22	0.331
1.0	0.312	0.185	0.21	0.326
1.1	0.318	0.185	0.2	0.32
1.2	0.324	0.184	0.194	0.314
1.3	0.328	0.184	0.186	0.31
1.4	0.333	0.183	0.18	0.3
1.5	0.337	0.182	0.174	0.295
1.75	0.346	0.18	0.16	0.28
2.0	0.354	0.178	0.15	0.265

From experiment, we have $\Delta M = M(N^*) - M(N) = 300$ Mev and
 $g_0^2/3 = 0.08$. (The factor of 3 is due to the fact that the coupling
 constant in our convention differs by a factor of 3 from the usual
 definition.) Since D_0 , $\langle \omega \rangle$ and η are all known functions of the cutoff

(see Figs. 3, 4 and 5). The comparison of f_0 with the experimental value fixes the cutoff: $\Lambda = 9m$ or $\eta = 1.25$.

The ratio $R = \Delta M (\text{calculated}) / \Delta M (\text{exp})$ is plotted as a function of Λ (in BeV) in Fig. 9. From this curve, we have $R = 0.75$ at $\Lambda = 9m = 1.24$ BeV. The value of χ as a function of η is given in Fig. 10.

If only terms in Eq. (5-3) with big coefficients are kept (corresponding to the graphs in Fig. 11), the general features of the exact solution are retained by the "first approximation" solution. The ordinary simplified "reciprocal bootstrap model"⁽⁸⁾ corresponds to dropping even more terms (given by the graphs within the parentheses in Fig. 11). This second approximation is fortuitously still quite good, but physically it is somewhat ambiguous, since owing to symmetry, the term omitted from the first line has exactly the same magnitude as the remaining term. We compare these two approximations with the exact solution in Table IV, where we write $\gamma_1 = f_0^2 / f_1^2$ and $\gamma_2 = f_2^2 / f_1^2$.

A comparison of this model with the $SU(4)$ model is given in the next section.

TABLE IV

Comparison of exact and approximate
solutions in the $SU(2) \times SU(2)$ model.

	<u>Exact Solution</u>	<u>First Approximation</u>	<u>Second Approximation</u>
η	1.25	1.3	1.315
Λ (BeV)	1.24	1.56	1.66
R	0.775	0.93	0.985
f_0^2	0.96	1.52	1.72
γ_1	3.14	2.81	2.14
γ_2	1.06	0	0
λ (N)	-0.39	-1/2	0
λ (N^*)	-0.05	---	---
x	0.311	0.48	0.304

c. The $SU(4)$ Model

In this model, we assume the solution satisfies the $SU(4)$ symmetry which includes the group $SU(2) \times SU(2)$ as a subgroup. This idea is similar to the one used by Wigner in nuclear physics⁽³⁹⁾. We may assign N^* and N to the representation 20, and assign mesons π, ρ and η to 15.⁽⁴⁾ (The number denotes the number of the states within the irreducible representation.)

(4) We have in mind here an $SU(4)$ model which is analogous to the $SU(6)$ model of Capps⁽¹⁵⁾, Belinfante and Cutkosky⁽¹⁶⁾. In other words, we assume that the spin (one) of the vector meson ρ combines with the orbital angular momentum (one) to form a **zero** total angular momentum.

The wave functions for N and N^* that result from this assignment may be written in the form (we leave the details and some other discussion about $SU(4)$ in Appendix III-b.)

$$\psi_{(N)} = \frac{1}{\sqrt{63}} \left[5 (N\pi) + \sqrt{3} (N\rho + N\eta) + 4\sqrt{2} (N^*\pi) \right],$$

$$\psi_{(N^*)} = \frac{1}{\sqrt{63}} \left[5 (N^*\pi) + \sqrt{15} (N\rho + N\eta) + 2\sqrt{2} (N\pi) \right]$$

Since N and N^* are assumed to be degenerate in this model, we have $x = 0$. The coupling constants are related by the Clebsch-Gordan coefficients, leading to $\gamma_1 = \gamma_2 = 25/8$. The probability (averaged over all baryons) that either ρ or η appears in the meson cloud of baryons is $6/15$, (Because in the representation 15, 6 states correspond to ρ and η .) While the probability that N contains $N + \rho$ or $N + \eta$ is $6/63$, which is quite small compared to the corresponding probability for N^* ($30/63$). We should therefore expect that in comparing the $SU(2) \times SU(2)$ and $SU(4)$ models, the values of γ_1 would not differ much, but the values of γ_2 would be quite different; this is really the case, and also agrees with the fact that $x > 0$. In Fig. 12 we compare the self-consistent $SU(2) \times SU(2)$ solution with the $SU(4)$ solution using the notation $r_i = \gamma_i(SU(2) \times SU(2)) / \gamma_i(SU(4))$. The experimental values of the masses of ρ and η are much bigger than that of pion (763 MeV and 548 MeV respectively). The actual situation might therefore be in between the $SU(4)$ and $SU(2) \times SU(2)$ models.

d. The $SU(2) \times SU(3)$ Model

In this model, we assume that the internal symmetry is the exact $SU(3)$ symmetry^(40,60). The baryons N, Λ, Σ and Ξ are assigned to the representation $\underline{8}$; Ω , Υ^* , Ξ^* , and N^* to $\underline{10}$; and the mesons π , K , $\bar{K}$ and η are assigned to $\underline{8}$. We use the averaged value (over the experimental masses of the states within the same irreducible representation) for the masses M , M^* and m^2 (^(*)) (i.e., $M = 1154$ MeV, $M^* = 1385$ MeV, $m = 412$ MeV), which correspond to $\underline{8}$ (baryon), $\underline{10}$ (baryon) and $\underline{8}$ (Meson) respectively.

We define $d = \sqrt{D_0}$, $f_F 5^{-1/2} = d \ g_{8,88}^A/4$, $f_D = d \ g_{8,888}^S/4$, $f_1 = d \ g_{10,88}/\sqrt{40}$ and $\sqrt{2} \ f_2 = d \ g_{10,108}/\sqrt{40}$, where A and S denote respectively the antisymmetric and symmetric coupling of the vertex $\underline{8}$ (baryon) $\rightarrow \underline{8}$ (baryon) + $\underline{8}$ (meson). The mixing angle θ is defined by $\theta = \tan^{-1} (g_{8,88}^A / g_{8,88}^S)$. We also write: $G^2/D_0 = (g_{8,88}^A)^2 + (g_{8,88}^S)^2$, $\gamma_1 = G^2/f_1^2$ and $\gamma_2 = f_2^2/f_1^2$.

The following bootstrap equations are obtained: (each term corresponds to, in the same order, a graph in Fig. 13).

$$f_D = f_D^3/10 + f_D f_F^2/30 + 4(1-x) f_D f_1^2/3 + 2(1-x) f_F f_1^2/3 + (5/6) (1-2x) f_1^2 f_2, \quad (5-9)$$

$$f_F = -f_F^3/30 + f_D^2 f_F/6 + 10(1-x) f_D f_1^2/3 + 25(1-2x) f_1^2 f_2/6 \quad (5-10)$$

(*) Note that we average over the square of the masses in the case of meson.

$$f_1 = f_1 \left[(4/15)(1+x) f_D (f_D + f_F) + f_1^2/12 + f_2(f_D + f_F)/3 + 2(1-x) f_2^2/3 \right] \quad (5-11)$$

$$f_2 = (1+2x) (f_D + f_F) f_1^2/6 + 2(1+x) f_1^2 f_2/3 + 11 f_2^3/10 \quad (5-12)$$

$$\begin{aligned} D_0^2/W_0 &= W(8) \equiv f_D^4/10 - f_F^4/150 + f_F^2 f_D^2/15 \\ &+ (2/3) (1-\eta x) f_1^2 f_D (f_D + f_F) \\ &+ (4/3) (1-x-\eta x/2) f_D f_1^2 (f_D + f_F) \\ &+ (5/24) (1-2x-\eta x) f_1^4 + (5/3) (1-x-3\eta x/2) f_2 f_1^2 (f_D + f_F) \\ &+ (5/3) (1-2x-2\eta x) f_1^2 f_2^2 \\ &= W(10) \equiv (4/15) (1+2x+2\eta x) f_1^2 f_D (f_D + f_F) \\ &+ f_1^4 (1+2x+\eta x)/12 + (2/3) (1+x+3\eta x/2) f_2 f_1^2 (f_D + f_F) \\ &+ (2/3) (1+\eta x) f_1^2 f_2^2 + (4/3) (1+x+\eta x/2) f_1^2 f_2^2 \\ &+ 22 f_2^4/10 \\ &= \frac{2W(8) + 5W(10)}{7} \quad (5-13) \\ &= (2/7) (f_D^2 + f_F^2/5 + 5 f_1^2 + 5 f_2^2) \end{aligned}$$

We found three solutions (see Appendix III c). The result is given in Table V, which indicates that two of them are not self-consistent. We check the eigenvalues of the potential matrices corresponding to the first solution and found the following result: (The meaning of the

TABLE V.
Comparison of Solutions of the
SU(2) x SU(3) Model.

x	0.229	0.138^a	0.298^a
$f_F 5^{-1/2}$	0.571	0	0.661
f_D	0.745	0	1.44
f_1	0.633	$3(2)^{1/2}$	0.778
$f_2 \sqrt{2}$	0.716	0	-1.33
$\lambda(8)$	-0.44, -0.04	-5, 11.9	-0.15, 0.46
$\lambda(10)$	-0.03	4.55	1.75

^a Evaluated at $\gamma = 1.1$.

notations are similar to those of the sec. a.)

$$\begin{aligned}
 \lambda(5/2, \underline{35}) &= 0.491 \text{ at } z = 1, \\
 \lambda(5/2, \underline{27}) &= 0.0039 \text{ at } z = -1, \\
 \lambda(5/2, \underline{10}) &= -0.035 \text{ at } z = -1, \\
 \lambda(5/2, \underline{8}) &= 0.024 \text{ at } z = -1, \\
 \lambda(3/2, \underline{35}) &= -0.0215 \text{ at } z = -1, \\
 \lambda(1/2, \underline{35}) &= 0.02 \text{ at } z = -1, \\
 \lambda(3/2, \underline{8}) &= 0.084 \text{ at } z = -1, \\
 \lambda(3/2, \underline{1}) &= 0.038 \text{ at } z = -1,
 \end{aligned}$$

$$\lambda(3/2, \underline{10}^*) = -0.0076 \quad \text{at } z = -1,$$

$$\lambda(1/2, \underline{1}) = 0.15 \quad \text{at } z = -1,$$

$$\lambda(1/2, \underline{10}^*) = 0.34 \quad \text{at } z = +1,$$

$$\lambda(3/2, \underline{27}) = 0.468 \quad \text{at } z = +1,$$

$$\lambda(1/2, \underline{27}) = 0.021 \quad \text{at } z = -1,$$

$$\lambda(1/2, \underline{10}) = 0.085 \quad \text{at } z = -1.$$

We conclude that this solution is self-consistent.

In order to compare this result with experiment, we want to remark that in the relativistic theory, one should use the Yukawa-type coupling in the $N - N - \pi$ vertex, since this type of coupling is renormalizable while the derivative-type coupling is not. However, it makes no difference in the static model. It is well-known that in the static model the coupling constant g in the case of derivative-type coupling is related to the relativistic coupling constant g_{rel} by (up to a dimensionless proportional constant which depends on the convention)

$$(g_m)^2 = g_{\text{rel}}^2 / R_0,$$

where R_0 is the ratio of the nucleon mass to the pion mass. The numerical value of R_0 is 6.7 in the $SU(2) \times SU(2)$ model but 2.8 in the

$SU(2) \times SU(3)$ model. We compare the coupling constant $g_{(NN\pi)}$ which is given by $(1/\sqrt{5}) (3f_D + f_F) D_0^{-\frac{1}{2}}$ with the modified experimental value

$$\frac{1}{3} \left[\frac{g_{(NN\pi)}}{2} \right]^2 = 0.08 \frac{6.7}{2.8}. \quad \text{We found that } g_{NN\pi} \text{ would fit the observed}$$

value if $\Lambda = 3.56 \text{ m} = 1.47 \text{ BeV}$. Note that in ordinary units (BeV)

this value of Λ is close to the cutoff required in the $SU(2) \times SU(2)$ model.

At this cutoff, the mixing angle is 37.5° to be compared with 41.8° in the $SU(6)$ model and to 33° in the Chew-type $SU(2) \times SU(3)$ model (see next section). The value of x and $R = \frac{(M^* - M)_{\text{cal.}}}{(M^* - M)_{\text{exp.}}}$ as functions of η and Λ are given in Figs. 10 and 9 respectively. In Table VI, the solution at several different values of η are given.

TABLE VI

The self consistent solution of the
 $SU(2) \times SU(3)$ Model as a function of η .

η	x	f_F	f_D	f_1	f_2	$g_{(NN\pi)}/2$	θ°
1.5	0.216	1.323	0.793	0.634	0.449	0.829	36.76
1.4	0.22	1.313	0.783	0.634	0.462	0.819	36.95
1.3	0.223	1.302	0.772	0.635	0.475	0.81	37.1
1.2	0.226	1.291	0.76	0.634	0.489	0.798	37.24
1.1	0.228	1.279	0.747	0.634	0.504	0.788	37.4
1.0	0.23	1.267	0.734	0.633	0.519	0.777	37.76
0.9	0.231	1.254	0.72	0.631	0.537	0.764	37.95
0.8	0.232	1.24	0.705	0.628	0.555	0.751	38.2
0.7	0.232	1.226	0.689	0.625	0.573	0.737	38.5
0.6	0.231	1.213	0.674	0.622	0.591	0.723	38.84
0.5	0.23	1.2	0.658	0.618	0.609	0.71	39.25

If one neglects some terms in the same way as was done in the $SU(2) \times SU(2)$ case (the underlined graphs in Fig. 12 represent the terms which are not neglected in the first approximation), one obtains

$\tan \theta = 5^{1/2} (1 + 6^{1/2})^{-1}$ or $\theta = 33^\circ$. This value is known to arise in a Chew-type theory (9-11). We compare the exact and approximate solutions in Table VII. A detailed comparison of this solution and the $SU(2) \times SU(2)$ solution is given in Table VIII.

TABLE VII.

Comparison of exact and approximate
solutions in the $SU(2) \times SU(3)$ model.

	<u>Exact Solution</u>	<u>First Approximation</u>	<u>Second Approximation</u>
η	1.08	1.107	1.117
Λ (BeV)	1.47	1.8	1.91
R	1.04	1.22	1.28
Mixing angle	37.5°	33°	33°
G^2	0.876	1.51	1.79
γ_1	2.19	1.94	1.62
γ_2	0.65	0	0
$g(NN\pi)^2 D_0/4$	0.616	1.05	1.25
$\lambda(8)$	-0.44, -0.04	-1/2, -0.21	-0.42, 0
$\lambda(10)$	-0.03	---	---
x	0.229	0.441	0.213

TABLE VIII

Comparison of the SU(2) x SU(2) and
SU(2) x SU(3) solutions. N = nucleon, $N^* = (3,3)$
resonance.

	Λ (BeV)	η	$g_{(N^* N \pi)} / g_{(N N \pi)}$	$g_{(N^* N^* \pi)} / g_{(N^* N \pi)}$	R
SU(2) x SU(2)	1.24	1.25	1.128 ^(*)	1.03	0.775
SU(2) x SU(3)	1.47	1.08	1.136 ^(*)	1.27	1.04

The experimental values of each state within the SU(3) IR's⁽⁴¹⁾ are
(in units of MeV):

$$\begin{aligned} &\Omega(1675), \Xi^*(1530), Y_1^*(1385), N^*(1238), \\ &\Xi(1318), \Sigma(1193), \Lambda(1115), N(939), \\ &\eta(538), K(495), \pi(137). \end{aligned}$$

The SU(2) coupling constants can be expressed in terms of the
SU(3) coupling constants. Let us write

$$\begin{aligned} g_{(a b \alpha)} &= g_{10,108} A(a b \alpha) + g_{10,88} B(a b \alpha) \\ &+ g_{8,88}^S C(aba) + g_{8,88}^A D(ab\bar{a}) \end{aligned}$$

The values of A, B, C, and D are given in Table IX.

(*) The experimental value is (approximately) 1.4⁽⁸⁾. In SU(4) theory, we have $g(N N \pi) : g(N^* N \pi) : g(N^* N^* \pi) = 1 : 4\sqrt{2}/5 (\approx 1.13) : 2$.

TABLE IX

Numerical values of A, B, C and D

<u>a</u>	<u>b</u>	<u>a</u>	<u>A(aba)</u>	<u>B(aba)</u>	<u>C(aba)</u>	<u>D(aba)</u>
Ω^-	Ω^-	η	$\sqrt{5}/10$			
Ξ^*	Ω	K	$\sqrt{5}/10$			
Ξ^*	Ξ^*	π	$\sqrt{10}/20$			
Ξ^*	Ξ^*	η	$\sqrt{10}/20$			
Y_1^*	Ξ^*	K	$\sqrt{10}/10$			
Y_1^*	Y_1^*	π	$\sqrt{10}/10$			
Y_1^*	Y_1^*	η	---			
N^*	Y_1^*	K	$\sqrt{10}/10$			
N^*	N^*	π	1/2			
N^*	N^*	η	$-\sqrt{5}/10$			
Ω	Ξ	$\bar{K}$		$\sqrt{10}/10$		
Ξ^*	Ξ	π		$\sqrt{5}/10$		
Ξ^*	Ξ	η		$-\sqrt{5}/10$		
Ξ^*	Σ	$\bar{K}$		$\sqrt{5}/10$		
Ξ^*	Λ	$\bar{K}$		$-\sqrt{5}/10$		
Y_1^*	Ξ	K		$-\sqrt{5}/10$		
Y_1^*	Σ	π		$\sqrt{5}/10$		
Y_1^*	Σ	η		$-\sqrt{30}/20$		
Y_1^*	Λ	π		$\sqrt{30}/20$		
Y_1^*	N	$\bar{K}$		$-\sqrt{5}/10$		
N^*	Σ	K		$-\sqrt{5}/5$		
N^*	N	π		$\sqrt{5}/5$		
Σ	Σ	π		1/2		
Σ	Λ	π				$-\sqrt{30}/20$
Σ	N	$\bar{K}$		$-3\sqrt{5}/20$		1/4

<u>a</u>	<u>b</u>	<u>a</u>	<u>A(aba)</u>	<u>B(aba)</u>	<u>C(aba)</u>	<u>D(aba)</u>
Λ	N	$\bar{K}$			$\sqrt{5}/20$	$1/4$
N	N	π			$3\sqrt{5}/20$	$1/4$
Ξ	Ξ	π			$-3\sqrt{5}/20$	$1/4$
Ξ	Ξ	η			$\sqrt{5}/20$	$1/4$
Ξ	Σ	$\bar{K}$			$3\sqrt{5}/20$	$1/4$
Ξ	Λ	$\bar{K}$			$-\sqrt{5}/20$	$-1/4$
Σ	Σ	η			$-\sqrt{30}/20$	
Λ	Λ	η			$\sqrt{10}/20$	
N	N	η			$\sqrt{5}/20$	$-1/4$

e. The SU(6) model^(15,16)

This model is a natural generalization of the SU(4) model. We assign the baryons to the representation 56 and the mesons to 35. Since SU(6) contains SU(2) x SU(3) as a subgroup, the multiplets of an irreducible representation can be classified according to its SU(2) x SU(3) content. The 56 includes a $J = 1/2$ baryon octet (B_8) and a $J = 3/2$ baryon decuplet (B_{10}). The 35 includes a p-wave pseudoscalar meson octet (P_8), a p-wave pseudoscalar meson singlet (P_1) and a vector meson octet (V_8) which have $l = 1$, $J = 0$ (l and J are orbital and total angular momentum respectively). The wave functions for B_8 and B_{10} can be written in the form⁽¹⁵⁾

$$\begin{aligned} \psi(B_8) &= \sqrt{2/5} \left[\sqrt{5/9} (P_8 B_8)_S + \sqrt{4/9} (P_8 B_8)_A \right] + \\ &\quad \sqrt{2/15} (V_8 B_8)_A + \sqrt{4/9} (P_8 B_{10}) + \sqrt{1/45} (P_1 B_8) \\ \psi(B_{10}) &= \sqrt{8/45} (P_8 B_8) + \sqrt{4/9} (P_8 B_{10}) + \sqrt{4/15} (V_8 B_{10}) + \\ &\quad \sqrt{1/9} (P_1 B_{10}) \end{aligned}$$

where the subscripts s and d denote the symmetric and antisymmetric octets. Therefore we have: $x = 0$, $\gamma_1 = 9/4$, $\gamma_2 = 5/4$ and $\theta = \tan^{-1} \sqrt{4/5} = 41.8^\circ$.

The fact that $x \approx 0.23$ in the $SU(2) \times SU(3)$ model is smaller than that of the $SU(2) \times SU(2)$ model may be explained in this way: In the $SU(6)$ model, the pseudoscalar meson octet appears in the meson cloud of baryons with an average probability of $24/35$, which is closer to unity than is the corresponding value ($3/5$) of the $SU(4)$ model. In other words the symmetry violation in the case of the $SU(2) \times SU(3)$ theory is less serious than that of the $SU(2) \times SU(2)$ theory. A comparison between the $SU(6)$ solution and the $SU(2) \times SU(3)$ solution is given in Fig. 14, where r_1 and r_2 have the same definition as in sec. b., and $r_3 = \sqrt{5/4} \tan \theta$. The ratios among $g_{N\pi\pi}$, $g_{N^*\pi\pi}$ and $g_{N^*\pi\pi}$ are the same as those in $SU(4)$ theory. (*)

f. Summary

As a check to the validity of the identity Eq. (4-7) gotten by using the linear D and pole approximation, we plot in Fig. 15 the ratios of the left-hand sides to the right hand sides of Eqs. (5-1) and (5-2) as R_1 and R_2 , respectively. (The case of $SU(2) \times SU(3)$ can be defined in the same way). The results show that these identities are badly violated.

The Bethe-Salpeter approach possesses an advantage over the N/D method for approximate bootstrap calculations, because the crossing symmetry can be incorporated more completely. The result in this model agrees very well with the empirical data. The calculation in Sec. d shows that the baryon supermultiplet (the broken 56) emerges automatically from the bootstrap dynamics even with a maximum perturbation from the $SU(6)$ symmetry (because the vector mesons and the pseudoscalar meson singlet are missing).

(*) This is not surprising since in both cases we use the Young diagram to represent the baryons.

CHAPTER VI

A Bootstrap Model of Y_1^* , Σ , Λ and π .

In this chapter, we shall consider a static bootstrap model which includes the three lightest baryons with strangeness minus one (Σ , Λ and Y_1^*). Since the baryons with strangeness zero are left out, this model is not a realistic one. There are five coupling constants and two mass differences. The bootstrap equations of this model are more complicated than those encountered before. Instead of trying to exhaust all possible solutions as we did before, we define a test function as a function of the coupling constants and the mass differences which is positive definite in general but zero at the position of a solution; then we choose at random some input position and use a computer to search for a new position at which the test function is a local minimum.

We found one solution in this way, which has the property that $m_{\Sigma} = m(Y_1^*)$, $g(Y_1^* Y_1^* \pi) : g(Y_1^* \Sigma \pi) : g(\Sigma \Sigma \pi) = \sqrt{10} : \sqrt{2} : 2$ and $g(Y_1^* \Lambda \pi) : g(\Sigma \Lambda \pi) = -\sqrt{2} : 1$: which indicates that this solution actually satisfies a higher symmetry than $SU(2) \times SU(2)$. The symmetry of this solution can be interpreted by a simple model in the following way: In this model, the baryons have an intrinsic angular momentum $1/2$; the total spin J of a baryon is the sum of this intrinsic spin $S = 1/2$ and an "orbital" angular momentum L which is an integer. We further assume that the properties of the baryons depend on L only, e.g., the masses of the baryons which have the same L are degenerate. The symmetry property of this model enables us to simplify the bootstrap equations and we need only pay attention to the "orbital" angular momentum of the baryons. The symmetry group of this model is $SU(2) \times SU(2) \times SU(2)$.

Let us apply this model to the case of Y_1^* , Σ , Λ and π . We shall assign Λ to the state with $L = 0$ and $I = 0$, Y_1^* with $L = I = 1$, $J = 3/2$, and Σ with $L = I = 1$, $J = 1/2$. The bootstrap equations involve one mass difference $\Delta = M(L = 1) - M(L = 0)$ and two coupling constants. Let us define $x = \frac{\Delta}{\langle \omega \rangle}$, $f = \sqrt{D_0} \epsilon(1, 0 \pi)^{1/3}$ and $g = \sqrt{D_0} \epsilon(1, 1 \pi)^{1/3}$; then the bootstrap equations are (corresponding to Fig. 16);

$$f = f^3 + g^2 f (1-x),$$

$$g = 2f^2 g(1+x) + (1/4)g^3,$$

$$D_0^2/W_0 = W(0) \equiv 9f^4 (1 - (2+\eta)x) + 9f^2 g^2 (1 - (2+2\eta)x)$$

$$= W(1) \equiv f^4 (1 + (2+\eta)x) + 2f^2 g^2 (1 + (1 + \frac{\eta}{2})x) + f^2 g^2 (1 + \eta x) + g^4/4$$

$$\left[= \frac{9}{10} (2f^2 + g^2) \right].$$

Considering x as a parameter, the vertex equations can be solved exactly: $f^2 = \frac{(4x-3)}{(8x^2-7)}$ and $g^2 = \frac{4(2x+1)}{(7-8x^2)}$. The requirements $f^2 \geq 0$ and $g^2 \geq 0$ give a restriction on the possible values of x : $3/4 \geq x \geq -1/2$.

Substituting the above expressions of the coupling constants into the normalization condition, we get the following equation ($\eta = 1 + \epsilon$):

$$F(x) \equiv (80 + 48\epsilon) x^3 + (32 + 8\epsilon) x^2 - (91 + 33\epsilon) x + 14 = 0,$$

which has one solution in the region $3/4 \geq x \geq -1/2$. The solution of $F(x) = 0$ can be expressed in the form

$$x = \lim_n x_n, \quad x_0 = \frac{14}{91 + 33\epsilon},$$

$$x_{n+1} = x_0 + 8x_n^2 \frac{(10 + 6\epsilon)x_n + 4 + \epsilon}{91 + 33\epsilon}, \quad n = 0, 1, 2, \dots$$

This formula gives a sufficiently accurate result at $n = 2$. The properties of the solution is given in Table X.

TABLE X

The properties of the solution of the
SU(2) x SU(2) x SU(2) Model.

ϵ	x	f^2	g^2	λ ($L = T = 1$)	f	g
0	0.168	0.344	0.789	-0.0578	0.586	0.888
0.1	0.161	0.347	0.779	-0.0558	0.589	0.883

The eigenvalues of the effective potential matrices corresponding to the channels without bound states are: $\lambda(T=2, L=2) = \alpha + \beta$, $\lambda(1,0) = \lambda(0,1) = -\alpha - 2\beta$, $\lambda(2,0) = \lambda(0,2) = \alpha - 2\beta$ and $\lambda(1,2) = \lambda(2,1) = -\alpha + \beta$; where we define α and β by $\alpha = f^2(1+z)$ and $\beta = g^2(1+z-x)/4$, and z is the parameter explained before. The result indicates a strong attractive potential at the channel $T = L = 2$. At $\eta = 1.1$, we have $\alpha + \beta = 0.512 + 0.544 z \approx 1$ at $z \approx 1$, and it is not clear whether this solution is self-consistent.

The coupling constants of the $(\Lambda, \Sigma, Y_1^*, \pi)$ model can be expressed in terms of $g_{1,0\pi}$ and $g_{1,1\pi}$:

$$g_{(Y_1^* Y_1^* \pi)} = \frac{\sqrt{10}}{3} g_{(1,1\pi)} = 2.79 \sqrt{D_0},$$

$$g_{(\Sigma \Lambda \pi)} = \frac{\sqrt{6}}{3} g_{(1,0\pi)} = -1.442 \sqrt{D_0},$$

$$g(Y_1^* \Sigma \pi) = \frac{\sqrt{2}}{3} g(1, 1\pi) = 1.25 \sqrt{D_0},$$

$$g(\Sigma \Sigma \pi) = \frac{2}{3} g(1, 1\pi) = 1.766 \sqrt{D_0},$$

$$g(Y_1^* \Sigma \pi) = \frac{\sqrt{12}}{3} g(1, 0\pi) = 2.03 \sqrt{D_0} \text{ at } \zeta = 0.1.$$

We should remark that this symmetry is due to the bootstrap dynamics and there is nothing fundamental about the $SU(2) \times SU(2) \times SU(2)$ model just described, although that model does give a short cut of calculation.

It is interesting that we can choose the signs of these coupling constants to be the same as the corresponding ones in the $SU(3)$ model of N^* , even the magnitudes are of the same order if we assume the same cutoff. Let us write

$$r_{(ab\gamma)} = \frac{g_{(ab\gamma)} [\text{this model}]}{g_{(ab\gamma)} [SU(3) \text{ model}]} \quad (\text{at } \eta = 1.1)$$

then we have $r(Y_1^* Y_1^* \pi) = 1.95$, $r(Y_1^* \Sigma \pi) = 1.4$, $r(Y_1^* \Lambda \pi) = 1.85$, $r(\Sigma \Sigma \pi) = 1.54$ and $r(\Sigma \Lambda \pi) = 1.77$.

Let us now discuss the problem of self-consistency. For the channels corresponding to the bound states, we have ($\eta = 1.1$) $\lambda(\Lambda) = -0.67$, $\lambda(\Sigma) = -0.79$, -0.0558 , $\lambda(Y_1^*) = -0.21$, -0.0558 . Note that some new eigenvalues appear. For the channels without bound states no new eigenvalues come out; and the results are: $\lambda(T = 2, J = 5/2) = \alpha + \beta$, $\lambda(1, 5/2) = -\alpha + \beta$, $\lambda(0, 5/2) = \alpha - 2\beta$, $\lambda(2, 1/2) = \alpha - 2\beta$ and $-\alpha + \beta$, $\lambda(2, 3/2) = \pm \alpha + \beta$ and $\lambda(0, 3/2) = \pm \alpha - 2\beta$. The results indicate that there are strong attractive potentials at the channels

($T=2, J=5/2$) and ($T=2, J=3/2$) as expected since they correspond to the $T=L=2$ channel of the $SU(2) \times SU(2) \times SU(2)$ model.

A similar concept can be applied to isotopic spin symmetry.

For example, let us consider a model which includes $\pi, \Lambda, \Sigma, Y_1^*$, and Y_2^* ; then there exists at least one solution which has the following property: $M(\Lambda)=M(\Sigma)$, $M(Y_1^*)=M(Y_2^*)$, $g(Y_2^* Y_2^* \pi) : g(Y_2^* Y_1^* \pi) : g(Y_1^* Y_1^* \pi) = 3:1:-\sqrt{5}$, $g(Y_2^* \Sigma \pi) : g(Y_1^* \Sigma \pi) : g(Y_1^* \Lambda \pi) = \sqrt{5}:-1:\sqrt{2}$, $g(\Sigma \Sigma \pi) : g(\Sigma \Lambda \pi) = \sqrt{2}:1$. Actually, if we write $[M(Y_1^*) - M(\Sigma)] / \langle \omega \rangle = x$, $\sqrt{D_0} = d$, $g(Y_2^* Y_2^* \pi) = 3\sqrt{2}f_2 d$, $g(Y_1^* \Sigma \pi) = -2f_1 d$, $g(\Sigma \Lambda \pi) = \sqrt{2}f_0 d$, then the f 's and x would satisfy the Eq's (5-5), (5-6), (5-7), and (5-8) which are the bootstrap equations of the $\pi-N-N^*$ model. This model is similar to the Pais' doublet scheme^(*). However, although the prediction that Σ and Λ are degenerate is good ($[M(\Sigma) - M(\Lambda)]^2 / [M(\Sigma) + M(\Lambda)]^2 = 0.5\%$), the prediction that there exists a Y_2^* near to Y_1^* contradicts with experiment.

(*) See J. J. Sakurai: "Invariance Principles and Elementary Particles" (pp. 287-288, Princeton University Press, Princeton, N.J., 1964) for references and an interesting discussion.

CHAPTER VII

A Static Bootstrap Model of Baryons
with Non-Degenerate Mesons

a. Introduction

In all of our previous models, we assume that the mesons are degenerate. However, if we want to consider a model which includes baryons with different strangeness, then the K-meson is needed. Now, we know that the K-meson is much heavier than the pion (495 MeV vs 140 MeV). Therefore, we need a static model with non-degenerate mesons in order to get a better description of the physical situation.

Since we do not bootstrap the mesons, the mass differences among the mesons are treated as parameters. The formulas of D and W have to be modified. The D integral corresponding to Fig. 2(a) can be expressed as

$$D_{dc\gamma}^{aba} = \int_0^{\Lambda} \frac{K^4 dK}{\omega_{\gamma} (\omega_{\gamma} - M_a + M_c) (\omega_{\gamma} - M_b + M_d)}$$

$$\omega_{\gamma} = \sqrt{K^2 + M_{\gamma}^2}.$$

We shall expand the integrand in terms of the mass differences of baryons and mesons. We shall assume the same cutoff for all different mesons; besides, we introduce another parameter ξ in order to take into account the effect of the outgoing meson α . Then we have (higher order terms are omitted):

$$D_{dc\gamma}^{aba} = \int_0^{\Lambda} \frac{k^4 dk}{\omega_o^3} \left[1 + \frac{M_a + M_b - M_c - M_d}{\omega_o} - \frac{3}{2} \frac{\Delta M_{\gamma}^2}{\omega_o^2} \right]$$

$$\begin{aligned}
& + \frac{(M_a - M_c)^2 + (M_b - M_d)^2 + (M_a - M_c)(M_b - M_d)}{\omega_0^2} \\
& - 2 \frac{\Delta M_\gamma^2 (M_a + M_b - M_c - M_d)}{\omega_0^3} \left. \vphantom{\frac{\Delta M_\gamma^2 (M_a + M_b - M_c - M_d)}{\omega_0^3}} \right\} (1 + \xi \frac{\Delta M_a^2}{\langle \omega^2 \rangle}) \\
& = D_0 (1 + \frac{M_a + M_b - M_c - M_d}{\langle \omega \rangle} - \frac{3 \Delta M_\gamma^2}{2 \langle \omega^2 \rangle} \\
& + \frac{(M_a - M_c)^2 + (M_b - M_d)^2 + (M_a - M_c)(M_b - M_d)}{\langle \omega^2 \rangle} \\
& - 2 \frac{\Delta M_\gamma^2 (M_a + M_b - M_c - M_d)}{\langle \omega^3 \rangle}) (1 + \xi \frac{\Delta M_a^2}{\langle \omega^2 \rangle}),
\end{aligned}$$

where $\omega_0 = \sqrt{K^2 + m^2}$,

$$\langle \omega^n \rangle = \int_0^\Lambda \frac{K^4 dK}{\omega^3} \left(\frac{1}{\omega_0} \right)^n,$$

$$\Delta M_a^2 = M_a^2 - m^2,$$

and m is the mass of the lightest meson.

The expression of K_a^{ba} (corresponding to Fig. 2c) can be obtained from the following identity:

$$K_a^{ba} D_{aa\beta}^{ba} = K_a^{a\beta} D_{aba}^{aa\beta}.$$

Let us define $K_0 = K_a^{a\beta}$ with $M_\beta = m$, then we have (higher order terms are omitted)

$$K_a^{ba} = K_0 \left[1 + 2 \frac{M_a - M_b}{\langle \omega \rangle} - \frac{\Delta M_a^2}{\langle \omega^2 \rangle} \left(\frac{3}{2} + \xi \right) \right].$$

Now it is straight forward calculation to get the form of W (up to the first order of the differences of baryon masses and the differences of the square of meson masses) from the definition:

$$\begin{aligned}
 W_{ba,c\gamma}^{ad} &= \frac{1}{4M} \left(K_a^{ba} \frac{\partial}{\partial M_a} D_{dc\gamma}^{aba} + K_a^{c\gamma} \frac{\partial}{\partial M_a} D_{dba}^{ac\gamma} \right) \\
 &= W_0 \left(1 + \frac{2M_a (1+\gamma) - (M_b + M_c) (1+\gamma/2) - M_d \gamma}{\langle \omega \rangle} \right. \\
 &\quad \left. - \left(\frac{3}{2 \langle \omega^2 \rangle} + \frac{2 \langle \omega \rangle}{\langle \omega^3 \rangle} \right) \frac{\Delta M_a^2 + \Delta M_\gamma^2}{2} \right)
 \end{aligned}$$

Let us define $a = M_a / \langle \omega \rangle$, $\Delta \alpha^2 = \Delta M_a^2 / \langle \omega^2 \rangle$ and use the approximation $\langle \omega^3 \rangle / \langle \omega \rangle \approx \langle \omega^2 \rangle$, then we have

$$D_{dc\gamma}^{aba} = D_0 (1 + a + b - c - d - 3\Delta \gamma^2/2 + \xi \Delta \alpha^2),$$

$$\begin{aligned}
 W_{ba,c\gamma}^{ad} &= W_0 (1 + 2a (1+\gamma) - (b+c) (1+\gamma/2) - d \gamma) \\
 &= 7 (\Delta \alpha^2 + \Delta \gamma^2)/4.
 \end{aligned}$$

Note that in this approximation the parameter ξ appears in D only.

b. Bootstrap Equations for a Static Model of baryons with non-degenerate mesons.

Since the $SU(2) \times SU(3)$ model of baryons (see Chap. V), works quite well, it is natural to ask whether we can calculate the mass splits among the particles which belong to the same $SU(3)$ representation by taking into account the mass differences among the mesons. Besides, it is interesting to know whether one can still find a self-consistent solution if some particles of the $SU(3)$ multiplets which are heavier

than others belonging to the same representation are left out. These two points are part of the motivation of our investigation of the static bootstrap model which includes N , N^* , Λ , Σ , Y_1^* , π and K . Another reason for our investigation is the greater simplicity of this model, compared with full $SU(3)$ multiplets.

We have 13 coupling constants and 4 mass differences (among the baryons) in this model, let us define

$$\begin{aligned}
 r &= \sqrt{D_0}, \quad \sqrt{10} f_{(Y_1^* Y_1^* \pi)} = r g_{(Y_1^* Y_1^* \pi)}, \quad f_{(Y_1^* \Sigma \pi)} = r g_{(Y_1^* \Sigma \pi)}, \\
 f_{(Y_1^* \Lambda \pi)} &= r g_{(Y_1^* \Lambda \pi)}, \quad f_{(\Sigma \Sigma \pi)} = r g_{(\Sigma \Sigma \pi)}, \quad f_{(\Sigma \Lambda \pi)} = r g_{(\Sigma \Lambda \pi)}, \\
 f_{(N^* N^* \pi)} &= r g_{(N^* N^* \pi)}, \quad f_{(N^* N \pi)} = r g_{(N^* N \pi)}, \quad f_{(N N \pi)} = \\
 r g_{(N N \pi)}, \quad \sqrt{10} f_{(N^* Y_1^* K)} &= \\
 r g_{(N^* Y_1^* K)}, \quad f_{(N Y_1^* K)} &= r g_{(N Y_1^* K)}, \quad f_{(N^* \Sigma K)} = \\
 r g_{(N^* \Sigma K)}, \quad f_{(N \Sigma K)} &= r g_{(N \Sigma K)}, \quad \text{and } \sqrt{6} f_{(N \Lambda K)} = r g_{(N \Lambda K)}.
 \end{aligned}$$

The bootstrap equations are expressed in the form of graphs in Fig. 17, where we use —, --- and -o-o-o-o to represent baryons, π and K -meson respectively. The coefficients of each term are denoted by $B_{(i,j)}$ (see Table XI). The meaning of the symbols are:

$$\frac{a \quad c \quad d \quad b}{\text{---}} = (1 + a + b - c - d) \varepsilon_{ac\pi} \varepsilon_{cd\pi} \varepsilon_{bd\pi},$$

$$\frac{a \quad c \quad d \quad b}{\text{---}} = (1 + a + b - c - d - 3K/2) \varepsilon_{cd\pi} \varepsilon_{acK} \varepsilon_{bdK},$$

$$\frac{a \quad c \quad d \quad b}{\text{---}} = (1 + a + b - c - d + \zeta K) \varepsilon_{ac\pi} \varepsilon_{cdK} \varepsilon_{bd\pi},$$

$$\frac{a \quad b \quad d \quad c}{\text{---}} = (1 + 2(1 + \eta)a - (b + c)(1 + \eta/2) - d\eta) \times$$

$$\varepsilon_{ab\pi} \varepsilon_{bd\pi} \varepsilon_{ac\pi} \varepsilon_{cd\pi},$$

$$\frac{a \quad b \quad d \quad c}{\text{---}} = (1 + 2(1 + \eta)a - (b + c)(1 + \eta/2) - d\eta - 7K/4) \times$$

$$\varepsilon_{abK} \varepsilon_{bd\pi} \varepsilon_{ac\pi} \varepsilon_{cdK},$$

$$K = \frac{M(K)^2 - M(\pi)^2}{\langle \omega^2 \rangle}.$$

TABLE XI

Numerical Values of $B_{(i,j)}$ (All numbers should be divided by a factor of 36).

j/i	1	2	3
1	6	1	-2
2	-12	1.21	110
3	12	0.25	1
4	30	1	20
5	4	4	2
6	-5	2.5	-10
7	5		2
8	0.5		0.5
9	2		-10
10	-10		-4
11	-2		5
12	2.5		-1
13	10		-1
14	55		4
15			-4

c. Some Remarks About the Model

Let us assume that this model (or a generalized model which would include more particles) represents approximately the actual physical situation. We can then check how well the linear approximation is satisfied by using the experimental values of the masses. Since the ratios

of the coupling constants and the ratios of the mass differences are not sensitive to η , we shall set η to be 1.2 (which is smaller than 1.25 of the $SU(2) \times SU(2)$ model but bigger than 1.08 of the $SU(2) \times SU(3)$ model). Let us define $\langle Y_1^* \rangle = \frac{M(Y_1^*) - M(N)}{\langle \omega \rangle}$, etc., and choose $\langle N^* \rangle$ to be 0.22 (close to the value in the $SU(2) \times SU(3)$ model), then we have $\langle Y_1^* \rangle = 0.325$, $\langle \Sigma \rangle = 0.19$, $\langle \Lambda \rangle = 0.13$, $K = 0.18$. These values are not very big compared with one. However, in some of the D's and W's, the linear approximation is badly violated. For example, we have $W_{N \pi}^{N Y_1^*} = -0.225$ and $W_{\Sigma K}^{N Y_1^*} = -0.36$. Since the Feynman integrals should go to zero if the masses of the virtual particles go to infinity (assuming smooth form factors), let us modify the linear approximation by setting the values of the D's and the W's to be zero if they are negative according to the linear approximation. Let us compare this model with the $SU(3)$ model (see sec. d of Ch. V); the corresponding values of the coupling constants and the mass differences of the $SU(3)$ model are given in Tables XII and XIII. (f's and $\langle a \rangle$'s.)

Our present model is a broken $SU(3)$ model. Can this symmetry breaking be considered as a small perturbation? In order to answer this question, we substitute the values of the $SU(3)$ solution into the bootstrap equations and see how well they are satisfied. Let us denote the ratio of the right hand side to the left hand side of a vertex equation by $R(a b \alpha)$. The values of $R(a b \alpha)$ and $R(a)$ are given in Tables XII and XIII. ($R(a) = W(a)/W(N)$.)

The above results indicate that most of the bootstrap equations are not satisfied very well, and therefore our model can not be considered as a small perturbation of the $SU(3)$ model. In order to see whether

it is possible to go continuously from the $SU(3)$ solution to a solution of our model, we add a term which is proportional to an adjustable parameter λ to each equation in such a way that they are satisfied by the $SU(3)$ solution if λ is one. We write the vertex equations in the form:

$$f_{(a\ b\ \alpha)}^2 = f_{(a\ b\ \alpha)} \sum_i (- - -) + \lambda C_{(a\ b\ \alpha)},$$

and the normalization equations in the form:

$$W(a) + \lambda C(a) = W(N).$$

The values of $C_{(a\ b\ \alpha)}$ and $C(a)$ are given in Tables XII and XIII.

TABLE XII

Numerical values of $f_{(a\ b\ \alpha)}$, $R_{(a\ b\ \alpha)}$ and $C_{(a\ b\ \alpha)}$

a	b	α	$f_{(a\ b\ \alpha)}$	$R_{(a\ b\ \alpha)}$	$C_{(a\ b\ \alpha)}$
Y_1^*	Y_1^*	π	0.437	73.8 (%)	0.139
Y_1^*	Σ	π	0.898	58	0.942
Y_1^*	$\wedge$	π	1.1	80.	0.674
Σ	Σ	π	1.155	57.9	1.56
Σ	$\wedge$	π	-0.833	69.5	0.589
N^*	N^*	π	2.185	95	0.563
N^*	N	π	1.796	97.8	0.19
N	N	π	1.598	100.47	-0.046
N^*	Y_1^*	K	0.437	50.9	0.26
N	Y_1^*	K	-0.898	71.6	0.636
N^*	Σ	K	-1.796	40	5.38
N	Σ	K	-0.443	57.6	0.231
N	$\wedge$	K	0.374	70.9	0.113

TABLE XIII

Numerical values of $\langle a \rangle$, $R(a)$ and $C(a)$

<u>a</u>	<u>$\langle a \rangle$</u>	<u>$R(a)$</u>	<u>$C(a)$</u>
Y_1^*	0.2256	45.8 (%)	3.227
Σ	0	27.8	4.295
$\wedge$	0	62	2.271
N^*	0.2256	77.6	1.334

We change λ from one to zero continuously and set K to zero.

The result is shown in Table XIV, where $r_{(a b \alpha)} = f_{(a b \alpha)}$, this solution / $f_{(a b \alpha)}$ SU(3) model. As λ decreases, the masses increase in such a way that some of them become so big that our approximation loses meaning. The masses of baryons satisfy the relation $M(Y_1^*) > M(N^*) > M(\Sigma) > M(\Lambda) > M(N)$ until $\lambda < 0.7$. The coupling constants of the type $f_{(a b \pi)}$ and the mass differences between baryons with the same strangeness (i.e., $\langle \Sigma - \Lambda \rangle$, $\langle Y_1^* - \Lambda \rangle$ and $\langle N^* - N \rangle$) are not very sensitive to λ compared with the coupling constants of the type $f_{(a b K)}$ and with the mass difference $\langle \Lambda - N \rangle$. The mass difference $\langle \Lambda - N \rangle$ increases while $f_{(a b K)}$'s decrease as λ decreases. The result indicates that a solution at $\lambda = 0$, if any exists, would be quite different from the SU(3) values (e.g., one would expect that some of the signs of the coupling constants may differ from the corresponding SU(3) values).

TABLE XIV

The properties of the solution as a
function of λ

λ	<u>1.0</u>	<u>0.9</u>	<u>0.8</u>	<u>0.7</u>	<u>0.6</u>	<u>0.5</u>	<u>0.4</u>	<u>0.35</u>
$\langle \Lambda \rangle$	0	0.035	0.06	0.15	0.25	0.41	0.67	0.92
$\langle \Sigma \rangle$	0	0.055	0.12	0.22	0.35	0.55	0.84	1.1
$\langle N^* \rangle$	0.2256	0.23	0.24	0.25	0.27	0.3	0.31	0.31
$\langle Y_1^* \rangle$	0.2256	0.278	0.34	0.44	0.57	0.79	1.1	1.4
$\langle \Sigma - \Lambda \rangle$	0	0.02	0.04	0.07	0.1	0.14	0.17	0.18
$\langle Y_1^* - \Lambda \rangle$	0.2256	0.243	0.26	0.29	0.32	0.38	0.43	0.48
$r(Y_1^* Y_1^* \pi)$	1	1.02	1.04	1.05	1.09	1.12	1.17	1.19
$r(Y_1^* \Sigma \pi)$	1	1.	1.	0.97	0.98	0.98	1.01	1.01
$r(Y_1^* \Lambda \pi)$	1	1.02	1.03	1.03	1.03	1.01	1.0	0.97
$r(\Sigma \Sigma \pi)$	1	1.01	1.01	0.99	1.	1.01	1.05	1.05
$r(\Sigma \Lambda \pi)$	1	1.01	1.01	1.0	0.99	0.96	0.93	0.9
$r(N^* N^* \pi)$	1	1.02	1.02	1.01	1.02	1.04	1.07	1.08
$r(N^* N \pi)$	1	1.03	1.07	1.12	1.16	1.2	1.23	1.23
$r(N N \pi)$	1	1.04	1.1	1.14	1.19	1.22	1.24	1.23
$r(N^* Y_1^* K)$	1	0.97	0.94	0.9	0.85	0.8	0.72	0.66
$r(N Y_1^* K)$	1	1.	1.	0.98	0.95	0.89	0.83	0.77
$r(N^* \Sigma K)$	1	0.97	0.92	0.87	0.81	0.75	0.68	0.62
$r(N \Sigma K)$	1	0.99	0.96	0.92	0.88	0.81	0.73	0.66
$r(N \Lambda K)$	1	0.98	0.94	0.88	0.81	0.7	0.59	0.52
ρ	2/3	0.64	0.62	0.59	0.59	0.62	0.68	0.73

d. The Properties of the Solution

Going back to the original problem in Sec. b, we find one solution (see Appendix IV for detail). The properties of the solution with different values of parameters are given in Table XV.

Let us compare this result with the one shown on Table XIV. The result with $\lambda = 0$ is far away from the one with $\lambda = 1$, the main differences are the sign of $r(N^* Y_1^* K)$ and the magnitudes of $r(N^* \Sigma K)$ and $\langle \Sigma \rangle$. Besides, the mass difference $\langle \Lambda - N \rangle$ increases rapidly as λ decreases. Therefore the model is not a simple extrapolation of previous result with $\lambda \rightarrow 0$.

Some of the masses of baryons are sensitive to the parameters while the coupling constants not. One of the coupling constants, $f(N^* Y_1^* K)$ has the opposite sign compared with the SU(3) value, and $f(N^* \Sigma K)$ is particularly small in this model.

The eigenvalues of the potential matrices corresponding to bound states are (since they are not sensitive to the parameters, we use the numbers listed in the first column of Table XV): $\lambda(Y_1^*) = 0.21, 0.09, -0.16, -0.42$; $\lambda(\Sigma) = 0.52, 0.002, -0.39, -0.98$; $\lambda(\Lambda) = 0.25, -0.59$; $\lambda(N^*) = 0.62, 0.11, -0.56$; $\lambda(N) = 0.35, 0.1, -0.11, -1$. All of them are smaller than one.

There are 27 potential matrices corresponding to channels without bound states. The solution indicates there are strong attractions in four channels, the largest eigenvalues of their potential matrices as functions of the parameter Z (see sec. b of Chap. 5) are shown on Table XVI. The four numbers within the parenthesis of $\lambda(Y, I, S, Z)$

TABLE XV

The properties of the solution as
functions of the parameters K and ξ .

K	0	0.05	0.1	0.2	0.2	0
$\xi \cdot K$	0.1	0.1	0.1	0.1	0.2	0
$\langle \wedge \rangle$	0.021	0.007	0.005	-0.01	-0.07	-
$\langle \Sigma \rangle$	0.247	0.228	0.224	0.202	0.115	-
$\langle N^* \rangle$	0.34	0.336	0.33	0.31	0.32	[0.314] (#)
$\langle Y_1^* \rangle$	0.138	0.129	0.129	0.12	0.068	-
$\langle \Sigma - \wedge \rangle$	0.226	0.221	0.219	0.212	0.185	(0.161) (#)
$\langle Y_1^* - \wedge \rangle$	0.117	0.122	0.124	0.13	0.138	(0.161)
$r(Y_1^* Y_1^* \pi)$	2.07	2.05	2.05	2.03	2.02	(1.95)
$r(Y_1^* \Sigma \pi)$	1.34	1.34	1.35	1.36	1.34	(1.4)
$r(Y_1^* \wedge \pi)$	1.61	1.63	1.63	1.65	1.75	(1.85)
$r(\Sigma \Sigma \pi)$	1.32	1.32	1.31	1.30	1.34	(1.54)
$r(\Sigma \wedge \pi)$	1.57	1.58	1.57	1.56	1.63	(1.77)
$r(N^* N^* \pi)$	1.77	1.79	1.78	1.78	1.67	[1.06]
$r(N^* N \pi)$	0.774	0.768	0.781	0.785	0.86	[1.23]
$r(N N \pi)$	1.26	1.26	1.26	1.27	1.24	[1.21]
$r(N^* Y_1^* K)$	-1.53	-1.55	-1.62	-1.77	-1.64	
$r(N Y_1^* K)$	1.48	1.52	1.6	1.75	1.47	
$r(N^* \Sigma K)$	0.0485	0.048	0.0435	0.03	0.055	
$r(N \Sigma K)$	1.55	1.57	1.64	1.98	1.56	
$r(N \wedge K)$	1.06	1.1	1.14	1.23	0.9	
ρ	0.46	0.45	0.45	0.45	0.39	(0.38)

The numbers within the square and round brackets refer respectively to the $\pi N N^*$ model ($\eta = 1.2$) and $\pi \Sigma \wedge Y_1^*$ model ($\eta = 1.1$).

denote the hypercharge Y , isotopic spin I , spin S , and Z respectively.

The other potential matrices are not very interesting. We shall give only their largest eigenvalues and the corresponding values of Z (z is varied from -1 to $+1$) in Table XVII

TABLE XVI

The biggest eigenvalues of the potential matrices corresponding to channels with strong attractions

Z	$(0,2,5/2,Z)$	$(0,2,3/2,Z)$	$(1,1/2,3/2,Z)$	$(-1,1/2,3/2,Z)^{(\neq)}$
-1	0^+	0.02	0.19	
-0.9				0.02
-0.6	0.16	0.17	0.13	
-0.5				0.16
-0.2	0.35	0.39	0.29	
-0.1				0.33
0.1				0.41
0.2	0.54	0.61	0.46	
0.3				0.5
0.5				0.58
0.6	0.72	0.83	0.63	
0.7				0.67
0.9				0.75
1	0.91	1.05	0.81	

$(\neq)$ The quantum numbers of this channel correspond to the baryon Ξ^* .

TABLE XVII

The values of the biggest eigenvalues
of the potential matrices.

Y	I	S	Z	$\lambda(Y, I, S, Z)$
0	1	5/2	1	0.39
2	1	3/2	1	0.52
2	1	1/2	1	0.53
1	3/2	5/2	1	0.63
1	1/2	5/2	1	0.17
-1	3/2	1/2	1	0.18
-1	3/2	3/2	1	0.19
0	0	5/2	-1	0.06
2	2	1/2	-1	0.025
2	1	5/2	-1	0.005
2	0	1/2	1	0.63
1	5/2	5/2	1	0.49
1	5/2	3/2	1	0.29
1	5/2	1/2	-1	0.085
-1	3/2	5/2	1	0.54
-1	1/2	5/2	1	0.098
0	0	3/2	1	0.51
0	2	1/2	-1	0.08
1	3/2	1/2	1	0.26
-1	1/2	1/2	1	0.23
2	2	5/2	1	0.21
2	2	3/2	1	0.38
2	0	3/2	-1	-0.026

e. Discussion

In order to interpret the results of the last section, let us compare the eigenvalues of the potential matrices with the solutions of other models.

In all of the models considered, there are strong attractions in some of the channels in which resonances were assumed to be absent. In order to compare them, we should not only find out what is the biggest eigenvalue of a particular potential matrix as Z changed from -1 to $+1$, but also find out where the same eigenvalue equals one by extrapolating Z . We get the following results (in the case of $SU(2) \times SU(3)$ model, we use the $SU(3)$ irreducible representation instead of the isotopic spin): in the present model, $\lambda(0, 2, 5/2, 1) = 0.91$, $Z = 1.05$, (this number refers to the value of Z where λ equals one.)
 $\lambda(0, 2, 3/2, 1) = 1.05$, $Z = 0.9$, $\lambda(1, 1/2, 3/2, 1) = 0.81$, $Z = 1.4$,
 $\lambda(-1, 1/2, 3/2, 1) = 0.8$, $Z = 1.4$; in the $SU(2) \times SU(2)$ model,
 $\lambda(1, 5/2, 5/2, 1) = 0.74$, $Z = 1.9$; in the $SU(2) \times SU(3)$ model,
 $\lambda(\underline{25}, 5/2, 1) = 0.49$, $Z = 3.04$, $\lambda(\underline{27}, 3/2, 1) = 0.47$, $Z = 3.14$
 (Note that these two $SU(3)$ channels contain all the channels which give strong attractions in the broken $SU(3)$ scheme); and in the $SU(2) \times SU(2) \times SU(2)$ model, $\lambda(0, 2, 5/2, 1) = \lambda(0, 2, 3/2, 1) = 1.06$, $Z = 0.9$.

It is interesting that this model and the one considered in the last chapter have some similar properties, that is, the corresponding coupling constants have similar magnitudes and both solutions indicate strong

attraction in the channel $T = 2, J = 3/2$. From the comparison made before, it seems that neither solutions are self-consistent, although the solution with all of the multiplets are self-consistent. However, the possibility that other solutions exist has not been ruled out, (although no others were found in a search).

Let us compare the coupling constants with the experiment. They are much more difficult to measure than the masses. We are interested in the ratio $\rho = g^2(Y_1^* \Sigma \pi) / g^2(Y_1^* \Lambda \pi)$. The branching ratio $R = \Gamma(Y_1^* \rightarrow \Sigma + \pi) / \Gamma(Y_1^* \rightarrow \Lambda + \pi)$ has been measured by several experimental groups⁽⁴¹⁾. The earlier measurements (before 1964) indicate that R is less than 5%, however, more recent measurement⁽⁴¹⁾ gives a much higher value $(14 \pm 3)_0\%$. In the limit of exact $SU(3)$ symmetry, we have $R = \rho = 2/3$. If we calculate R by using the $SU(3)$ coupling constants but the experimental masses, then R turns out to be 0.16 (Glashow and Sakurai⁽⁶⁵⁾). A bootstrap calculation by Wali and Warnock⁽⁶⁴⁾ gives $R = 0.1$. Experimentally, ρ is related to R by $R = 0.24\rho$. The earlier measurements of R indicate $\rho < 20\%$, while the recent one gives $\rho \approx 58^{+12}_{-0}\%$. The values of ρ for different models are given in Tables XIII and XIV. However, all these numbers are within the big uncertainty of the available experimental data.

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Appendix I Some useful isotopic spin and SU(3) crossing matrices

Some useful isotopic spin and SU(3) crossing matrices are given in this appendix. The definition is given in sec. d of chapter III. Each matrix is labeled, left to right, and up to down, by decreasing dimensions of the allowed irreducible representations. In the case of isotopic spin symmetry, the isospins of the baryons Λ , N , Σ , N^* , Y_2^* , N' are respectively 0, 1/2, 1, 3/2, 2, 5/2. We use π to denote a self-conjugate meson with isotopic spin one, and K to denote a non-self-conjugate meson with isotopic spin 1/2 ($\bar{K}$ is the antiparticle of K). In the case of SU(3) symmetry, we consider only a self-conjugate meson which belongs to the 8 representation. To find the potential due to the exchange of a particle of isotopic spin T , read down the T column of the corresponding crossing matrix. (Do the same thing in the case of SU(3).)

Process	Crossing Matrix
$N + \pi \rightarrow N + \pi$	$\begin{bmatrix} 1/3 & 2/3 \\ 4/3 & -1/3 \end{bmatrix}$
$N^* + \pi \rightarrow N + \pi$	$\begin{bmatrix} -2/3 & -\sqrt{10}/6 \\ -\sqrt{10}/3 & 2/3 \end{bmatrix}$

$$N^* + \pi \rightarrow N^* + \pi$$

$$\begin{vmatrix} 1/10 & 2/5 & 1/2 \\ 3/5 & 11/15 & -1/3 \\ 3/2 & -2/3 & 1/6 \end{vmatrix}$$

$$\Lambda + \pi \rightarrow \Sigma + \pi$$

$$-1$$

$$\Lambda + \pi \rightarrow Y_2^* + \pi$$

$$+1$$

$$\Sigma + \pi \rightarrow \Sigma + \pi$$

$$\begin{vmatrix} 1/6 & 1/2 & 1/3 \\ 5/6 & 1/2 & -1/3 \\ 5/3 & -1 & 1/3 \end{vmatrix}$$

$$N' + \pi \rightarrow N' + \pi$$

$$\begin{vmatrix} 1/21 & 2/7 & 2/3 \\ 8/21 & 31/35 & -4/15 \\ 4/3 & -2/5 & 1/15 \end{vmatrix}$$

$$Y_2^* + \pi \rightarrow \Sigma + \pi$$

$$\begin{vmatrix} -1/2 & -3\sqrt{5}/10 \\ -\sqrt{5}/2 & 1/2 \end{vmatrix}$$

$$Y_2^* + \pi \rightarrow Y_2^* + \pi$$

$$\begin{vmatrix} 1/15 & 1/3 & 3/5 \\ 7/15 & 5/6 & -3/10 \\ 7/5 & -1/2 & 1/10 \end{vmatrix}$$

$$N' + \pi \rightarrow N^* + \pi$$

$$\begin{vmatrix} -2/5 & -\sqrt{14}/5 \\ -3\sqrt{14}/10 & 2/5 \end{vmatrix}$$

$$\Sigma + K(\bar{K}) \rightarrow \Sigma + K(\bar{K})$$

$$\begin{vmatrix} 1/3 & 2/3 \\ 4/3 & -1/3 \end{vmatrix}$$

$$N + K(\bar{K}) \rightarrow N + K(\bar{K})$$

$$\begin{vmatrix} 1/2 & 1/2 \\ 3/2 & -1/2 \end{vmatrix}$$

$$N^* + K(\bar{K}) \rightarrow N + K(\bar{K})$$

$$-1$$

$$\Sigma + K(\bar{K}) \rightarrow \Lambda + K(\bar{K})$$

$$-1$$

$$N^* + \pi \rightarrow \Lambda + K$$

$$-\sqrt{6}/2$$

$$N + \pi \rightarrow \Lambda + K$$

$$+\sqrt{6}/2$$

$$\Lambda + \pi \rightarrow N^* + \bar{K}$$

$$-2/\sqrt{6}$$

$$\Lambda + \pi \rightarrow N + \bar{K}$$

$$+2/\sqrt{6}$$

$$N^* + K(\bar{K}) \rightarrow N^* + K(\bar{K})$$

$$\begin{vmatrix} 1/4 & 3/4 \\ 5/4 & -1/4 \end{vmatrix}$$

$$N^* + \pi \rightarrow \Sigma + K$$

$$\begin{vmatrix} \sqrt{30}/12 & \sqrt{10}/4 \\ 5\sqrt{3}/6 & -1/2 \end{vmatrix}$$

$$N + \pi \rightarrow \Sigma + K$$

$$\begin{vmatrix} -1/2 & -\sqrt{6}/6 \\ -1 & \sqrt{6}/6 \end{vmatrix}$$

$$\Sigma + \pi \rightarrow N^* + \bar{K}$$

$$\begin{vmatrix} \sqrt{30}/15 & \sqrt{3}/3 \\ \sqrt{10}/3 & -1/3 \end{vmatrix}$$

$$Y_1^* + \pi \rightarrow N + \bar{K}$$

$$\begin{vmatrix} -2/3 & -2/3 \\ -2\sqrt{6}/3 & \sqrt{6}/3 \end{vmatrix}$$

$$\underline{10} + \underline{8} \rightarrow \underline{8} + \underline{8}$$

	<u>27</u>	<u>10</u>	<u>8^s</u>	<u>8^a</u>
<u>27</u>	1/10	$-\sqrt{2}/6$	-2/5	$2\sqrt{5}/15$
<u>10</u>	$-9\sqrt{2}/20$	-1/2	$-\sqrt{2}/5$	$-\sqrt{10}/5$
<u>8^s</u>	-27/20	$-\sqrt{2}/4$	2/5	$\sqrt{5}/5$
<u>8^a</u>	$9\sqrt{5}/20$	$-\sqrt{10}/4$	$\sqrt{5}/5$	0

$$\underline{10} + \underline{8} \rightarrow \underline{10} + \underline{8}$$

	<u>35</u>	<u>27</u>	<u>10</u>	<u>8</u>
<u>35</u>	1/8	9/40	1/4	2/5
<u>27</u>	7/24	37/40	-1/12	-2/15
<u>10</u>	7/8	-9/40	3/4	-2/5
<u>8</u>	7/4	-9/20	-1/2	1/5

$$\underline{8} + \underline{8} \rightarrow \underline{8} + \underline{8}$$

	<u>27</u>	<u>10</u>	<u>10[*]</u>	<u>8^{s,s}</u>	<u>8^{s,a}</u>	<u>8^{a,s}</u>	<u>8^{a,a}</u>	<u>1</u>
<u>27</u>	7/40	1/12	1/12	1/5	0	0	1/3	1/8
<u>10</u>	9/40	1/4	1/4	2/5	$\sqrt{5}/5$	$\sqrt{5}/5$	0	-1/8
<u>10[*]</u>	9/40	1/4	1/4	2/5	$-\sqrt{5}/5$	$-\sqrt{5}/5$	0	-1/8
<u>8^{ss}</u>	27/40	1/2	1/2	-3/10	0	0	-1/2	1/8
<u>8^{sa}</u>	0	$\sqrt{5}/4$	$-\sqrt{5}/4$	0	-1/2	1/2	0	0
<u>8^{as}</u>	0	$\sqrt{5}/4$	$-\sqrt{5}/4$	0	1/2	-1/2	0	0
<u>8^{aa}</u>	9/8	0	0	-1/2	0	0	1/2	-1/8
<u>1</u>	27/8	-5/4	-5/4	1	0	0	-1	1/8

Appendix II

Discussion of some relations among the coupling constants
derived by using the approximate N/D method

The equations (5-1) and (5-2) can be put in the form:

$$(4g_1^2 + g_0^2)^2 = (4g_1^4 + 16g_1^2g_2^2 + 48g_1^2g_0^2 + 40g_0^2g_1g_2 + g_0^4)/9,$$

$$(g_1^2 + g_2^2)^2 = (12g_1^2g_2^2 + 121g_2^4/25 + 4g_0^2g_1^2 + 10g_0^2g_1g_2 + g_1^4)/9.$$

Let us define x and y by $x = g_0/g_1$ and $y = g_2/g_1$, then these two equations can be reduced to

$$2x^4 + 6x^2 + 35 - 10xy - 4y^2 = 0, \quad (\alpha)$$

$$4.16y^4 + 6y^2 + 8 - 10xy - 4x^2 = 0. \quad (\beta)$$

We plot these two equations in Fig. 18. It is clear that they don't have any solution.

The equations (5-3) and (5-4) can be reduced to the form:

$$8x^2y^2 - 81y^2 - 135x^2 - 350xy + 20 = 0, \quad (\gamma)$$

$$170.24x^2y^2 - 98.28y^2 - 81x^2 - 350xy + 32 = 0. \quad (\delta)$$

We plot these two equations in Figs. 19a and b. They don't have any solution. However, they almost touch each other within

a small region ($x \approx -0.15$, $y \approx 0.9$). (The curves which are symmetrical with respect to the origin are omitted. Note that we use y instead of x as the abscissa.) The solution of the bootstrap model (see sec. b of Ch. V) corresponds to $x = 1.78$, $y = 1.08$. Therefore the "solution" of Eqs. γ and δ is far away from the physical solution. (Note also that the signs of x 's are different.)

Appendix III The method of solving bootstrap equations in the models of the (3,3) resonance

a. The SU(2)XSU(2) model

In this simple model, it turns out that we can solve the vertex equations exactly with x as a parameter. Then the normalization condition determines the value of x . This method does not depend on the linear approximation since x is considered as a parameter.

Let us reproduce the vertex equations from Ch. V here; they are:

$$f_0 = (f_0^3 + 32f_0f_1^2(1-x) + 20f_1^2f_2(1-2x))/9, \quad (1)$$

$$f_1 = (f_1) (4f_0^2(1+x) + f_1^2 + 5f_0f_2 + 4f_2^2(1-x))/9, \quad (2)$$

$$f_2 = (5f_0f_1^2(1+2x) + 8f_1^2f_2(1+x) + 121f_2^3/25)/9. \quad (3)$$

The eq.(2) can be factorized. The factor $f_1 = 0$ leads to no solution, so we can replace eq. (2) by

$$1 = (f_1^2 + 4f_0^2(1+x) + 5f_0f_2 + 4f_2^2(1-x))/9.$$

We prefer to rewrite the above equation in the form:

$$f_1^2 = 9 - 4f_0^2(1+x) - 5f_0f_2 - 4f_2^2(1-x). \quad (2')$$

Substituting the expression for f_1^2 into eqs. (1) and (3), we have

$$\begin{aligned} 9(f_0 - 32f_0(1-x) + 20f_2(1-2x)) &= f_0^3 - (32f_0(1-x) + 20f_2(1-2x))X \\ &\quad (4f_0^2(1+x) + 5f_0f_2 + 4f_2^2(1-x)), \end{aligned} \quad (1')$$

$$9(f_2 - 5f_0(1+2x) - 8f_2(1+x)) = 121f_2^3/25 - (5f_0(1+2x) + 8f_2(1+x))x \\ (4f_0^2(1+x) + 5f_0f_2 + 4f_2^2(1-x)) \quad (3')$$

It can be checked that the condition $f_2 = 0$ leads to one and only one solution: $f_2 = 0 = f_0$, $f_1 = 3$, and $x = 3/(10+5\eta)$.

Let us define a new variable $t = f_0/f_2$. (This can be done since we need not worry about the case $f_2 = 0$ anymore.)

Now the eqs. (1') and (3') can be rewritten in the form:

$$9f_2^2 = \frac{t - 32t(1-x) - 20(1-2x)}{t^3 - (32t(1-x) + 20(1-2x))(4t^2(1+x) + 5t + 4(1-x))} = \\ \frac{1 - 5t(1+2x) - 8(1+x)}{4.84 - (5t(1+2x) + 8(1+x))(4t^2(1+x) + 5t + 4(1-x))}.$$

Finally, we end up with a 4th degree equation:

$$t^4 10(1+2x)(3+4x) + t^3 2(160x^2 + 106x - 78) - t^2 120(1-2x)(3+x) \\ + t 8(-40x^2 + 75.28x - 11.49) + 16(1-2x)(2.1 + 10x) = 0, \quad (4)$$

together with two subsidiary conditions gotten from the requirements $f_2^2 > 0$ and $f_1^2 > 0$.

The eq. (4) can be solved exactly by Ferrari's method^(*).

The normalization condition can be put in the form:

$$1 = R \equiv \frac{1}{5}(f_0^2 + 8f_1^2 + 4f_2^2)/W(N).$$

The result which corresponds to the case $\eta = 1$ is shown in Fig.20.

(*) Any textbook on higher algebra.

b. The SU(4) model^(*)

In this section, we shall discuss briefly the group SU(4) and show how to calculate the Clebsch-Gordan (CG) coefficients which are used in Ch. V.

The group SU(4) includes SU(2)XSU(2) as a subgroup. We shall relate these two SU(2) groups to the isotopic spin symmetry and the rotational invariance. Let us define the generators of these two SU(2)'s (in the four-dimensional representation), by τ_i and σ_i ($i=1,2,3$) respectively, then the 15 (4^2-1) generators of SU(4) can be written as

$$\tau_i \times 1 \quad i=1,2,3;$$

$$1 \times \sigma_i \quad i=1,2,3;$$

and $\tau_i \times \sigma_j \quad i,j=1,2,3.$

We shall diagonalize $\tau_3 \times 1$ and $1 \times \sigma_3$, then each state of an irreducible representation (IR) $\underline{a}$ is specified uniquely by the following quantum numbers: I, I_3, J and J_3 . Let us denote this state by the symbol $\left| \begin{smallmatrix} I & I_3 \\ J & J_3 \end{smallmatrix} \right\rangle \underline{a}$. The simplest IR is $\underline{4}$.

One can determine all IR's from the direct product of several

$\underline{4}$'s. We shall use the following symbols to denote the four states of $\underline{4}$: $\left| \begin{smallmatrix} + \\ + \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix} \right\rangle$, $\left| \begin{smallmatrix} + \\ - \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{smallmatrix} \right\rangle$, $\left| \begin{smallmatrix} - \\ - \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{smallmatrix} \right\rangle$, $\left| \begin{smallmatrix} - \\ + \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{smallmatrix} \right\rangle$

(*) This section is based, in part, on an unpublished calculation done by Prof. R. E. Cutkosky.

and $\begin{vmatrix} + \\ - \end{vmatrix} = \begin{vmatrix} 1/2-1/2 \\ 1/2-1/2 \end{vmatrix}$. The 20 representation is represented by the Young's diagram (66) $\square\square\square$, and the adjoint representation (15) by $\begin{vmatrix} \square & \square \\ \square & \square \end{vmatrix}$. The weight diagrams of 20 and 15 are shown in Fig. 21.

The wave function of each state of the 20 representation can be expressed as the linear combinations of the product of the three 4 spinors, and it is totally symmetric in these three indices. The wave function of the state with the highest weight is given by $\begin{vmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{vmatrix} \underline{20} = \begin{vmatrix} + \\ +/1 \end{vmatrix} \begin{vmatrix} + \\ +/2 \end{vmatrix} \begin{vmatrix} + \\ +/3 \end{vmatrix}$ (we shall drop the subscripts in the following discussion). The wave functions of all other states can be derived from this state by using lowering operators. For example, let us define $\tau_{\pm}$ by $\tau_{\pm} = \tau_1 \pm i\tau_2$ (same for $\sigma_{\pm}$), then we have

$$\begin{aligned} \begin{vmatrix} 3/2 & 1/2 \\ 3/2 & 3/2 \end{vmatrix} \underline{20} &\propto \tau_- \begin{vmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{vmatrix} \\ &\propto \begin{vmatrix} - \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} + \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} - \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} + \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} - \\ + \end{vmatrix}, \end{aligned}$$

$$\text{i.e., } \begin{vmatrix} 3/2 & 1/2 \\ 3/2 & 3/2 \end{vmatrix} \underline{20} = \frac{1}{\sqrt{3}} \left(\begin{vmatrix} - \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} + \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} - \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} + \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} + \\ + \end{vmatrix} \begin{vmatrix} - \\ + \end{vmatrix} \right).$$

Operating with σ_- on the state $\begin{vmatrix} 3/2 & 1/2 \\ 3/2 & 3/2 \end{vmatrix} \underline{20}$, we get

$$\begin{vmatrix} 3/2 & 1/2 \\ 3/2 & 1/2 \end{vmatrix} \underline{20} = \frac{1}{3} (A+B),$$

where $B \equiv \begin{vmatrix} + & + & - \\ + & - & + \end{vmatrix} + \begin{vmatrix} + & + & - \\ - & + & + \end{vmatrix} + \begin{vmatrix} + & - & + \\ - & + & + \end{vmatrix} + \begin{vmatrix} + & - & + \\ + & + & - \end{vmatrix}$

$$+ \begin{vmatrix} - & + & + \\ + & + & - \end{vmatrix} + \begin{vmatrix} - & + & + \\ + & - & + \end{vmatrix},$$

and $A \equiv \begin{vmatrix} + & + & - \\ + & + & - \end{vmatrix} + \begin{vmatrix} + & - & + \\ + & - & + \end{vmatrix} + \begin{vmatrix} - & + & + \\ + & + & + \end{vmatrix}.$

Considering the symmetry property of the wave function, the most general form of the wave function of the state $\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} \underline{20}$

is: $\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} \underline{20} = aA + bB.$

The condition $\begin{vmatrix} 3/2 & 1/2 \\ 3/2 & 1/2 \end{vmatrix} \underline{20} \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} \underline{20} = 0$ gives $b = -a/2 = \sqrt{2}/6.$

Let us assume that the direct product of a and b gives

$$a \times b = c + \dots,$$

then one can write

$$\begin{vmatrix} I & I_3 \\ J & J_3 \end{vmatrix} \begin{vmatrix} a & b & c \\ I_a I_{a_3} & I_b I_{b_3} & I_c I_{c_3} \\ J_a J_{a_3} & J_b J_{b_3} & J_c J_{c_3} \end{vmatrix} = \sum_{\substack{I_a, I_{a_3} \\ I_b, I_{b_3} \\ J_a, J_{a_3} \\ J_b, J_{b_3}}} \begin{vmatrix} I_a I_{a_3} & I_b I_{b_3} & I_c I_{c_3} \\ J_a J_{a_3} & J_b J_{b_3} & J_c J_{c_3} \end{vmatrix} \begin{vmatrix} I_a I_{a_3} & I_b I_{b_3} \\ J_a J_{a_3} & J_b J_{b_3} \end{vmatrix} \begin{vmatrix} I_c I_{c_3} \\ J_c J_{c_3} \end{vmatrix}.$$

The fact that we have $SU(4) \supset SU(2) \times SU(2)$ enables us to write the CG coefficient in the form

$$\begin{vmatrix} a & b & c \\ I_a I_{a_3} & I_b I_{b_3} & I_c I_{c_3} \\ J_a J_{a_3} & J_b J_{b_3} & J_c J_{c_3} \end{vmatrix} = \begin{vmatrix} a & b & c \\ I_a J_a & I_b J_b & I_c J_c \end{vmatrix}$$

$$\begin{vmatrix} I_a I_{a_3} & I_b I_{b_3} & I_c I_{c_3} \\ J_a J_{a_3} & J_b J_{b_3} & J_c J_{c_3} \end{vmatrix} \begin{vmatrix} I_c I_{c_3} \\ J_c J_{c_3} \end{vmatrix},$$

where $\langle I_a I_{a_3} I_b I_{b_3} | II_3 \rangle$ is the SU(2) CG coefficient, and we

shall call $\langle \begin{smallmatrix} a & b \\ I_a J_a & I_b J_b \end{smallmatrix} \parallel \begin{smallmatrix} c \\ I_c J_c \end{smallmatrix} \rangle$ the isoscalar factor of SU(4)

which is free form I_{a_3}, J_{a_3} , etc.

We are interested in those CG coefficients which have the general form $\langle \begin{smallmatrix} a & 15 & a \\ \gamma_a & \gamma_b & \gamma_c \end{smallmatrix} \rangle$ where γ_a means the appropriate quantum numbers. According to the Wigner-Eckart theorem, they are proportional to the corresponding matrix elements of the generators. The proportionality constant can be determined from the normalization condition. We have used the squares of the following isoscalar factors in Ch. V:

$$\begin{aligned} A &= \left\langle \begin{smallmatrix} 20 & 15 \\ N^* & \pi \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N^* \end{smallmatrix} \right\rangle, \quad B = \left\langle \begin{smallmatrix} 20 & 15 \\ N & \rho \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N^* \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} 20 & 15 \\ N & \gamma \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N^* \end{smallmatrix} \right\rangle, \\ D &= \left\langle \begin{smallmatrix} 20 & 15 \\ N & \pi \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N^* \end{smallmatrix} \right\rangle, \quad a = \left\langle \begin{smallmatrix} 20 & 15 \\ N & \pi \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N \end{smallmatrix} \right\rangle, \\ b &= \left\langle \begin{smallmatrix} 20 & 15 \\ N & \rho \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} 20 & 15 \\ N & \gamma \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N \end{smallmatrix} \right\rangle, \quad d = \left\langle \begin{smallmatrix} 20 & 15 \\ N & \pi \end{smallmatrix} \parallel \begin{smallmatrix} 20 \\ N \end{smallmatrix} \right\rangle, \end{aligned}$$

which satisfy the normalization conditions

$$A^2 + 2B^2 + D^2 = 1 \quad \text{and} \quad a^2 + 2b^2 + d^2 = 1.$$

(The notation N^* represents the quantum number $I=j=3/2$, other symbols have similar meanings.) We shall calculate these

isoscalar factors in the next paragraph.

Using the definition of isoscalar factor and the Wigner-Eckart theorem, we have

$$\begin{aligned}
 & \left\langle \begin{array}{cc} \underline{20} & \underline{15} \\ 3/2 & 3/2 \end{array} \begin{array}{cc} 1 & 0 \\ 3/2 & 3/2 \end{array} \begin{array}{cc} \underline{20} & \\ 3/2 & 3/2 \end{array} \right\rangle = A \left\langle 3/2 \ 3/2 \ 1 \ 0 \mid 3/2 \ 3/2 \right\rangle^2 \\
 & = 3A/5 \\
 & = R \left\langle \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \begin{array}{cc} \underline{20} & \\ 3/2 & 3/2 \end{array} \mid \tau_3^x \sigma_3 \mid \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle \\
 & = R \left\{ \left| \begin{array}{c} + \\ + \end{array} \right\rangle_1 \left| \begin{array}{c} + \\ + \end{array} \right\rangle_2 \left| \begin{array}{c} + \\ + \end{array} \right\rangle_3 \sum_{i=1}^3 \tau_3^{(i)} \sigma_3 \left| \begin{array}{c} + \\ + \end{array} \right\rangle_1 \left| \begin{array}{c} + \\ + \end{array} \right\rangle_2 \left| \begin{array}{c} + \\ + \end{array} \right\rangle_3 \right\} \\
 & = 3R,
 \end{aligned}$$

where $\sigma_j^{(i)}$ ($\tau_j^{(i)}$) are the Pauli matrices which operate on the i -th (iso-) spinor. One can do the same thing for the other CG coefficients:

$$\begin{aligned}
 & \left\langle \begin{array}{cc} \underline{20} & \underline{15} \\ 3/2 & 3/2 \end{array} \begin{array}{cc} 1 & 0 \\ 3/2 & 3/2 \end{array} \begin{array}{cc} \underline{20} & \\ 3/2 & 3/2 \end{array} \right\rangle = B \left\langle 3/2 \ 3/2 \ 1 \ 0 \mid 3/2 \ 3/2 \right\rangle^2 \\
 & = B \sqrt{\frac{3}{5}} \\
 & = R \left\langle \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \begin{array}{cc} \underline{20} & \\ 3/2 & 3/2 \end{array} \mid I_3 \mid \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle \\
 & = 3R,
 \end{aligned}$$

$$\left\langle \begin{array}{cc} \underline{20} & \underline{15} \\ 1/2 & 1/2 \end{array} \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \begin{array}{cc} \underline{20} & \\ 3/2 & 3/2 \end{array} \right\rangle = D \left\langle 1/2 \ 1/2 \ 1 \ 1 \mid 3/2 \ 3/2 \right\rangle^2$$

$$\begin{aligned}
&= D \\
&= R \left\langle \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \left| \begin{array}{c} \tau_+ \sigma_+ \\ \tau_+ \sigma_+ \end{array} \right| \begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array} \right\rangle \\
&= 2\sqrt{2} R \cdot \\
&\text{(Note that } \tau_- \sigma_- \left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle = \sum_{i=1}^3 \hat{\tau}_+^{(i)} \hat{\sigma}_+^{(i)} \left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle = \left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle \text{)} \\
&= (\sqrt{2})^2 \left(\left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle + \left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle + \left| \begin{array}{cc} 3/2 & 3/2 \\ 3/2 & 3/2 \end{array} \right\rangle \right) \cdot \text{The normalization}
\end{aligned}$$

condition $A^2 + 2B^2 + D^2 = 1$ gives the value of $R (=1/\sqrt{63})$. Finally, we have $A=5/\sqrt{63}$, $B=\sqrt{5}/\sqrt{21}$ and $D=2\sqrt{2}/\sqrt{63}$. The isoscalar factors a, b and d can be calculated in the same way, they are: $a = 5/\sqrt{63}$, $b=1/\sqrt{21}$ and $d=4\sqrt{2}/\sqrt{63}$. Since only the square of each isoscalar factor is used in Ch. V, we don't want to discuss the phase problem here.

c. The $SU(2) \times SU(3)$ model

We have 4 coupling constants and one mass difference in this model. We shall proceed in a way which is similar to that of sec.a.

The Eq.(5-11) can be factorized. The factor $f_1 = 0$ leads to no solution, and we can replaced Eq.(5-11) by

$$1 = 4(1+x)f_D(f_D+f_F)/15 + f_2(f_D+f_F)/3 + 2(1-x)f_2^2/3 + f_1^2/12 \quad (5-11')$$

Let us define f'_D, f'_F and R by

$$f_D = f'_D \varepsilon_2, \quad f_F = f'_F \varepsilon_2$$

and $f_F/f_D = R$. The relation between R and the mixing angle θ is $R = \sqrt{5} \tan \theta$. This equation can be written in the form:

$$f_1 = 12(1-f_2A),$$

where

$$A = 4(1+x)f'_D(f'_D + f'_F)/15 + (f'_D + f'_F)/3 + 2(1-x)/3.$$

For technical reasons, we replace Eq. (5-10) by a new equation gotten by taken linear combination of Eqs. (5-9) and (5-10):

$$5f_D - f_F = f_D^3/2 + f_D f_F^2/6 + f_F^3/30 - f_D^2 f_F/6 + 10(1-x)f_1^2(f_D + f_F)/3 \quad (5-10')$$

Substituting the expression for f_1^2 into Eqs. (5-9), (5-10'), and (5-12), we get the following equations:

$$\begin{aligned} f_2^2 = H_1 &= \left[1 - 8(1+x) - 2(1+2x)(f'_D + f'_F) \right] / \left\{ 1.1 - \left[8(1+x) + \right. \right. \\ &\quad \left. \left. + 2(1+2x)(f'_D + f'_F) \right] A \right\} \\ = H_2 &= \left[5f'_D - f'_F - 40(1-x)(f'_D + f'_F) \right] / \left\{ (1/30)(15f'_D - 5f_D^2 f'_F \right. \\ &\quad \left. + 5f'_D f_F^2 + f_F^3) - 40(1-x)(f'_D + f'_F) A \right\} \\ = H_3 &= \left[f'_F - 40(1-x)f'_D - 50(1-2x) \right] / \left\{ -f'_F/30 + f_D^2 f_F^2/6 - \right. \\ &\quad \left. - \left[40(1-x)f'_D + 50(1-2x) \right] \right\} A \end{aligned}$$

Finally we have two equations $H_2 = H_1$ and $H_2 = H_3$, together with two subsidiary conditions gotten from the following two requirements: $f_1^2 > 0$ and $f_2^2 > 0$. Taking x and R as parameters, we can rewrite these two equations in the following

form:

$$F_1 \equiv 1 + C_1(x, R) f_D' + C_2(x, R) f_D'^2 + C_3(x, R) f_D'^3 = 0,$$

$$\text{and } F_2 \equiv 1 + C_1(x, R) f_D' + C_2(x, R) f_D'^2 + C_3(x, R) f_D'^3 = 0.$$

Taking the difference of these two equations, we have

$$F_3 \equiv 1 + C_1'' f_D' + C_2'' f_D'^2 = 0,$$

$$\text{where } C_1'' = (C_2 - C_2') / (C_1 - C_1'),$$

$$\text{and } C_2'' = (C_3 - C_3') / (C_1 - C_1').$$

Since we have the identity $F_1 \equiv F_3 [1 + (C_1 - C_1'') f_D'] + f_D'^2 [C_2 - (C_1 - C_1'') C_1''] + f_D'^3 [C_3 - (C_1 - C_1'') C_2'']$, we can replace

$$F_1 = 0 \text{ by } f_D' = - [C_2 - (C_1 - C_1'') C_1''] / [C_3 - (C_1 - C_1'') C_2''] .$$

The normalization condition is $1 = F_4 \equiv \frac{2}{7} (f_D^2 + f_F^2/5 + 5f_1^2 + 5f_2^2) / W(\underline{g})$.

Considering x and R as parameters, the functions $F_3(x, R)$ and $F_4(x, R)$ are slowly varying functions of R and x when

$|R|$ is much greater than 1. It is easy to check that there is no solution in the case $|R| > 2$. However, when $|R|$ is of the order one, then F_3 and F_4 change rapidly and we need a very careful analysis with the help of a computer. In Fig. 22, the curves show the values of x and R which satisfy $F_4 = 1$ and the subsidiary conditions (we put η equal 1); considering

R as a function of x (related by the relation $F_4(x, R) = 1$), we plot the values of F_3 as a function of x in Fig. 23. The result shows that there are two solutions.

The cases $f_2 = 0$ and $f_D = 0$ have to be considered separately.

It can be shown that both conditions lead to the same solution: $f_D = f_F = f_2 = 0$, $f_1^2 = 12$, and $x = (3/7)/(2+3)$.

This "scanning" method shows where the solutions are.

In order to get a precise result, one can use methods similar to Newton's method with the help of an electronic computer.

Appendix IV The method of solving
bootstrap equations in the broken $SU(3)$ model.

We have 13 coupling constants ($f(ab\pi)$'s and $f(abK)$'s) and 4 mass differences among the baryons (Δ 's) in this model. Due to the complexity of this problem, instead of trying to find out all possible solution, we use the "random" method mentioned in chapter six (see also sec. f of chapter three). We try to solve some of the vertex equations exactly, this can be done in several different ways. What we have done is to solve the five vertex equations corresponding to $f(abK)$'s exactly. The advantage of this method is that these equations are linear and homogeneous in these five coupling constants. Assuming that not all of $f(abK)$'s are zero, a five dimensional determinant involving the coefficients of the $f(abK)$'s has to vanish. This determinant is an equation of second degree of the $f(ab\pi)$'s and the above-mentioned condition can be used to expressed one of the $f(ab\pi)$'s as a (double-valued) function of other $f(ab\pi)$'s and Δ 's. Only ratios of $f(abK)$'s are determined in this way; however, the absolute magnitude can

be calculated by solving one of the other vertex equations exactly. In this way, we reduce the number of unknowns from 17 to 11.

The method described above is very general and can be used in any kind of approximation. In our case, it is better to write the Δ 's in this way: $\langle \Lambda - N \rangle$, $\langle N^* - N \rangle$, $\langle \Sigma - \Lambda \rangle$ and $\langle Y_1^* - \Lambda \rangle$; then the coefficients of the vertex equations corresponding to $f(abK)$'s are independent of $\langle \Lambda - N \rangle$. Therefore we are able to reduce one more unknown by solving one more vertex equation exactly. In other words, we use the $f(ab\pi)$'s except one of them and Δ 's except $\langle \Lambda - N \rangle$ as unknowns, all other coupling constants and Δ 's are expressed explicitly as functions of them.

A Fortran IV program written by Prof. R. E. Cutkosky has been very useful in the "random" search procedure.

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CAPTIONS OF FIGURES

- 1) The vertex (a) and normalization (b) equations.
- 2) The vertex (a) and normalization (b) equations of the static model. Dotted lines and solid lines denote respectively the mesons and the baryons.
- 3) A plot of γ as a function of the cutoff (Λ/m).
- 4) A plot of $(12\pi^2 D_0)$ as a function of the cutoff (Λ/m).
- 5) A plot of $\langle \omega \rangle / m$ as a function of the cutoff (Λ/m).
- 6) One of the terms in B.
- 7) A triangle Feynman diagram.
- 8) The bootstrap equations of the SU(2)XSU(2) model of the (3,3) resonance. We use the dotted, solid and double lines to represent the pion, nucleon and (3,3) resonance respectively. We put terms with identical contributions into a parenthesis.
- 9) A plot of $R = (M^* - M)_{\text{calculated}} / (M^* - M)_{\text{observed}}$ versus $\Lambda(\text{BeV})$ for the SU(2)XSU(2) model (solid line) and the SU(2)XSU(3) model (dashed line).
- 10) Curves showing x as a function of γ (solid line for the SU(2)XSU(2) model, dashed line for the SU(2)XSU(3) model).

- 11) Graphs which represent terms in the bootstrap equations with big coefficients. (A double line represents $\mathbb{H}^*$, a solid line N , and a dashed line π .)
- 12) A comparison of the $SU(2)XSU(2)$ solution with the $SU(4)$ model.
- 13) The bootstrap equations of the $SU(2)XSU(3)$ model. We use the dotted, solid and double lines to denote the 8 (meson), 8(baryon), and 10(baryon) respectively.
- 14) A comparison of the $SU(2)XSU(3)$ solution with the $SU(6)$ model.
- 15) A check of an "identity" between coupling constants in the $SU(2)XSU(2)$ model (solid line) and the $SU(2)XSU(3)$ model (dashed line).
- 16) The bootstrap equations of Y_1^*, Σ, Λ and π in the $SU(2)XSU(2)XSU(2)$ model. The pions are denoted by the dotted lines, the baryons with $L=0$ and 1 are denoted by single and double lines respectively.
- 17) The bootstrap equations of a model which includes $\pi, K, N, N^*, \Sigma, \Lambda$, and Y_1^* . We use the symbols ---, $\circ\circ\circ$ and — to denote π, K and baryons respectively. The coefficient of each term is denoted by $B(i,j)$, and the notation Y means Y_1^* .

- 18) A graph which shows that the Eqs. (5-1) and (5-2) (corresponding to α and β respectively) have no solution.
- 19) A graph which shows that the Eqs. (5-3) and (5-4) (corresponding to γ and δ respectively) have no solution (a). The region where they almost have a solution is shown separately (b).
- 20) A graph which indicates the existence of two solutions of the $SU(2) \times SU(2)$ model.
- 21) The weight diagrams of 15 and 20 of $SU(4)$.
- 22) A graph which shows the values of x and R satisfying $F_4=1$ and the auxiliary conditions.
- 23) A graph which indicates the existence of two solutions of the $SU(2) \times SU(3)$ model.

FIG. 1

$$\begin{array}{c} \alpha \\ \diagup \\ \text{---} a \quad b \end{array} = \sum_{c,d,\gamma} \begin{array}{c} \alpha \\ \diagup \\ \text{---} a \quad c \quad d \quad b \\ \quad \quad \quad \diagdown \\ \quad \quad \quad \gamma \end{array}$$

(a)

$$1 = \sum_{b,c,d,\alpha,\gamma} \begin{array}{c} \alpha \\ \diagup \\ \text{---} a \quad c \quad d \quad b \quad a \\ \quad \quad \quad \diagdown \\ \quad \quad \quad \gamma \end{array}$$

(b)

$$K_a^{ba} = \begin{array}{c} \alpha \\ \diagup \\ \text{---} a \quad b \end{array}$$

(c)

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG. 9

FIG. 10

$$\text{---} \diagup \text{---} = \text{---} \diagup \text{---} + \left(\text{---} \diagup \text{---} \right)$$

$$\text{=---} \diagup \text{---} = \text{=---} \diagup \text{---}$$

$$1 = \text{---} \diagup \text{---} + \left(\text{---} \diagup \text{---} + \text{---} \diagup \text{---} \right)$$

$$1 = \text{=---} \diagup \text{---}$$

FIG. 11

FIG. 12

FIG. 13

FIG. 14

FIG. 15

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \\
 & \text{Diagram 4} = (\text{Diagram 5} + \text{Diagram 6}) + \text{Diagram 7} \\
 & 1' = \text{Diagram 8} + \text{Diagram 9} \\
 & = \text{Diagram 10} + (\text{Diagram 11} + \text{Diagram 12}) \\
 & \quad + \text{Diagram 13} + \text{Diagram 14}
 \end{aligned}$$

FIG. 1 6

$$\begin{aligned}
\sqrt{10}f(Y\bar{Y}\pi) = & 2 \frac{\text{Diagram } B(1,10)}{B(1,10)} + \frac{\text{Diagram } B(1,13)}{B(1,13)} + \frac{\text{Diagram } B(1,14)}{B(1,14)} + \frac{\text{Diagram } B(3,2)}{B(3,2)} + 2 \frac{\text{Diagram } B(3,4)}{B(3,4)} + 2 \frac{\text{Diagram } B(3,6)}{B(3,6)} \\
& + 2 \frac{\text{Diagram } B(3,9)}{B(3,9)} + \frac{\text{Diagram } B(3,11)}{B(3,11)} \\
f(Y\Sigma\pi) = & \frac{\text{Diagram } B(1,5)}{B(1,5)} + \frac{\text{Diagram } B(1,6)}{B(1,6)} + \frac{\text{Diagram } B(1,7)}{B(1,7)} + \frac{\text{Diagram } B(1,8)}{B(1,8)} + \frac{\text{Diagram } B(3,5)}{B(3,5)} + \frac{\text{Diagram } B(3,6)}{B(3,6)} \\
& + \frac{\text{Diagram } B(3,8)}{B(3,8)} + \frac{\text{Diagram } B(3,9)}{B(3,9)} + \frac{\text{Diagram } B(3,10)}{B(3,10)} + \frac{\text{Diagram } B(3,11)}{B(3,11)} + \frac{\text{Diagram } B(3,13)}{B(3,13)} + \frac{\text{Diagram } B(3,15)}{B(3,15)} \\
f(Y\Lambda\pi) = & \frac{\text{Diagram } B(1,2)}{B(1,2)} + \frac{\text{Diagram } B(1,4)}{B(1,4)} + \frac{\text{Diagram } B(3,3)}{B(3,3)} + \frac{\text{Diagram } B(3,4)}{B(3,4)} + \frac{\text{Diagram } B(3,6)}{B(3,6)} + \frac{\text{Diagram } B(3,13)}{B(3,13)} \\
& + \frac{\text{Diagram } B(3,14)}{B(3,14)} + \frac{\text{Diagram } B(3,15)}{B(3,15)} \\
f(\Sigma\Sigma\pi) = & 2 \frac{\text{Diagram } B(1,9)}{B(1,9)} + \frac{\text{Diagram } B(1,12)}{B(1,12)} + \frac{\text{Diagram } B(1,11)}{B(1,11)} + 2 \frac{\text{Diagram } B(3,5)}{B(3,5)} + 2 \frac{\text{Diagram } B(3,7)}{B(3,7)} + \frac{\text{Diagram } B(3,11)}{B(3,11)} +
\end{aligned}$$

FIG. 17 (continued)

(continued)

$$\begin{aligned}
 & + \frac{\Sigma \quad \Sigma}{B(3,12)} + 2 \frac{\wedge \quad Y}{B(3,15)} \\
 f(\Sigma \wedge \pi) = & \frac{\Sigma \quad N \quad N \quad \wedge}{B(1,1)} + \frac{N^* \quad N}{B(1,3)} + \frac{\wedge \quad \Sigma}{B(3,1)} + \frac{Y \quad Y}{B(3,6)} + \frac{\Sigma \quad \Sigma}{B(3,7)} + \frac{Y \quad \Sigma}{B(3,10)} \\
 & + \frac{\wedge \quad Y}{B(3,14)} + \frac{\Sigma \quad Y}{B(3,15)} \\
 f(N^* N^* \pi) = & 2 \frac{\Sigma \quad Y}{B(1,6)} + \frac{\Sigma \quad \Sigma}{B(1,12)} + \frac{Y \quad Y}{B(1,14)} + \frac{N^* \quad N^*}{B(2,2)} + 2 \frac{N \quad N^*}{B(2,4)} + \frac{N \quad N}{B(2,6)} \\
 f(N^* N \pi) = & \frac{N^* \quad \Sigma \quad \wedge \quad N}{B(1,3)} + \frac{Y \quad \wedge}{B(1,4)} + \frac{Y \quad \Sigma}{B(1,7)} + \frac{\Sigma \quad Y}{B(1,8)} + \frac{\Sigma \quad \Sigma}{B(1,9)} + \frac{Y \quad Y}{B(1,10)} \\
 & + \frac{N \quad N^*}{B(2,3)} + \frac{N^* \quad N^*}{B(2,4)} + \frac{N \quad N}{B(2,5)} + \frac{N^* \quad N}{B(2,6)} \\
 f(N N \pi) = & 2 \frac{N \quad \wedge \quad \Sigma \quad N}{B(1,1)} + 2 \frac{\wedge \quad Y}{B(1,2)} + 2 \frac{\Sigma \quad Y}{B(1,5)} + \frac{\Sigma \quad \Sigma}{B(1,11)} + \frac{Y \quad Y}{B(1,13)} + \frac{N \quad N}{B(2,1)}
 \end{aligned}$$

FIG. 17 (continued)

(continued)

$$\begin{aligned}
 & + \frac{\text{Diagram 1}}{B(2,6)} + 2 \frac{\text{Diagram 2}}{B(2,5)} \\
 \sqrt{10}f(N*YK) = & \frac{\text{Diagram 3}}{B(1,14)} + \frac{\text{Diagram 4}}{B(1,10)} + \frac{\text{Diagram 5}}{B(1,6)} + \frac{\text{Diagram 6}}{B(1,7)} + \frac{\text{Diagram 7}}{B(1,4)} \\
 f(NYK) = & \frac{\text{Diagram 8}}{B(1,10)} + \frac{\text{Diagram 9}}{B(1,13)} + \frac{\text{Diagram 10}}{B(1,8)} + \frac{\text{Diagram 11}}{B(1,5)} + \frac{\text{Diagram 12}}{B(1,2)} \\
 f(N*\Sigma K) = & \frac{\text{Diagram 13}}{B(1,6)} + \frac{\text{Diagram 14}}{B(1,8)} + \frac{\text{Diagram 15}}{B(1,12)} + \frac{\text{Diagram 16}}{B(1,9)} + \frac{\text{Diagram 17}}{B(1,3)} \\
 f(N\Sigma K) = & \frac{\text{Diagram 18}}{B(1,7)} + \frac{\text{Diagram 19}}{B(1,5)} + \frac{\text{Diagram 20}}{B(1,9)} + \frac{\text{Diagram 21}}{B(1,11)} + \frac{\text{Diagram 22}}{B(1,1)} \\
 \sqrt{6}f(N\wedge K) = & \frac{\text{Diagram 23}}{B(1,4)} + \frac{\text{Diagram 24}}{B(1,2)} + \frac{\text{Diagram 25}}{B(1,3)} + \frac{\text{Diagram 26}}{B(1,1)} \\
 W(Y) = (1/12) (2 (& \frac{\text{Diagram 27}}{B(1,2)} + \frac{\text{Diagram 28}}{B(1,4)} + \frac{\text{Diagram 29}}{B(1,5)} + \frac{\text{Diagram 30}}{B(1,6)} + \frac{\text{Diagram 31}}{B(1,7)}
 \end{aligned}$$

FIG. 17 (continued)

(continued)

$$\begin{aligned}
 & + \text{diagram } B(1,8) + \text{diagram } B(1,10) + \text{diagram } B(1,10) + \text{diagram } B(1,13) + \text{diagram } B(1,14) + \text{diagram } B(3,2) \\
 & + \text{diagram } B(3,3) + 2 \text{diagram } B(3,4) + \text{diagram } B(3,4) + \text{diagram } B(3,5) + 2 \text{diagram } B(3,6) + 2 \text{diagram } B(3,6) \\
 & + \text{diagram } B(3,8) + 2 \text{diagram } B(3,9) + \text{diagram } B(3,9) + \text{diagram } B(3,10) + \text{diagram } B(3,11) + \text{diagram } B(3,13) \\
 & + \text{diagram } B(3,14) + 2 \text{diagram } B(3,15) \\
 = W(\Sigma) = & (1/6) (2 (\text{diagram } B(1,1) + \text{diagram } B(1,3) + \text{diagram } B(1,5) + \text{diagram } B(1,6) + \text{diagram } B(1,7) \\
 & + \text{diagram } B(1,8) + 2 \text{diagram } B(1,9) + 2 \text{diagram } B(1,9) + \text{diagram } B(1,11) + 2 \text{diagram } B(1,12) + \text{diagram } B(3,1) \\
 & + \text{diagram } B(3,5) + \text{diagram } B(3,5) + \text{diagram } B(3,6) + \text{diagram } B(3,7) + \text{diagram } B(3,7) + \text{diagram } B(3,8))
 \end{aligned}$$

FIG. 17 (continued)

(continued)

$$\begin{aligned}
 & + \text{diagram } B(3,9) + 2 \text{diagram } B(3,10) + 2 \text{diagram } B(3,11) + \text{diagram } B(3,12) + \text{diagram } B(3,13) + \text{diagram } B(3,14) \\
 & + 2 \text{diagram } B(3,15) + 2 \text{diagram } B(3,15)) \\
 = W(\wedge) = & (1/2) (2 (\text{diagram } B(1,1) + \text{diagram } B(1,2) + \text{diagram } B(1,3) + \text{diagram } B(1,4) + \text{diagram } B(3,1) \\
 & + 2 \text{diagram } B(3,3) + 2 \text{diagram } B(3,4)) \\
 = W(N^*) = & (1/16) (2 (\text{diagram } B(3,14) + \text{diagram } B(3,15) + \text{diagram } B(1,3) + \text{diagram } B(1,4) + \text{diagram } B(1,6) + \text{diagram } B(1,6) + \text{diagram } B(1,7) \\
 & + \text{diagram } B(1,8) + \text{diagram } B(1,9) + \text{diagram } B(1,10) + \text{diagram } B(1,12) + \text{diagram } B(1,14) + \text{diagram } B(2,2))
 \end{aligned}$$

FIG. 17 (continued)

(continued)

$$\begin{aligned}
 & + \text{B(2,3)} + \text{B(2,4)} + 2 \text{B(2,4)} + \text{B(2,5)} + 2 \text{B(2,6)}) \\
 = W(N) = (1/4) (& 2 (\text{B(1,1)} + \text{B(1,1)} + \text{B(1,2)} + \text{B(1,2)} + \text{B(1,3)} \\
 & + \text{B(1,4)} + \text{B(1,5)} + \text{B(1,5)} + \text{B(1,7)} + \text{B(1,8)} + \text{B(1,9)} \\
 & + \text{B(1,10)} + \text{B(1,11)} + \text{B(1,13)} + \text{B(2,1)} + \text{B(2,3)} + \text{B(2,4)} \\
 & + \text{B(2,5)} + 2 \text{B(2,5)} + 2 \text{B(2,6)}))
 \end{aligned}$$

FIG. 17

FIG. 18

FIG. 19 (a)

FIG. 19 (b)

FIG. 20

FIG. 21

FIG. 22

FIG. 23