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Some Recent Developments in the Theory of Phase Transitions

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

LA-3805-MS
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Report written: October 1967

Report distributed: December 7, 1967

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Leonard W. Gruenberg

These notes are based on lectures given at LASL in June 1967.

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SOME RECENT DEVELOPMENTS IN THE THEORY OF PHASE TRANSITIONS

by

Leonard W. Gruenberg†

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These notes are based on lectures given at LASL in June 1967. Included is a discussion of (a) the Landau theory of second-order phase transitions, generalized to take into account fluctuations, (b) the breakdown of superfluidity in one and two dimensions, and (c) the scaling laws. The lectures were based in part on the review article by L. P. Kadanoff, et al, Reviews of Modern Physics 39, 395 (1967).

Chapter I. THE LANDAU THEORY OF PHASE TRANSITIONS

A. Thermodynamic Properties.

We are dealing here with second-order phase transitions. This includes critical points in liquids as well as superfluid, ferromagnetic, and certain ferroelectric transitions.

Landau[1] assumes that the ordered phase can be characterized by an order parameter $\eta(r)$ which can be regarded as a thermodynamic variable. The order parameter approaches zero continuously as the temperature increases and approaches the transition temperature. Thus, in the vicinity of the transition, the order parameter is small. Landau assumes that the free energy is an analytic function of the order parameter so that for $T \lesssim T_c$, we can make a power series expansion.

Although most of what I will say is quite general, it will be useful to keep a particular model in mind. A particularly simple one is the Ising model of a ferromagnet. The spin at each lattice site can only point up or down. The order parameter is $M_z(r)$ (magnetization) and the free energy density $g(r)$ can be written as

$$1.1)\quad g(M_z,T,H) = g_o(T,H) + a\,M_z^{\,2} + \frac{b}{2}\,M_z^{\,4} + c(\nabla M_z)^2 - H\,M_z$$

where g_o is the free energy per unit volume which would exist if there were no magnetization and H is the external magnetic field. The odd powers of M_z are missing because in the absence of a field, the energy cannot depend on the direction of the magnetization.

If the system is in thermal equilibrium, the magnetization will be some function of T and H. To determine the equilibrium value of M_z, we minimize the free energy with respect to small variations in M_z. If we make a small variation $M_z \to M_z + \delta M_z$, the free energy will vary as follows:

$$1.2)\quad \delta G = \int d^3r\,(2\,\delta M_z)\left[\frac{1}{2}H + a\,M_z + b\,M_z^{\,3} - c\,\nabla^2 M_z\right]$$

where the last term is found by making a partial integration.

Since $\delta M_z(r)$ can be anything, the quantity in brackets must vanish in order to make $\delta G = 0$

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(hopefully $\delta^2 G > 0$). Thus

$$1.3)\quad -c\,\nabla^2 M_z + M_z(a + b\,M_z^2) = \frac{1}{2} H$$

Assuming H is zero and $c > 0$, the best solutions will be

$$1.4)\quad M_z = 0 \quad \text{or} \quad M_z = (-a/b)^{1/2}$$

We want the first of these to minimize the free energy for $T > T_c$ and the second for $T < T_c$. This will happen if a changes sign at $T = T_c$. Landau makes the simplest possible assumption:

$$1.5)\quad a(T) = a'(T - T_c)$$

and also

$$1.6)\quad b(T) = b \qquad c(T) = c$$

Presumably, these are the first terms in expansions in powers of $T - T_c$, but near the transition it is all right to drop the higher-order terms.

Using (1.4) and (1.5), we find that

$$1.7)\quad M_z = (a'/b)^{1/2}\,(T_c - T)^{1/2} \quad ; \quad T > T_c$$

and, making use of (1.1), we can obtain

$$1.8)\quad g = g_o - \frac{a'^2}{2b}(T_c - T)^2$$

From this it can be seen that the transition is a second-order one of the Ehrenfest kind - there is no latent heat but there is a specific heat jump:

$$1.9)\quad C(T_c-) - C(T_c+) = T_c\,a'^2/b$$

At $T = T_c$, when a field is present the magnetization can be determined by Eq. (1.3), which leads to

$$1.10)\quad M_z(T_c,H) = (H/2b)^{1/3}$$

B. <u>The Effect of Fluctuations.</u>

An assumption made implicitly by Landau is that thermal fluctuations are small. However, since $a(T_c) = 0$, Eq. (1.1) tells us that a fluctuation δM_z will only increase the free energy by an amount of order $(\delta M_z)^4$ when $H = 0$. This suggests that fluctuations play an important role near T_c, a fact which is well known experimentally.

Ginzburg[(2)] has suggested a test for the validity of the Landau theory. He pointed out that the concept of an order parameter as a thermodynamic variable is only meaningful if fluctuations in it are small compared to its average value. Thus, he requires that $\langle\overline{\delta M_z^2}\rangle$ must be much less than $\langle M_z\rangle^2$ where $\langle\ \rangle$ represents a thermal average and the bar a spatial average.

The fluctuations can be estimated in the following manner: According to thermodynamics, the probability of a fluctuation δM_z proportional to $\exp\left[-\beta\left\{G(M_z + \delta M_z) - G(M_z)\right\}\right]$. To find the average fluctuation, we must sum over all possible fluctuations, using the above as a weighting factor. This can be cone most simply by introducing the Fourier transform of $\delta M_z(r)$

$$1.11)\quad \delta\tilde{M}_z(q) = \int d^3r\; \delta M_z(r)\, e^{-i\vec{q}\cdot\vec{r}}$$

The spatial average of the mean-square fluctuation can be expressed in terms of $\delta\tilde{M}_z$:

$$1.12)\quad \langle\overline{\delta M_z^2}\rangle \equiv \frac{1}{V}\int d^3r\; \langle\{\delta M_z(r)\}^2\rangle = \int \frac{d^3r}{(2\pi)^3}\,\chi(q)$$

where V is the volume

$$1.13)\quad \chi(q) = \frac{1}{V}\langle|\delta\tilde{M}_z(q)|^2\rangle$$

In order to evaluate $\chi(q)$ it is necessary to find the increase in free energy δg due to a fluctuation $\delta\tilde{M}_z(q)$. From Eq. (1.1), we find that this is given by

$$1.14)\quad \delta g(q) = (a + 3b\,\langle M_z\rangle^2 + cq^2)\,|\delta\tilde{M}_z(q)|^2$$

$\chi(q)$ can now be found by using the probability distribution for fluctuations:

$$1.15) \quad \chi(q) = [Z(q)]^{-1} \sum_{\delta\widetilde{M}_z} |\delta\widetilde{M}_z(q)|^2 \exp\left(-\beta|\delta\widetilde{M}_z(q)|^2 \left\{cq^2 + 2a'(T_c - T)\right\}\right)$$

where

$$1.16) \quad Z(q) = \sum_{\delta\widetilde{M}_z} \exp\left[-\beta|\delta\widetilde{M}_z|^2 \left\{cq^2 + 2a'(T_c - T)\right\}\right]$$

(In order to arrive at Eq. (1.15), the equilibrium value $\langle M\rangle^2 = -a/b$ has been substituted into Eq. (1.14). To find the thermodynamic average of any quantity, it is necessary to average over all fluctuations, even those for which $q' \neq q$, but these factors cancel out of numerator and denominator.) The calculation can be simplified considerably by noting that Eqs. (1.15) and (1.16) can be combined to yield

$$1.17) \quad \chi(q) = -\left[cq^2 + 2a'(T_c - T)\right]^{-1} \frac{1}{Z}\frac{\partial Z}{\partial \beta}$$

To find Z(q), it is necessary to integrate over all $\delta\widetilde{M}_z(q)$. Since $\delta\widetilde{M}_z$ may be complex, we must integrate over both the real and the imaginary part of it. Since $|\delta\widetilde{M}_z|^2 = (\delta M_z^{(1)})^2 + (\delta M_z^{(2)})^2$, we get two identical factors. Each factor involves the integral of a Gaussian function, and this leads to

$$1.18) \quad Z(q) = \pi/[cq^2 + 2a'(T_c - T)]\beta$$

so that

$$1.19) \quad \chi(q) = kT/[cq^2 + 2a'(T_c - T)]V$$

Before making use of $\chi(q)$ to find the average mean square fluctuation it is worth while noting that $\chi(q)$ is the Fourier transform of the function g(R) which describes the spatial correlations of the order parameter. We can thus find g(R):

$$1.20) \quad g(r - r') \equiv \langle \delta M_z(r)\, \delta M_z(r')\rangle = V\int \frac{d^3q}{(2\pi)^3}\chi(q)\, e^{iq\cdot(r - r')} = \frac{VkT}{8\pi c}\frac{e^{-|r - r'|/\xi(T)}}{|r - r'|}$$

where

$$1.21) \quad \xi(T) = (c/2a)^{1/2}(T_c - T)^{-1/2} \quad ; T < T_c$$

Note that the range of correlations is determined by $\xi(T)$; at $T = T_c$, $\xi(T)$ diverges and g(R) falls off as 1/R. These results are valid only for $T < T_c$. For $T > T_c$ it is only necessary to set $\langle M_z\rangle = 0$ in Eq. (1.14), and then a result identical to Eq. (1.20) is obtained, except that the coherence length is now given by

$$1.22) \quad \xi_+(T) = (c/a)^{1/2}(T - T_c)^{-1/2}$$

The average mean-square fluctuation can now be found from Eqs. (1.12) and (1.19). However, the resulting integral diverges because of the behavior of the integrand for large values of q. Large q means small distances; we assumed to start that $M_z(r)$ was slowly varying - this is what enabled us to keep only the lowest order derivative $(\nabla M_z)^2$ in the free energy expansion. It is therefore not surprising that theory breaks down at small distances. A reasonable approximation is to cut off the integrals at $q = [\xi(T)]^{-1}$ since variations in M_z which occur over distances shorter than $\xi(T)$ are not well accounted for. Inserting Eq. (1.19) into Eq. (1.12), we then obtain

$$1.23) \quad \langle \overline{\delta M_z(r)^2}\rangle = \frac{kT}{2\pi^2 c}\frac{1}{\xi(T)}\left(1 - \frac{\pi}{4}\right)$$

Thus $\langle\overline{\delta M_z{}^2}\rangle$ varies as $\sqrt{T_c - T}$ while $\langle M_z\rangle^2$ goes to zero as $(T_c - T)$. Hence, near T_c, there must be a region in which $\langle\overline{\delta M_z{}^2}\rangle > \langle M_z\rangle^2$. Ginzburg argues that the Landau theory cannot be correct in this temperature range.

Using Eqs. (1.7), (1.9), and (1.23), we find that the Landau theory can only be valid in the temperature range

$$1.24) \quad 1 \gg (T_c - T)/T_c \gg [k/\xi_o{}^3 \Delta C]^2$$

where k is Boltzmann's constant, ΔC the specific heat jump and ξ_o a temperature independent coherence length;

$$1.25) \quad \xi_o = [(T_c - T)/T_c]^{1/2}\xi(T)$$

The first part of the inequality is a requirement on account of the initial assumption that M_z be small.

In many cases, the coherence length ξ_o is determined by the range of the interparticle forces. Thus, a long-range force (e.g., ferroelectrics) can mean that it is possible to satisfy Eq. (1.24) while a short-range force (e.g., helium) means that it is impossible. In a superconductor ξ_o is large (10^{-4} - 10^{-6} cm usually) even though the force is short range, and the Landau theory applies near T_c, except in a narrow temperature range $\Delta T \approx 10^{-14}$ degrees.

It can also be shown that $\chi(q)$ is the magnetic susceptibility, i.e., the linear response to a magnetic field $H_o \cos qx$. From Eq. (1.10) we see that for $q = 0$ (constant external field), the susceptibility diverges as $(T_c - T)^{-1}$ when the transition is approached from below. For $T_c > T$, it can be shown that the susceptibility also diverges as $(T - T_c)^{-1}$.

C. Phase Fluctuations in Superfluids and the Breakdown of Long-Range Order in One and Two Dimensions

In superfluids (He^4 below the lambda point and superconductors below their transition temperatures) the order parameter η is complex. It is easy to generalize the Landau theory so as to take this into account. It is customary to write

$$1.26) \quad \eta(r) = |\Psi(r)| e^{i\varphi(r)}$$

with φ real. In the absence of perturbations (i.e., rotations in helium and magnetic fields in superconductors) the free energy density is again of the form given by Eq. (1.1). The equilibrium situation then corresponds to $\varphi(r)$ = constant, since a gradient in the phase leads to an increase in the free energy. Consequently, the thermodynamic properties are still given by the results of Section A. In considering the importance of fluctuations, it is necessary to take into account phase fluctuations. These behave quite differently from the amplitude fluctuations: a uniform phase change costs no energy; consequently, long wave-length phase fluctuations are very soft. It has been shown,[3] in fact, that these fluctuations are so strong in one- and two-dimensional systems that they effectively suppress superfluidity.

In order to understand how this comes about, let us study the phase correlation function $\langle\varphi(r)\varphi(r')\rangle$. Its Fourier transform is analogous to $\chi(q)$ of Eq. (1.13), and so, we define

$$1.27) \quad \chi_\varphi(q) = \frac{1}{V}\langle|\varphi(q)|^2\rangle$$

We proceed as in Section B. The increase in free energy density is simply

$$1.28) \quad \delta g(q) = c\,|\Psi|^2 q^2 |\varphi^2(q)|$$

Hence the average fluctuation is given by

$$1.29) \quad \chi_\varphi(q) = [VZ'(q)]^{-1} \sum_{\varphi(q)} |\varphi(q)|^2 \exp[-\beta\, c\, |\Psi|^2 q^2 |\varphi(q)|^2]$$

where

$$1.30) \quad Z'(q) = \sum_{\varphi(q)} \exp[-\beta\, c\, |\Psi|^2 q^2 |\varphi(q)|^2$$

This may be evaluated as in Section B. The result is

$$1.31) \quad \chi_\varphi(q) = (\xi_\varphi(T)/q^2)V$$

where

$$1.32) \quad \xi_\varphi(T) = kT/c\,|\Psi|^2$$

Finally, we take the Fourier transform

$$1.33) \quad g_\varphi(r - r') = \frac{V}{(2\pi)^3}\int d^3q\, e^{iq\cdot(r - r')}\chi(q)$$

In three-dimensional systems, $1/q^2$ is just the Fourier transform of the Coulomb potential and we find

$$1.34) \quad g_\varphi(R) = \xi_\varphi(T)/R$$

Thus, the phase correlation function falls off much more slowly than does the amplitude correlation function (compare with Eq. (1.20)).

In one- and two-dimensional systems the integrand of Eq. (1.33) diverges when $q \to 0$. This implies that $g_\varphi(R)$ diverges at large distances. This has been used by Rice[3] and by Kane and Kadanoff[3] to show that off-diagonal long-range order cannot exist in one and two dimensional systems. These proofs have been made more convincing by the work of Hohenberg[4] who has shown that this result follows from rigorous sum-rules. A simplified demonstration of this result can be provided by evaluating the thermal average of $\langle e^{i\varphi(x)} \rangle$. This can be evaluated, again, by using the general technique for studying fluctuations. With the aid of Eq. (1.28), we find

$$1.35) \quad \langle e^{i\varphi(x)} \rangle = e^{-\sum_q \xi_\varphi(T)/4\pi q^2}$$

In one and two dimensions, as already mentioned, the sum diverges at long wavelengths ($q \to 0$). This means that the thermal average of the order parameter which is proportional to $\langle e^{i\varphi(x)} \rangle$ must vanish.

Chapter II. THE SCALING LAWS

A. Derivation

Although the Landau theory does not give an accurate picture of the behavior of a system near its critical point, it is certain that certain qualitative features of the theory are correct. In particular, I want to emphasize three of these features:

(1) Near the critical point, fluctuations become very large. It is likely that they play an important role in determining the thermodynamic behavior of the system.

(2) At temperatures above and below the critical temperatures there are characteristic lengths $\xi_\pm$ which give the range over which fluctuations are correlated. These lengths diverge at the critical point.

(3) At the critical point, the correlation functions have, effectively, an infinite range, i.e., they fall off as some power of R^{-1}.

As a result of these features, it is thought that the critical behavior should not depend on the details of the interparticle forces, at least when they have a short range. The scaling laws[5] represent an attempt to exploit this fact. Their "derivation" (which is more of a conjecture than a proof) relies on dimensional arguments as well as points (1), (2), and (3) above. Although the scaling laws are quite general, it is useful to have a particular model in mind. Let us consider the Ising model. The spins σ_i are localized on a lattice with lattice constant $a_o = 1$ (for simplicity). The free energy density g can be expressed as a function of two dimensionless parameters $\epsilon = (T - T_c)/T_c$ and $h = \mu_o H/kT$.

The critical point is given by $\epsilon = h = 0$. The behavior near the critical point can be characterized by nine parameters which are defined in Table II of Kadanoff's article which is included below. The scaling laws lead to relations between these parameters which are presumed to be valid for all phase transitions. All told there are seven equations relating to these nine parameters which can presumably take on different values for different systems.

Now consider a near-sighted observer who cannot see individual cells in the ferromagnet; the smallest cell he can see is one with dimension L which contains L^d localized spins where d is the number of dimensions. We will assume that $L \gg 1$ (he is quite near-sighted) and also that $L \ll \xi$. The latter can always be satisfied since ξ diverges at the critical point.

Suppose the observer makes a guess at how far the system is from the critical point. He can do this by measuring the correlation length; the relevant question is: over how many cells does the correlation extend? The observer will think it extends over ξ/L cells. He will therefore guess that he is further from the critical point than will another observer who can actually see a unit cell, since the latter will know that ξ cells are correlated. Another way of saying this is that the near-sighted observer will think that the parameters ϵ and h are larger than their true values. He will assign to them values $\bar{\epsilon}(L,\epsilon)$ and $\bar{h}(L,h)$.

But if the system is right at the critical point, the two observers must agree, since in this case the correlation length is infinite and hence appears infinite to both of them. Thus $\bar{\epsilon} = \bar{h} = 0$ when $\epsilon = h = 0$.

Let us assume that

$$2.1) \quad \bar{\epsilon} = L^y \epsilon$$

$$\bar{h} = L^x h$$

TABLE II. Parameters describing phase transitions.

Physical quantity	Range of variables: $\epsilon = (T - T_c)/T_c$	h	Behavior of quantity	Parameter describing quantity
$\langle p \rangle$	> 0	0	$\langle p \rangle = 0$	
	< 0	0	$\langle p \rangle \sim \pm\lvert\epsilon\rvert^{\beta}$	β
	0	$\neq 0$	$\sim \pm\lvert h\rvert^{1/\delta}$	δ
$\chi = \partial\langle p \rangle/\partial h\rvert$	> 0	0	$\sim \epsilon^{-\gamma}$	γ
	< 0	0	$\sim \lvert\epsilon\rvert^{-\gamma'}$	γ'
$g(r,r') = \langle p_r\, p_{r'} \rangle - \langle p \rangle^2$	0	0	$\sim \lvert r - r'\rvert^{-d+2-\eta}$	η
ξ = range of $g(r,r')$	> 0	0	$\sim \epsilon^{-\nu}$	ν
	< 0	0	$\sim \epsilon^{-\nu'}$	ν'
C_h = specific heat at constant h	> 0	0	$a\epsilon^{-\alpha} + b$	α
	< 0	0	$a'\lvert\epsilon\rvert^{-\alpha'} + b$	α'
	or > 0	0	$A \log \epsilon^{-1} + B$	$\alpha = 0$
	< 0	0	$A' \log \lvert\epsilon\rvert^{-1} + B'$	$\alpha' = 0$

This clearly satisfies the above requirements on $\bar{\epsilon}$ and $\bar{h}$, although they are not the most general relations that will do so. What is most remarkable is that all nine parameters can be determined from x and y alone if it is assumed that the two observers find that except for their disagreement about ϵ and h, they are in complete agreement about the behavior of the system.

To see what is meant by this cryptic remark, consider the free energy. The near-sighted observer will think that a unit cell will have a free energy $g(\bar{\epsilon},\bar{h})$. A closer inspection would tell him that this cell (which really contains L^d spins) has a free energy $L^d\, \tilde{g}(\epsilon,h)$. The assumption made here is that the two g's are the same; the only difference is in the variables (ϵ,h). But clearly the free energy must be the same, no matter how it is measured. Making use of (2.1), we find

$$2.2)\quad g(L^y \epsilon,\ L^x h) = L^d g(\epsilon,h)$$

A second relation which turns out to be quite useful is the relation between the average magnetization in the two pictures. This will be proportional to the average spin $\langle\sigma\rangle$ on a given sight. The near-sighted observer will describe this in terms of a variable $\langle\mu\rangle$ which is the average spin in the cell of dimension L. The assumption made is that if $\langle\sigma\rangle = f(\epsilon,h)$ then $\langle\mu\rangle = f(\bar{\epsilon},\bar{h})$. To find a relation between these two quantities, suppose that the field h is changed to $h + \delta h$. This will lead to a change in free energy density $\delta g/kT = \langle\sigma\rangle\delta h$, or a change in free energy in the larger cell of $\delta G/kT = L^d\langle\sigma\rangle\delta h$. But the near-sighted observer will measure a free energy change per unit cell of $\langle\mu\rangle\delta\bar{h}$. These two expressions for δG must be equal. Making use of the relation between δh and $\delta\bar{h}$, we find that $\langle\sigma\rangle = L^{x-d}\langle\mu\rangle$ so that

$$2.3)\quad \langle\sigma\rangle \equiv f(\epsilon,h) = L^{x-d}\, f(L^y\epsilon,\ L^x h)$$

Equations (2.2) and (2.3) must be satisfied for <u>all</u> $L \ll \xi$. This imposes strong restrictions on the functions f and g. These restrictions will be satisfied if these functions can be written as

$$2.4a)\quad g(\epsilon,h) = \lvert\epsilon\rvert^{+d/y}\, \bar{g}(\epsilon/h^{y/x})$$

$$2.4b)\quad f(\epsilon,h) = \lvert\epsilon\rvert^{(d-x)/y}\, \bar{f}(\epsilon/h^{y/x})$$

The desired relations between exponents follow from these equations.

(1) Suppose $\epsilon < 0$ and $h = 0$. Then, in the limit $\epsilon \to 0$, by definition (see Table) $f(\epsilon,h) \sim \lvert\epsilon\rvert^{\beta}$. Examining Eq. (2.4b), we see that

2.5) $\beta = (d-x)/y$

(2) When $\epsilon = 0$. $f(\epsilon,h) \sim h^{1/\delta}$. To find δ, we note that Eq. (2.4b) can also be written as

2.6) $$f(\epsilon,h) = h^{(d-x)/x} [(|\epsilon|/h^{y/x})^{(d-x)/y} f(\epsilon/h^{y/x})]$$

When $\epsilon \to 0$, the quantity in brackets must approach a constant value $C = \lim_{z \to 0} [zf(z)]$. Hence

2.7) $\delta = x/d-x$

(3) The coefficients γ and γ' are defined as the degree of the singularity in the susceptibility above and below T_c. From Eq. (2.4a) we find, after taking a derivative

2.8) $$\chi = |\epsilon|^{(d-2x)/y} \lim_{z \to 0} [z\overline{f}'(z)]$$

Hence

2.9) $\gamma = \gamma' = (2x-d)y$

(4) The form of the singularity in the specific heat can be found most easily by differentiating the expression for the free energy density twice. Keeping only the most singular term leads to the expression ($h = 0$)

2.10) $$C(T) \approx \frac{1}{T_c} |\epsilon|^{-2+d/y} \overline{g}(\infty)$$

Thus

2.11) $\alpha = \alpha' = 2 - \frac{d}{y}$

The case $d/y = 2$ needs special consideration. It was shown by Widom and Kadanoff[5] that a logarithmic singularity may then ensue with one and the same coefficient for $\epsilon < 0$ and $\epsilon > 0$.

(5) The last two scaling laws are related to the behavior of the correlation function $g(r - r',\epsilon,h) = \langle \delta\sigma(r)\ \delta\sigma(r') \rangle$. We have already shown that $\langle \sigma \rangle = L^{(x-d)} \langle \mu \rangle$. From this and the scaling hypothesis, it follows that

2.12) $$g(R,\epsilon,h) = L^{2(x-d)}\ g(\tfrac{R}{L},\tilde{\epsilon},\tilde{h})$$

From this, it may be seen that

2.13) $$g(R,\epsilon,h) = |\epsilon|^{2(d-x)/y}\ \tilde{g}(R|\epsilon|^{1/y},\ \epsilon/h^{y/x})$$

It follows that the coherence length ξ varies as $|\epsilon|^{-1/y}$, or, in other words

2.14) $\upsilon = \upsilon' = 1/y$

From Eq. (2.12), it can be seen that when $\epsilon = h = 0$, we must have

2.15) $$g(R,0,0) = R^{2(x-d)} \equiv R^{-(d-2+\eta)}$$

Hence

2.16) $\eta = d + 2 - 2x$

Equations (2.5), (2.7), (2.9), (2.11), (2.14), and (2.16) constitute the scaling laws. A comparison of these predictions with experiment and with exact calculation is given by Kadanoff et al. I will restrict my discussion to the superfluid transition.

B. Applicability of the Scaling Laws to the Superfluid Transition

Recently, Josephson[6] has shown that the observed variation of the superfluid density[7] and the specific heat[8] near the lambda point are in agreement with the scaling laws.

Clow and Reppy[7] interpreted their observation of a $(T_c - T)^{2/3}$ dependence of ρ_s as a demonstration that $\beta = 1/3$ (where $|\Psi| \sim |\epsilon|^{\beta}$). Josephson begins by pointing out that this is not necessarily correct since ρ_s may not be equal to $|\Psi|^2$ near the lambda point. ρ_s can be defined by relating the superfluid kinetic energy to gradient term in the expression for the free energy. This leads to

2.17) $\rho_s = 2\ m^2\ c|\Psi|^2$

In the Landau theory, c is assumed not to vanish at T_c. Josephson argues that this may not be true;

instead, he suggests that $c \sim \xi^{\eta}$ where ξ is the coherence length and η the coefficient defined by Eq. (2.15). Since $\xi \sim |\epsilon|^{-\nu'}$ it follows that $c \sim |\epsilon|^{-\eta\nu}$ so that from Eq. (2.17), we see that the temperature dependence of ρ_s is given by

$$2.18) \quad \rho_s \sim |\epsilon|^{2\beta-\eta\nu'}$$

Hence experiment shows that

$$2.19) \quad 2\beta - \eta\nu' = 2/3$$

By combining Eqs. (2.5), (2.11), (2.15), and (2.16), it can be shown that these scaling laws lead to

$$2.20) \quad 2\beta - \eta\nu' = \frac{1}{3}(2 - \alpha')$$

The observed temperature dependence of the specific heat is logarithmic so that $\alpha' = 0$. This agrees with Eq. (2.20).

This discussion hinges on the relation

$$2.21) \quad c \sim \xi^{\eta}$$

To prove this, Josephson points out that at T_λ, the correlation function for phase fluctuations $g_\varphi(R)$ must vary as $R^{-(1+\eta)}$. This will be true providing its Fourier transform $\chi_\varphi(q)$ varies as $q^{-(2-\eta)}$ at $T = T_\lambda$. But we have seen (Eq. (1.31)) that according to the Landau theory χ_φ varies as $1/cq^2$. Josephson suggests that this is valid below the lambda point for long wavelengths but that for short wavelengths, it breaks down and $\chi_\varphi \sim q^{-(2-\eta)}$. He argues that the long wavelength expression is valid providing $q < 1/\xi(T)$. Equating the two expressions of $\chi_\varphi(q)$ at $q = 1/\xi(T)$ leads to Eq. (2.21).

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