IS-2194

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Research and Development Report

CONTRIBUTION TO THE THEORY OF OPTICAL
REFLECTION AT A METAL SURFACE

by

Joseph M. Keller

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October, 1969

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at
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Robert S. Hansen, Director
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Printed in the United States of America
Available from

Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151

Price: Printed Copy \$3.00; Microfiche \$0.65

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IS-2194

CONTRIBUTION TO THE THEORY OF OPTICAL REFLECTION AT A
METAL SURFACE

Joseph M. Keller

ABSTRACT

Kliwer and Fuchs have formulated a theory for optical reflection at the surface of a free electron metal, including consideration of the nonlocal nature of the conductivity. For the case of p polarization and diffuse scattering of electrons at the surface, the theory leads to a set of three coupled integro-differential equations. The major part of this report is a study of properties of the kernels of these equations. By using these properties the equations may be reduced to two uncoupled equations, one of which can be solved, at least formally, by the Wiener-Hopf technique. An expression for the surface impedance of the metal has not been determined.

I. INTRODUCTION

Fuchs and Kliwer¹ have formulated a theory for optical reflection

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¹ K. L. Kliwer and Ronald Fuchs, Phys. Rev. 172, 607 (1968) (referred to as KF), and private communications.

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at a metal surface, including effects of non-local conductivity on the metal electrons. The behavior of the electrons is described by a Boltzmann equation, in which relaxation is to a local distribution, different from a Fermi distribution. The expressions obtained for the distribution functions are integrals over the electric fields existing throughout the metal. The corresponding Maxwell equations for the fields in the metal are coupled integro-differential equations.

The integral kernels are a set of closely related functions. By studying the properties of these kernels and their inter-relations, I have succeeded in reducing the problem to the solution of two uncoupled equations. One of these involves only the magnetic field (curl of the electric field), and the other only the oscillating charge density (divergence of the electric field). For the case of p polarization of the electromagnetic wave and diffuse reflection of electrons at the surface, the equation for the magnetic field is soluble by standard Wiener-Hopf techniques. I have been unable to solve the other equation.

The metal-vacuum surface is taken to be the $z=0$ plane, with the metal on the positive z side. The plane of polarization is chosen to be the x - z plane. All fields and currents are assumed to have a space-

time dependence

$$F(\underline{r}) = F(z) e^{i(k_x x - \omega t)} .$$

Likewise, the one-particle distribution function is written

$$f = f_0 + f_1(\underline{v}, z) e^{i(k_x x - \omega t)} \quad (1)$$

where f_0 is the equilibrium distribution (in principle a Fermi function, but actually a step function in the usual zero temperature limit).

The fundamental parameters describing the (free) electrons in the metal are a relaxation time τ and a (Fermi) velocity v . The product τv (the mean free path) provides a natural unit of length. Throughout this report all lengths will be considered as multiples of τv ; and k (k_x) will represent the wave vector multiplied by τv . In the distribution function and the integrals describing currents the (circular) frequency only occurs in the combination $\omega = \omega \tau$. The magnitude of the electron charge is represented by e .

For p polarization, there are x and z components of the electric field $\underline{E}$, and the magnetic field $\underline{H}$ has a y component. Assuming diffuse reflection of electrons at the metal-vacuum interface, Kliwer and Fuchs find for $f_1(\underline{v}, z)$, the correction to the distribution function,

$$v_z < 0$$

$$f_1'(\underline{v}, z) = -(e/v_z) (\partial f_0 / \partial e_k) \int_z^\infty ds [v_x E_x(s) + v_z E_z(s) + v G(s)] e^{\xi(s-z)} ;$$

(2a)

$$v_z > 0$$

$$f_1''(v, z) = (e/v_z) (\partial f_0 / \partial e_k) \int_0^z ds [v_x E_x(s) + v_z E_z(s) + vG(s)] e^{-\xi(z-s)} \\ + v \int_0^\infty ds G(s) e^{-\xi(z+s)}. \quad (2b)$$

As already mentioned, it has been assumed that the distribution relaxes to a local distribution different from the equilibrium distribution, and the quantity G is a measure of the difference between these two distributions². The quantity ξ is defined by

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² My G , which has the dimensions of a field, is equal to KF's G divided by ev .

$$\xi = (1-ia) (v/v_z) + ik_x (v_x/v_z).$$

Current densities are obtained by multiplying the distribution function by the electron charge and the appropriate velocity component, then integrating over velocities. The quantity G is determined (self-consistently) by a similar integration over the distribution function. These basic expressions are

$$2G(z) = \int_0^\infty ds \{-iE_x(s) K_x(z-s) + E_z(s) K_z(z-s) + G(s)[K(z-s) + K(z+s)]\}. \quad (3)$$

$$J_x(z) = (3/2) \sigma \int_0^\infty ds \{-E_x(s) K_{xx}(z-s) + iE_z(s) K_{xz}(z-s) \\ -iG(s) [K_x(z-s) + K_x(z+s)]\}. \quad (4a)$$

$$J_z(z) = (3/2) \sigma \int_0^\infty ds \{ iE_x(s) K_{xz}(z-s) + E_z(s) K_{zz}(z-s) + G(s) [K_z(z-s) + K_z(z+s)] \}. \quad (4b)$$

σ is the standard conductivity of the metal, expressed in terms of the (free) electron density n or the plasma frequency ω_p as

$$\sigma = ne^2\tau/m = \omega_p^2\tau/4\pi.$$

II. PROPERTIES OF THE KERNELS

A. Defining Relations.

I have used several different integral representations of the kernels to study their properties. Although they are not all necessary, I include them for completeness. The kernels (with different names) are defined by Kliwer and Fuchs¹ in terms of integrals over an angle θ . I have found it convenient to introduce as an alternative variable of integration $x = \sec\theta$. From here on I shall simply write k instead of k_x . With

$$y = k \sqrt{x^2 - 1} |z|,$$

$$K(z) = \int_1^\infty dx x^{-1} \exp[-(1-ia)x|z|] J_0(y). \quad (5a)$$

The other kernels, along with the names given them by KF, are

$$K_x(z) = \bar{K}'(z) = \int_1^\infty dx x^{-2} \sqrt{x^2 - 1} \exp[-(1-ia)x|z|] J_0'(y), \quad (5b)$$

$$K_z(z) = \bar{K}(z) = -\text{sgn}(z) \int_1^\infty dx x^{-2} \exp[-(1-ia)x|z|] J_0(y), \quad (5c)$$

$$K_{xx}(z) = K''(z) = \int_1^\infty dx x^{-3} (x^2-1) \exp[-(1-ia)x|z|] J_0''(y), \quad (5d)$$

$$K_{xz}(z) = K'(z) = \text{sgn}(z) \int_1^\infty dx x^{-3} \sqrt{x^2-1} \exp[-(1-ia)x|z|] J_0'(y), \quad (5e)$$

$$K_{zz}(z) = \tilde{K}(z) = \int_1^\infty dx x^{-3} \exp[-(1-ia)x|z|] J_0(y). \quad (5f)$$

J_0 is the Bessel function of zero order; ' indicates differentiation with respect to the argument.

A closely related representation is obtained by using as the variable of integration y , the argument of the Bessel function. The argument of the exponential becomes

$$v = k^{-1} (1-ia) \sqrt{y^2 + k^2 z^2}.$$

$$K(z) = \int_0^\infty dy y (y^2 + k^2 z^2)^{-1} e^{-v} J_0(y), \quad (6a)$$

$$K_x(z) = \int_0^\infty dy y^2 (y^2 + k^2 z^2)^{-3/2} e^{-v} J_0'(y), \quad (6b)$$

$$K_z(z) = -kz \int_0^\infty dy y (y^2 + k^2 z^2)^{-3/2} e^{-v} J_0(y), \quad (6c)$$

$$K_{xx}(z) = \int_0^\infty dy y^3 (y^2 + k^2 z^2)^{-2} e^{-v} J_0''(y), \quad (6d)$$

$$K_{xz}(z) = kz \int_0^\infty dy y^2 (y^2 + k^2 z^2)^{-2} e^{-v} J_0'(y), \quad (6e)$$

$$K_{zz}(z) = k^2 z^2 \int_0^\infty dy y (y^2 + k^2 z^2)^{-2} e^{-v} J_0(y). \quad (6f)$$

The kernels K_z and K_{xz} are odd functions of their arguments; the other four kernels are even functions.

B. Laplace Transforms and Another Integral Representation.

The well-known¹ two-sided Laplace transforms of the kernels are simply expressible in terms of three related functions. I use p as the transform variable and define

$$P = \sqrt{p^2 - k^2},$$

$$L = \ln\left(\frac{1 - ia + P}{1 - ia - P}\right).$$

$$F_1(p) = P^{-1} L. \quad (7a)$$

$$F_2(p) = P^{-3} [L - 2P/(1 - ia)]. \quad (7b)$$

$$F_3(p) = P^{-5} [L - 2P/(1 - ia) - 2P^3/3(1 - ia)^3]. \quad (7c)$$

The choice of phase of P is irrelevant, so long as one does not change in mid-formula. The functions $F_n(p)$ are all analytic at $p = \pm k$. In fact they are analytic throughout the region $-1 \leq \text{Re } p \leq 1$. Their only singularities are logarithmic branch points at $p = \pm Q$, where

$$Q = \sqrt{k^2 + (1 - ia)^2}$$

is a complex number in the fourth quadrant. The branch lines extend to infinity away from the imaginary axis. As p approaches infinity, $F_1(p)$ vanishes as $1/p$ and $F_2(p)$ and $F_3(p)$ vanish as $1/p^2$. L is defined to be zero where P is zero.

In terms of the functions $F_n(p)$, the transforms of the kernels are

$$K(p) = F_1(p), \quad (8a)$$

$$K_x(p) = -(1-ia) k F_2(p), \quad (8b)$$

$$K_z(p) = (1-ia) p F_2(p), \quad (8c)$$

$$K_{xx}(p) = 3(1-ia)^2 k^2/2 F_3(p) + [(1-ia)^2 - k^2]/2 F_2(p) - \frac{1}{2} F_1(p), \quad (8d)$$

$$K_{xz}(p) = \frac{1}{2} k p [3(1-ia)^2 F_3(p) - F_2(p)], \quad (8e)$$

$$K_{zz}(p) = 3(1-ia)^2 k^2/2 F_3(p) + [(1-ia)^2 - \frac{1}{2} k^2] F_2(p). \quad (8f)$$

Of course the original functions of z can be recovered from the Laplace transforms by the inverse transformation

$$2\pi i f(z) = \int_{-i\infty}^{i\infty} dp e^{pz} f(p).$$

In applying this process to the kernels, or preferably to the functions F_n , a particularly simple form is obtained for $z > 0$ ($z < 0$) by deforming the contour of integration around the left (right) branch cut. The difference in the value of L on the two sides of the cut contributes a factor $2\pi i$. The resulting inverse Laplace transforms of the F_n functions are

$$F_1(z) = \int_Q^\infty dp p^{-1} e^{-|z|p}, \quad (9a)$$

$$F_2(z) = \int_Q^\infty dp p^{-3} e^{-|z|p}, \quad (9b)$$

$$F_3(z) = \int_Q^\infty dp p^{-5} e^{-|z|p}, \quad (9c)$$

In these expressions the phase of P is no longer arbitrary, and it has been chosen so that the real part of P is positive.

It is evident that the kernels may be written in a similar fashion.

$$K(z) = \int_Q^\infty dp \, p^{-1} e^{-|z|p}. \quad (10a)$$

$$K_x(z) = - (1-ia) k \int_Q^\infty dp \, p^{-3} e^{-|z|p}. \quad (10b)$$

$$K_z(z) = - (1-ia) \operatorname{sgn}(z) \int_Q^\infty dp \, p \, p^{-3} e^{-|z|p}. \quad (10c)$$

$$K_{xx}(z) = \frac{1}{2} \int_Q^\infty dp \, \{ 3 (1-ia)^2 k^2 p^{-5} + [(1-ia)^2 - k^2] p^{-3} - p^{-1} \} e^{-|z|p}. \quad (10d)$$

$$K_{xz}(z) = - \frac{1}{2} k \operatorname{sgn}(z) \int_Q^\infty dp \, p [3(1-ia)^2 p^{-5} - p^{-3}] e^{-|z|p}. \quad (10e)$$

$$K_{zz}(z) = \frac{1}{2} \int_Q^\infty dp \, \{ 3(1-ia)^2 k^2 p^{-5} + [2(1-ia)^2 - k^2] p^{-3} \} e^{-|z|p}. \quad (10f)$$

C. Limiting Behavior of the Kernels.

For sufficiently large values of $|z|$, the entire contribution to the integrals in (10) arises from the immediate neighborhood of the lower limit. The asymptotic behavior of the kernels is thus

$$K(z) \sim (1-ia)^{-1} |z|^{-1} e^{-Q|z|}. \quad (11a)$$

$$K_x(z) \sim -(1-ia)^{-2} k |z|^{-1} e^{-Q|z|}. \quad (11b)$$

$$K_z(z) \sim -(1-ia)^{-2} Q z^{-1} e^{-Q|z|}. \quad (11c)$$

$$K_{xx}(z) \sim (1-ia)^{-3} k^2 |z|^{-1} e^{-Q|z|}. \quad (11d)$$

$$K_{xz}(z) \sim -(1-ia)^{-3} k Q z^{-1} e^{-Q|z|}. \quad (11e)$$

$$K_{zz}(z) \sim (1-ia)^{-3} Q^2 |z|^{-1} e^{-Q|z|}. \quad (11f)$$

The behavior of the kernels for small values of $|z|$ can be obtained from Eqs. (6) or (10). The value of $F_1(z)$ for small $|z|$ is found by using $q = |z|p$ as a variable of integration.

$$\begin{aligned} F_1(z) &= \int_{Q|z|}^{\infty} dq (q^2 - k^2 z^2)^{-\frac{1}{2}} e^{-q} \\ &= \int_{Q|z|}^1 dq (q^2 - k^2 z^2)^{-\frac{1}{2}} - \int_{Q|z|}^1 dq (q^2 - k^2 z^2)^{-\frac{1}{2}} (1 - e^{-q}) \\ &\quad + \int_1^{\infty} dq (q^2 - k^2 z^2)^{-\frac{1}{2}} e^{-q}. \end{aligned}$$

The first of these integrals is elementary. For $z = 0$, the sum of the remaining integrals is the negative of Euler's constant γ ($= 0.577\dots$).

So

$$F_1(z) = -\ln \left[\frac{1}{2}(Q + 1-ia) |z| \right] - \gamma + O(z). \quad (12a)$$

The remaining functions are all finite at $|z| = 0$

$$dF_2(z)/dz = - (1-ia)^{-1} \operatorname{sgn}(z) + O(z \ln z). \quad (12b)$$

$$F_2(z) = k^{-2} [(1-ia)^{-1} Q - 1] + O(z). \quad (12c)$$

$$dF_3(z)/dz = -1/3 (1-ia)^{-3} \operatorname{sgn}(z) + O(z). \quad (12d)$$

$$F_3(z) = 1/3 k^{-2} (1-ia)^{-3} Q - 2/3 k^{-4} [(1-ia)^{-1} Q - 1] + O(z). \quad (12e)$$

The F functions are all even functions of their argument z , whose absolute values are all monotone decreasing functions of z^2 . $F_1(z)$ is logarithmically infinite at the origin, and the other two functions are finite at the origin with discontinuous first derivatives.

The behavior of the kernels for small arguments is as follows.

$$K(z) = -\ln \left[\frac{1}{2}(Q + 1 - ia) |z| \right] - \gamma + O(z). \quad (13a)$$

$$K_x(z) = k^{-1} (1 - ia - Q) + O(z). \quad (13b)$$

$$K_z(z) = -\operatorname{sgn}(z) + O(z \ln z). \quad (13c)$$

$$K_{xx}(z) = \frac{1}{2} \ln \left[\frac{1}{2}(Q + 1 - ia) |z| \right] - \gamma + O(z). \quad (13d)$$

$$K_{xz}(z) = \frac{1}{2} k z \ln |z| + O(z). \quad (13e)$$

$$K_{zz}(z) = \frac{1}{2} + O(z). \quad (13f)$$

D. The F Functions Expressed in Terms of Modified Bessel Functions.

The F functions all satisfy simple second order linear inhomogeneous differential equations. For example, from Eq. (9a) (for $z \neq 0$)

$$\begin{aligned} dF_1(z)/dz &= -\operatorname{sgn}(z) \int_Q^\infty dp \, p \, P^{-1} e^{-|z|p} \\ &= -\operatorname{sgn}(z) \int_Q^\infty dp \, e^{-|z|p} \, dP/dp \\ &= (1 - ia) \operatorname{sgn}(z) e^{-Q|z|} - z \int_Q^\infty dp \, P e^{-|z|p}. \end{aligned}$$

$$d^2 F_1(z)/dz^2 = \int_Q^\infty dp \, p^2 p^{-1} e^{-|z|p}.$$

So

$$[d^2/dz^2 + z^{-1} d/dz - k^2] F_1(z) = (1-ia) |z|^{-1} e^{-Q|z|}. \quad (14a)$$

Likewise (for all z),

$$[d^2/dz^2 - z^{-1} d/dz - k^2] F_2(z) = (1-ia)^{-1} [|z|^{-1} e^{-Q|z|} - 2\delta(z)], \quad (14b)$$

$$[d^2/dz^2 - 3z^{-1} d/dz - k^2] F_3(z) = (1-ia)^{-3} [|z|^{-1} e^{-Q|z|} - 2/3\delta(z)]. \quad (14c)$$

Consider Eq. (14a). This is a useful relation because the solutions of the corresponding homogeneous equations are modified Bessel functions of zero order. Consequently, a particular integral of Eq. (14a) may be written down at once, namely

$$\begin{aligned} \text{P.I.} = & -(1-ia) [K_0(k|z|) \int_0^{|z|} dz' I_0(kz') e^{-Qz'} \\ & + I_0(kz) \int_{|z|}^\infty dz' K_0(kz') e^{-Qz'}]. \end{aligned}$$

This function does not equal $F_1(z)$, as $F_1(z)$ is singular at the origin, and P.I. is not.

To see that P.I. is regular at the origin, observe that it may be written as

$$\begin{aligned}
 \text{P.I.} = & -(1-ia) \left\{ I_0(kz) \int_0^\infty dz' K_0(kz') e^{-Qz'} \right. \\
 & \left. + \int_0^{|z|} dz' [K_0(k|z|) I_0(kz') - I_0(kz) K_0(kz')] e^{-Qz'} \right\}
 \end{aligned}$$

$I_0(0)$ equals unity, and the first integral is a number independent of z . For sufficiently small values of $|z|$, the Bessel functions in the second integral may be approximated by their small argument limits, and the exponential functions by unity. The integrand is, to a sufficient approximation, $\ln(z'/|z|)$, and the second integral vanishes in the limit as $z \rightarrow 0$.

The solution to Eq. (14a) that is identical with $F_1(z)$ is obtained by adding to the particular integral the appropriate multiple of $K_0(k|z|)$ and the result is

$$\begin{aligned}
 F_1(z) = & K_0(k|z|) - (1-ia) \left[K_0(k|z|) \int_0^{|z|} dz' I_0(kz') e^{-Qz'} \right. \\
 & \left. + I_0(kz) \int_{|z|}^\infty dz' K_0(kz') e^{-Qz'} \right]. \quad (15a)
 \end{aligned}$$

One may not, for this purpose, add any multiple of $I_0(kz)$ to the particular integral because I_0 grows exponentially for large arguments.

It is of interest to verify that (15a) has the same asymptotic behavior as (9a) for large arguments. In the limit of large $|z|$

$$\begin{aligned}
\int_{|z|}^{\infty} dz' K_0(kz') e^{-Qz'} &\sim \sqrt{\pi/2k} \int_{|z|}^{\infty} dz' z'^{-\frac{1}{2}} \exp[-(k+Q)z'] \\
&\sim \sqrt{\pi/2k|z|} (k+Q)^{-1} \exp[-(k+Q)|z|].
\end{aligned}$$

For the rest of (15a) I have to distinguish two cases.

Case 1: $k > \text{Re } Q$. $[k^2(a^2-1) > a^2]$.

Divide the integral from 0 to $|z|$ into a part from 0 to w , and a second part from w to $|z|$; w is a number large compared with $1/k$ and independent of z . The first integral is a number independent of z . The second integral is approximately

$$\begin{aligned}
\int_w^{|z|} dz' I_0(kz') \exp(-Qz') &\approx \int_w^{|z|} dz' (2\pi kz')^{-\frac{1}{2}} \exp[(k-Q)z'] \\
&\approx (2\pi k)^{-\frac{1}{2}} \exp[(k-Q)|z|] \int_0^{|z|-w} du (|z|-u)^{-\frac{1}{2}} \\
&\quad \exp[-(k-Q)u] \\
&\approx (2\pi k |z|)^{-\frac{1}{2}} (k-Q)^{-1} \exp[(k-Q)|z|].
\end{aligned}$$

By combining these results with the asymptotic forms of the Bessel functions, one verifies that the asymptotic form of P.I. is indeed identical with that of $F_1(z)$. The bare Bessel function $K_0(k|z|)$ added to the particular integral in (15a) does not contribute to the asymptotic behavior, since it decays according to the steeper exponential $\exp(-k|z|)$.

Case 2: $\text{Re } Q > k$. $[a^2 > k^2(a^2-1)]$.

In this case the function $K_0(k|z|)$ decays more slowly than $F_1(z)$, and so it must be cancelled out, at least for large values of $|z|$, by the particular integral. For $\text{Re } Q > k$, the integral from 0 to $|z|$ may be written as the difference of the integral from 0 to infinity and the integral from $|z|$ to infinity. By using Bessel's integral representation for I_0 , the former integral is

$$\begin{aligned} \int_0^\infty dz I_0(kz) e^{-Qz} &= (2\pi)^{-1} \int_0^\infty dz e^{-Qz} \int_{-\pi}^{\pi} d\varphi \exp(-kz \sin\varphi) \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} d\varphi (Q + k \sin\varphi)^{-1} \\ &= (Q^2 - k^2)^{-\frac{1}{2}}. \end{aligned}$$

So for $\text{Re } Q > k$

$$F_1(z) = (1-ia) \int_{|z|}^{\infty} dz' [K_0(k|z|) I_0(kz') - I_0(kz) K_0(kz')] e^{-Qz'} \quad (15b)$$

and there is no further difficulty in verifying that (15b) has the correct asymptotic form.

Starting from Eq. (14b), one can obtain a similar expression for $F_2(z)$:

$$\begin{aligned} F_2(z) &= k^{-1} [(1-ia)^{-1}Q - 1] |z| K_1(k|z|) \\ &\quad - (1-ia)^{-1} |z| [K_1(k|z|) \int_0^{|z|} dz' I_1(kz') z'^{-1} e^{-Qz'} \\ &\quad + I_1(kz) \int_{|z|}^{\infty} dz' K_1(kz') z'^{-1} e^{-Qz'}] \quad (15c) \end{aligned}$$

which, for $\text{Re } Q > k$ reduces to

$$F_2(z) = (1-ia)^{-1} |z| \int_{|z|}^{\infty} dz' [K_1(k|z|) I_1(kz') - I_1(kz) K_1(kz')] \times z'^{-1} e^{-Qz'}. \quad (15d)$$

E. Relations Between Kernels.

Equations (8) express the four even kernels as linear combinations of the three F functions. The latter can be eliminated to show that

$$2(K_{zz} - K_{xx}) = K - (1-ia)k^{-1} K_x. \quad (16)$$

Bessel's differential equation provides another relation between the kernels as described by Eqs. (5) or (6):

$$K_{zz}(z) - K_{xx}(z) = K(z) + (kz)^{-1} K_{xz}(z). \quad (17)$$

The derivatives of all the kernels are calculable.

$$dK(z)/dz = (1-ia) K_z(z) - 2z^{-1} K_{zz}(z). \quad (18a)$$

$$dK_x(z)/dz = -k K_z(z). \quad (18b)$$

$$dK_z(z)/dz = (1-ia) K(z) - k K_x(z) - 2\delta(z). \quad (18c)$$

$$dK_{xx}(z)/dz = z^{-1} K_{zz}(z) + k K_{xz}(z). \quad (18d)$$

$$dK_{xz}(z)/dz = -(1-ia) K_x(z) + k K_{xx}(z) \quad (18e)$$

$$= z^{-1} K_{xz}(z) + k K_{zz}(z). \quad (18f)$$

$$dK_{zz}(z)/dz = (1-ia) K_z(z) + k K_{xz}(z). \quad (18g)$$

Of these equations, (a) follows directly from (6), (b) from (8), (c), (e),

and (g) from (5). Equations (16) and (17) were used along with the equations of (18) already described to obtain (d) and (f).

Finally, by differentiating $F_2(z)$, a relation is obtained involving the exponential function explicitly.

$$\begin{aligned} dF_2(z)/dz &= \operatorname{sgn}(z) \int_Q^\infty dp e^{-|z|p} d/dp (P^{-1}) \\ &= - (1-ia)^{-1} \operatorname{sgn}(z) e^{-Q|z|} + z F_1(z) \end{aligned}$$

which is equivalent to

$$(1-ia)z K(z) = K_z(z) + \operatorname{sgn}(z) e^{-Q|z|}. \quad (19)$$

III. THE INTEGRAL EQUATIONS AND THEIR UNCOUPLING

I introduce the notation

$$C(z) = dE_x/dz - ikE_z \quad (20a)$$

$$D(z) = ikE_x + dE_z/dz \quad (20b)$$

for (the y component of) the curl and the divergence respectively of the electric field in the metal. For p polarization, Faraday's law gives for (the y component of) the magnetic field

$$H(z) = -i (c/av) C(z) \quad (21)$$

in my units (z and k dimensionless). The remaining Maxwell equations are

$$dC/dz + (av/c)^2 E_x = -4\pi i \omega^{-1} (av/c)^2 J_x, \quad (22a)$$

$$-ikC + (av/c)^2 E_z = -4\pi i \omega^{-1} (av/c)^2 J_z. \quad (22b)$$

Within the metal, the current densities are given by expressions (4). Equations (3) and (22) thus form a set of three coupled integral equations which describe the variation of the electromagnetic fields within the metal. There is also the boundary condition that J_z must vanish at the boundary $z = 0$.

It turns out to be simpler to work with equations obtained by differentiating both members of Eqs. (22), or rather with the longitudinal and transverse parts of the equations so obtained. Thus we are led to calculate the divergence and curl of the current densities. If we use Eqs. (18) to eliminate the derivatives of the kernels that arise, we find from (4)

$$\begin{aligned} \text{div } J &= ikJ_x + dJ_z/dz \\ &= (3/2)\sigma(1-ia) \int_0^\infty ds \{-iE_x(s)K_x(z-s) + E_z(s)K_z(z-s) \\ &\quad + G(s) [K(z-s) + K(z+s)]\} - 3\sigma G(z). \end{aligned} \quad (23)$$

The integral in this expression is precisely that of Eq. (3). So

$$\text{div } J = -3ia\sigma G, \quad (24a)$$

and from Eqs. (22)

$$\begin{aligned} D &= -4\pi i\omega^{-1} \text{div } J \\ &= -3(\omega_p\tau)^2 G. \end{aligned} \quad (24b)$$

G is just a multiple ($-v/3\sigma$) of the charge density, and its inclusion in the Boltzmann equation is equivalent to allowing relaxation to a situa-

tion with a nonvanishing local charge density.

Now use Eq. (18b) to eliminate K_z from (23). An integration by parts transforms (23) to the form

$$\begin{aligned} \text{div } J + 3\sigma G(z) &= 3(1-ia)\sigma G(z) \\ &= (3\sigma/2k)(1-ia) \left[-E_z(0)K_x(z) + \int_0^\infty ds \{ -[(ikE_x(s) + dE_z(s)/ds)K_x(z-s) \right. \\ &\quad \left. + kG(s) [K(z-s) + K(z+s)]] \right]. \end{aligned} \quad (25)$$

The divergence of the current density is expressed by an integral involving the electric field only through its divergence. So the divergence of the field or current satisfies an integral equation that does not involve the transverse part of the field. This equation is

$$\begin{aligned} 2kD(z) &= 3(\omega_p \tau)^2 [E_z(0)K_x(z) + \int_0^\infty ds D(s) K_x(z-s)] \\ &\quad + k \int_0^\infty ds D(s) [K(z-s) + K(z+s)]. \end{aligned} \quad (26)$$

I proceed to calculate the (y component of the) curl of the current density from Eqs. (4). In the resulting integral expression, the coefficient of G vanishes by virtue of Eq. (18b). The derivative of K_{xz} , which appears in the coefficient of E_z in this integral, can be eliminated by use of Eq. (18e). Then the coefficient of E_z takes the form

$$i[k(K_{xx} - K_{zz}) - (1-ia)K_x] = iz^{-1}K_{xz} = ik\mathcal{K}$$

[the equality by Eqs. (16) and (17)]. Equations (18b and g) can be used to show that the coefficient of E_x is proportional to the derivative of

this same quantity. As a result

$$\begin{aligned}
 \text{curl } J &= dJ_x(z)/dz - ikJ_z \\
 &= (3/2)\sigma \int_0^\infty ds [E_x(s) d\mathcal{K}(z-s)/ds + i E_z(s) k \mathcal{K}(z-s)] \\
 &= (3/2)\sigma \{-E_x(0) \mathcal{K}(z) - \int_0^\infty ds [dE_x(s)/ds - ikE_z(s)] \mathcal{K}(z-s)\}.
 \end{aligned}
 \tag{27}$$

The curl of the current density is expressed in terms of an integral that involves the electric field only through its curl. So Maxwell's equation for the curl of the electric field becomes an integro-differential equation that does not involve the longitudinal part of the field.

$$\begin{aligned}
 (d^2/dz^2 + g^2)C(z) &= -4\pi i \omega^{-1} (av/c)^2 \text{curl } J \\
 &= ib [E_x(0) \mathcal{K}(z) + \int_0^\infty ds C(s) \mathcal{K}(z-s)].
 \end{aligned}
 \tag{28}$$

The constants in this equation are³

$g^2 = (av/c)^2 - k^2$ (the square of the z component of the wave vector of the electromagnetic wave in vacuum); and

$$b = (3a/2) (\omega_p \tau v/c)^2.$$

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³ The constant b is twice the constant α of G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. A195, 336 (1948). It is related to KF's κ as $b = (\tau v)^3 i \kappa$.

Properties of the kernel $\mathcal{K}$ follow from the relation

$$\mathcal{K}(z) = \frac{1}{2} [(1-ia)^2 F_2(z) - F_1(z)]. \quad (29)$$

The asymptotic behavior for large argument is

$$\mathcal{K}(z) \sim -(1-ia)^{-3} Q z^{-2} e^{-Q|z|}.$$

For small values of z

$$\mathcal{K}(z) = \frac{1}{2} \left\{ \ln \left[\frac{1}{2} (Q + 1-ia) |z| \right] + \gamma + k^{-2} (1-ia) (Q - 1+ia) \right\} + O(z).$$

Equations (26) and (28) are the two uncoupled integral integrations that describe the behavior of the fields within the metal.

IV. THE WIENER-HOPF SOLUTION

Equation (28) can be solved, at least formally, by the Wiener-Hopf technique. The equation is only valid for z positive. It is convenient in this section to define $C(z)$ to be zero for negative values of z .

Define a new function $M(z)$ by

$$M(z) = 0 \text{ for } z > 0$$

$$= ib[E_x(0) \mathcal{K}(z) + \int_0^\infty ds C(s) \mathcal{K}(z-s)] \text{ for } z < 0. \quad (30)$$

Add Eqs. (28) and (30) and take the two-sided Laplace transform of the sum. In terms of the transform variable p , the result is

$$M(p) - h = ibE_x(0) \mathcal{K}(p) + [ib\mathcal{K}(p) - p^2 - g^2] C(p). \quad (31)$$

The quantity $[dC(z)/dz + pC(z)]$, evaluated at $z = 0$, has been abbreviated as h . $M(p)$ is a function regular in the half space to the left of Q . $C(z)$ must fall off for large z as some finite power of z times

$\exp(-Qz)$, and so $C(p)$ must be regular in the half space to the right of $-Q$. $\mathcal{K}(p)$ is known to be regular in the strip $-\text{Im } Q < \text{Im } p < \text{Im } Q$, and so Eq. (31) must be valid in this strip.

The Wiener-Hopf method requires breaking the coefficient of $C(p)$ into two factors, each of which is regular in a right or left half plane. For this purpose, write

$$ib \mathcal{K}(p) - p^2 - g^2 = (N^2 - p^2) U(p),$$

where N is any number with a positive real part. For the present it is useful to think of $\text{Re } N > \text{Re } Q$. $U(p)$ approaches unity as p becomes arbitrarily large in any direction. And I shall now show that $U(p)$ has no roots in a strip of finite width centered about the imaginary axis.

There is no question of roots of U for arbitrarily large imaginary values of p . For finite pure imaginary values of p , the real part of $\mathcal{K}(p)$, and hence the imaginary part of $U(p)$, is negative and bounded away from zero. For consider $\mathcal{K}(p)$ as p starts from the value k , goes along the real axis to the origin, and then up or down the imaginary axis to $\pm i\infty$. Along this entire path P is pure imaginary, say $P = iq$ where q starts out zero and increases monotonely. For small P and q

$$\mathcal{K}(p) = -(2/3)(1 - ia)^{-1} + O(P)$$

which has a negative real part; and by differentiating $q^3 \mathcal{K}(p)$ one can show that the real part of this function for pure imaginary P (real q) is a monotone decreasing function of q . Thus, except possibly in the limit $P \rightarrow i\infty$ ($q \rightarrow \infty$), the imaginary part of $U(p)$ is bounded away from zero,

which demonstrates the existence of a strip of finite width centered on the imaginary axis in which $U(p)$ has no roots.

Throughout this strip $\ln U(p)$ is an analytic function, which we may write in the form

$$\begin{aligned} \ln U(p) &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} dz \frac{\ln U(z)}{z-p} + \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} dz \frac{\ln U(z)}{p-z} \\ &= -\ln U_-(p) + \ln U_+(p). \end{aligned}$$

d is a positive number limited by the half width of the strip. I adopt the convention of writing denominators in these integrands so that their real parts are always positive. I have thus defined the pair of functions $U_+(p)$ regular in the half plane $\operatorname{Re} p > -d$, and $U_-(p)$ regular in the half plane $\operatorname{Re} p < d$.

Now multiply both members of Eq. (31) by $U_-(p)/(N-p)$. The result is

$$\begin{aligned} (N-p)^{-1} U_-(p) [M(p) - h - (p^2 + g^2) E_X(0)] &= (N+p) U_+(p) [E_X(0) + C(p)] \\ &= A(p). \end{aligned} \tag{32}$$

The first equality in (32) is valid in the strip of width $2d$ about the imaginary axis. But by the usual Wiener-Hopf argument the first member of (32) is analytic in a left half plane and thus extends the definition of $A(p)$ from a function defined and analytic over a strip to a function defined and analytic over the left half plane. Likewise the second member of (32) is analytic in a right half plane and thus extends the definition of $A(p)$ to a function defined and analytic everywhere.

The nature of the function $A(p)$ can be determined by examining it for large values of $\text{Re } p$. In this limit, $C(p) \sim C(z=0)/p$ and $\ln U_+(p) \sim R/p$ where R is a constant. So

$$A(p) = (p + N + R) E_x(0) + C(0),$$

and

$$C(p) = -E_x(0) + (N + p)^{-1} U_+(p)^{-1} [(p + N + R)E_x(0) + C(0)]. \quad (33)$$

Previously I asserted that the value of N was immaterial. It is easy to show, by differentiating, that in fact $(N + p)U_+(p)$ (and consequently $N + R$) is independent of N .

If in the limit $\omega_p \tau \gg 1$ one is willing to drop the second integral in Eq. (26), this equation may be solved in the same manner.

V. UNIQUENESS OF THE FIELDS

From the original integral equations [Eqs. (22)] in the electric fields, I have in the preceding sections derived equations [Eqs. (26) and (28)] for the divergence and curl of the electric field. I now consider two related questions: to what extent is the electric field in the metal determined by its divergence and curl; and is it guaranteed that the field so determined satisfies the original equations?

The electric field has no y component, and no dependence on y . The x dependence of the field is as e^{ikx} . With the boundary condition that the field must vanish for large z , it is easy to show that the field is expressible in terms of its divergence $D(z)$ and (the y component of) its

curl $C(z)$ as

$$\begin{aligned}
 E_x(z) &= B e^{-kz} + \frac{1}{2} \int_0^z ds [C(s) - iD(s)] e^{-k(z-s)} \\
 &\quad - \frac{1}{2} \int_z^\infty ds [C(s) + iD(s)] e^{-k(s-z)}, \\
 E_z(z) &= iB e^{-kz} + \frac{1}{2}i \int_0^z ds [C(s) - iD(s)] e^{-k(z-s)} \\
 &\quad + \frac{1}{2}i \int_z^\infty ds [C(s) + iD(s)] e^{-k(s-z)}.
 \end{aligned} \tag{34}$$

B is a constant not determined by the derivatives of the electric field. It must be chosen to satisfy the remaining boundary condition, i.e., that the normal current density vanishes at the metal surface. By setting z equal to zero in Eqs. (33) we see that

$$\begin{aligned}
 E_x(0) &= B - \frac{1}{2} [C(k) + iD(k)], \\
 E_z(0) &= i\{B + \frac{1}{2} [C(k) + iD(k)]\};
 \end{aligned}$$

$C(k)$ and $D(k)$ are the Laplace transforms of C and D with transform variable k .

To show to what extent any solution of Eqs. (26) and (28) is a solution of Eqs. (22), I outline a procedure by which Eqs. (22) might be derived from Eqs. (26) and (28). The manipulations of the kernels, so essential in section III, are all clearly reversible, and therefore irrelevant here. The problem is one of integration.

Let $X_L(z)$ and $X_R(z)$ represent respectively the left and right members of Eq. (22a), similarly let $Y_L(z)$ and $Y_R(z)$ represent the corres-

ponding members of Eq. (22b). These functions are all defined for non-negative values of z and vanish for sufficiently large values of z . For each subscript, define

$$\Gamma(z) = dX(z)/dz - ikY(z),$$

$$\Delta(z) = ikX(z) + dY(z)/dz.$$

Integrations by parts yield the following identities, suggested by Eqs. (33).

$$\begin{aligned} \frac{1}{2} \int_0^z ds [\Gamma(s) - i\Delta(s)] e^{-k(z-s)} - \frac{1}{2} \int_z^\infty ds [\Gamma(s) + i\Delta(s)] e^{-k(s-z)} \\ = X(z) - \frac{1}{2} [X(0) - iY(0)] e^{-kz}, \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{1}{2}i \int_0^z ds [\Gamma(s) - i\Delta(s)] e^{-k(z-s)} + \frac{1}{2}i \int_z^\infty ds [\Gamma(s) + i\Delta(s)] e^{-k(s-z)} \\ = Y(z) - \frac{1}{2}i [X(0) - iY(0)] e^{-kz}. \end{aligned}$$

So it follows from Eqs. (26), (28), and (35) that

$$X_L(z) - \frac{1}{2} [X_L(0) - iY_L(0)] e^{-kz} = X_R(z) - \frac{1}{2} [X_R(0) - iY_R(0)] e^{-kz}, \quad (36)$$

$$Y_L(z) - \frac{1}{2}i [X_L(0) - iY_L(0)] e^{-kz} = Y_R(z) - \frac{1}{2}i [X_R(0) - iY_R(0)] e^{-kz}.$$

From either of Eqs. (36) it follows that

$$X_L(0) + iY_L(0) = X_R(0) + iY_R(0).$$

Equations (35) imply Eqs. (22) except for components of fields and

currents proportional to e^{-kz} . One would expect that the constant B could be chosen to resolve this difficulty. But B has already been selected to guarantee no normal current density at the metal surface, and it is not clear that these two responsibilities for B are compatible. If not, the implication must be that the original theory is not entirely self-consistent from a physical standpoint.

ACKNOWLEDGMENTS

I am grateful for many discussions with Drs. B. C. Carlson, R. Fuchs, K. Kliewer, T. Weber and H. J. Weiss. In particular, the very useful representation of section IIB was suggested by a discussion on asymptotic behavior with Dr. Weber.