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Subroutine RKLDEQ for the Integration of First Order Ordinary

Differential Equations on the IBM 360/75

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Abstract

The FORTRAN subroutine RKLDEQ for the integration of first order ordinary differential equations using the Runge-Kutta-Gill method has been adapted for use on the IBM 360/75 computer. This memorandum gives a description of the use of this subroutine and the results of some test calculations. For the equations and step sizes used in this study, the observed cumulative truncation errors were proportional to the fourth power of the step size, in agreement with the theoretical behavior of a fourth-order method. Round off errors were not observed for any step size.

Since this subroutine is used in an unconventional way with the calculation of the derivatives being done in the calling program, a sample program is displayed in the text and must be carefully studied if the subroutine is to be used correctly. Familiarity with Gill's fourth-order Runge-Kutta method is also desirable.

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I. Introduction:

The subroutine RKLDEQ⁽¹⁾ for the CDC 1604A computer was originally written in FORTRAN-62 as a single-precision FUNCTION subprogram, using Gill's⁽²⁾ version of the fourth-order Runge-Kutta method.⁽³⁾ It has been adapted for use on the IBM 360/75 using the FORTRAN/360 programming language⁽⁴⁾, as a double-precision subroutine with data transmission through blocks of labelled COMMON. Detailed directions for its use are given in Section II.

Some test calculations were done with the subroutine on the 360/75 and the results of these are discussed in Section III. FORTRAN listings of the program for these calculations and of the subroutine are given in the appendix.

II. Use of the Subroutine:

Given a system of n first order ordinary differential equations of the form

$$y'_i = f_i(x, y_1, y_2, \dots, y_n) \quad i = 1, 2, \dots, n \quad (1)$$

where the prime indicates differentiation with respect to the independent variable x, and given a set of values

$$y_i(x), \quad i = 1, 2, \dots, n, \quad (2)$$

the subroutine RKLDEQ is used to calculate a set of values

$$y_i(x+h), \quad i = 1, 2, \dots, n \quad (3)$$

and must be called four times in order to do this. Since the procedure is complicated it is best illustrated by an actual example program, as shown in Fig. I.

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Before the subroutine is called the first time the initial values $y_i(x)$ are stored in the array Y, the derivatives f_i are stored in the array F, the step size h is stored in the location H, $h/2$ is stored in the location HALF, the number of equations n is stored in the location N and the independent variable x is stored in the location X in the COMMON block labelled DIST. The fixed point variable IWS will have the value 1 on return from the subroutine until the fourth return, when it is set to 0. At the return from the fourth call of the subroutine, the array Y contains the values $y_i(x+h)$ and the location X contains the value $x + h$.

Since the derivative functions f_i are evaluated twice at $x + h/2$ and twice at $x + h$, the fixed point variable KWS in the COMMON block KS is used as an indicator that the value in location X has been altered by RKLDEQ. If the derivative functions include functions of the independent variable that would be time consuming to calculate it is recommended that the indicator KWS be tested in the calling program and an appropriate branch set up to avoid recalculating the x -dependent parts of the functions f_i . This technique is demonstrated in Fig. I, a program for solving the equations*

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= e^{-x}y_1 \end{aligned} \quad (4)$$

* It is to be noted that a faster procedure for this system would be achieved if a third dependent variable were included:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 y_3 \end{aligned}$$

$y_3' = -y_3$, $y_3(0) = 1$. The point of the example in Fig. I is the illustration of the use of the indicator KWS.

subject to the initial conditions

$$y_1(0) = 1$$

$$y_2(0) = 0$$

for $0 \leq x \leq 1$ with a step size h of 0.01.

III. Test calculations:

Three simple systems of two equations were integrated with step sizes varying from .1 to .005 over the range $0 \leq x \leq 1$:

$$\begin{array}{lll} y_1' = y_2 & y_1(0) = 0 & y_1 = \sin x \\ y_2' = -y_1 & y_2(0) = 1 & y_2 = \cos x \end{array} \quad (a)$$

$$\begin{array}{lll} y_1' = y_1 & y_1(0) = 1 & y_1 = e^x \\ y_2' = 3y_2 & y_2(0) = 1 & y_2 = e^{3x} \end{array} \quad (b)$$

$$\begin{array}{lll} y_1' = -y_1 & y_1(0) = 1 & y_1 = e^{-x} \\ y_2' = y_1 - y_2/2 & y_2(0) = 0 & y_2 = 2(e^{-x/2} - e^{-x}) \end{array} \quad (c)$$

The exact solutions were computed using the library double precision subroutines and the errors were computed from

$$\epsilon = y(\text{approx}) - y(\text{exact}).$$

The errors for cases (a) thru (c) are displayed in Table I and the errors for case (a) are plotted in Fig. 2 on a log plot. The straight lines in the figure indicate the function

$$\epsilon = Ah^4$$

and the computed errors fall on these lines for $h \leq .05$.

Table I. Computed Errors

Case a.

$y_1 - \sin x$					
x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	-1.655(10^{-7})	-1.029(10^{-8})	-2.622(10^{-10})	-1.636(10^{-11})	-1.022(10^{-12})
.4	-3.167	-1.951	-4.946	-3.081	-1.922
.6	-4.347	-2.650	-6.697	-4.150	-2.585
.8	-5.026	-3.025	-7.558	-4.684	-2.914
1.0	-5.070	-2.994	-7.390	-4.561	-2.830

$y_2 - \cos x$					
x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	0.194(10^{-7})	0.164(10^{-8})	0.486(10^{-10})	0.318(10^{-11})	0.204(10^{-12})
.4	1.038	0.731	1.995	1.273	0.805
.6	2.470	1.656	4.406	2.789	1.756
.8	4.379	2.865	7.527	4.744	2.980
1.0	6.612	4.261	11.098	6.975	4.374

Case b.

$y_1 - e^x$					
x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	-0.187(10^{-6})	-0.122(10^{-7})	-0.320(10^{-9})	-0.202(10^{-10})	-0.128(10^{-11})
.4	-0.458	-0.298	-0.782	-0.493	-0.311
.6	-0.838	-0.546	-1.434	-0.904	-0.570
.8	-1.365	-0.889	-2.335	-1.472	-0.928
1.0	-2.084	-1.358	-3.564	-2.247	-1.416

$$y_2 - e^{3x}$$

x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	-0.058(10 ⁻³)	-0.041(10 ⁻⁴)	-0.112(10 ⁻⁶)	-0.072(10 ⁻⁷)	-0.046(10 ⁻⁸)
.4	-0.210	-0.148	-0.409	-0.262	-0.166
.6	-0.573	-0.406	-1.119	-0.717	-0.454
.8	-1.392	-0.985	-2.718	-1.742	-1.102
1.0	-3.170	-2.244	-6.191	-3.967	-2.511

Case c.

$$y_1 - e^{-x}$$

x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	1.483(10 ⁻⁷)	0.889(10 ⁻⁸)	2.220(10 ⁻¹⁰)	1.376(10 ⁻¹¹)	0.856(10 ⁻¹²)
.4	2.429	1.456	3.635	2.253	1.402
.6	2.983	1.788	4.464	2.767	1.721
.8	3.256	1.952	4.873	3.021	1.879
1.0	3.332	1.998	4.988	3.091	1.923

$$y_2 - 2(e^{-x/2} - e^{-x})$$

x	h = .1	h = .05	h = .02	h = .01	h = .005
.2	-2.868(10 ⁻⁷)	-1.718(10 ⁻⁸)	-4.288(10 ⁻¹⁰)	-2.657(10 ⁻¹¹)	-1.653(10 ⁻¹²)
.4	-4.680	-2.803	-6.995	-4.335	-2.697
.6	-5.724	-3.428	-8.555	-5.301	-3.298
.8	-6.221	-3.726	-9.296	-5.761	-3.584
1.0	-6.335	-3.794	-9.465	-5.865	-3.650

References

- 1) H. E. Ferguson, Runge-Kutta-Gill Differential Equation Solver, COOP Writeup, Feb. 1, 1962.
- 2) S. Gill, A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine, Proc. Cambridge Philos. Soc. 47, 96-108 (1951)
- 3) M. J. Romanelli, Runge-Kutta Methods for the Solution of Ordinary Differential Equations, Chap. 9 in Mathematical Methods for Digital Computers, A. Ralston and H. Wilf eds., John Wiley and Sons, New York, 1960.
- 4) ORCID Memo 1/360

**FTN,L,E,M.	
PROGRAM RK1	RK1 100
COMMON/FUNC/Y(10),F(10),Q(10),H,NEQ,IWS,HALF/DIST/X/KS/KWS	RK1 105
DATA (Y=1.0,00),(H=.010,2,0,.0050),(X=00),(KWS=1)	RK1 110
C	RK1 115
C INITIAL CONDITIONS Y(1)=1, Y(2)=0, X=0	RK1 120
C STEP SIZE H = .01	RK1 125
C DOUBLE PRECISION	RK1 130
C	RK1 135
C IMPLICIT REAL*8 (A-H,O-Z)	RK1 140
C EXPMX=1.	RK1 145
C LINES=0	RK1 150
C DO 106 N=1,100	RK1 155
100 IF(KWS)102,101	RK1 160
C	RK1 161
C BRANCH AROUND CALCULATION OF EXP(-X), IF KWS NON-ZERO	RK1 162
C	RK1 163
101 EXPMX=DEXP(-X)	RK1 165
102 F(1)=Y(2)	RK1 170
F(2)=EXPMX*Y(1)	RK1 175
CALL RKLDEQ	RK1 180
IF(IWS)100,103	RK1 185
C	RK1 190
C GO BACK UNLESS IWS = 0	RK1 195
C	RK1 200
C PRINT RESULTS	RK1 205
C	RK1 210
103 IF(LINES)105,104	RK1 215
104 PRINT 1 \$ LINES=50	RK1 220
1 FORMAT('1 X'10X1HY13X2HY'/1X)	RK1 225
105 PRINT 2,X,(Y(I),I=1,2)	RK1 230
2 FORMAT(1XOPF6.2,1P2E14.5)	RK1 235
106 LINES=LINES-1	RK1 240
STOP	RK1 245
END	RK1 250

Fig. 1. FORTRAN/360 Program Using RKLDEQ.

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Fig. 2. Computed Error as a Function of Step Size.

SUBROUTINE RKLDEQ	RKLD 100
COMMON/FUNC/Y(10),F(10),Q(10),H,N ,IWS,HALF/DIST/X/KS/KWS	RKLD 105
DATA (NT=0)	RKLD 110
IMPLICIT REAL*8 (A-H,O-Z)	RKLD 115
NT=NT+1 \$ GO TO (100,200,300,400),NT	RKLD 120
100 DO 101 J=1,N	RKLD 125
101 Q(J)=0 \$ A=0.5	RKLD 130
102 X=X+HALF \$ KWS=0 \$ GO TO 500	RKLD 135
200 A=.2928932188634525 \$ KWS=1 \$ GO TO 500	RKLD 140
300 A=1.707106781136548 \$ GO TO 102	RKLD 145
400 DO 401 J=1,N	RKLD 150
401 Y(J)=Y(J)+.3333333333333333*(HALF*F(J)-Q(J))	RKLD 155
NT=IWS=0 \$ KWS=1 \$ GO TO 600	RKLD 160
500 ABH=A*H \$ DO 501 J=1,N	RKLD 165
AQ=A*Q(J) \$ T=ABH*F(J) \$ Y(J)=T-AQ+Y(J)	RKLD 170
501 Q(J)=T+T+Q(J)-3.*AQ \$ IWS=1	RKLD 175
600 END	RKLD 180

Fig. 3. FORTRAN/360 Subroutine RKLDEQ.

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