HIGH ENERGY SCATTERING OF HADRONS

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May, 1971

Contract No. AT (11-1)-264

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Meinem verehrten Lehrer

Herrn Professor Werner Heisenberg

zum siebzigsten Geburtstage gewidmet

Zusammenfassung

Nach einer kurzen Einleitung über die Entwicklung der Dispersions-
theorie wird zunächst der Zusammenhang zwischen der asymptotischen
Form von Streuamplituden bei hohen Energien und den Singularitäten
der Partialwellenamplituden in der komplexen ℓ -Ebene des gekreuzten
Kanals betrachtet. Einschränkungen des Charakters und der analytischen
Eigenschaften dieser singulären Flächen durch Analytizitätseigenschaften
und Unitaritätsbedingungen in beiden Kanälen werden abgeleitet und deren
Einfluss auf das Verhalten der Amplitude bei hohen Energien besprochen.
Insbesondere werden komplexe Trajektorien eingeführt und ihr Zusammenhang
mit Stossparameter Darstellungen aufgezeigt. Die Betrachtungen beziehen
sich hauptsächlich auf Probleme der Diffraktionstreuung und auf damit
zusammenhängende Fragen.

I. INTRODUCTION: Dispersion Theory.

Twenty years ago in Göttingen, I wrote an article in collaboration with Lüders and Thirring which was dedicated to Heisenberg on the occasion of his fiftieth birthday.¹ In this paper, we explored the possible implications of covariant perturbation theory and of the renormalization scheme for the physics of strongly interacting particles. Although we saw some precursors of later developments, we also found that the methods of perturbation theory, which worked so well in quantum electrodynamics, were completely insufficient for hadronic interactions. We indicated that a new mathematical formalism was required which incorporated the general invariance principles. Indeed, a few years later, such a formalism began to take shape with the development of relativistic dispersion theory.

Dispersion Relations for the forward scattering of light had been known for quite some time,² and they were rederived on the basis of relativistic quantum electrodynamics.³ Although analytic methods had been used before in order to make the connection between causality and forward dispersion relations,^{2,3} for the extension of dispersion theory to hadronic interactions it became essential to recognize the importance of the complex plane and to view physical amplitudes as boundary values of analytic functions.⁴ In particular, certain related amplitudes were seen to be different boundary values of one and the same analytic function of two or more complex variables. It is true that these crossing properties can be read off from perturbation theoretical

models, but they were introduced at this time directly on a more general basis. Perturbation theory was so much in disrepute that it would not have been sufficiently convincing for the acceptance of these results. The further exploration of the analytic properties of amplitudes changed this view, and we learned to use perturbation theory as a valuable guide to extend our understanding of the singularity structure of Green's functions.⁵ Particularly, the artificial limitations of general proofs of analytic properties from the basic assumptions of field theory could be judged very well on the basis of Feynman graphs.⁶ The latter were also most helpful for the understanding of those singularities which are not directly related to the physical intermediate states of a given channel.^{6,7} These singularities describe the structure effects of particles considered as composite systems of other particles;⁸ they emerge from unitarity and crossed channel properties. We have described these singularities ten years ago in an article entitled "The Compound Structure of Elementary Particles", which was written on the occasion of Heisenberg's sixtieth birthday.⁹ These structure singularities or anomalous thresholds may well be of importance in connection with scaling properties of off-shell amplitudes at nonasymptotic energies, and for higher order Green's functions.

On the phenomenological side, the prompt and astonishing success of forward pion-nucleon dispersion relations¹⁰ gave a wellcome boost to the analytic approach to particle physics, as did sensible pole dominance methods and other approximate dispersion calculations for low energy scattering.¹¹ On the other hand, high energy limits came into focus with the advent of complex angular momentum methods. These provided a tool to correlate the asymptotic expansion in one channel with the low energy properties of the appropriate crossed channels. They also brought a connection between the

classification of hadrons and the analytic methods.¹²

Approximate higher symmetries had been developed rather independent of dispersion theoretical methods.¹³ But then a very interesting connection was discovered with the help of superconvergent dispersion relations.¹⁴ It was found that, for amplitudes which vanish sufficiently fast at infinity, these dispersion relations could be saturated with a finite number of resonances. Consistency requires that the saturation is made with sets of resonances which fall into the lower representations of rest symmetry groups like¹⁵ $U(6) \otimes U(6)$, and it turns out that vertex symmetries of collinear $U(6)$ ¹⁶ emerge automatically.¹⁷ Since these symmetries are broken in reality, their association with an approximate scheme within the dispersion theoretical framework seems most natural. In any case, we have here a certain connection between the analytic methods of dispersion theory and the group theoretical structure which is characteristic for the quark model.

The approximation of amplitudes with only resonance states has in recent years been extended by the introduction of meromorphic functions which satisfy crossing and which have Regge asymptotic behaviour in directions away from the real axes.¹⁸ The resonances in one channel are generated by the controlled divergence of the infinite sum of resonances in the corresponding crossed channel (mathematical duality).¹⁹ Meromorphy makes it possible to write down explicit forms with factorized residues even for production amplitudes.²⁰ But the ghosts which have been seen before with superconvergence relations are still present in spite of the exponentially increasing degeneracy of states which factorization requires, although in

certain models they reappear in the form of Tachyons. Nevertheless, there are attempts to use meromorphic amplitudes as a basis for a new approximation scheme which would bring in the unitarity constraints.

Of special interest are the implications of dispersion theory and complex angular momentum methods for high energy scattering. Unfortunately, what we have here is only a general scheme rather than a complete theory. We know from basic assumptions that there must be Regge poles, branch points and possibly other singularities in the complex angular momentum plane.²¹ But at present, we have no reliable method to calculate the actual singularity structure. In general there are many constraints, and for certain situations these can be quite restrictive.

There are several models for diffraction scattering which use quasiclassical notions or intuitive abstractions from field theory.²² These are often considered as an end in themselves because they can be easily explained and understood. On the other hand, the methods of dispersion theory are more mathematical and more abstract, and sometimes rather subtle to handle. Nevertheless, we think that this framework is a very comprehensive one for the description of hadrons and hence that we should try to understand diffraction scattering on this basis. Of course, as we have pointed out on previous occasions,²¹ it is not impossible that the dispersion scheme itself is too narrow and that there is something wrong with the basic assumptions. To find such a discrepancy would be a most important discovery in itself and may give a clue for a more concise theory.

In the following we describe some of the constraints dispersion theory imposes upon the high energy limits of amplitudes. We are particularly interested in diffraction scattering, which is a most important and central problem. It is also of high current interest in view of the new data now becoming available from Serpukhov, CERN and NAL.

II. COMPLEX ANGULAR MOMENTUM

1. Partial Wave Amplitudes

We consider first the high energy behavior of scattering amplitudes as described by Regge trajectories in the complex angular momentum plane of the crossed channel. The unique analytic interpolation of partial wave amplitudes is a direct consequence of the existence of single variable dispersion relations with a finite number of subtractions.²² Let us use invariant amplitudes $F(s,t)$ and $\bar{F}(s,t)$ for particle and antiparticle scattering,* and introduce the combinations

$$F_{\pm}(s,t) = F(s,t) \pm \bar{F}(s,t) ; \quad (2.1)$$

they satisfy the reality condition

$$F_{\pm}^*(s^*, t^*) = \bar{F}_{\pm}(s, t) , \quad (2.2)$$

and they have the symmetry property

$$F_{\pm}(s, t) = \pm F_{\pm}(u, t) , \quad (2.3)$$

where $u = -s-t+\Sigma$ (Σ = sum of squares of external masses). For appropriate values of t (like real $t \leq 0$), we have the dispersion relations

*For simplicity of writing, we use the Feldverein model.

$$F_+(v, t) = F_+(v_0, t) + \frac{2(v^2 - v_0^2)}{\pi} \int_0^\infty dv' \frac{v' A_+(v', t)}{(v'^2 - v^2)(v'^2 - v_0^2)}, \quad (2.4)$$

$$F_-(v, t) = \frac{v}{v_0} F_-(v_0, t) + \frac{2v(v^2 - v_0^2)}{\pi} \int_0^\infty dv' \frac{A_-(v', t)}{(v'^2 - v^2)(v'^2 - v_0^2)}, \quad (2.5)$$

where the symmetric variable $v \equiv \frac{1}{2}(s-u)$ has been introduced for simplicity.

On the basis of the dispersion relations we obtain continued partial wave amplitudes

$$F_\pm(t, \lambda) = \frac{1}{\pi} \int_{v_0}^\infty dv \frac{1}{2q^2(t)} Q_\lambda\left(1 + \frac{v}{2q^2(t)}\right) A_\pm(v, t), \quad (2.6)$$

with

$$A_\pm(v, t) = A_s(v, t) \pm A_u(v, t). \quad (2.7)$$

The absorptive parts A_s and A_u determine analytic functions which uniquely interpolate the physical partial wave amplitudes for $\ell > 1$:

$$F_\pm(t, \lambda = \ell) = F_\ell(t), \quad \ell = \text{even/odd}. \quad (2.8)$$

Since Q_λ is a meromorphic function with only simple poles at negative integer values of λ , we have in Eq. (2.6) a direct correspondence between the asymptotic expansion of absorptive parts $A_\pm(v,t)$ for $v \rightarrow \infty$, and the singularities of $F_\pm(t,\lambda)$ in the complex λ -plane. For sufficiently large values of $\text{Re}\lambda$, these functions are always given by the representation (2.6), and the continuation to smaller values of λ can be done with the help of the asymptotic expansion of $A_\pm(v,t)$. Hence the singular surfaces of $F_\pm(t,\lambda)$ are determined by the high energy part

$$F_{b^2\pm}(t,\lambda) \propto \int_{b^2}^{\infty} dv v^{-\lambda-1} A_\pm(v,t) \quad (2.9)$$

of $F_\pm(t,\lambda)$, with b^2 taken as large as we please. We see that the power behavior of an asymptotic term in the expansion of $A_\pm$ corresponds generally to the position of the related singularity in the complex λ -plane; logarithmic factors are related to the character of the singular surfaces.

We are here particularly interested in the restrictions imposed upon the character of λ -plane singularities by the unitarity condition in the t -channel. We have discussed this subject on several previous occasions²³,

but there are reasons to resume and to combine these earlier treatments.*

2. Continued Unitarity

The continued unitarity condition, in the case of an elastic cut for $t_0 \leq t \leq t_1$, may be written in the form**

$$F(t+i0, \lambda) - F(t-i0, \lambda) = 2i\rho(t)\bar{F}(t+i0, \lambda)\bar{F}(t-i0, \lambda), \quad (2.10)$$

and hence the continuation of $F(t, \lambda)$ around the branch point t_0 into the second sheet is given by

$$F^{\text{II}}(t, \lambda) = F^{\text{I}}(t, \lambda) + 2i\rho(t). \quad (2.11)$$

Here and in Eq. (2.10), we have

$$\rho(t) = \left(\frac{t-t_0}{t} \right)^{1/2}. \quad (2.12)$$

* Prior to the introduction of complex angular momentum methods, it has already been pointed out by Gribov that a high energy behavior like $F(s, t) \sim iC(t)s$ is not allowed for $t_0 \leq t < t_1$. This interesting result is however much weaker than the absence of a fixed pole at $\lambda = 1$. For example, an asymptotic behavior of the form $F(s, t) \sim iC(t)s + \beta(t)S^{\alpha(t)}$ with $\text{Re}\alpha(t) < 1$ for $t \leq 0$ and $\text{Re}\alpha(t) > 1$ for $t \geq t_0$ would be compatible with the theorem of Ref. 24, but it would be forbidden if a fixed pole at $\lambda = 1$ is not allowed.

** Here and in the following we ignore signature indices whenever they are not relevant.

The characteristic feature of this condition is not only the boundedness which it implies for real $t_0 \leq t < t_1$, where

$$S(t, \lambda) S^*(t, \lambda^*) = 1 \quad (2.13)$$

with

$$S(t, \lambda) = 1 + 2i\rho(t) F(t, \lambda), \quad (2.14)$$

but also the more detailed information that $F(t, \lambda)$ has a square root branch point at $t=t_0^*$. These features, combined with the fact that $F(t, \lambda)$ is an analytic function of two complex variables, impose many restrictions upon the singular surfaces of this function. In their most general form, these restrictions are a consequence of the continuity theorem.²⁵ Roughly speaking, this theorem states that a singularity of a function of several complex variables cannot simply disappear somewhere if one follows them through the inside of the domain of definition of the function.

The most simple example is a fixed pole of $F(t, \lambda)$ at the point $\lambda=\lambda_0$. If λ_0 is real, we see of course directly from the boundedness for $t_0 \leq t < t_1$

*There is a superimposed winding point, but the corresponding cut should be drawn to the left. It does not interfere with our considerations. In the physical sheet, it can be removed by multiplication with $(t-t_0)^{-\lambda} = (2q(t))^{-2\lambda}$. See 01.

in Eq. (2.13) that it is forbidden. But for complex poles the situation in Eq. (2.13) is not so straightforward. On the other hand, the continuity theorem immediately gives the result. We see from Eq. (2.11) that

$$\lim_{\lambda \rightarrow \lambda_0} F^{\text{II}}(t, \lambda) = \frac{1}{2i\rho(t)}, \quad (2.15)$$

and hence the pole appears to get lost in the continuation from sheet I to sheet II. If this continuation is possible, then we have a contradiction and the fixed pole is forbidden. In order to see this more directly, we write for λ in a neighborhood of λ_0 :

$$F^{\text{II}}(t, \lambda) = \frac{1}{2\pi i} \oint_C d\lambda' \frac{F^{\text{II}}(t, \lambda')}{\lambda' - \lambda}$$

which is possible since F^{II} is regular there as seen from Eq. (2.15). Let us now continue this representation through the elastic branch cut, appropriately deforming the closed contour C in order to avoid other singularities. If this continuation is not blocked, and we will see later that it may be blocked by very specific shielding cuts, we obtain

$$F(t, \lambda_0) = \frac{1}{2\pi i} \oint_{C'} d\lambda' \frac{F(t, \lambda')}{\lambda' - \lambda_0}$$

Hence there can be no pole at $\lambda = \lambda_0$.

The arguments given above are not restricted to fixed poles in the λ -plane. They apply also for singular surfaces $\lambda = \alpha(t)$ which have the following properties:

$$1) \lim_{\lambda \rightarrow \alpha(t)} F(t, \lambda) = \infty$$

$$2) \alpha(t) = \alpha^{\text{II}}(t), \text{ i.e. the analytic function } \alpha(t) \text{ has no branch point at } t = t_0$$

From Eq. (2.11) and the condition 1), we obtain

$$F^{\text{II}}(t, \lambda = \alpha(t)) = \frac{1}{2i\rho(t)}, \quad (2.16)$$

and if the limit $\lambda \rightarrow \alpha(t)$ and the continuation $F \leftrightarrow F^{\text{II}}$ can be interchanged, we see that Eq. (2.16), together with the condition 2), would lead to

$$F(t, \lambda = \alpha^{\text{II}}(t)) = - \frac{1}{2i\rho(t)}, \quad (2.17)$$

which contradicts condition 1).

So far, in our discussion, we had in mind trajectories $\alpha(t)$ which are pole surfaces and branch point surfaces with t -independent character. However, the arguments can be generalized to many other situations which we do not want to discuss here in detail. We mention only the case of an essential singularity of $F(t, \lambda)$ at a point $\lambda = \alpha(t)$ with $\alpha^{\text{II}}(t) = \alpha(t)$.

We know that there must then be sequences $\{\lambda_n(t)\} \rightarrow \alpha(t)$ for $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} F(t, \lambda_n(t)) = \infty.$$

If there is a sequence $\{\lambda_n(t)\}$ with $\lambda_n^{\text{II}}(t) = \lambda_n(t)$ for almost all λ_n , then we have again a contradiction with the unitarity condition.

So far, we have considered only the implications of elastic unitarity in the t-channel. The arguments can however be generalized²⁶ to systems with many coupled channels. As long as all thresholds are different, we can always find a Riemann sheet where a singularity would have to disappear suddenly unless specially arranged shielding cuts intervene and prevent the continuation. There are some possible exceptions for nonfactorizing singularities, but they are not of direct physical interest for hadronic interactions. Also coupled spin channels with identical threshold require special treatment. Here we do not want to elaborate on these details.

3. Shielding Cuts

As we have already indicated above, there are ways to prevent a contradiction with the unitarity condition, even though a singular surface $\alpha(t)$ satisfies the condition 1). If the character of the singularity is t -dependent, we may have in some cases the possibility to adjust this t -dependence in order to get consistency with the unitarity condition. But it is difficult then to comply with the known analytic properties of $F(t, \lambda)$. Here we do not want to elaborate on this possibility. Rather, we concentrate on two methods which are of importance for high energy scattering.

a) A well known solution to the unitarity problem is the introduction of a branch point in $\alpha(t)$ at $t = 0$, with a cut along the real axis for $t \geq t_0$. Then we have $\alpha^{\text{II}}(t) \neq \alpha(t)$. This branch point is related to the unitarity cut of $F(t, \lambda)$, and for Regge poles one can show that it is of the form²²

$$\alpha(t) = \alpha(t_0) + \text{const.} (t_0 - t)^{\alpha(t_0) + \frac{1}{2}} + \dots$$

b) The other possibility is the presence of shielding cuts which make the limits $\lambda \rightarrow \alpha(t)$ and the continuation through the cut $t \geq t_0$ non-interchangeable. Let us denote these branch point surfaces of $F(t, \lambda)$ by $t = t_s(\lambda)$ or $\lambda = \alpha_s(t)$, $\alpha_s(t_s(\lambda)) = \lambda$. In order to see the required

character of these surfaces, we consider the function $\varphi(t, \lambda)$, which is defined by*

$$\varphi^{-1}(t, \lambda) = F^{-1}(t, \lambda) + i\rho(t), \quad (2.18)$$

and which does not have the unitarity branch point at $t = t_0$:

$$\varphi^{\overline{\Pi}}(t, \lambda) = \varphi(t, \lambda). \quad (2.19)$$

If $F(t, \lambda) \rightarrow \infty$ for $\lambda \rightarrow \alpha(t)$, we find

$$\varphi^{-1}(t, \lambda = \alpha(t)) = i\rho(t). \quad (2.20)$$

We assume here $\alpha^{\overline{\Pi}}(t) = \alpha(t)$. Since $\varphi(t, \lambda)$ does not have the elastic branch cut corresponding to $(t - t_0)^{1/2}$, the latter must be supplied by the shielding cut trajectory $t_s(\lambda)$. Hence we have the requirement

$$t_s(\lambda = \alpha(t)) = t_0 + \dots \quad (2.21)$$

so that the branch cut on the right hand side of Eq. (2.20) is generated by

*Again there is a left hand cut due to the factor $q^{2\lambda}(t)$, which does not disturb us.

$$(t_s(\lambda) - t)^{1/2} \rightarrow (t_0 - t)^{1/2} \quad (2.22)$$

for $\lambda \rightarrow \alpha(t)$. We see that shielding cuts must be of square-root type.

As an oversimplified example for the way a shielding cut works, we give an explicit function which has a pole at $\lambda = \alpha_0(t)$ with $\alpha_0^{\text{II}}(t) = \alpha_0(t)$, and which is bounded at this point in the second sheet. We write

$$f(t, \lambda) = \frac{(t_s(\lambda) - t)^{1/2} + (t_0 - t)^{1/2}}{\lambda - \alpha_0(t)} \quad (2.23)$$

with $t_s(\alpha_0(t)) = t_0$. Then we have in the physical sheet for $\lambda \rightarrow \alpha_0(t)$

$$f(t, \lambda) \rightarrow \frac{2(t_0 - t)^{1/2}}{\lambda - \alpha_0(t)}, \quad (2.24)$$

whereas in the second sheet we obtain

$$\lim_{\lambda \rightarrow \alpha_0(t)} f^{\text{II}}(t, \lambda) \rightarrow t'_s(\alpha_0(t)), \quad (2.25)$$

which is finite.

4. Applications

There are many physical implications of the restrictions discussed here. The most familiar is perhaps the fact that a positive signature fixed pole at $\lambda = 1$, which gives a high energy limit for $t \leq 0$ of the form

$$\overline{F}_+(s, t) \sim i \beta(t) s \quad (2.26)$$

corresponding to a non-shrinking diffraction peak, must be accompanied by an appropriate shielding cut $\alpha_s(t)$ with $\alpha_s(t_0) = 1$. A high energy behavior of the form (2.26) is typical for semiclassical models which are a priori unrelated to the t -channel exchanges. But via the required shielding cut, and possible Regge poles which generate this shielding cut, they are indirectly connected with the low energy properties of the t -channel reaction.

As a further application, we mention the branch point trajectories which are associated with Regge poles. In the most simple cases, the position of these branch points is given by

$$\alpha_n(t) = n \alpha\left(\frac{t}{n^2}\right) - n + 1, \quad n \geq 2, \quad (2.27)$$

where $\alpha(t)$ is a pole trajectory.²⁷ By themselves, these cuts act as shielding cuts for the nonsense wrong signature fixed poles of the amplitude $F(t, \lambda)$, which are due to the poles of Q_λ in Eq. (2.6).²⁸ Since the pole trajectory $\alpha(t)$ has a branch point at $t = t_0$, we see from Eq. (2.27) that the functions $\alpha_n(t)$ generally do not have one at this threshold. Hence we see from our previous discussions that these trajectories α_n must be singular surfaces such that $F(t, \lambda)$ remains bounded for $\lambda \rightarrow \alpha_n(t)$. But

quite apart from this general restriction, they must be square-root branch points in order to act as shielding cuts²³ for the fixed poles at nonsense wrong signature points.

From our discussions, we see that t-channel unitarity imposes certain restrictions upon the singular surfaces of the partial wave amplitude $F(t, \lambda)$, and hence also upon the high energy limits of $F(s, t)$ in the s-channel. It is true that the connection with physical high energy limits requires analytic continuations between the regions $t \leq 0$ and $t \geq t_0$. It should be noted that this is not a very restrictive requirement in view of the analytic properties of $F(t, \lambda)$ as a function of two complex variables.* Of course, there may be, for example, very sophisticated natural boundaries which in principle could block an analytic connection, but we must recall that such boundaries themselves must comply with the unitarity condition and also with the known analytic properties of $F(t, \lambda)$. We know that for

*For example, one may think of writing $F_+(s, t) \rightarrow i\beta(t)s$ for $s \rightarrow \infty$ with $\beta(t) \propto \theta(t_0 - t)$. This would correspond to a term in the meromorphic expansion of $F_+(t, \lambda)$ which is proportional to

$$\frac{\theta(t_0 - t)}{\lambda - 1}$$

This term generally is not compatible with the analytic properties of the partial wave amplitudes in the finite t-plane. Although we are dealing with an asymptotic expansion of $F(s, t)$ for $s \rightarrow \infty$, the t-dependence appearing in the individual terms of this expansion is related to the t-dependence of the partial wave amplitude $F(t, \lambda)$, which is an analytic function.

$l > 1$, $l = \text{even/odd}$ the function $F(t, \lambda)$ coincides with the physical partial wave amplitudes in the t -channel. For $l = 0, 1$ we cannot prove this coincidence, but this does not detract from the existence of a continuation of $F(t, \lambda)$ into the region $\text{Re } \lambda < 1$. In fact, as we have seen in Eq. (2.6), the asymptotic expansions of the absorptive parts A_s and A_u completely determine this continuation. If it should happen that $F_+(t, 0) \neq F_0(t)$ and/or $F_-(t, 1) \neq F_1(t)$, then we have only an additional real polynomial of the form

$$\Delta F_0(t) + \Delta F_1(t) P_1 \left(1 + \frac{s}{2q^2(t)} \right)$$

in the asymptotic expansion of $F(s, t)$.^{*} In principle, the functions $\Delta F_0(t)$ and $\Delta F_1(t)$ are also determined by $F(t, \lambda)$, but at present we have no method for their explicit construction.^{**}

^{*} Elsewhere we have given a detailed discussion of these terms, in particular in view of their connection with "elementary poles". 23)

^{**} It is important to note that the presence of a term proportional to $\Delta F_1(t)s$ in the high energy limit of $F(s, t)$ implies the existence of a contribution in the absorptive part which grows at least like s . Hence there must be a singularity in the amplitude $F_+(t, \lambda)$ at $\lambda = 1$ for $t = 0$ 29). This follows from s -channel unitarity for $t = 0$. Using s -channel unitarity also for $t < 0$, one can give some argument for 30)

$$A_s(s, 0) \sim O(s \lg s).$$

A finite term $\Delta F_1(t)$ can be related to an "elementary" vector meson.

III. HIGH ENERGY LIMITS

1. General Bounds.

The restrictions for the high energy limit of the amplitude, which follow from direct channel properties, can in many cases be formulated independent of the complex angular momentum description. However, many of the constraints obtained from unitarity and analyticity are particularly interesting in connection with complex angular momentum methods. The usual axiomatic s-channel bounds have been discussed in great detail in the recent literature.³¹ Therefore we concentrate here on the interplay of these bounds with the crossed channel Regge description considered in the previous Section.

Most of the general bounds for $F(s,t)$ in the s-channel are a consequence of the analyticity of $F(s,t)$ as a function of two complex variables and of unitarity. Actually only analyticity in a domain

$$[(s,t) : \text{cut } s\text{-plane} \otimes |t| < t_0] \quad (3.1)$$

is required, which can even be proven from the axioms of field theory. The important practical step is the truncation of the partial wave series. From unitarity and polynomial boundedness in the domain (3.1), we obtain for $t < t_0$ and $s \rightarrow \infty$

$$F(s,t) \sim \sum_{l=0}^L (2l+1) \overline{F}_l(s) P_l\left(1 + \frac{t}{2q^2s}\right), \quad (3.2)$$

with

$$L = \frac{1}{2} \sqrt{as} \lg s, \quad a = t_0^{-1}, \quad (3.3)$$

because partial waves with $\ell > L$ decrease exponentially and become negligible for $s \rightarrow \infty$. With Eq. (3.2) and the unitarity relations

$$\rho^2 |F_\ell(s)|^2 \leq \rho \operatorname{Im} F_\ell(s) \leq 1, \quad (3.4)$$

$\rho = \left(\frac{s-s_0}{s}\right)^{1/2}$, we get the Froissart bound³²

$$|F(s, 0)| \leq \frac{\pi}{4} a s (\lg s)^2 \quad (3.5)$$

for $s \rightarrow \infty$. It implies

$$\sigma_{\text{tot}} \leq 4\pi a s (\lg s)^2. \quad (3.6)$$

With the same assumptions and the Schwarz inequality, we find also that asymptotically*

$$\sigma \geq \frac{16\pi}{s} \frac{|F(s, 0)|^2}{\frac{a}{4} s (\lg s)^2}. \quad (3.7)$$

*
$$\begin{aligned} \sigma_{\ell\ell} &\sim \frac{16\pi}{s} \sum_0^L (2\ell+1) |F_\ell|^2 \geq \frac{16\pi}{s} \left(\sum_0^L (2\ell+1) |F_\ell| \right)^2 \left[\sum_0^L (2\ell+1) \right]^{-1} \\ &\geq \frac{16\pi}{s} |F(s, 0)|^2 \left[\frac{a}{4} s (\lg s)^2 \right]^{-1}. \end{aligned}$$

Other bounds, which we will need later, are

$$|F(s, t)| \leq \frac{q}{4} s (\lg s)^2 I_0(\sqrt{qt} \lg s) \quad (3.8)$$

for $0 \leq t < t_0$, and

$$|F(s, t)| \leq \frac{q}{4} s (\lg s)^2 \quad (3.9)$$

for $t \leq 0$.

2. Complex Trajectories.

In Section II, we have considered the properties of Regge trajectories from the point of view of the t -channel, and we have pointed out that the position of a singularity $\lambda = \alpha(t)$ of $F(t, \lambda)$ in the complex λ -plane corresponds to the power behavior of the related term in the asymptotic expansion of $F(s, t)$ for $s \rightarrow \infty$. Let us now see how the analytic properties of the trajectory function $\alpha(t)$ affect the high energy behavior. We are particularly interested in diffraction scattering, and hence in the neighbourhood of $(t, \lambda) = (0, 1)$.

The singular surfaces $\alpha(t)$ do not inherit the left-hand branch points of $F(t, \lambda)$.³³ This follows from Eqs. (2.6) and (2.9), which show that only the asymptotic properties of A_s and A_u are relevant for the trajectory functions. A priori, we may therefore assume that the $\alpha(t)$ are regular functions near $t = 0$:

$$\alpha(t) = \alpha(0) + \alpha'(0)t + \dots \quad (3.10)$$

For poles and branch-point trajectories, we obtain then a high energy contribution of the form

$$F(s, t) \propto s^{\alpha(t)},$$

which gives at most a logarithmic shrinkage of the diffraction peak. We have

$$\int_0^1 dt |f(s, t)|^2 \propto (\alpha' \lg s)^{-1}, \quad (3.11)$$

with

$$\phi(s, t) = \frac{F(s, t)}{F(s, 0)} \quad (3.12)$$

Faster shrinkage can be obtained if there is a branch point in $\alpha(t)$ at $t = 0$. Such branch point must not be present in the continued partial wave amplitude $F(t, \lambda)$, and therefore it is only possible if we have two or more trajectories of the same character which cross at $t = 0$.³⁴ Then $\alpha(t)$ near $t = 0$ can be expanded in the form

$$\alpha(t) = \alpha(0) + \sum_{j=0}^{\infty} c_j t^{\frac{j}{n}} \quad (3.13)$$

with $n \geq 2$ being an integer. There are n trajectories which are branches of the same analytic function,³⁵ and all these branches must appear in $F(t, \lambda)$ in a completely symmetric fashion.

Of special interest for diffraction scattering are crossing trajectories with the behavior

$$\alpha(t) = 1 + c t^{\frac{m}{n}} \quad (3.14)$$

near $t = 0$. We are interested in exponents $\frac{m}{n} < 1$. For $\frac{m}{n} = \frac{1}{2}$, we have

$\alpha(t)$ real for $t \geq 0$ and two complex conjugate singularities for $t < 0$.

This case complies with the bounds (3.8) and (3.9) for $c < \sqrt{a}$. However,

for $0 < \frac{m}{n} < \frac{1}{2}$ and $\frac{1}{2} < \frac{m}{n} < 1$, we must have $n \geq 3$ and $m < \frac{1}{2}n$ or $m < n$, and hence there is always an asymptotic term like

$$s^{1+b|t|^{\frac{m}{n}}} \quad (3.15)$$

with $\text{Re } b > 0$. Such terms violate at least one of the bounds (3.8) or (3.9), and we are left with the possibility $\frac{m}{n} = \frac{1}{2}$. Trajectories of the form

$$\alpha(t) = 1 + \text{const.} \sqrt{t} \quad (3.16)$$

near $t = 0$ can give rise to diffraction peaks which shrink according to

$$\int_{-s}^0 dt |f(s,t)|^2 \propto (\lg s)^{-2}, \quad (3.17)$$

and they have many interesting properties.³⁷ One is that they satisfy the bootstrap-like condition

$$\alpha(t) = \alpha_n(t), \quad \alpha_n(t) = n \alpha\left(\frac{t}{n^2}\right) - n + 1, \quad (3.18)$$

where $\alpha_n(t)$ is the branch point trajectory generated by the iteration of poles $\alpha(t)$. Other properties will be mentioned later.

In order to see more directly when trajectories of the form (3.16) are required, we make specific Ansätze for the high energy limits of the forward positive and negative signature amplitudes as defined in Eq. (2.1). In accordance with the dispersion relations (2.4) and (2.5), we write³⁸

$$F_+(s,0) \propto i s (\lg s)^{\beta_+} + \frac{\pi}{2} \beta_+ s (\lg s)^{\beta_+-1} + \dots, \quad (3.19)$$

$$\bar{F}_-(s,0) \propto -\frac{2}{\alpha} \frac{1}{\beta_-+1} s (\lg s)^{\beta_-+1} + i s (\lg s)^{\beta_-} + \dots \quad (3.20)$$

General bounds like (3.5) and (3.7) imply the restrictions³¹

$$\beta_+ \leq 2 \quad \text{and} \quad \beta_- \leq \frac{1}{2} \beta_+ \quad (3.21)$$

Using the requirement $\sigma_{el} \leq \sigma_{tot}$, it is a simple matter to show that the shrinkage of the diffraction peak must be faster than $(\lg s)$ if we have

$$\beta_+ > 1 \quad (3.22)$$

for the positive signature, and if

$$2\beta_- + 2 - \beta_+ > 1 \quad (3.23)$$

for the negative signature trajectory. In these cases we conclude from our previous considerations that the appropriate singular surface $\alpha(t)$ must have a branch point at $t = 0$. Here we assume that the shrinkage is generated by the trajectory function, which is generally the case for pole-and branch point surfaces.*

We note that for $\beta_- = \beta_+ = 0$ and $\alpha_-(0) = 1$, complex trajectories must be present in both the negative and the positive^{36, 39} signature amplitude in order to comply with positivity conditions for $\text{Im } F(s,t)$ and $\text{Im } \bar{F}(s,t)$ for $0 \leq t < t_0$. Of course, we may have trajectories of the form (3.16) also in cases where they are not explicitly required.

*There are possible exceptions in the case of essential singularities.

3. Impact Parameter Representation.

Another interesting aspect of complex trajectories like (3.16) is their simple relation to Bessel-function representations.⁴⁰ In general, we can write the amplitude $F(s,t)$ for large values of s in the form

$$F(s,t) \sim s \int_0^1 d\xi \psi(\xi, s) J_0(\xi \sqrt{-at} \lg s), \quad (3.24)$$

which is the usual impact parameter representation with

$$b = \xi \sqrt{a} \lg s$$

and

$$\psi(\xi, s) = \xi (\lg s)^{\frac{1}{2}} \frac{\eta(b(\xi), s) e^{2i\delta(b(\xi), s)} - 1}{2i\rho(s)}. \quad (3.25)$$

The maximal impact parameter $b_{\max} = \sqrt{a} \lg s$ corresponds to $L = \frac{1}{2} \sqrt{as} \lg s$ as given in Eq. (3.3).

Near $(t, \lambda) = (0, 1)$, a rather general class of continued partial wave amplitudes with complex trajectories can be written in the form^{41, 42}

$$F_{\pm}(t, \lambda) \propto \int_0^1 d\xi \frac{\rho_{\pm}(\xi, t)}{[(\lambda-1)^2 - \xi^2 at]^{1/2}}. \quad (3.26)$$

We can make this Ansatz compatible with t -channel unitarity in several ways.

The most direct one is to choose the weight functions appropriately.*

* See Ref. 36 for explicit examples.

Otherwise we may consider the denominator in Eq. (3.26) as a degenerate superposition of a complex pole trajectory and a square-root branch point trajectory. Further away from $t = 0$, the pole trajectory must then acquire the appropriate branch point at $t = t_0$ in order to comply with unitarity. As usual, we assume then that this threshold of $\alpha(t)$ can be neglected as far as the high energy behavior of $F(s, t)$ is concerned. For ρ_- we must also require the condition

$$\int_0^1 d\xi \xi^{-1} \rho_-(\xi, 0) = 0 \quad (3.27)$$

in order to prevent singularities of the partial wave amplitude which are not allowed.

As an example, let us consider the case where $\beta_+ = \beta_- = 0$ in Eqs. (3.19) and (3.20). Then *the* forward amplitude is dominated by the real part of $F_-(s, t)$, and we have different asymptotic total cross-sections σ and $\bar{\sigma}$ for particle and antiparticle scattering.* Hence we write

$$F_-(s, 0) \sim -\frac{2}{\pi} s(\lg s) \frac{\sigma - \bar{\sigma}}{16\pi} + i s \frac{\sigma - \bar{\sigma}}{16\pi} + \dots, \quad (3.28)$$

and with $\rho_-(\xi, t) = \rho_-(\xi)$, we have ⁴²

* Whatever may be the outcome of present experiments on K-meson-nucleon total cross-sections and on the K_S^0 regeneration amplitude, the Serpukhov data have at least shown that $\sigma = \bar{\sigma}$ should not be taken for granted.

$$F(s, t) \sim \frac{2}{\pi} s (lgs) \int_0^1 d\xi \left(\int_0^\xi dx \frac{\rho_-(x)}{x} \right) J_0(\xi \sqrt{-at} lgs) \\ + i s \int_0^1 d\xi \rho_-(\xi) J_0(\xi \sqrt{-at} lgs) + \dots \quad (3.29)$$

For fixed $t \leq 0$ and $s \rightarrow \infty$, the first term dominates, and we can calculate the asymptotic integrated cross-section

$$\sigma^{(-)} \sim \frac{16a}{s^2} \int_{-s}^0 dt |F(s, t)|^2 \quad (3.20)$$

in terms of ρ_- :

$$\sigma^{(-)} \sim \frac{32\pi}{a} \frac{4}{\pi^2} \int_0^1 d\xi \xi^{-1} \left(\int_0^\xi dx \frac{\rho_-(x)}{x} \right)^2 \quad (3.31)$$

Except for a factor, the quantity $\sigma^{(-)}$ is the charge exchange (e.g. $\pi\pi$, πN) or regeneration (KN) cross-section. In analogy with the derivation of Eq. (3.7), we find the lower bound ($s \rightarrow \infty$)⁴³

$$\sigma^{(-)} \geq \frac{(\sigma - \bar{\sigma})^2}{\pi^3 a} \quad (3.32)$$

There is also the upper bound

$$\sigma^{(-)} \leq 4 \max(\sigma, \bar{\sigma}) \quad (3.33)$$

It follows that the ratio

$$R \equiv \frac{4\pi^3 a \sigma^{(-)}}{(\sigma - \bar{\sigma})^2} = \frac{\frac{1}{2} \int_0^1 d\xi \xi^{-1} \left(\int_0^\xi dx \frac{\rho_-(x)}{x} \right)^2}{\left[\int_0^1 d\xi \int_0^\xi dx \frac{\rho_-(x)}{x} \right]^2} \quad (3.34)$$

varies between the limits

$$1 \leq R \leq \frac{4\pi^3 a \max(\sigma, \bar{\sigma})}{(\sigma - \bar{\sigma})^2} \quad (3.35)$$

Of particular interest is the boundary case $R \rightarrow 1+0$, which gives rise to a unique asymptotic^{form} of the amplitude^{44, 42} for $s \rightarrow \infty$ and $\tau = -at(\lg s)^2$ fixed. From Eqs. (3.28) and (3.34), we obtain

$$\int_0^\xi dx \frac{\rho_-(x)}{x} = \frac{\sigma - \bar{\sigma}}{16\pi} (-2\xi) \Theta(1 - \xi - 0), \quad (3.36)$$

and hence

$$F_-(s, t) \sim -\frac{2}{\pi} s (\lg s)^2 \frac{\sigma - \bar{\sigma}}{16\pi} \frac{2J_1(\sqrt{s})}{\sqrt{s}} + i s \frac{\sigma - \bar{\sigma}}{16\pi} \left\{ -\frac{2J_1(\sqrt{s})}{\sqrt{s}} + 2J_0(\sqrt{s}) \right\} \quad (3.37)$$

This is an example of a limiting situation where the general constraints are sufficient to determine the form of the amplitude for large energies.

In the interval $0 \leq t < t_0$, we have the inequalities $\text{Im } F(s, t) \geq 0$ and $\text{Im } \bar{F}(s, t) \geq 0$, and they imply that the negative signature part of these amplitudes cannot dominate. Consequently there must also be a related complex positive signature trajectory so that, for large values of s , there is a contribution to the amplitude $F_+(s, t)$ which is of the form

$$i s \int_0^1 d\xi \rho_+(\xi) J_0(\xi \sqrt{-at} \lg s). \quad (3.38)$$

However, it is not expected that the term (3.38) is dominating the amplitude for $t \leq 0$, at least in the range of presently accessible energies. It is probable that there is also a regular trajectory $\alpha(t)$ with $\alpha(0) = 1$, or a fixed pole at $\lambda = 1$ with the appropriate shielding cuts.⁴⁵

The relevant feature of the representations (3.29) and (3.38) is the finite support of the weight functions for non-zero values of the parameter $\xi \leq 1$. In contrast, an asymptotic form corresponding to a Regge pole like

$$-\beta e^{-\frac{i\pi\alpha(t)}{2}} s^{\alpha(t)},$$

with $\alpha(t) = 1 + \alpha'(0)t$, would correspond to an s -dependent weight function in Eq. (3.24), which is given by

$$\psi(\xi, s) \sim \beta \frac{a}{2\alpha'} (i \lg s - \frac{u}{2}) \xi e^{-\xi^2 \frac{2a \lg s}{4\alpha'}} \quad (3.39)$$

for $\lg s \gg 1$. For $\xi > 0$, its support vanishes in the limit $s \rightarrow \infty$.

A fixed pole at $\lambda = 1$ would correspond to

$$\psi(\xi, s) \sim i\beta \delta(\xi - \varepsilon), \quad \varepsilon \rightarrow 0_+ \quad (3.40)$$

There are other situations where a unique form of the asymptotic amplitude is obtained. The following case is quite familiar: We assume that the inelastic cross-section approaches a limiting form

$$\sigma_{inel} \sim \pi a (\lg s)^2, \quad (3.41)$$

which corresponds to the maximal value which is compatible with the bound

$l \leq L = \frac{1}{2} \sqrt{as} \lg s$ for the relevant partial waves in the expansion of $F(s, t)$. Then we obtain^{41, 42, 46}

$$F_+(s, -\frac{3}{a(\lg s)^2}) \sim i \frac{a}{8} s (\lg s)^2 \frac{2 J_1(\sqrt{s})}{\sqrt{s}} \quad (3.42)$$

for fixed $\tau = -at(\lg s)^2$. This amplitude gives $\sigma_{tot} \sim 2\sigma_{el}$, and if it is extended to fixed values of t , it corresponds to a fully absorbing

disc of radius $\sqrt{a}$.^{*} In the complex angular momentum plane it corresponds to

$$F_+(t, \lambda) \propto [(\lambda-1)^2 - at]^{-3/2} \quad (3.43)$$

near $(t, \lambda) = (0, 1)$. Further away from this point the form (3.43) requires modification or shielding cuts.

^{*}A form corresponding to Eq. (3.42) except for a factor of two is obtained by requiring full saturation of the Froissart bound. Then we have

$$\sigma_{\text{tot}} \sim \sigma_{\text{el}} \sim 4\pi a (\lg s)^2.$$

IV. Concluding Remarks

Empirically, we know that high energy collisions are accompanied by the production of hadrons with an average multiplicity which appears to be slowly increasing with energy. Via s-channel unitarity, the absorptive part of the elastic amplitude is related to these production processes. It is therefore of great interest to connect models for the production amplitudes with the Ansätze made for the description of diffraction scattering. However, we may expect that in the sum over intermediate states at high energy many details of the actual production amplitudes will be washed out. In this sense, diffraction scattering should perhaps be considered more as a constraint for these amplitudes, although it is expected to depend upon the more general features like the increase of the multiplicity, etc. . Because of the rather direct connection of the diffraction amplitude with the low energy properties of the crossed channel, we may hope that an understanding will be possible without a complete theory of production processes.

We have considered in this article only hadronic interactions and we have explored asymptotic expansions within this framework. We hope that the energy region of up to about 1000 GeV, which is now becoming available, will bring some clarification of the uncertainties which at present cloud the experimental situation of diffraction scattering. Some important questions are: do total cross-sections for particle and antiparticle scattering remain constant and different? does shrinkage stop or not? do charge exchange and regeneration cross-sections continue to fall or do they

level off or even start to increase? There are of course many other questions.

We should also keep in mind the possibility that the strong interaction asymptotics is only relevant for an intermediate energy region.

It is not impossible that there is a new regime at higher energies where electromagnetic or even weak interactions come into play and are no more small corrections to the strong ones. However, this possibility does not detract from the importance of exploring the hadronic asymptotics within the framework of dispersion theory. Any indication of deviations from the assumption of this framework would be of greatest interest. In the study of high energy limits within the framework of dispersion theory, it is a principal aim to get some idea about the physics in small dimensions.

Once we have a reasonable understanding of the asymptotic expansions of hadron amplitudes, we may go back to quantum field theory in the hope of obtaining new insights. The recent developments of operator product expansions⁴⁷ may be very helpful in order to establish a more intimate relationship between high energy limits and the space-time formalism of field theory.

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