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Lectures on DUAL RESONANCE MODELS - P. H. Frampton

PART FOUR: INTERNAL SYMMETRY*

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The diagram illustrates the experimental setup. A participant is seated at a table, looking at a video screen. On the screen, there is a starting point (a large circle) and a target (a small circle). The distance between them is labeled 'Distance'. The participant's hand is positioned at the starting point and is labeled 'Hand'.

INTRODUCTION

We now consider in turn the incorporation of the three internal symmetries relevant to hadron physics, namely those associated with conservation of isospin, strangeness, and baryon number.

The treatment of isospin is simplest. By introducing a multiplicative factor one can preserve the fundamental properties of cyclic symmetry and factorisability of the multiparticle dual amplitude. At the same time the observed exchange degeneracy and absence of exotics is introduced in a natural way. The multiplicative factor for the $SU(2)$ (isospin) case is a trace of Pauli matrices.

To include strangeness, we may generalize the multiplicative factor by extending the Pauli matrices to the Gell-Mann λ - matrices of $SU(3)$. This does not necessarily imply that there is $SU(3)$ mass degeneracy in the Born Amplitude.

Next we briefly analyse the factorization of unequal intercepts in the N -point function, and a solution is found where the intercepts depend quadratically on the internal quantum numbers of the external particles.

Baryon number cannot be dealt with by a multiplicative approach because the pole structure is different when baryons are present. One method is to use the rubber string picture. The

mesonic dual amplitudes may be derived from calculational rules where a meson is pictured either as a linear rubber string with a quark at one end ($\sigma=0$) and an antiquark at the other end ($\sigma=\pi$), or equivalently as a circular string ($0 \leq \sigma < 2\pi$) with quarks at these points. Baryons may then be imagined as circular strings with quarks at $\sigma=0$, $2\pi/3$ and $4\pi/3$; this leads to the pole structure expected from Harari-Rosner quark diagrams, and to generalizations to exotic mesons and baryons.

A more rigorous treatment of the rubber string for free mesons reveals that the elimination of the ghost states, through the generalized projective gauge conditions, may be re-interpreted as being a direct consequence of the general covariance of the classical action of the string under arbitrary re-parametrizations of the two-dimensional world-sheet mapped out by the string as it propagates in space-time.

4a. Multiplicative Internal Symmetry Factor

We begin with a discussion of isospin invariance, which we may regard as an exact symmetry of the strong interactions. That is, we assume that the mass differences between different members of the same isospin multiplet are entirely electromagnetic in origin.

When all the external mesons are isoscalars we write the full S-matrix as a sum over inequivalent cyclic permutations. Each particular configuration contains poles only in the planar channels (i.e. channels corresponding to groups of adjacent external particles in the planar diagram). The principal properties of B_N that we shall need are that it is cyclic symmetric and that it satisfies factorization; for example at a three-particle ground-state pole in B_6 one knows that

$$\lim_{\alpha_{123} \rightarrow 0} \alpha_{123} B(p_1 p_2 p_3 p_4 p_5 p_6) = B_4(s_{12}, s_{23}) B_4(s_{45}, s_{56}). \quad (4-1)$$

We do not need to know the detailed form of B_N . We must introduce isospin in such a way that the following requirements are satisfied:

- i) factorization and bootstrap consistency
- ii) cyclic symmetry, if the external particles are identical
- iii) absence of poles with isospin > 1 . (this is the input of an experimental result).

The solution ⁽¹⁾ is to multiply each term in the sum over permutations by a simple isospin multiplicative factor. The factorization has to be proved but this turns out to be remarkably simple.

Let us begin with the case where all N external particles have isospin $T=1$ with isospin labels τ_{a_i} ($i = 1, 2, \dots, N$; $a_i = 1, 2, 3$). Then the proposal is to write

$$T_N^{a_1 a_2 \dots a_N} = \sum \frac{1}{2} \text{Tr} (\tau_{a_1} \tau_{a_2} \dots \tau_{a_N}) B_N(p_1 p_2 \dots p_N) \quad (4.2)$$

where the sum is over all $\frac{1}{2} (N-1)!$ inequivalent permutations. The τ_{a_i} are Pauli matrices. The cyclic symmetry (ii) follows at once from the cyclic property of the trace. The absence of exotics is ensured by the fact that the products of 2×2 matrices are again 2×2 matrices and hence can be expanded as linear combinations of isoscalar and isovector. Factorization follows from the fundamental identity for the Pauli matrices

$$\begin{aligned} \frac{1}{2} \text{Tr}(\tau_{a_1} \tau_{a_2} \tau_{a_3} \dots \tau_{a_N}) = \\ \sum_{k=0,1,2,3} \frac{1}{2} \text{Tr}(\tau_{a_1} \tau_{a_2} \dots \tau_{a_M} \tau_k) \frac{1}{2} \text{Tr}(\tau_k \tau_{a_{M+1}} \dots \tau_{a_N}) \end{aligned} \quad (4.3)$$

where we have written

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.4)$$

To prove the factorization property we need use only the fact that

$$\tau_i \tau_j = -\tau_j \tau_i = i\tau_k \quad (4.5)$$

and

$$\tau_i^2 = \tau_0 \quad (4.6)$$

to re-write

$$\tau_L = (\tau_{a_1} \tau_{a_2} \dots \tau_{a_M}) \quad (4.7)$$

$$\tau_R = (\tau_{a_{M+1}} \tau_{a_{M+2}} \dots \tau_{a_N}) \quad (4.8)$$

and thence

$$\tau_L = \alpha_1^L \tau_1 + \alpha_2^L \tau_2 + \alpha_3^L \tau_3 + \alpha_0^L \tau_0 \quad (4.9)$$

$$\tau_R = \alpha_1^R \tau_1 + \alpha_2^R \tau_2 + \alpha_3^R \tau_3 + \alpha_0^R \tau_0 \quad (4.10)$$

whereupon

$$\frac{1}{2} \text{Tr} (\tau_{a_1} \tau_{a_2} \dots \tau_{a_N}) = \frac{1}{2} \text{Tr} (\tau_L \tau_R) \quad (4.11)$$

$$= \alpha_1^L \alpha_1^R + \alpha_2^L \alpha_2^R + \alpha_3^L \alpha_3^R + \alpha_0^L \alpha_0^R \quad (4.12)$$

$$= \sum_{k=0,1,2,3} \frac{1}{2} \text{Tr} (\tau_L \tau_k) \frac{1}{2} \text{Tr} (\tau_k \tau_R) \quad (4.13)$$

where we observed that

$$\text{Tr} (\tau_i \tau_j) = 2 \delta_{ij} \quad (4.14)$$

This procedure has the immediate consequence of exchange degenerate isocalar - isovector pairs of trajectories since we may write

$$\tau_a \tau_b = \delta_{ab} + i \epsilon_{abx} \tau_x \quad (4.15)$$

After summing over permutations one finds that the $T=0$ and $T=1$ have opposite signatures. (This is obvious for a two-particle channel, for example).

The multiplicative approach for isospin may be regarded as very satisfactory. The only note of consternation (which might ultimately prove serious) is that it leads to π - η mass degeneracy.

To incorporate strange mesons we can extend the matrices to 3×3 matrices corresponding to the broken $SU(3)$ symmetry of Gell-Mann-Ne'emann. [Refs. 2,3 reprinted in ref. 4] In fact for the Gell-Mann matrices⁽²⁾

$$\begin{aligned}
 \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
 \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & \lambda_0 &= \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned} \tag{4.16}$$

we find once again

$$\text{Tr} (\lambda_i \lambda_j) = 2 \delta_{ij} \tag{4.17}$$

which ensures the factorsability. Furthermore only singlet and octet representations will be present, thus ensuring absence of unwanted (exotic) particles with $T > 1$ or $|S| > 1$. The assignment

of, say, the pseudoscalar mesons will be according to the scheme

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \eta - \frac{1}{\sqrt{2}} \pi^0 & \pi^- & K^0 \\ \pi^+ & \frac{1}{\sqrt{6}} \eta + \frac{1}{\sqrt{2}} \pi^0 & K^+ \\ K^0 & K^- & -\sqrt{\frac{2}{3}} \eta \end{pmatrix} \quad (4.18)$$

Concerning the multiplicative internal symmetry factors it should be noted that

(i) Although SU(2) mass degeneracy (isospin) is essential, it is not necessary to assume SU(3) mass degeneracy. Indeed it is unlikely that we should begin with SU(3) mass degeneracy in the Born amplitude because it is so badly broken in Nature. It is, however, fortunately consistent with the multiplicative approach that the strange ($S=\pm 1$) octet members have a different Regge intercept than the non-strange ($S=0$) members, since by strangeness conservation they always occur in different planar channels. The only note of consternation is that it seems difficult to avoid the (undesirable) equality of π and η masses.

(ii) The uniqueness of the form of the multiplicative factor, assuming the requirements listed above, has been demonstrated for SU(2) by Tornqvist⁽⁵⁾ and for SU(3) by several authors. [Refs. 6-8].

(iii) The approach described here is readily re-interpreted in terms of the quark model, and in particular in terms of the planar Harari-Rosner quark diagrams which we described much earlier.

4b. Factorization of Unequal Intercepts

One possible way to introduce internal symmetry, which we now discuss briefly, is to make the intercepts depend on the internal quantum numbers of the external trajectories. We shall be content simply to indicate how restrictive the requirement of factorization is on the form of such a dependence; the question of ghosts will not be treated, but it seems certain that for the conventional generalized Euler B function ghosts will be present for all cases unless at least one intercept is set equal to one.

We have written the N-point function in the form

$$B_N = \int_0^1 \prod_{j=1}^{N-2} du_{1j} u_{1j}^{-\alpha_{1j}-1} \prod_{2 \leq i < j \leq (N-1)} (1 - u_{1i} u_{1i+1} \dots u_{1j-1})^{-2p_i \cdot p_j - C_{ij}} \quad (4.19)$$

in which

$$C_{ij} = \alpha_{ij}(0) + \alpha_{i+1,j-1}(0) - \alpha_{i+1,j}(0) - \alpha_{i,j-1}(0) \quad (4.20)$$

For example, if the intercept $\alpha_n(0)$ depends only on the number n of external particles coupling to the trajectory this gives

$$C_{ij} = + \alpha_{j-i+1}(0) - 2\alpha_{j-i}(0) + \alpha_{j-i-1}(0) \quad (4.21)$$

More generally we would like to write

$$-2p_i \cdot p_j - C_{ij} = -2\hat{p}_i \cdot \hat{p}_j \quad (4.22)$$

where $\hat{p}_i$ is a (4+D) dimensional vector. In such a case we may

factorize simply by adding D extra dimensions to the harmonic operators.

Let us assume that [References 9-11]

$$\alpha_{ij}(0) = B + \sum_{k=i}^j d_k + \left(\sum_{k=i}^j e_k \right)^2 \quad (4.23)$$

$$= B + D_{ij} + E_{ij} \quad (4.24)$$

where d_k, e_k depend on the internal quantum numbers of particle k .

[For example, we may take d_k, e_k proportional to the total quark numbers ($v=N_q + N_{\bar{q}}$), in which case the dependence must be at least quadratic to avoid stable high-mass exotics since the scalar states would have masses satisfying

$$M_v - M_{v-1} \xrightarrow{v \rightarrow \infty} 0 \quad (4.25)$$

if $\alpha(0)$ depends only linearly on v but this difference ($M_v - M_{v-1}$) can remain finite for $v \rightarrow \infty$ in the case of $\alpha(0)$ depending quadratically on v .]

Inserting our ansatz for $\alpha_{ij}(0)$ one finds

$$C_{ij} = 2e_i e_j \quad (4.26)$$

Now certain consistency conditions must be met. Consider first that the number of external particles N be held fixed.

Then there is the requirement that $\alpha_{ij}(0) = \alpha_{j-1,N}(0)$. In particular

$$\sum_{k=1}^{N-1} d_k + \left(\sum_{k=1}^{N-1} e_k \right)^2 = d_N + e_N^2 \quad (4.27)$$

It follows that

$$d_N = -\sqrt{E_{1,N}} e_N + \frac{1}{2} (D_{1,N} + E_{1,N}) \quad (4.28)$$

More generally we use

$$D_{1,j} = D_{1,N} - D_{j+1,N} \quad (4.29)$$

$$E_{1,j} = E_{1,N} - 2\sqrt{E_{1,N} E_{j+1,N}} + E_{j+1,N} \quad (4.30)$$

to find that the requirement

$$\alpha_{1j}(0) = D_{1j} + E_{1j} = \alpha_{j+1,N}(0) = D_{j+1,N} + E_{j+1,N} \quad (4.31)$$

implies

$$D_{1,N} - 2D_{j+1,N} + E_{1,N} - 2\sqrt{E_{1,N} E_{j+1,N}} = 0 \quad (4.32)$$

Combining the results we see that

$$D_{j+1,N-1} = -\sqrt{E_{1,N} E_{j+1,N-1}} \quad (4.33)$$

for all j . The most general solution of this is to take

$$d_k = -\sqrt{E_{1,N}} e_k \quad 2 \leq k \leq (N-1) \quad (4.34)$$

Similarly we find

$$d_1 = \frac{1}{2} D_{1,N} + \frac{1}{2} E_{1,N} - \sqrt{E_{1,N}} e_1 \quad (4.35)$$

This leads to

$$\alpha_{ij} = B + D_{ij} + E_{ij} \quad (4.36)$$

$$= B - \sqrt{E_{ij}} (\sqrt{E_{1,N}} - \sqrt{E_{ij}}) + \frac{1}{2} (D_{1,N} + E_{1,N}) (\delta_{1i} + \delta_{1j}) \quad (4.37)$$

Because of the arbitrariness in the cycle labelling consistency now requires

$$D_{1,N} + E_{1,N} = 0 \quad (4.38)$$

whereupon

$$\alpha_{ij}(0) = B + \left(\sum_{k=i}^j e_k \right)^2 - \sqrt{E_{1,N}} \sum_{k=i}^j e_k \quad (4.39)$$

is a consistent possibility for fixed N.

When we vary N, however, one soon finds contradictions unless the quantity e_k is conserved, that is

$$\sqrt{E_{1,N}} = \sum_{k=1}^N e_k = 0 \quad (4.40)$$

We should add the remarks:

(i) Suppose that we take all external particles to be identical. Then we must have, say, $d_k = e_k = \pm 1$, and consistency is possible only for N even.

(ii) We may factorize the extra piece $2e_i e_j$ in $2\hat{p}_i \cdot \hat{p}_j$ by adding one additional (fifth) dimension. This increases the density of hadronic states $\rho(m)$ as a function of mass m .

Recall that

$$\lim_{m \rightarrow \infty} \left[\frac{m}{\ln g(m)} \right] = T_H(d) = \frac{\sqrt{6}}{2\pi} \frac{1}{\sqrt{\alpha' d}} \quad (4.41)$$

where T_H is the Hagedorn temperature and $d \equiv (4+D)$ is the dimensionality of oscillators. Putting $\alpha' = 1 \text{ GeV}^{-2}$ one finds $T_H(4) = 195 \text{ MeV}$, $T_H(5) = 174 \text{ MeV}$. Both are close to that suggested by Hagedorn's fits to transverse momentum distributions; if one takes the numerology very seriously the extra dimension improves the agreement. [Reference 9]

4c. The Rubber String Model

The operator formalism is suggestive of a quantum system with an infinite number of degrees of freedom, for example black-body radiation inside a cavity or a one-dimensional elastic continuum. It is by now traditional to discuss a massless rubber string. [References 12-16]

The discussion of the string, that is given in the present section, will be at such a level that we use the string simply to set up calculational rules (analogous to Feynman rules) that reproduce the generalized Euler B function. We shall here be

non-rigorous about the method of embedding the string in Minkowski space-time; in a later section we shall give a more detailed treatment, using Riemannian geometry, of the two-dimensional world-sheet mapped out in space-time by the propagation of a one-dimensional string. The price paid for the higher rigour in the second discussion will be that, although we obtain a beautiful understanding of the level structure, we shall not deal at all with the (non-local) interactions. In the present less rigorous first discussion we shall include interactions and will already gain some insight into how our third and last internal symmetry, baryon-number, might be incorporated.

Consider a set of N mass points arranged along a one-dimensional line, and with nearest-neighbour harmonic-oscillator interactions. The classical energy will now be in the form of a sum of kinetic and potential terms

$$E = \frac{1}{2} \sum_{i=1}^N \dot{x}_\mu^{(i)} \dot{x}_\mu^{(i)} + \frac{1}{2} \sum_{i=1}^N (x_\mu^{(i)} - x_\mu^{(i-1)})^2 \quad (4.42)$$

where $x_\mu^{(i)}$ is the position of the i th particle. Taking a continuum limit to a string of length π gives

$$E = \frac{1}{2} \int_0^\pi d\sigma \left(\dot{x}_\mu(\sigma) \dot{x}_\mu(\sigma) + \frac{\partial x_\mu}{\partial \sigma} \frac{\partial x_\mu}{\partial \sigma} \right) \quad (4.43)$$

The relevant equation of motion is the wave equation,

$$\ddot{x}_\mu(\sigma) - \frac{\partial^2}{\partial \sigma^2} x_\mu(\sigma) = 0 \quad (4.44)$$

and the solution of this can be expanded in terms of normal modes by

$$X_\mu(\sigma) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}} (a_\mu^{(n)} + a_\mu^{(n)\dagger}) \cos n\sigma \quad (4.45)$$

with $[a_\mu^{(n)}, a_\nu^{(n)\dagger}] = -\delta_{mn} g_{\mu\nu}$ and where we have selected those solutions which respect the boundary condition

$$\frac{\partial X_\mu(\sigma)}{\partial \sigma} = 0 \quad \text{at } \sigma = 0, \pi \quad (4.46)$$

corresponding to the vanishing tension at the ends of the massless string.

We can now use this field $X_\mu(\sigma)$ to set up calculational rules, based on diagrams, that faithfully reproduce the generalized Euler B functions.

The calculational rules for the N-meson Born amplitude are [Reference 16]

- (1) We take two of the external lines (chosen arbitrarily) and interpret them as the incoming and outgoing rubber string ($\tau = -\infty$ and $\tau = +\infty$) which maps out the region $0 \leq \sigma \leq \pi$ and $-\infty \leq \tau \leq +\infty$ in the complex $W = (\tau + i\sigma)$ plane (see Figure 1)
- (2) The remaining (N-2) scalars are emitted and absorbed from the two sides $\sigma=0$ and $\sigma=\pi$ of the basic infinite strip; one scalar must be at a fixed τ -value, say $\tau=0$ but the remaining (N-3) lines must be integrated over all possible τ - orderings which respect the cyclic ordering of the external lines.

(3A) For each emitted or absorbed line we write a factor $T_0(p)$ or $T_{\pi}(p)$ corresponding to $\sigma=0$ or $\sigma=\pi$ respectively in

$$T_{\sigma}(p) = \exp(ip_{\mu} X_{\mu}(\sigma)) \quad (4.47)$$

(3B) Internal propagation of the rubber string is described by a factor (for unit intercept)

$$P(s=p_{\mu}p^{\mu}) = \frac{1}{R-s-1} = i \int_0^{\infty} d\tau e^{i(h_0-1)\tau} \quad (4.48)$$

where p_{μ} is the total four momentum.

Consider first the four-meson Born amplitude, in the configuration corresponding to Fig. 1(a). We have the amplitude

$$i \int_0^{\infty} d\tau e^{-i(s+1)\tau} \langle 0 | T_0(p_2) e^{iR} T_0(p_3) | 0 \rangle \quad (4.49)$$

$$= \int_0^1 dx x^{-s-2} \langle 0 | \exp(i\sqrt{2} \sum_{n=1}^{\infty} \frac{p_2 \cdot a_n}{\sqrt{n}}) x^R \exp(i\sqrt{2} \sum_{n=1}^{\infty} \frac{p_3 \cdot a_n}{\sqrt{n}}) | 0 \rangle$$

$$= \int_0^1 dx x^{-\alpha(s)-1} (1-x)^{-\alpha(p_2 \cdot p_3)} \quad (4.50)$$

$$= \int_0^1 dx x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} = B(-\alpha(s), -\alpha(t)) \quad (4.51)$$

where we have substituted $x = e^{-i\tau}$.

By interchanging $p_{2\mu}$ and $p_{3\mu}$ as in Fig. 1(b) and integrating τ_3 from $-\infty$ to 0 we arrive at

$$i \int_{-\infty}^0 d\tau e^{i(u+1)\tau} \langle 0 | T_0(p_3) e^{-iR} T_0(p_2) | 0 \rangle$$

$$= \int_0^1 dx x^{-\alpha(u)-1} (1-x)^{-\alpha(t)-1} \quad (4.52)$$

$$= B(-\alpha(t), -\alpha(u)) \quad (4.53)$$

More interesting, since the direct connection to the operator factorization is slightly less immediate, is to consider the $B(-\alpha(u), -\alpha(s))$ term which arises from the sum of two time orderings as depicted in Fig. 2. The amplitude for this permutation is then

$$i \int_0^\infty dz \left[e^{-iz(u+1)} \langle 0 | T_\pi(p_3) e^{iRz} T_0(p_2) | 0 \rangle + e^{iz(s+1)} \langle 0 | T_0(p_2) e^{iRz} T_\pi(p_3) | 0 \rangle \right] \quad (4.54)$$

$$= \int_0^1 dx \left[x^{-u-2} \langle 0 | \exp\left(i\sqrt{2} \sum_{n=1}^\infty \frac{p_3 \cdot a^{(n)}}{\sqrt{n}} (-1)^n\right) x^R \exp\left(i\sqrt{2} \sum_{n=1}^\infty \frac{p_2 \cdot a^{(n)}}{\sqrt{n}}\right) | 0 \rangle + x^{-s-2} \langle 0 | \exp\left(i\sqrt{2} \sum_{n=1}^\infty \frac{p_2 \cdot a^{(n)}}{\sqrt{n}}\right) x^R \exp\left(i\sqrt{2} \sum_{n=1}^\infty \frac{p_3 \cdot a^{(n)}}{\sqrt{n}} (-1)^n\right) | 0 \rangle \right]$$

$$= \int_0^1 dx \left[x^{-\alpha(u)-1} (1+x)^{-\alpha(t)-1} + x^{-\alpha(s)-1} (1+x)^{-\alpha(t)-1} \right] \quad (4.55)$$

Now we combine these two terms by using $-\alpha(t)-1 = \alpha(s) + \alpha(u)$ and substituting $x = (\frac{1-y}{y})$ in the first, $x = (\frac{y}{1-y})$ in the second to find

$$\int_0^{1/2} dy y^{-\alpha(s)-1} (1-y)^{-\alpha(u)-1} + \int_{1/2}^1 dy y^{-\alpha(s)-1} (1-y)^{-\alpha(u)-1}$$

$$= B(-\alpha(u), -\alpha(s)) \quad (4.56)$$

Now that we understand that the two τ -orderings fit together in this way we may write formally a full vertex

$$T(p) = T_0(p) + T_\pi(p) \quad (4.57)$$

and then obtain the sum over all permutations by writing

$$\begin{aligned} & \langle 0 | (T(p_2) P(s) T(p_3) + T(p_3) P(u) T(p_2)) | 0 \rangle \\ &= 2 \left(B(-\alpha(s), -\alpha(u)) + B(-\alpha(u), -\alpha(s)) \right) \end{aligned} \quad (4.58)$$

where the eight contributing diagrams are indicated in Fig.3.

Does this procedure continue to work in the multiparticle extensions? The answer is yes, it does. We shall give some details of the $N=5$ case only, since it already gives a non-trivial combination over τ -orderings. Let us keep particles 1 and 5 as the ends of the rubber strip; then there are 48 possible distributions of the remaining three lines (analogous to the 8 of Figure 3). These fall into two basic types, the first of which is exemplified by

Figure (4a), whose amplitude is

$$\begin{aligned} & (-i) i \int_{-\infty}^0 d\tau_1 e^{-i\tau_1(s_{12}+1)} \int_0^{\infty} d\tau_2 e^{-i\tau_2(s_{45}+1)} \\ & \langle 0 | T_0(p_2) e^{i\tau_1 R} T_0(p_3) e^{i\tau_2 R} T_0(p_4) | 0 \rangle \\ &= \int_0^1 dx_1 dx_2 x_1^{-s_{12}-1} x_2^{-s_{45}-1} \end{aligned} \quad (4.59)$$

$$\begin{aligned} & \langle 0 | e^{i\sqrt{2} p_2 \sum \frac{a_n}{\sqrt{n}}} x_1^R e^{i\sqrt{2} p_3 \sum \frac{a_n}{\sqrt{n}}} e^{i\sqrt{2} p_4 \sum \frac{a_n}{\sqrt{n}}} x_2^R e^{i\sqrt{2} p_5 \sum \frac{a_n}{\sqrt{n}}} | 0 \rangle \\ &= \int_0^1 dx_1 dx_2 x_1^{-\alpha(s_{12})-1} x_2^{-\alpha(s_{45})-1} (1-x_1)^{-\alpha(s_{23})-1} (1-x_2)^{-\alpha(s_{34})-1} (1-x_1 x_2)^{-\alpha(s_{15})+\alpha(s_{23})+\alpha(s_{34})} \end{aligned} \quad (4.60)$$

$$= B_5(p_1, p_2, p_3, p_4, p_5) \quad (4.61)$$

The second type of configuration is indicated in Fig. 4(b), and here we must add three τ -orderings to give the amplitude

$$\begin{aligned}
 & (-i) i \int_{-\infty}^0 d\tau_1 \int_0^{\infty} d\tau_2 \cdot \\
 & * \left[e^{i\tau_1(s_{12}+1)} e^{-i\tau_2(s_{45}+1)} \langle 0 | T_{\pi}(p_2) e^{-i\tau_1 R} T_0(p_3) e^{i\tau_2 R} T_0(p_4) | 0 \rangle \right. \\
 & + e^{i\tau_1(s_{13}+1)} e^{-i\tau_2(s_{45}+1)} \langle 0 | T_0(p_3) e^{-i\tau_1 R} T_{\pi}(p_2) e^{i\tau_2 R} T_0(p_4) | 0 \rangle \\
 & \left. + e^{i\tau_1(s_{13}+1)} e^{-i\tau_2(s_{25}+1)} \langle 0 | T_0(p_3) e^{-i\tau_1 R} T_0(p_4) e^{i\tau_2 R} T_{\pi}(p_2) | 0 \rangle \right] \\
 & = \int_0^1 dx_1 dx_2 \left[x_1^{-\alpha_{12}-1} x_2^{-\alpha_{45}-1} (1+x_1)^{-2p_2 \cdot p_3} (1-x_2)^{-2p_3 \cdot p_4} (1+x_1 x_2)^{-2p_2 \cdot p_4} \right. \\
 & \quad + x_1^{-\alpha_{13}-1} x_2^{-\alpha_{45}-1} (1+x_1)^{-2p_2 \cdot p_3} (1+x_2)^{-2p_2 \cdot p_4} (1-x_1 x_2)^{-2p_3 \cdot p_4} \\
 & \quad \left. + x_1^{-\alpha_{13}-1} x_2^{-\alpha_{25}-1} (1-x_1)^{-2p_3 \cdot p_4} (1+x_2)^{-2p_2 \cdot p_4} (1+x_1 x_2)^{-2p_2 \cdot p_3} \right]
 \end{aligned}
 \tag{4.62}$$

Now we use, (for unit intercept)

$$-2p_2 \cdot p_3 = -\alpha_{45} + \alpha_{12} + \alpha_{13} \tag{4.63}$$

$$-2p_3 \cdot p_4 = -\alpha_{34} - 1 \tag{4.64}$$

$$-2p_2 \cdot p_4 = -\alpha_{13} + \alpha_{25} + \alpha_{45} \tag{4.65}$$

and we make the changes of variables denoted by

$$\left. \begin{aligned} u_{12} &= \frac{x_1}{1+x_1} \\ u_{45} &= \frac{x_2(1+x_1)}{1+x_1x_2} \end{aligned} \right\} \quad \begin{aligned} (4.66a) \\ (4.66b) \end{aligned}$$

$$\left. \begin{aligned} u_{12} &= (1+x_1)^{-1} \\ u_{45} &= \frac{x_2(1+x_1)}{(1+x_2)} \end{aligned} \right\} \quad \begin{aligned} (4.67a) \\ (4.67b) \end{aligned}$$

$$\left. \begin{aligned} u_{12} &= (1+x_1x_2)^{-1} \\ u_{45} &= \frac{1+x_1x_2}{1+x_2} \end{aligned} \right\} \quad \begin{aligned} (4.68a) \\ (4.68b) \end{aligned}$$

respectively in the three terms of the amplitude. After a little algebra one now finds that the three pieces fit together precisely to give the full integration region in

$$\int_0^1 du_{12} \int_0^1 du_{45} u_{12}^{-\alpha_{12}-1} (1-u_{12})^{-\alpha_{13}-1} (1-u_{45})^{-\alpha_{34}-1} u_{45}^{-\alpha_{45}-1} (1-u_{12}u_{45})^{-\alpha_{52}+\alpha_{13}+\alpha_{34}} \quad (4.69)$$

$$= B_5(p_2 p_1 p_3 p_4 p_5) \quad (4.70)$$

as required. The three terms give respectively the three domains of integration

$$0 \leq u_{12} \leq \frac{1}{2} ; \quad 0 \leq u_{45} \leq 1 \quad (4.71)$$

$$\frac{1}{2} \leq u_{12} \leq 1 ; \quad 0 \leq u_{45} \leq 1 ; \quad u_{12} u_{45} \leq \frac{1}{2} \quad (4.72)$$

$$\frac{1}{2} \leq u_{12} \leq 1 ; \quad \frac{1}{2} \leq u_{45} \leq 1 ; \quad u_{12} u_{45} \geq \frac{1}{2} \quad (4.73)$$

which combine nicely into the unit square $0 \leq u_{12}, u_{45} \leq 1$.

By repeated application of the two basic calculations (Fig.4(a) and Fig.4(b)) one finds that the 48 string diagrams add together to give each of the 12 inequivalent permutations added with equal coefficients (the coefficients are all equal to 2).

A similar result presumably obtains for the generalization to N external particles. (see Reference 16, where such a conjecture is made)

We should now add some general remarks.

(i) As mentioned at the beginning, these rubber string ideas as presented so far are no more than calculational rules, and we have not directly confronted the problem of how the string is embedded in Minkowski space-time. We give a more satisfactory treatment of this point later.

(ii) We have treated two external lines asymmetrically, and have not needed to exhibit any rubber string character for the remaining $(N-2)$ ground state particles. The final result, however, was always fully cyclic symmetric and clearly independent of the particular choice of external state to treat preferentially.

(iii) Everywhere we used unit intercept $\alpha(0)=1$, since the calculational rules are most simple for this case. One can, if necessary, modify the rules in a somewhat ad hoc manner (analogous to adding

a fifth dimension) to deal with the cases $\alpha(0) \neq 1$ [For details, see Frye, 17]

(iv) The way in which the different τ -orderings combine neatly together becomes less surprising when we note that the boundary of the infinite strip is nothing more than a re-mapping of the Koba-Nielsen circle. To be explicit, suppose we consider the unit circle in a Z -plane with $Z_1=1$, $Z_{N-1}=i$, $Z_N=-1$ and $Z_i=e^{i\theta_i}$ ($2 \leq i \leq N-2$, $0 \leq \theta_i < \frac{\pi}{2}$) together with its mapping into an x -plane with $x_1=0$, $x_{N-1}=1$ and $x_N=\infty$ ($0 \leq x_i \leq 1$ (real); $2 \leq i \leq N-2$). Now we map these into a W -plane strip with $W_1=-\infty$, $W_{N-1}=+\infty$, $W_N=i\pi$. The appropriate transformations are found to be

$$x = \frac{i(1-z)}{(1+z)} \quad (4.74a)$$

$$z = \frac{i-x}{i+x} \quad (4.74b)$$

$$W = \ln\left(\frac{x}{1-x}\right) \quad (4.75a)$$

$$x = \frac{e^W}{1+e^W} \quad (4.75b)$$

$$z = \frac{1+e^W(1+i)}{1+e^W(1-i)} \quad (4.76a)$$

$$W = \ln\left(\frac{i(1-z)}{(1-i)+z(1+i)}\right) \quad (4.76b)$$

With these transformations in mind, we can understand that the tortuous summations over τ -orderings are no more than the integration of Koba-Nielsen variables on the unit circle (or real axis) suitably respecting the cyclic ordering of the external particles.

(v) We may make duality transformations directly on the strip. For example in Figure 5a we may transform from the first to the second configurations by the strip-preserving transformation.

$$W' = -\ln (1-e^W) \quad (4.77)$$

It is, of course, very tempting to identify these string diagrams directly with Harari-Rosner diagrams. One edge of the strip is now a quark, and the other is an antiquark, as indicated by the arrows in Fig. 5. The duality transformation we have just made is then interpreted as proceeding from the S-channel representation to the t-channel representation, equal by duality, as indicated in Fig. 5(b).

4d Baryons and Exotics

When we have recognized a correspondence between the strip diagrams and the Harari-Rosner diagrams, a natural question to ask is: can we exploit this connection to arrive at an amplitude for meson-baryon scattering? Of course, at the present level we ignore the presence of half-integer spins, and consider spinless ground-state baryons. Baryon number will be incorporated only in the sense that baryons contain three "quarks", while mesons contain only two.

We shall show that once an ansatz has been made for the configuration of quarks on the rubber band for baryons, then the configurations of all higher quark numbers (exotic mesons and baryons) are completely determined.

The ansatz for baryons [Reference 18, see also 19, 20; the baryonic amplitudes were first written by Mandelstam, 21, without explicit use of the rubber string] will be that the three quarks are situated symmetrically at positions $\sigma=0, 2\pi/3, 4\pi/3$ on a circular string, as indicated in Fig. 6(a). We should emphasize that this is only an assumption, taken as one of the simplest generalizations of the mesonic case.

We first consider the diagram of Figure 7(a), for meson-baryon scattering. The baryon strip carries quarks at $\sigma=0, 2\pi/3, 4\pi/3$. We use a field containing both cosine and sine modes of the form

$$\begin{aligned} X_\mu(\sigma, \tau) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}} \left[(a_\mu^{(n)} e^{-n\tau} + a_\mu^{(n)\dagger} e^{n\tau}) \cos n\sigma \right. \\ \left. + (b_\mu^{(n)} e^{-n\tau} + b_\mu^{(n)\dagger} e^{n\tau}) \sin n\sigma \right] \end{aligned} \quad (4.78)$$

and then the amplitude is given by

$$\int_0^\infty d\tau_3 \langle 0 | e^{i p_{2,0} X(0,0)} e^{i p_{3,0} X(0,\tau_3)} | 0 \rangle = B(-\alpha_s, -\alpha_t) \quad (4.79)$$

That is, we obtain the Euler B function. This is because the calculation is identical to that for the mesonic diagram of Fig. 1(a).

We may now look for the vertex to describe baryon emission⁽¹⁹⁾ by making the strip-preserving transformation.

$$W' = \ln (e^W - 1) \quad (4.80)$$

This gives the W' -plane of Fig.7(b). The points have been mapped according to

$$W_1 = -\infty \rightarrow W'_1 = i\pi \quad (4.81)$$

$$W_2 = 0 \rightarrow W'_2 = -\infty \quad (4.82)$$

$$W_3 = \tau_3 (0 \leq \tau_3 \leq \infty) \rightarrow W'_3 = \tau'_3 (-\infty \leq \tau'_3 \leq \infty) \quad (4.83)$$

$$W_4 = \infty \rightarrow W'_4 = \infty \quad (4.84)$$

In particular we note that the passive quark lines have been mapped into the curved focusing lines $C_{\pm}$ given by

$$\cos \sigma'_{\pm}(\tau) = - \frac{3 + (4e^{2\tau'} - 3)^{1/2}}{4e^{\tau'}} \quad (4.85)$$

selected by

$$\sigma'_{\pm}(\infty) = \pi \pm \frac{1}{3} \pi \quad (4.86)$$

for the original lines $\sigma_{\pm} = \pi \pm \frac{1}{3} \pi$ respectively.

To check this write

$$W_{\pm} = \pi i \pm \frac{i\pi}{3} + \tau \quad (4.87)$$

whereupon

$$W'_{\pm} = \ln \left[\left(-\frac{1}{2} \mp \frac{i\sqrt{3}}{2} \right) e^{\tau} - 1 \right] \quad (4.88)$$

$$e^{\tau'_{\pm} + i\sigma'_{\pm}} = e^{\tau} \left(-\frac{1}{2} \mp \frac{i\sqrt{3}}{2} \right) - 1 \quad (4.89)$$

$$e^{2\tau'_{\pm}} = 1 + e^{\tau} + e^{2\tau} \quad (4.90)$$

$$e^{\tau} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{4e^{2\tau'_{\pm}} - 3} \quad (4.91)$$

$$\cos \sigma_{\pm}' = e^{-\tau_{\pm}'} \left(-\frac{1}{2} e^{\tau} - 1 \right) \quad (4.92)$$

$$= - \left[\frac{3 \pm \sqrt{4e^{2\tau_{\pm}'} - 3}}{4e^{\tau_{\pm}'}} \right] \quad (4.93)$$

as required.

To ensure that the calculation of the diagram Fig.7(b) gives the same result, namely $B(-\alpha_s, -\alpha_t)$, as Fig.7(a) we deduce that the baryon vertex is given by

$$e^{ip_1 \cdot X(\sigma, \tau)} \quad (4.94)$$

precisely as for the mesonic case. This deduction will enable us to proceed to more complicated processes with more baryonic external lines.

First we must discuss the other permutations of the meson-baryon scattering case. The tu -term (1324) is obtained simply by interchanging $p_2 \leftrightarrow p_3$ in the st -diagram (1234) and clearly gives $B(-\alpha_t, -\alpha_u)$, again an Euler B function. The us -term, Fig.7(c), is more interesting, since in this case baryons in the u -channel build up baryons in the s -channel by duality, and we expect the new (sine) modes in the baryon to be excited. Indeed the calculation gives now

$$\begin{aligned} & \int_{-\infty}^0 d\tau_3 \langle 0 | e^{ip_3 \cdot X(\theta, \tau_3)} e^{ip_2 \cdot X(0, 0)} | 0 \rangle \\ & + \int_0^{\infty} d\tau_3 \langle 0 | e^{ip_2 \cdot X(0, 0)} e^{ip_3 \cdot X(\theta, \tau_3)} | 0 \rangle \end{aligned} \quad (4.95)$$

with $\theta = 4\pi/3$. We will, however, leave θ free to see how the general four-point function depends on the angular separation of the active quarks (for $\theta = \pi$ we should regain the Euler B function, for example.)

The amplitude becomes

$$\int_0^1 dx x^{-\alpha_u-1} \langle 0 | \exp \left(\sqrt{2} i p_3 \cdot \sum_{n=1}^{\infty} \frac{a_n^{(1)}}{\sqrt{n}} \cos n\theta \right) x^R \exp \left(\sqrt{2} i p_2 \cdot \sum_{n=1}^{\infty} \frac{a_n^{(1)}}{\sqrt{n}} \right) | 0 \rangle + (\alpha_s \leftrightarrow \alpha_u)$$

$$= \int_0^1 dx x^{-\alpha_u-1} \exp \left[p_2 \cdot p_3 \sum_{n=1}^{\infty} \frac{x^n}{n} (e^{in\theta} - e^{-in\theta}) \right] + (\alpha_s \leftrightarrow \alpha_u) \quad (4.96)$$

$$= \int_0^1 dx x^{-\alpha_u-1} (1 - 2x \cos \theta + x^2)^{-\frac{1}{2}(1+\alpha_s)} + (\alpha_s \leftrightarrow \alpha_u) \quad (4.97)$$

Now make the change of variables

$$u = (1 - 2x \cos \theta + x^2)^{-1/2} \quad (4.98)$$

$$v = xu \quad (4.99)$$

and use $\alpha_s + \alpha_t + \alpha_u = -1$ (for unit intercepts) to rewrite the amplitude

$$\int_0^1 \frac{1}{2 \sin \theta} \frac{du}{v - u \cos \theta} u^{-\alpha_s-1} v^{-\alpha_u-1}$$

$$+ \int_0^1 \frac{1}{2 \sin \theta} \frac{dv}{u - v \cos \theta} v^{-\alpha_s-1} u^{-\alpha_u-1}$$

$$= \int_0^1 \frac{du}{v - u \cos \theta} u^{-\alpha_s-1} v^{-\alpha_u-1} \quad (4.100)$$

$$= 2 \int_0^1 du \int_0^1 dv u^{-\alpha_s-1} v^{-\alpha_u-1} \delta(u^2 + v^2 - 2uv \cos \theta - 1) \quad (4.101)$$

where we have noticed that

$$u^2 + v^2 - 2uv \cos \theta = 1 \quad (4.102)$$

If we now make the change of variable

$$u = z \left[z^2 + (1-z)^2 - 2z(1-z) \cos \theta \right]^{1/2} \quad (4.103)$$

we may re-write the amplitude

$$\int_0^1 dz \quad z^{-\alpha_s-1} (1-z)^{-\alpha_u-1} \left[1 - 4z(1-z) \cos^2 \frac{\theta}{2} \right]^{\frac{\alpha_s+\alpha_u}{2}} \quad (4.104)$$

so that in addition to the usual branch points of the integrand at $Z = 0, 1, \infty$ there are new branch points situated at

$$Z = \frac{1}{2} \pm \frac{1}{2} i \left(\sec^2 \frac{\theta}{2} - 1 \right)^{1/2} \quad (4.105)$$

In particular, for the particular case in which we are interested the meson-baryon amplitude, with $\theta = 4\pi/3$, becomes

$$\int_0^1 dx \quad x^{-\alpha_s-1} (1-x)^{-\alpha_u-1} (1-x+x^2)^{\frac{\alpha_s+\alpha_u}{2}} \quad (4.106)$$

This amplitude was first written by Mandelstam⁽²¹⁾ without the use of the rubber string picture.

To check the consistency of the approach we may re-calculate the (us)-diagram by using the configuration of Fig.7(d). This is obtained from Fig.7(c) by the strip preserving duality transformation

$$W' = \ln (e^W - 1) \quad (4.107)$$

The interesting feature now is that in the W' - plane one meson (with momentum $p_{3\mu}$) is emitted from a focusing curve C_+ . The appropriate amplitude is

$$i \int_0^\infty d\tau_3 \left[1 + \left(\frac{d\sigma'_+(\tau'_3)}{d\tau'_3} \right)^2 \right]^{1/2} \langle 0 | e^{i p_1 \cdot x(0, \pi)} e^{i p_3 \cdot x(\tau'_3, \sigma'_+(\tau'_3))} | 0 \rangle \quad (4.108)$$

with $\sigma'_+(\tau_3)$ given by the explicit equation for C_+ derived above.

The vacuum expectation value becomes (with $x=e^{-\tau_3}$)

$$\langle 0 | e^{i\sqrt{2}p_3 \sum \frac{a_n^{(+)}}{\sqrt{n}} (-1)^n} x^R e^{i\sqrt{2}p_3 \sum \frac{a_n^{(+)}}{\sqrt{n}} \cos n\sigma'_+} | 0 \rangle$$

$$= \exp \left[p_1 p_3 \sum_{n=1}^{\infty} \frac{x^n}{n} (-1)^n (e^{in\sigma'_+} + e^{-in\sigma'_+}) \right] \quad (4.109)$$

$$= \left[(1 + x e^{i\sigma'_+}) (1 + x e^{-i\sigma'_+}) \right]^{p_1 p_3} \quad (4.110)$$

$$= (1 + 2x \cos \sigma'_+ + x^2)^{p_1 p_3} \quad (4.111)$$

$$= \left(1 - \frac{1}{2} x^2 - \frac{x}{2} \sqrt{4-3x^2} \right)^{p_1 p_3} \quad (4.112)$$

$$= \left(-\frac{1}{2} x + \frac{1}{2} \sqrt{4-3x^2} \right)^{2p_1 p_3} \quad (4.113)$$

where in the last step we simply noticed that

$$\left(-\frac{1}{2} x + \frac{1}{2} \sqrt{4-3x^2} \right)^2 = 1 - \frac{1}{2} x^2 - \frac{x}{2} \sqrt{4-3x^2} \quad (4.114)$$

To calculate the arc length note that

$$\sin \sigma'_+ = \frac{\sqrt{3}}{4} \left(e^{-\tau'_3} \pm e^{-\tau'_3} \sqrt{4e^{2\tau'_3} - 3} \right) \quad (4.115)$$

and hence

$$\cos \sigma'_+ \frac{d\sigma'_+}{d\tau'_3} = \frac{\sqrt{3}}{4\sqrt{4e^{2\tau'_3} - 3}} \left[3e^{-\tau'_3} \pm e^{-\tau'_3} \sqrt{4e^{2\tau'_3} - 3} \right] \quad (4.116)$$

$$= \frac{\sqrt{3}}{\sqrt{4e^{2\tau'_3} - 3}} \cos \sigma'_+ \quad (4.117)$$

whereupon

$$\left[1 + \left(\frac{d\sigma'_+}{d\tau'_3} \right)^2 \right]^{1/2} = \frac{2}{\sqrt{4-3x}} \quad (4.118)$$

Collecting results the amplitude is

$$2 \int_0^1 \frac{dx}{\sqrt{4-3x^2}} x^{-\alpha_s-1} \left(-\frac{1}{2}x + \frac{1}{2}\sqrt{4-3x^2} \right)^{-\alpha_u-1} \quad (4.119)$$

Putting $y = \left(-\frac{1}{2}x + \frac{1}{2}\sqrt{4-3x^2} \right)$ we notice that

$$x^2 + y^2 + xy = 1 \quad (4.120)$$

$$x + 2y = \sqrt{4-3x^2} \quad (4.121)$$

and hence the original (us)- amplitude may be regained, namely

$$2 \int_0^1 dx \int_0^1 dy x^{-\alpha_s-1} y^{-\alpha_u-1} \delta(x^2+y^2+xy-1) \quad (4.122)$$

This confirms that the pictorial idea of two quarks focusing to a point vertex is consistent at the four-point level.

Knowledge of the baryon emission vertex enables us to generalize further to include more baryon external lines, whereupon (as is well known) exotic intermediate states become essential. Here we can directly investigate how the extra quarks of such states should be situated on the rubber string. [Reference 22]

The amplitude for baryon-antibaryon scattering is indicated by Fig.8(a) in the W-plane. By making the strip-preserving duality transformation

$$W' = \ln (e^{-W}-1)^{-1} \quad (4.123)$$

we arrive at the configuration of Fig.8(b) which exhibits an

exotic meson in the t -channel. The t -channel pole arises from $(\tau_3' - \tau_4') \rightarrow \infty$ corresponding to $(\tau_3 - \tau_2) \rightarrow 0$ and so we are interested in the asymmetric positions of the two quarks and two antiquarks in the intermediate state. It is not necessary to make an explicit calculation (the precise equations are, however, contained in reference 22). The required information can be obtained by comparison to Figs. 8(c), 8(d) for meson-baryon scattering where a similar quark "loop" occurs. The comparison reveals that the quarks are at $\sigma=0, \frac{2\pi}{3}$ and the antiquarks at $\sigma=\pi, \frac{5\pi}{3}$. By scattering these exotic mesons, now as external legs, off of baryons (Fig. 8(e)) we arrive at exotic baryon intermediate states with four quarks at $\sigma = 0$ (two), $\frac{2\pi}{3}, \frac{4\pi}{3}$ plus an antiquark at $\sigma=\pi$.

Following such an iterative approach one finds that the following rule obtains, (c.f. Fig. 9a)

All quarks are placed at the vertices of one equilateral triangle (T_1) of a regular starred hexagon inscribed into the circular rubber band, and all antiquarks are placed on the other equilateral triangle (T_2). There are the restrictions that for each $q(\bar{q})$ at a vertex of $T_1(T_2)$ there must appear either one $q(\bar{q})$ each at the other two vertices of $T_1(T_2)$ or one $\bar{q}$ (q) at the diametrically opposed vertex of $T_2(T_1)$.

Examples for total quark numbers four and five are given in Figures 9(b) and 9(c) respectively.

We should add two remarks:

- (a) Once we have agreed upon the quark configuration $\sigma=0, \frac{2\pi}{3}, \frac{4\pi}{3}$

for baryons, the above Star of David rule followed. Nevertheless it is quite possible to envisage other generalizations from two to three quarks such as that indicated schematically in Fig.6(b).

(b) It is not possible to use a simple multiplicative factor for baryon number similar to that which we discussed earlier for isospin and hypercharge. To see this point, consider the non-planar tree diagram of Fig.10 (which must be present for consistent factorization); this is a five point function with poles in six non-planar, rather than five planar, channels and clearly therefore cannot be adequately described by the mesonic B_5 . Instead we must make a more essential modification as we have done above.

4e More on the Rubber String

We shall now give a more mathematically rigorous treatment of the rubber string model. In particular we shall be concerned with how the string is embedded into Minkowski space-time. This procedure might ultimately lead to a much simpler reformulation of the dual models. For the moment, it only provides us with a useful, different viewpoint of the spectrum of physical states. ~~Ref: The underlying idea of Nambu, Reference 223; see also References 24-28] This is to take as the action of the freely propagating rubber~~ string the area of the world sheet mapped out in space time. Defining two coordinates $\xi_1 = \sigma$, $\xi_2 = \tau$ to parametrise the sheet we write

$$S = \frac{1}{2\pi} \int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau \sqrt{-g} \quad (4.124)$$

with

$$g = |g_{ab}| \quad (4.125)$$

$$g_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi_a} \frac{\partial x^\nu}{\partial \xi_b} \quad (4.126)$$

and $x^\mu(\sigma, \tau)$ is the space-time position regarded as a field on variables σ, τ .

We introduce the notation

$$\dot{x}_\mu = \frac{\partial x_\mu}{\partial \tau} \quad (4.127)$$

$$x'_\mu = \frac{\partial x_\mu}{\partial \sigma} \quad (4.128)$$

whereupon

$$S = \int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau \mathcal{L} \quad (4.129)$$

$$= \frac{1}{2\pi} \int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} \quad (4.130)$$

To see that this is the area mapped out by the world sheet consider the infinitesimal

$$dx^\mu = \dot{x}^\mu d\tau + x'^\mu d\sigma \quad (4.131)$$

The corresponding area element is

$$dA = \left[- (dx^\mu \wedge dx^\nu) (dx^\mu \wedge dx^\nu) \right]^{1/2} \quad (4.132)$$

where

$$dx^\mu \wedge dx^\nu = d\sigma d\tau [\dot{x}^\mu x'^\nu - \dot{x}^\nu x'^\mu] \quad (4.133)$$

and hence

$$dA = d\sigma d\tau \left[(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2 \right]^{1/2} \quad (4.134)$$

as required.

This action satisfies

i) Poincare invariance.

ii) General covariance under arbitrary redefinition of the coordinates σ, τ . This follows because the action has been constructed with $\sqrt{-g}$, the inner metric of the sheet. It is also physically clear because the area cannot depend on the choice of parametrization. This general covariance will be shown to be the origin of the generalized projective gauge conditions. Clearly the form of the action is closely analogous to the use of $\sqrt{-g}$, $\bar{g} = |g^{\mu\nu}|$ in general relativity (e.g. Reference 29); note that probably no clearer exposition of the latter exists than the original by Einstein [Reference 30 translated and reprinted in Reference 31] in what is perhaps the most celebrated article contributed yet to theoretical physics since Newton. From the point of view of general covariance in the case of the two-dimensional sheet, we might equally consider using the outer metrics

$$h_{ab}^{\alpha} = g_{\mu\nu} \frac{\partial \hat{x}^{\mu}}{\partial \xi^a} \frac{\partial \hat{x}^{\nu}}{\partial \xi^b} \quad (4.135)$$

where $\hat{n}_\alpha^\mu$ ($\alpha = 1, 2$) are unit vectors normal to the sheet.

However, the action we have written is the simplest possibility and it possesses the third property that iii) it gives rise to equations of motion of second order.

Concerning the general covariance (ii), it is here very useful to bear in mind the simpler situation of a point particle propagating in space-time. To make a manifestly covariant description there we may introduce a four component space-time position $x_\mu(\tau)$ regarded as a field on τ and then introduce an action

$$S = \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{x}^2} \quad (4.136)$$

which is the length of the world-line. This action is covariant under general coordinate changes $\tau \rightarrow f(\tau)$; physically this corresponds to the fact that only three independent dynamical variables exist since for example x_i ($\tau = x_0 = t$), $i=1,2,3$, completely specifies the shape of the world-line.

To find the equations of motion and the boundary conditions we apply Hamilton's principle of stationary action, keeping fixed the initial and final $\tau = \tau_i, \tau_f$ configurations of string. This gives

$$0 = \delta S = \int d\sigma d\tau \delta \mathcal{L}(\dot{x}_\mu, x'_\mu) \quad (4.137)$$

$$= \int d\sigma \int d\tau \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \frac{\partial}{\partial \tau} (\delta x_\mu) + \frac{\partial \mathcal{L}}{\partial x'_\mu} \frac{\partial}{\partial \sigma} (\delta x_\mu) \right] \quad (4.138)$$

$$= \int d\tau \left(S_{\mu}^{\tau} \frac{\partial \mathcal{L}}{\partial x'_{\mu}} \right)_{\sigma=0}^{\sigma=\pi} - \int d\tau \int d\sigma \left(\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu}} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x'_{\mu}} \right) \quad (4.139)$$

Hence the equations of motion are

$$\frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu}} \right) + \frac{\partial}{\partial \sigma} \left(\frac{\partial \mathcal{L}}{\partial x'_{\mu}} \right) = 0 \quad (4.140)$$

and the boundary conditions are

$$\frac{\partial \mathcal{L}}{\partial x'_{\mu}} = 0 \quad \text{at } \sigma = 0, \pi \quad (4.141)$$

The equations of motion are, more explicitly,

$$0 = \frac{\partial}{\partial \tau} \left(\frac{(\dot{x} \cdot x') x'_{\mu} - (x')^2 \dot{x}_{\mu}}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \right) + \frac{\partial}{\partial \sigma} \left(\frac{(\dot{x} \cdot x') \dot{x}_{\mu} - (\dot{x}^2) x'_{\mu}}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \right) \quad (4.142)$$

Conjugate momentum densities are defined by

$$P^{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu}} \quad (4.143)$$

and

$$P^{\mu}_{\sigma} = \frac{\partial \mathcal{L}}{\partial x'_{\mu}} \quad (4.144)$$

the total momentum of the string being given by

$$P^{\mu} = \int_0^{\pi} P^{\mu} d\sigma \quad (4.145)$$

Similarly we can define densities for the generators of the homogeneous Lorentz group by

$$m^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \quad (4.146)$$

$$m_\sigma^{\mu\nu} = x^\mu p_\sigma^\nu - x^\nu p_\sigma^\mu \quad (4.147)$$

The total Lorentz generator for the string is

$$M^{\mu\nu} = \int_0^\pi d\sigma m^{\mu\nu} \quad (4.148)$$

The equations of motion show that these densities are all locally conserved, namely

$$\frac{\partial}{\partial \tau} p^\mu + \frac{\partial}{\partial \sigma} p_\sigma^\mu = 0 \quad (4.149)$$

$$\frac{\partial}{\partial \tau} m^{\mu\nu} + \frac{\partial}{\partial \sigma} m_\sigma^{\mu\nu} = 0 \quad (4.150)$$

and the boundary conditions yield

$$\left. \begin{aligned} p_\sigma^\mu &= 0 \\ m_\sigma^{\mu\nu} &= 0 \end{aligned} \right\} \text{ at } \sigma = 0, \pi \quad (4.151a)$$

$$(4.151b)$$

The general covariance enables us to select at our convenience a particular choice of σ, τ parametrization, i.e. a particular gauge, in order that we simplify (linearise) the Lagrangian density and the equations of motion. We therefore choose a coordinate system in which the inner metric g^{ab} is both diagonal, and traceless (i.e. scale-invariant) by demanding

$$g_{12} = g_{21} = 0 \quad (4.152)$$

$$g_{11} + g_{22} = 0 \quad (4.153)$$

That is,

$$\dot{x} \cdot x' = 0. \quad (4.154)$$

$$\dot{x}^2 + x'^2 = 0. \quad (4.155)$$

This linearizes the Lagrangian density to

$$\mathcal{L} = \frac{1}{2\pi} \sqrt{-g} = \frac{1}{4\pi} (\dot{x}^2 - x'^2) \quad (4.156)$$

The equations of motion become

$$\ddot{x}_\mu - x''_\mu = 0 \quad (4.157)$$

while the boundary conditions become $x'_\mu = 0$ at $\sigma = 0, \pi$.

The Poincare generators are now

$$P^\mu = \frac{1}{2\pi} \dot{x}_\mu \quad (4.158)$$

$$P^\mu_\sigma = -\frac{1}{2\pi} x'_\mu \quad (4.159)$$

$$M^{\mu\nu} = \frac{1}{2\pi} (x_\mu \dot{x}_\nu - x_\nu \dot{x}_\mu) \quad (4.160)$$

$$M^{\mu\nu}_\sigma = \frac{1}{2\pi} (x_\mu x'_\nu - x_\nu x'_\mu) \quad (4.161)$$

As mentioned already, we expect constraints between the components of P_μ and x_μ and that therefore we shall not regard them as independent dynamical variables in setting up the classical canonical formalism. Indeed with our particular choice of gauge

these constraints read

$$X' \cdot P = 0 \quad (4.162)$$

$$P^2 + \frac{1}{(2\pi)^2} (X')^2 = 0 \quad (4.163)$$

Our procedure will be to eliminate certain components from the equations of motion, by using these gauge conditions.

The particular identification of σ, τ is a matter only of our convenience. For example, in the point particle case mentioned above, it seems very natural to identify the coordinate τ with x_0 = time, although this is not essential. In the present case there is a slight calculational advantage in using light-cone variables, since there the gauge constraints separate rather neatly, but again this is not essential.

For any d-vector A_μ ($\mu = 0, 1, 2, 3, 4, \dots, (d-2), (d-1)$) we denote

$$A_\pm = \frac{1}{\sqrt{2}} (A_0 \pm A_L) \quad (4.164)$$

with A_L (L for longitudinal) = A_{d-1} . The remaining transverse components will be denoted by

$$A_i \quad (i = 1, 2, \dots, d-2) \quad (4.165)$$

To be specific we will identify τ with the light-cone component x^+ by

$$x^+ = 2 P^+ \tau \quad (4.166)$$

where P^+ , a component of the total four-momentum P^μ , is inserted for convenience only (we shall see shortly that P^+ is a constant of the motion). The identification of σ will be such that

$\sigma (0 \leq \sigma \leq \pi)$ is π times that fraction of the total string momentum along the τ -direction carried by the portion between $\sigma = 0$ and $\sigma = \sigma$. This definition implies that the momentum density component $\mathcal{P}^+$ is constant along the string and is given by

$$P^+ = \mathcal{P}^+ / \pi \quad (4.167)$$

We now outline the principal steps and results in this light-cone language. [see Goddard et al, Reference 28]

Using the fact that now $x'_+ = 0$ the gauge constraints become

$$\frac{x'_- \mathcal{P}^+}{\pi} = \underline{x}' \cdot \underline{\mathcal{P}} \quad (4.168)$$

and

$$\underline{\mathcal{P}}^2 + \frac{\underline{x}'^2}{(2\pi)^2} = \frac{2P_+ P_-}{\pi} \quad (4.169)$$

We now choose to eliminate P_- and x_+ from the equations of motion.

We have immediately

$$P_- = \frac{\pi}{2P_+} \left(\underline{\mathcal{P}}^2 + \frac{\underline{x}'^2}{(2\pi)^2} \right) \quad (4.170)$$

Let us introduce the center of mass coordinate

$$q_-(\tau) = \frac{1}{\sqrt{2}\pi} \int_0^\pi d\sigma \ x_-(\sigma, \tau) \quad \text{(definition)} \quad (4.171)$$

then x_- is eliminated through

$$x_- = \sqrt{2} q_-(\tau) + \frac{\pi}{P_+} \int_0^\pi d\sigma' \left(\frac{\sigma'}{\pi} - \theta(\sigma' - \sigma) \right) \underline{x}' \cdot \underline{\mathcal{P}} \quad (4.172)$$

To see this write

$$x'_\perp = \frac{\pi}{p_+} \tilde{x}'_\perp \tilde{p}_\perp \quad (4.173)$$

$$x_\perp(\sigma, \tau) = x_\perp(\pi, \tau) - \int_0^\pi d\sigma' \frac{\pi}{p_+} \tilde{x}'_\perp \tilde{p}_\perp \Theta(\sigma' - \sigma) \quad (4.174)$$

Now re-write

$$\sqrt{2}\pi q_\perp(\tau) = \int_0^\pi d\sigma' x_\perp(\sigma', \tau) = \sigma' x_\perp(\sigma', \tau) \Big|_0^\pi - \int_0^\pi d\sigma' \sigma' x'_\perp(\sigma', \tau) \quad (4.175)$$

$$= \pi x_\perp(\pi, \tau) - \int_0^\pi d\sigma' \sigma' x'_\perp(\sigma', \tau) \quad (4.176)$$

to arrive at the final form since

$$x_\perp(\pi, \tau) = \sqrt{2} q_\perp(\tau) + \frac{1}{\pi} \int_0^\pi d\sigma' \sigma' x'_\perp(\sigma', \tau) \quad (4.177)$$

To write the hamiltonian note that

$$P \cdot x = 2P_+ P_- \tau + P_+ x_- - P_- \tilde{x} \quad (4.178)$$

so that for translations in τ the generator is

$$H = 2P_+ P_- = \pi \int_0^\pi d\sigma (\tilde{p}_\perp^2 + \tilde{x}'^2 / (2\pi)^2) \quad (4.179)$$

We are now ready to write the classical equations of motion for the independent dynamical variables $(\tilde{x}, \tilde{p})$. These

follow at once from

$$\frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \right) + \frac{\partial}{\partial \sigma} \left(\frac{\partial \mathcal{L}}{\partial x'_\mu} \right) = 0 \quad (4.180)$$

with $L = \frac{1}{2} (\dot{x}^2 - x'^2)$ to be

$$\dot{\mathcal{P}} = \frac{1}{2\pi} \dot{x}'' \quad (4.181)$$

and

$$\dot{x} = 2\pi \mathcal{Q} \quad (4.182)$$

The equations demand that the Poisson brackets are

$$\{x^i, x^j\} = \{P^i, P^j\} = 0 \quad (4.183)$$

$$\{x^i(\sigma), P^j(\sigma')\} = \delta^{ij} \delta(\sigma - \sigma') \quad (4.184)$$

by using the form for H given earlier and requiring that

$$\dot{x} = \{x, H\} \quad (4.185)$$

$$\dot{\mathcal{P}} = \{\mathcal{P}, H\} \quad (4.186)$$

as the Hamilton equations of motion.

In addition we know that $P_+ = \text{constant}$, and that

$$\dot{q}_- = \frac{1}{\sqrt{2} P_+} H \quad (4.187)$$

Hence

$$\dot{q}_{-} = \frac{1}{\sqrt{2} P_{+}} H \tau + q_{-0} \quad (4.188)$$

The following Poisson brackets are appropriate to q_{-0} and P_{+}

$$\{ \sqrt{2} q_{-}, P_{+} \} = -1 \quad (4.189)$$

and

$$\{ P_{+}, \underline{x} \} = \{ P_{+}, \underline{p} \} = \{ q_{-}, \underline{x} \} = \{ q_{-}, \underline{p} \} = 0 \quad (4.190)$$

Before making the transition to quantum mechanics, it is convenient to avoid the use of the continuous coordinates σ and τ by making the Fourier expansion into normal modes according to

$$x_{\mu}(\sigma, \tau) = \sqrt{2} \left[q_{\mu}^0 + \alpha_{\mu}^0 \tau + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_{\mu}^n}{n} \cos n\sigma e^{-in\tau} \right] \quad (4.191)$$

This exploits the wave equation and boundary conditions satisfied by $x_{\mu}(\sigma, \tau)$. Of course, we expect that the α_{μ}^n will be identifiable (after quantization) with the harmonic-oscillator-like operators of the operator formalism.

The independent variables are now the constants of the motion α_n and q_0 . The plus components all vanish:

$$q_{0+} = \alpha_n^{+} = 0 \quad (n \neq 0) \quad (4.192)$$

except for

$$\alpha_0^+ = \sqrt{2} P_+ \quad (4.193)$$

On the other hand the minus components can be re-expressed in terms of the transverse components by

$$\alpha_n^- = \frac{1}{\alpha_0^+} L_n \quad (4.194)$$

where

$$L_n = \frac{1}{2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \alpha_{-n} \cdot \alpha_{n+m} \quad (4.195)$$

To derive this expression note that

$$\underline{X}' = -i\sqrt{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n \sin n\sigma e^{-inz} \quad (4.196)$$

$$\underline{P} = \frac{\dot{\underline{X}}}{2\pi} = \frac{\sqrt{2}}{2\pi} \left[\alpha_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n \cos n\sigma e^{-inz} \right] \quad (4.197)$$

Now use the relationship

$$\sqrt{2} i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^-}{n} \cos n\sigma e^{-inz} = \frac{\pi}{P_+} \int_0^\pi d\sigma' \left(\frac{\sigma'}{\pi} - \theta(\sigma' - \sigma) \right) \underline{X}' \cdot \underline{P} \quad (4.198)$$

and project out α_n^- by operation with
to deduce that

$$\int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma e^{inz} \cos n\sigma$$

$$\alpha_n^- = \frac{1}{2\sqrt{2}P_+} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \alpha_{-n} \cdot \alpha_{n+m} = \frac{1}{\alpha_0^+} L_n \quad (4.199)$$

as required. Here we have used

$$\int_0^\pi d\sigma \frac{\sigma'}{\pi} \sin A\sigma' \cos B\sigma' = \int_0^\pi d\sigma' \sin A\sigma' \cos B\sigma'$$

$$= -\frac{1}{2} \left[\frac{\cos(A+B)\sigma}{A+B} - \frac{\cos(A-B)\sigma}{A-B} \right] \quad (4.200)$$

The Poisson brackets of the independent variables in normal modes are

$$\{\alpha_n^i, \alpha_m^j\} = -in \delta_{n+m,0} \delta^{ij} \quad (4.201)$$

$$\{q_0^i, \alpha_0^j\} = \delta^{ij} \quad (4.202)$$

$$\{q_0^i, q_0^j\} = 0 \quad (4.203)$$

$$\{q_0^-, \alpha^+\} = -1 \quad (4.204)$$

$$\{q_0^-, \alpha_n^i\} = \{q_0^-, q_0^i\} = \{\alpha_0^+, \alpha_n^i\} = \{\alpha_0^+, q_0^i\} = 0 \quad (4.205)$$

For the dependent modes one finds the generalized projective algebra (for Poisson brackets)

$$\{L_n, L_m\} = -i(n-m)L_{n+m} \quad (4.206)$$

$$\{L_n, \alpha_m^i\} = -im \alpha_{m+n}^i \quad (4.207)$$

$$\{q_0^i, L_m\} = \alpha_m^i \quad (4.208)$$

We take the hamiltonian to be L_0 . It is given by

$$H = L_0 = 2P_+P_- = \tilde{P}^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_n \cdot \tilde{\alpha}_n^* \quad (4.209)$$

The total momentum is

$$P^\mu = \frac{1}{2\pi} \int_0^\pi d\sigma \dot{x}^\mu = \frac{1}{\sqrt{2}} \alpha_0^\mu \quad (4.210)$$

and the Lorentz generators are

$$M^{\mu\nu} = \int_0^\pi d\sigma (P^\nu x^\mu - P^\mu x^\nu) \quad (4.211)$$

$$= \frac{1}{2\pi} \int_0^\pi d\sigma (\dot{x}^\nu x^\mu - \dot{x}^\mu x^\nu) \quad (4.212)$$

$$= \frac{1}{\pi} \int_0^\pi d\sigma \left(q_0^\mu + \alpha_0^\mu \tau + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} \cos n\sigma e^{-in\tau} \right).$$

$$\left(\alpha_0^\nu + \sum_{n' \neq 0} \alpha_{n'}^\nu \cos n'\sigma e^{-in'\tau} \right) - (\mu \leftrightarrow \nu) \quad (4.213)$$

$$= (q_0^\mu \alpha_0^\nu - q_0^\nu \alpha_0^\mu) + i \sum_{n \neq 0} \left(\frac{\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\nu \alpha_{-n}^\mu}{n} \right)$$

$$(4.214)$$

We are now ready to proceed to the quantum mechanics of our massless relativistic string. We accomplish this by imposing the Dirac quantum conditions (see Chapter 4 of Dirac's book, reference 32) i.e. by replacing

$$i \{ \text{Poisson bracket} \} \rightarrow [\text{commutator}] \quad (4.215)$$

This gives immediately the canonical commutation relations for the independent dynamical variables.

The crucial point is that in expressing the dependent variables in terms of the independent ones, there now may occur a serious ambiguity in ordering the (non-commutative) operators. In the present case this happens for L_0 in α_0^- . We define

$$\alpha_0^- = \frac{1}{\alpha_0^+} [L_0 - \alpha(0)] \quad (4.216)$$

where

$$L_0 = \sum_{n=1}^{\infty} \alpha_n^+ \cdot \alpha_n + \tilde{P}^2 \quad (4.217)$$

is normal ordered. The interpretation of $\alpha(0)$ is, as our choice of symbol indicates, the leading trajectory intercept. To see this note that the invariant squared mass is

$$m^2 = 2P_+ P_- - \tilde{P}^2 = \sum_{n=1}^{\infty} \alpha_n^+ \cdot \alpha_n - \alpha(0) \quad (4.218)$$

so that $(-\alpha(0))$ is the squared mass of the ground state. We can already see by counting that since there are only transverse states we will run into trouble with covariance unless the first excited vector state is massless (i.e. $\alpha(0) = 1$). We shall, however, display this in a slightly more general manner by writing the Lorentz generators $M^{\mu\nu}$. The canonical components of $M^{\mu\nu}$ are

$$M^{ij} = q_0^i \alpha_0^j - q_0^j \alpha_0^i - i \sum_{n=1}^{\infty} \left(\frac{\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i}{n} \right) \quad (4.219)$$

$$M^{i+} = -M^{+i} = q_0^i \alpha_0^+ \quad (4.220)$$

$$M^{+-} = -\frac{1}{2} (q_0^- \alpha_0^+ + \alpha_0^+ q_0^-) \quad (4.221)$$

But the components $M^{i-} = -M^{-i}$ contain the non-canonical variables L_n which satisfy (as we know from our earlier discussion of the operator formalism)

$$[L_n, L_m] = (n-m) L_{n+m} + \left(\frac{d-2}{12} \right) n(n^2-1) \delta_{n+m,0} \quad (4.222)$$

where d = dimension of space-time. The expression for M^{i-} is

$$M^{i-} = A^{i-} + B^{i-} \quad (4.223)$$

where

$$A^{i-} = \frac{q_0^i}{\alpha_0^+} (L_0 - \alpha(0)) - q_0^- \alpha_0^i \quad (4.224)$$

$$B^{i-} = -\frac{i}{\alpha_0^+} \sum_{n=1}^{\infty} \left(\frac{\alpha_{-n}^i L_n - L_{-n} \alpha_n^i}{n} \right) \quad (4.225)$$

Now the commutator $[M^{i-}, M^{j-}]$ should vanish according to the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = g^{\mu\rho} M^{\nu\sigma} + g^{\nu\sigma} M^{\mu\rho} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} \quad (4.226)$$

There is, however, a contribution proportional to $(d-2)$ in $[B^{i-}, B^{j-}]$ of the form [Using $[AB, CD] = [A, C]BD + A[B, C]D + C[A, D]B + CA[B, D]$]

$$- \frac{1}{\alpha_0^{d+2}} \left(\frac{d-2}{12} \right) \sum_{n=1}^{\infty} \frac{n(n^2-1)}{n^2} [\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i] \quad (4.227)$$

arising from the anomaly in $[L_n, L_{-n}]$. Also there is a term in $[A^{i-}, A^{j-}]$ proportional to $\alpha(0)$ namely

$$+ \frac{2}{\alpha_0^{d+2}} \sum_{n=1}^{\infty} \frac{\alpha(0)}{n} (\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i) \quad (4.228)$$

Computation of the full commutator gives the answer

$$[M^{i-}, M^{j-}] = - \frac{2}{\alpha_0^{d+2}} \sum_{n=1}^{\infty} \left[\left(1 - \frac{d-2}{24} \right) n + \left(\frac{d-2}{24} - \alpha(0) \right) \frac{1}{n} \right] * (\alpha_{-n}^i \alpha_n^j - \alpha_n^i \alpha_{-n}^j) \quad (4.229)$$

which vanishes only if $\alpha(0)=1$ and $d=26$. This confirms the completeness of the transverse states under these conditions, as was demonstrated earlier; the use of the light-cone quantization here is most closely related to the infinite momentum limit discussed there. The present treatment gives us some more insight into the anomaly term of the generalized projective algebra: the anomaly term is (i) a purely quantum mechanical effect and (ii) a result of the infinite number of degrees of freedom which make normal ordering essential to avoid divergent matrix elements.

The most important general result of this sub-section is that the decoupling of spurious states of the dual model is a direct consequence of the general covariance of the classical action of the rubber string. This observation might lead not only to a reformulation of existing models but eventually to the building of improved ghost-free models with more realistic resonance spectra. [see References 33-36]

SUMMARY

We have shown how a simple multiplicative approach to internal symmetry can work well for isospin and for strangeness. Both cyclic symmetry and factorizability can be maintained, together with the absence of unwanted states with exotic quantum numbers. Although the λ - matrices of $SU(3)$ were used to accomodate strangeness, it was not necessary to have exact $SU(3)$ mass degeneracy in the Born amplitude. This last fact is important because $SU(3)$ symmetry is strongly broken in Nature and one is anxious not to attribute any large (symmetry breaking) effects to the perturbatively unitary corrections.

A method of introducing baryons, while overlooking the existence of half-integer spin, is to picture them as circular rubber strings with three quarks placed symmetrically on the circumference. This leads to non-planar tree diagrams having an essentially different pole structure from the mesonic case. Such an ansatz for baryons, together with the already-known configuration for mesons, dictates by projective invariance the configurations of all exotic hadrons with total quark number greater than three.

The observation that links the ghost-eliminating gauges of the unit intercept dual model to the general covariance of the classical action of a rubber string is an interesting one. Indeed, one may hope on the basis of this idea that he can perhaps simplify the formulation of the theory from the postulation of a complicated set of on-mass-shell tree amplitudes to the postulation of a limited number of axioms in the language of the string.

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FIGURE 1

FIGURE 2

FIGURE 3

(b)

FIGURE 34U017

FIGURE 5

(a)

(b)

FIGURE 6

FIGURE 7

FIGURE 8

(a)

(b)

(c)

FIGURE 9

FIGURE 10