Accelerator Department
BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, New York

CRISP 72-34
ISABELLE PROJECT

BNL 17017

INTERSECTING STORAGE ACCELERATOR NOTES

LONG-RANGE BEAM-BEAM INTERACTION OF CROSSING BEAMS

E. Keil
CERN, Geneva, Switzerland

C. Pellegrini
Laboratori Nazionali di Frascati, Frascati, Italy

and

A.M. Sessler*
Lawrence Berkeley Laboratory, Berkeley, California

June 26, 1972

ABSTRACT

The linear tune shift of a particle crossing an azimuthally uniform cylindrically symmetric Gaussian beam is evaluated taking into account the β variation through the crossing region.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

*Research supported by the U.S. Atomic Energy Commission.

RECEIVED BY TFC
JUN 28 1972

1. Introduction

In storage rings having low- β crossing regions, small crossing angle, and either unbunched beams, long bunches or many closely spaced bunches, there can be significant beam-beam interaction even when the beams do not spatially overlap. In particular, because β increases quadratically with distance, s , from the crossing point, while the beam-generated gradient decreases inversely with s^2 , the beam interaction is equally strong throughout the free space crossing region.

In this note we present the general formalism required to compute the beam-induced change in frequency of a particle crossing a uniform beam and then evaluate the effect for the case of cylindrically symmetric beams. Formulas for the tune shift are given for a number of limiting cases; in the general situation numerical evaluation of two integrals is required.

2. Problem Definition

We shall study the weak-strong interaction of two beams. In particular, we compute $\Delta\nu_x$ and $\Delta\nu_y$ for a single particle crossing a prescribed strong beam. In making this calculation we shall:

- a. Neglect closed orbit deviations coming from beam-beam interactions.
- b. Assume that wall effects are negligible; i.e., we calculate beam fields as if the beam is in free space.
- c. Assume the strong beam is unbounded; i.e., that the bunch length, l_b , is much greater than the interaction length l .
- d. Assume the beams cross vertically.
- e. Assume that the beam interaction region is free space and, furthermore, that the beams no longer interact beyond the magnets marking the end of the interaction region.
- f. Assume that derivatives of the beam profile with distance along the beam can be neglected when computing beam fields.

3. Formalism

The tune shift is given by

$$\Delta v_{(x,y)} = \frac{1}{4\pi} \int_{-L/2}^{L/2} \beta_{(x)}(s) k_{(x)}(s) ds, \quad (1)$$

where

$$k_{(x)}(s) = \frac{e(1 + \beta^2)}{m_0 c^2 \gamma \beta^2} \frac{\partial \mathcal{E}_{(x,y)}}{\partial (x/y)}, \quad (2)$$

and s is distance measured, from the crossing point, along the equilibrium orbit of the particle intersecting the strong beam. In (2),

$$\mathcal{E}_{(x,y)}(x,y) = - \frac{\partial V(x,y)}{\partial (x/y)}, \quad (3)$$

with $V(x,y)$ determined from:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V(x,y) = - 4\pi\rho \left(\frac{x}{a}, \frac{y}{b} \right). \quad (4)$$

In (4) the beam density function is ρ , which is assumed to depend on s only through the dependence of beam dimensions a and b upon s . In fact,

$$a = a_0 \sqrt{\frac{\beta_x(s)}{\beta_{0x}}}, \quad (5)$$

$$b = b_0 \sqrt{\frac{\beta_y(s)}{\beta_{0y}}},$$

The dependence of $\beta(s)$ upon s is given by:

$$\beta_{(x,y)}(s) = \beta_{0(x,y)} + \frac{s^2}{\beta_{0(x,y)}}, \quad (6)$$

where $\beta_{0(x,y)}$ is the β -value at the crossing point.

We shall not attempt the solution of Eqs. (1) through (6) in the general case, but, rather, specialize the problem to a situation in which more progress can be made by analytical methods.

4. Cylindrically Symmetric Gaussian Beam: Formulas

For a cylindrically symmetric beam Eq. (4) may be replaced by Gauss' theorem and the field perpendicular to the beam is given by

$$\mathcal{E}_r(t) = -\frac{(4\pi)}{2\pi\kappa} \int_0^r \rho(r') r' dr' \quad . \quad (7)$$

We need the fields $\mathcal{E}_x$ and $\mathcal{E}_y$, and for this purpose introduce the coordinate systems of Fig. 1.

The fields $\mathcal{E}_x$, and $\mathcal{E}_y$, are given by

$$\begin{aligned} \mathcal{E}_x &= \mathcal{E}_r \cos \varphi \quad , \\ \mathcal{E}_y &= \mathcal{E}_r \sin \varphi \quad . \end{aligned} \quad (8)$$

From Fig. 1, we have the transformation relating the beam and the test particle coordinate systems:

Fig. 1. Coordinate systems. The angle between the beams is α ; r - φ is a cylindrical coordinate system about the beam direction.

$$\begin{aligned}x' &= x \\y' &= y \cos \alpha + s \sin \alpha \\s' &= s \cos \alpha - y \sin \alpha ,\end{aligned}\tag{9}$$

and

$$\begin{aligned}x &= x' \\y &= y' \cos \alpha - s' \sin \alpha \\s &= s' \cos \alpha + y' \sin \alpha .\end{aligned}\tag{10}$$

Hence

$$\begin{aligned}e_x &= e_{x'} = e_r(r) \cos \varphi \\e_y &= e_{y'} \cos \alpha = e_r(r) \sin \varphi \cos \alpha .\end{aligned}\tag{11}$$

We now specialize to a Gaussian beam; i.e.

$$\rho(r) = \frac{\lambda}{2\pi\sigma^2} \exp(-r^2/2\sigma^2) ,\tag{12}$$

where σ is the rms half-width of the beam and hence a function of s' .

Note that

$$2\pi \int_0^\infty \rho(r) r dr = \lambda ,\tag{13}$$

so that λ is the charge per unit length of the strong beam.

From (7) and (12)

$$e_r(r) = \frac{2\lambda}{r} [1 - \exp(-r^2/2\sigma^2)] ,\tag{14}$$

and, hence,

$$\begin{aligned}e_x &= \frac{2\lambda \cos \varphi}{r} [1 - \exp(-r^2/2\sigma^2)] , \\e_y &= \frac{2\lambda \sin \varphi \cos \alpha}{r} [1 - \exp(-r^2/2\sigma^2)] .\end{aligned}\tag{15}$$

In evaluating (15) one recalls that

$$\begin{aligned}\sin \varphi &= y'/r \\ \cos \varphi &= x'/r \\ r^2 &= x'^2 + y'^2\end{aligned}\tag{16}$$

and the primed quantities are related to the unprimed variables by (9).

We need $k_{(x/y)}(s)$, which using (2), (15), (16), and (9), may be written in the form:

$$k_x = \left[\frac{4e\lambda}{m_0 c^2 \gamma \beta^2} \right] \frac{\partial}{\partial x} \left\{ \frac{x}{r^2} [1 - \exp(-r^2/2\sigma^2)] \right\}, \quad (17)$$

$$k_y = \left[\frac{4e\lambda \cos \alpha}{m_0 c^2 \gamma \beta^2} \right] \frac{\partial}{\partial y} \left\{ \frac{y'}{r^2} [1 - \exp(-r^2/2\sigma^2)] \right\}, \quad (18)$$

$$\text{with} \quad y' = y \cos \alpha + s \sin \alpha, \quad (19)$$

$$r^2 = x^2 + (y \cos \alpha + s \sin \alpha)^2. \quad (20)$$

Actually, we want k_x and k_y evaluated at $x = y = 0$; i.e. on the trajectory of the test particle. We obtain

$$k_x = \left[\frac{4e\lambda}{m_0 c^2 \gamma \beta^2} \right] \frac{1}{s^2 \sin^2 \alpha} \left[1 - \exp \left(- \frac{s^2 \sin^2 \alpha}{2\sigma^2} \right) \right], \quad (21)$$

and

$$k_y = \left[\frac{4e\lambda \cos^2 \alpha}{m_0 c^2 \gamma \beta^2} \right] \left[\left(\frac{1}{s^2 \sin^2 \alpha} + \frac{1}{\sigma^2} \right) \exp \left(- \frac{s^2 \sin^2 \alpha}{2\sigma^2} \right) - \frac{1}{s^2 \sin^2 \alpha} \right]. \quad (22)$$

In Eqs. (21) and (22) it must be remembered that σ is a function of s' .

In fact

$$\sigma = \sigma_0 \sqrt{\frac{\beta(s')}{\beta_0}}, \quad (23)$$

with

$$\beta(s') = \beta_0 + \frac{s'^2}{\beta_0}, \quad (24)$$

and

$$s' = s \cos \alpha. \quad (25)$$

Combining (1), (6), (21) through (25), we obtain

$$\Delta v_x = \frac{e\lambda\beta_0(1+\beta^2)}{2\pi m_0 c^2 \gamma \beta^2 \sin^2 \alpha} \int_{-t/2}^{t/2} \left(1 + \frac{s^2}{\beta_0^2} \right) \frac{ds}{s^2} \left\{ 1 - \exp \left[- \frac{s^2 \sin^2 \alpha}{2\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] \right\}. \quad (26)$$

and

$$\Delta v_y = \frac{e\lambda (1 + \beta^2) \beta_0 \cos^2 \alpha}{2\pi m_0 c^2 \gamma \beta^2} \int_{-l/2}^{l/2} \left(1 + \frac{s^2}{\beta_0^2} \right) \left\{ \left[\frac{1}{s^2 \sin^2 \alpha} + \frac{1}{\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] \right. \\ \times \exp \left[-\frac{s^2 \sin^2 \alpha}{2\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] \\ \left. - \frac{1}{s^2 \sin^2 \alpha} \right\} ds \quad . \quad (27)$$

Formulas (26) and (27) cannot be evaluated analytically. If we specialize to the case of small crossing angle ($\alpha \ll 1$) and introduce a change of variables, they take the form:

$$\Delta v_x = \left[\frac{e\lambda(1 + \beta^2)}{2\pi m_0 c^2 \gamma \beta^2 \alpha^2} \right] I_x \left(\frac{l}{2\beta_0}, \frac{\beta_0 \alpha}{\sigma_0} \right) , \quad (28)$$

$$\Delta v_y = \frac{e\lambda(1 + \beta^2)}{2\pi m_0 c^2 \gamma \beta^2 \alpha^2} I_y \left(\frac{l}{2\beta_0}, \frac{\beta_0 \alpha}{\sigma_0} \right)$$

where

$$I_x(\xi, \eta) = \int_{-\xi}^{\xi} dx \left(1 + \frac{1}{x^2} \right) \left\{ 1 - \exp \left[-\frac{1}{2} \eta^2 x^2 / (1 + x^2) \right] \right\} , \quad (29)$$

$$I_y(\xi, \eta) = \int_{-\xi}^{\xi} dx \left(1 + x^2 \right) \left\{ \left[\frac{1}{x^2} + \frac{\eta^2}{(1 + x^2)} \right] \exp \left[-\frac{1}{2} \eta^2 x^2 / (1 + x^2) \right] - \frac{1}{x^2} \right\} .$$

This last form is convenient for numerical evaluation, as the dependence upon all the machine parameters has been reduced to a dependence upon only two parameters ξ and η .

5. Cylindrically Symmetric Gaussian Beams: Special Cases

5.1 Constant β

For a machine without low β in the crossing region we can readily obtain a closed expression for Δv_x . This case of negligible variation of β is obtained from (28) and (29) by taking the limit $\xi \rightarrow 0$. (Since β_0 determines the rate of change of β , we must take $\beta_0 \gg 1$.) In this limit

$$I_x(\xi \rightarrow 0, \eta) = - \left(\frac{2}{\xi} \right) [1 - \exp(-\eta^2 \xi^2 / 2)] + \sqrt{2\pi} \eta \operatorname{erf}(\eta \xi / \sqrt{2}) . \quad (30)$$

If, furthermore, the beams are separated at the ends of the interaction region then $l\alpha \gg \sigma_0$; i.e. $\xi\eta \gg 1$ and

$$I_x \rightarrow \sqrt{2\pi} \eta . \quad (31)$$

From (28) and (31) we obtain:

$$(\Delta v)_{\text{const}\beta} = \frac{e\lambda(1 + \beta^2) \beta_0}{\sqrt{2\pi} m_0 c^2 \gamma \beta^2 \alpha \sigma_0} . \quad (32)$$

For $\beta \approx 1$, and replacing λ by $(N_B e / L_B)$, we obtain

$$(\Delta v_x)_{\text{const}\beta} = \sqrt{\frac{2}{\pi}} \frac{N_B r_0 \beta_0}{L_B \gamma \alpha \sigma_0} . \quad (33)$$

Note that in this situation -- in which β doesn't increase as one goes away from the crossing point -- the Δv_x is independent of the length of the interaction region.

5.2 Long-Range Contribution to Δv

From (26) and (27) it is clear that for $|s| \gg \beta_0$ the integrals become trivial. Introducing a cut-off $(s_0/2) \gg \beta_0$ we have the contribution to $\Delta v(\frac{x}{\beta})$ for $|s| \geq (s_0/2)$:

$$(\Delta v_x)_l = \frac{e\lambda\beta_o(1+\beta^2)}{\pi m_c^2 \gamma \beta^2 \sin^2 \alpha \beta_o^2} \left\{ 1 - \exp \left[-\frac{1}{2} \left(\frac{\beta_o}{\sigma_o} \tan \alpha \right)^2 \right] \right\} \int_{s_o/2}^{l/2} ds ,$$

$$(\Delta v_y)_l = \frac{e\lambda\beta_o \cos^2 \alpha (1+\beta^2)}{\pi m_c^2 \gamma \beta^2 \beta_o^2} \left\{ \left[\frac{1}{\sin^2 \alpha} + \frac{\beta_o^2}{\sigma_o^2 \cos^2 \alpha} \right] \exp \left[-\frac{1}{2} \left(\frac{\beta_o \tan \alpha}{\sigma_o} \right)^2 \right] - \frac{1}{\sin^2 \alpha} \right\} \int_{s_o/2}^{l/2} ds . \quad (34)$$

For $\beta \approx 1$, $\alpha \ll 1$, and replacing λ by $(N_B e / L_B)$, we obtain

$$(\Delta v_x)_l = \frac{N_B r_o (l - s_o)}{\pi \gamma \alpha^2 \beta_o L_B} [1 - e^{-\frac{1}{2}(\alpha \beta_o / \sigma_o)^2}] . \quad (35)$$

$$(\Delta v_y)_l = \frac{N_B r_o (l - s_o)}{\pi \gamma \beta_o L_B} \left\{ \left[\frac{1}{\alpha^2} + \left(\frac{\beta_o}{\sigma_o} \right)^2 \right] e^{-\frac{1}{2}(\alpha \beta_o / \sigma_o)^2} - \frac{1}{\alpha^2} \right\} . \quad (36)$$

These formulas, Eqs. (35) and (36), are the contributions to $\Delta v_{(x)}$ from the region between $(s_o/2)$ and $(l/2)$ and between $(-s_o/2)$ and $(-l/2)$. The contribution from $(-s_o/2)$ to $(s_o/2)$ must be evaluated numerically. For low- β machines $\beta_o \ll 1$ and thus a major part of $\Delta v_{(x)}$ is contained in (35) and (36). By choosing $(s_o/2)$ at the point of beam separation, the contribution from $(-s_o/2)$ to $(s_o/2)$ is just the (local) part of Δv as it is included in the usual calculations.

5.3 $\eta^2 \ll 1$

In this limit -- which corresponds to low- β and small crossing angle -- the integrals of Eq. (29) may be readily evaluated. Thus

$$I_x(\xi, \eta) \approx \int_{-\xi}^{\xi} dx \left(1 + \frac{1}{x^2} \right) \left\{ 1 - 1 + \frac{\eta^2}{2} \left(\frac{x^2}{1+x^2} \right) - \dots \right\} dx , \quad (37)$$

since the exponent is bounded by $(\eta^2/2)$ and consequently can be approximated by the series expansion, one finds

$$I_x(\xi, \eta) \rightarrow \xi \eta^2, \quad (38)$$

and from (28), with λ replaced by $(N_B e / L_B)$ and $\beta \approx 1$,

$$(\Delta v_x)_{\text{small } \eta} = \frac{N_B r_o \beta_o \ell}{2\pi \gamma L_B \sigma_o^2}. \quad (39)$$

6. Examples and Discussion

6.1 The ISR

For the ISR, β varies sufficiently slowly through the interaction region that Δv is adequately approximated by Eq. (33). Taking

$$\frac{N_B}{L_B} = \frac{N}{2\pi R}, \quad (40)$$

$N = 2 \times 10^{14}$, $R = 150$ m, $\gamma = 30$, $\alpha = 0.25$, $\sigma_o = (h_{\text{eff}}/2/\pi)$; $h_{\text{eff}} = 0.5$ cm, and $\beta_o = 13$ m, one obtains

$$\left. \Delta v_x \right|_{\text{ISR}} = 3.5 \times 10^{-4}$$

6.2 ISA High-Luminosity Mode

For this mode $\ell = 20$ m, $\beta_o = 0.7$ m, $\alpha = 2 \times 10^{-4}$, $\sigma_o^2 = [(7 \times 10^{-2})(2 \times 10^{-2})/3] \text{ cm}^2$. From (28) and (29)

$$\xi = \frac{\ell}{2\beta_o} = 14,$$

$$\eta = \frac{\beta_o \alpha}{\sigma_o} = 0.3, \quad (42)$$

and since $\eta^2 \ll 1$ we can employ (39) to evaluate Δv . Taking $\gamma = 200$, and $(N_B/L_B) = (6 \times 10^{14}/2 \times 10^3 \text{ m})$, one finds

$$\left. \Delta v_x \right|_{\text{ISA}} = 0.12 \quad (43)$$

Equation (43) results from the assumptions of unbunched beams interacting over the full (20 meter) free space; neither assumption needs to be maintained in future designs.

6.3 Discussion

In general, the limiting cases of Section 5 are not adequate for the design of modern (low- β) storage rings. For example, the ISA modes with larger crossing angle have an η too large to allow the use of (39). It is thus necessary to numerically evaluate, and tabulate, the integrals of Eq. (29). It is the authors' intention to complete this task subsequent to the ISA Summer Study, but prior to preparing the material of this note for publication.

It is interesting to note that (35) reduces to (39) when $\eta^2 \ll 1$; i.e. in this case, the per-unit-length short-range contribution to Δv_x is of the same form (and magnitude) as the per-unit-length long-range contribution.

Although the analysis has been done for cylindrically symmetric Gaussian beams, it is reasonable to expect (because of Gauss' law) that the long-range contribution to Δv would be essentially the same for other beam cross sections. (Of course, the local term is quite sensitive to beam profile.)

Finally, we wish to emphasize that it is not at all clear that the linear tune shift, Δv , is an adequate measure of the destructive influence of strong beams on each other, or even of a strong beam on a weak beam. Limited experimental observations on electron-positron storage rings, and some numerical studies, suggest that, if anything, long-range interactions are more severe than the interaction of beams crossing head-on. If so, the conventional $\Delta v \approx 0.02$ may be too liberal a criterion.

It should also be noted that long-range interactions may be harmful in connection with coherent modes. It is possible that the coupling of coherent and incoherent effects may be responsible for the reduced limits observed in strong beam-strong beam interaction (in contrast with weak-strong interaction). It was with these effects in mind that SPEAR and PEP were designed for single short bunch operation; a design philosophy that, at least from

the present point of view, is pleasantly conservative.

7. Acknowledgment

The authors wish to extend their thanks to M.Q. Barton for his kind encouragement, helpful comments, and gracious assistance with administrative details; all of which greatly expedited this paper.