THE CONCEPT OF TRANSITION OPERATOR BOSON
AND ITS APPLICATION TO AN
EXACTLY SOLUBLE MODEL*

R. M. Dreizler
Institut für Theoretische Physik
Universität Frankfurt/Main, West Germany
and

Abraham Klein
Department of Physics, University of Pennsylvania
Philadelphia, Pennsylvania 19104

UPR-0012N

7/5/72

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED
DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED
leg

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

THE CONCEPT OF TRANSITION OPERATOR BOSON
AND ITS APPLICATION TO AN EXACTLY SOLUBLE MODEL*

by

R. M. Dreizler
Institut für Theoretische Physik
Universität Frankfurt/Main
West Germany

and

Abraham Klein
Department of Physics, University of Pennsylvania
Philadelphia, Pennsylvania 19104

UPR 00012N

7/5/72

ABSTRACT

In phenomenological descriptions of nuclear vibrations a close connection is implied or expressed between the most collective transition operator and the harmonic oscillator (boson) variables which are introduced to describe the collective motion. With the help of a widely used example - the monopole vibration model of Meshkov, Lipkin, and Glick - it is shown how this relationship suggests, for the microscopic case, a new and flexible class of boson representations. As shown, these are capable of yielding accurate descriptions of the system not only in the extremes of weak and strong coupling, but in the usually more inaccessible intermediate regions as well.

*Supported in part by the U.S. Atomic Energy Commission.

I. Introduction

The expansion of pairs of fermion operators in terms of boson degrees of freedom provides a bridge between the individual particle aspects and the collective model description of nuclei. At present most of the work in this area employs the traditional methods of Beliaev and Zelevinsky⁽¹⁾ or of Marumori et al.⁽²⁾ Questions of convergence of these expansions, the satisfaction of the Pauli principle within the framework of the boson space etc. have been amply discussed in the literature.⁽³⁾

One aspect of boson expansions, however, has perhaps not been stressed sufficiently so far: Boson expansions are only determined up to unitary equivalence. The ensuing freedom in selecting a particular expansion with improved convergence properties for a given physical situation has so far not been explored to any large extent. One might keep in mind, in particular, the question of finding an expansion with adequate convergence properties for situations reasonably far removed from the ideal vibrational limit.

In this note we will introduce a more flexible expansion through application to a simplified model of a many fermion system - (the model of Lipkin, Meshkov, and Glick⁽⁴⁾).

The key to selecting the particular expansion emerges from the following philosophy. Collective motion (in nuclei) is characterized by two dominant features:

- (1) Low-lying bands of states with specific energy spacings.
- (2) Marked selection rules for electromagnetic transitions among the members of each band.

Regularities of this nature imply a relatively simple structure for the Hamiltonian and/or the transition operators (in a suitable basis). Examples in which the choice of basis simplifies the structure of the Hamiltonian have been given in the literature^(5,6), and seem particularly convenient for phenomenological analysis. On the other hand, one could select an expansion where the transition operator of interest has the simplest possible form (to be defined). In an ideal situation one could even select the transition operator directly as the boson degree of freedom. The remaining operators will then be determined through the commutation relations of pairs of fermion operators as in the method of Beliaev and Zelevinsky.

After a brief summary of the model in Section II, we will demonstrate the introduction of such an expansion (which we called the transition operator boson expansion⁽⁷⁾) in Section III. Section IV contains the "physical aspects" of selecting the expansion according to a given dynamical situation, governed by the Hamiltonian of the system. Results and conclusions are presented in Section V.

II. The Model

The model of MGL⁽⁴⁾ consists of N identical fermions, which are distributed among two N-fold degenerate single particle levels separated by the energy ϵ . The Hamiltonian of this system,

$$\hat{H} = \frac{1}{2} \epsilon \sum_{p,\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} - \frac{1}{2} |V| \sum_{p,p',\sigma} a_{p\sigma}^\dagger a_{p'\sigma}^\dagger a_{p'-\sigma} a_{p-\sigma}, \quad (2.1)$$

contains only a monopole two-body interaction. The index $\sigma = \pm 1$ refers to the upper and lower single particle levels, the index $p = 1, 2, \dots, N$ labels the degeneracy within each of the levels.

In terms of the quasispin operators

$$\begin{aligned} \hat{J}_0 &= \frac{1}{2} \sum_{p,\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma}, \\ \hat{J}_+ &= \sum_p a_{p+}^\dagger a_{p-} = \hat{J}_-^\dagger, \end{aligned} \quad (2.2)$$

which satisfy the familiar SU(2) commutation relations

$$\begin{aligned} [\hat{J}_+, \hat{J}_-] &= 2 \hat{J}_0, \\ [\hat{J}_0, \hat{J}_\pm] &= \pm \hat{J}_\pm, \end{aligned} \quad (2.3)$$

The Hamiltonian can be expressed as

$$\hat{H} = \epsilon \hat{J}_0 - \frac{1}{2} |V| (\hat{J}_+^2 + \hat{J}_-^2). \quad (2.4)$$

As the Casimir operator of the group SU(2),

$$\hat{C}^{(2)} = \hat{J}^2 = \hat{J}_+ \hat{J}_- + \hat{J}_0^2 - \hat{J}_0, \quad (2.5)$$

commutes with this Hamiltonian, one can separate the N-body states into noninteracting bands. Each band spans a representation of the group and is labelled by the (quasispin) quantum number J .

The groundstate for the limit $|V| = 0$ consists of the fully occupied lower level

$$|0\rangle = \prod_{p=1}^N a_{p-}^\dagger |\text{vac}\rangle \quad (2.6)$$

As this state is annihilated by the operator $\hat{J}_-$ and as it carries the label $J_0 = -\frac{1}{2}N$ (each particle contributes the quasispin $-1/2$), the value of J is $\frac{1}{2}N$. Other members of the "groundstate band" are obtained by the application of the raising operator J_+

$$|J = \frac{1}{2}N, J_0 = -\frac{1}{2}N + n\rangle = |n\rangle = A(N,n) \hat{J}_+^n |0\rangle \quad (2.7)$$

In this limit we have a "harmonic" spectrum.

The interaction part of the Hamiltonian contains only the square of the raising and lowering operators and it does not connect states with even n and odd n . As a consequence one observes in the "strong coupling limit", which is characterized by

$$\delta = \frac{|V|N}{\epsilon} \gg 1 \quad ,$$

a doublet structure for the low-lying states of the groundstate band. Each member of these doublets is respectively a strong mixture of basis states with even or odd n .

The transition between these two limiting situations covers

only a fairly narrow range of values of the relative interaction strength, centering about $\delta \approx 1$. The presence of such a "phase transition" connecting two physically distinct regions renders the model a respectable caricature of the real world.

III. The Transition Operator Boson Expansion

We reiterate our aim: We are looking for a boson expansion with a simple structure for the collective operator. We must then ask which is the collective operator of the model of MGL. For an attractive interaction, we see that it is the operator

$$\hat{J}_x = \frac{1}{2} (\hat{J}_+ + \hat{J}_-) .$$

In terms of the Cartesian components of the quasispin, the Hamiltonian (2.1) has the form

$$\hat{H} = \epsilon \hat{J}_0 - |V| (\hat{J}_x^2 - \hat{J}_y^2) . \quad (3.1)$$

For the weak coupling limit, either $\hat{J}_x$ or $\hat{J}_y$ could be considered as a candidate, but for growing $|V|$, the energy will be minimized by increasing $\langle \hat{J}_x^2 \rangle$ at the cost of $\langle \hat{J}_y^2 \rangle$.

The commutation relations for the Cartesian components are

$$[\hat{J}_x, \hat{J}_y] = i \hat{J}_0 , \quad (3.2)$$

$$[\hat{J}_y, \hat{J}_0] = i \hat{J}_x , \quad (3.3)$$

$$[\hat{J}_0, \hat{J}_x] = i \hat{J}_y . \quad (3.4)$$

Introducing the collective variables

$$\begin{aligned}\hat{X} &= \sqrt{\frac{\omega}{2}} (B^+ + B) , \\ \hat{P} &= i \sqrt{\frac{1}{2\omega}} (B^+ - B) ,\end{aligned}\tag{3.5}$$

where the boson operators obey the relation

$$[B, B^\dagger] = 1\tag{3.6}$$

in order to insure the canonical commutation relation

$$[\hat{X}, \hat{P}] = i ,\tag{3.7}$$

one directly verifies that the ansatz

$$\hat{J}_x = \hbar \hat{X}, \hat{J}_y = -\hat{P}, \hat{J}_0 = -\hbar\tag{3.8}$$

satisfies the first commutation relation (3.2). The quantity $\hbar$ is a constant to be determined subsequently. In order to satisfy the remaining conditions (3.3) and (3.4), the transcription (3.8) will have to be extended. One way of proceeding is to keep the simple form for either $\hat{J}_x$ or $\hat{J}_y$. As it turns out, the choice

$$\hat{J}_y = -\hat{P}$$

appears to be the best to make from a physical point of view, as it corresponds directly to the diminishing importance of the collective momentum in the limit of large interaction strength.

After extending the boson transcription in a minimal way by adding terms proportional to $\hat{X}^2$ and $\hat{P}^2$, etc., it is realized after some manipulations that a closed form of the solution is possible. The transformation

$$\begin{aligned}
\hat{J}_x &= n (\sin \hat{X} \vartheta(\hat{P}^2) + \vartheta(\hat{P}^2) \sin \hat{X}) = n \{ \sin \hat{X}, \vartheta(\hat{P}^2) \} , \\
\hat{J}_y &= - \hat{P} , \\
\hat{J}_0 &= - n \int \cos \hat{X}, \vartheta(\hat{P}^2) ,
\end{aligned} \tag{3.9}$$

is indeed an exact solution of the commutators (3.2) and (3.3).

The commutator (3.4) in turn determines the structure of the function $\vartheta(\hat{P}^2)$. With the ansatz of a power series expansion

$$\vartheta(\hat{P}^2) = 1 + \sum_{n=1}^{\infty} C_{2n} \hat{P}^{2n} \tag{3.10}$$

one finds e.g. the recursion relations

$$\begin{aligned}
n^2 \sum_{n \leq m} [\binom{2n+2m}{2k+1} + \binom{2m}{2k+1-2n} + \binom{2n}{2k+1-2m}] \\
\times (2 - \delta_{n,m}) C_{2n} C_{2m} = - \delta_{k,0}
\end{aligned} \tag{3.11}$$

for the expansion coefficients in terms of the constant $\mathcal{N}$.

The original suggestion of the transition operator expansion is one of other possible solutions that can be obtained by cyclic permutation of the angular momentum operators and/or the canonical transformation $\hat{X} \longleftrightarrow -\hat{P}$.

The constant $\mathcal{N}$ represents the only handle we have for insisting that we are dealing with a system of fermions. It must then be determined from the Casimir operator (2.5). Evaluation of this operator with the ansatz (3.9) and the expansion (3.10) gives the relation

$$\frac{1}{2} N(\frac{1}{2} N + 1) = \hat{P}^2 + n^2 \sum_{n \leq m} C_{2n} C_{2m} \sum_{k=0}^{n+m} \left[\left\{ \binom{2n}{2k-2m} + \binom{2m}{2k-2n} + \binom{2n+2m}{2k} \right\} \hat{P}^{2k} + \hat{P}^{2n+2m} \right] (2 - \delta_{n,m}) . \quad (3.12)$$

This equation establishes a relation between the constants N and $\mathcal{N}$.

If we carry the expansion (3.10) to fourth order and rename

$$C_2 = a \quad , \quad C_4 = b$$

we find consistently that $b \sim a^2$. The set of Eqs. (3.11) gives to order a^2

$$b + \frac{1}{2} a^2 = 0 \quad ,$$

$$8 n^2 a + 16 n^2 b + 4 n^2 a^2 + 1 = 0 \quad , \quad (3.13)$$

with the solution

$$a = 1 - \left(1 + \frac{1}{4n^2} \right)^{\frac{1}{2}} \cong -\frac{1}{8n^2} + \frac{1}{128n^4} \quad ,$$

$$b \cong -\frac{1}{128n^4} . \quad (3.14)$$

In Eq. (3.12) we find (as expected) cancellation of all the $\hat{P}$ -dependence and from the constant term the condition

$$M = (N + 1)^2 = 16 n^2 + 8 a n^2 . \quad (3.15)$$

Combination of Eqs. (3.13) and (3.15) finally gives

$$a = \frac{M + 1}{M} - \left[\frac{4}{M} + \frac{(M + 1)^2}{M^2} \right]^{\frac{1}{2}} \cong -\frac{2}{M} + \frac{4}{M^2} + \dots ,$$

and

$$n = \frac{1}{4} \left[\frac{2M}{2 + a} \right]^{\frac{1}{2}} \cong \frac{\sqrt{M}}{4} \left(1 + \frac{1}{2M} - \frac{5}{8M^2} + \dots \right) . \quad (3.16)$$

From either Eq. (3.14) or (3.16) we recognize that the series (3.10) converges very rapidly for large values of N . This is because $\hat{P}^2$ is $O(N)$ for small coupling and decreases to $O(1)$ in the strong coupling region.

IV. The Hamiltonian

The ansatz (3.9) for the three basic operators of the model and the expansion (3.10) to order $b = -\frac{1}{2}a^2$ give the following expression for the model Hamiltonian (2.1)

$$\begin{aligned}
 H = & \left[-2\epsilon n \cos \hat{X} - 4n^2 |V| \sin^2 \hat{X} \right. \\
 & + \left[-\epsilon n a \left\{ \cos \hat{X}, \hat{P}^2 \right\} - |V| (-\hat{P}^2 \right. \\
 & + 2a n^2 \left\{ \sin \hat{X}, \left\{ \sin \hat{X}, \hat{P}^2 \right\} \right\}) \left. \right] \\
 & + \left[\frac{1}{2} \epsilon n a^2 \left\{ \cos \hat{X}, \hat{P}^4 \right\} - |V| a^2 n^2 \right. \\
 & \times \left(\left\{ \sin \hat{X}, \hat{P}^2 \right\}^2 - \left\{ \sin \hat{X}, \left\{ \sin \hat{X}, \hat{P}^4 \right\} \right\} \right) \left. \right] . \quad (4.1)
 \end{aligned}$$

This expression constitutes the approximate (to order $\hat{P}^4$) collective Hamiltonian of the model. The first square bracket represents the collective potential energy. The condition for its minimum is

$$\frac{dV}{dX} = 2\epsilon n \sin X - 8|V| n^2 \sin X \cos X = 0 \quad (4.2)$$

The solutions of this equation are

$$\sin X_0 = 0 \quad \text{or} \quad X_0 = 0 \pmod{2\pi} , \quad (4.3)$$

which pertains to the weak coupling situation ($\delta < 1$), or

$$\cos X_0 = \frac{\epsilon}{4n|V|} = \frac{N}{4n} \frac{1}{\delta} \quad (4.4)$$

which is possible for $\delta > 1$.

The potential energy surface has a single minimum at $X_0 = 0$ for $\delta < 1$. For the regime $\delta \geq 1$ it develops a double minimum which is symmetric with respect to the origin and is separated by a maximum at $X = 0$. The double minimum of the potential energy expresses the doublet structure of the low-lying levels in the strong coupling region in a very pictorial fashion.

Previous treatments of the model based on "deformed Hartree-Fock"⁽⁸⁾ have produced variants of Eq. (4.2), but in conjunction with the collective variables introduced here, we have the possibility of studying oscillations of the system without any equilibrium value of X_0 . From these we will consider in some detail the weak coupling situation characterized by $X_0 = 0$ and the extreme strong coupling situation, which is characterized according to Eq. (4.4) by

$$\cos X_0 = 0 \quad X_0 = \frac{\pi}{2} \pmod{2\pi} .$$

Expansion of Eq. (4.1) about $X_0 = 0$ to quartic terms in the collective variable yields the expression

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_2 + \widehat{H}_4$$

with

$$\hat{H}_0 = -2\epsilon n$$

$$\hat{H}_2 = (\epsilon n - 4n^2 |V|) \hat{X}^2 + (|V| - 2\epsilon n) \hat{P}^2,$$

$$\begin{aligned} \hat{H}_4 = & \left(-\frac{\epsilon n}{12} + \frac{4}{3} n^2 |V| \right) \hat{X}^4 + \left(\frac{1}{2} n a \right. \\ & \left. - 2a n^2 |V| \right) \{ \hat{X}^2, \hat{P}^2 \} - 4a n^2 \hat{X} \hat{P}^2 \hat{X} + \epsilon n a^2 \hat{P}^4. \end{aligned} \quad (4.5)$$

The coefficient of the quadratic term of the potential energy vanishes at the point where Eq. (4.4) starts to be satisfied. On the other hand the coefficient of the quartic term is positive and growing linearly with $|V|$. For this reason one should not take H_2 as the unperturbed approximation. (RPA approximation) and treat H_4 as a perturbation, if one wishes to approach the situation $\delta \approx 1$ without any breakdown of the theory.

For this case we note that the zero order Hamiltonian

$$H_{20} = \epsilon n \hat{X}^2 - 2a\epsilon n \hat{P}^2 \quad (4.6)$$

defines a frequency which is fixed. The problem of diagonalizing the total Hamiltonian H in the basis defined by H_{20} is very nearly the problem considered before by the authors and their colleagues.⁽⁹⁾

To demonstrate this explicitly, we introduce the boson operators as defined in Eq. (3.5), which yields for the Hamiltonian H_{20} the expression

$$H_{20} = \epsilon n \left[\left(\frac{\omega}{2} + \frac{a}{\omega} \right) (B^{\dagger 2} + B^2) + \left(\omega - \frac{2a}{\omega} \right) (B^{\dagger} B + \frac{1}{2}) \right] \quad (4.7)$$

We define the frequency by setting the coefficient of the term $(B^{\dagger 2} + B^2)$ equal to zero and obtain

$$\omega^2 = -2a \quad (4.8)$$

With this definition for the frequency we have for the full (quartic) Hamiltonian (4.5)

$$\begin{aligned} H = & h_{00} + h_{20} (B^{\dagger 2} + B^2) + h_{11} (B^{\dagger} B) \\ & + h_{40} (B^{\dagger 4} + B^4) + h_{31} (B^{\dagger 3} B + B^{\dagger} B^3) \\ & + h_{22} (B^{\dagger 2} B^2) \quad , \end{aligned} \quad (4.9)$$

with the coefficients

$$\begin{aligned} h_{00} &= -\epsilon n \left[2 - \omega - \frac{\omega^2}{4} \right] + |V| \left[\frac{1}{2\omega} - 2n^2 \omega + 2n^2 \omega^2 \right] \\ h_{20} &= -\epsilon n \frac{\omega^2}{2} - |V| \left[\frac{1}{2\omega} + 2n^2 \omega - 2n^2 \omega^2 \right] \\ h_{11} &= 2\epsilon n \omega + |V| \left[\frac{1}{\omega} - 4n^2 \omega + 8n^2 \omega^2 \right] \\ h_{40} &= \frac{1}{6} \epsilon n \omega^2 - \frac{2}{3} |V| n^2 \omega^2 \\ h_{31} &= -\frac{1}{3} \epsilon n \omega^2 + \frac{4}{3} |V| n^2 \omega^2 \\ h_{22} &= 4|V| n^2 \omega^2 \quad . \end{aligned} \quad (4.10)$$

If one uses the expressions (3.16) for a and n one recovers to leading order the corresponding equations of Section IV in Ref. 9.

The expansions of the sine and cosine functions in the

collective variable introduces the (if ever slight) disadvantage, that contributions arising from normal ordering of the terms X^n with $n > 4$ are neglected. These contributions can be obtained with some additional labor by inserting the boson operators directly into the Hamiltonian (4.1) and expanding the functions of the boson operators in normal order. This results (to fourth order in the boson variables) in the coefficients

$$h_{00} = -\epsilon n e^{-\frac{\omega}{4}} \left\{ 2 + a \left(\frac{1}{\omega} + \frac{1}{2} \right) + b \left(\frac{3}{2\omega^2} + \frac{3}{2\omega} + \frac{1}{8} \right) + |V| \left\{ \frac{1}{2\omega} - n^2 (2(1-e^{-\omega}) + 2a \left(\frac{1}{\omega} + 1 \right) \times (1-e^{-\omega}) + b \left(\frac{2}{\omega} + 1 - \left(\frac{6}{\omega} + 1 \right) e^{-\omega} \right) \right\} \right\},$$

$$h_{20} = \epsilon n e^{-\frac{\omega}{4}} \left\{ \frac{\omega}{2} + a \left(\frac{1}{\omega} + \frac{1}{4} + \frac{\omega}{8} \right) + b \left(\frac{3}{\omega^2} + \frac{15}{8\omega} + \frac{3}{8} + \frac{\omega}{32} \right) \right\} - |V| \left\{ \frac{1}{2\omega} + n^2 (2\omega e^{-\omega} + 2a \left(-\frac{1}{\omega} + e^{-\omega} \left(\frac{1}{\omega} + 1 + \omega \right) \right) + b \left(-\frac{2}{\omega} + \left(\frac{6}{\omega} + 6 + \omega \right) e^{-\omega} \right) \right\},$$

$$h_{11} = \epsilon n e^{-\frac{\omega}{4}} \left\{ \omega + a \left(-\frac{2}{\omega} + \frac{1}{2} + \frac{\omega}{4} \right) + b \left(-\frac{6}{\omega^2} - \frac{9}{4\omega} + \frac{3}{4} + \frac{\omega}{16} \right) + |V| \left\{ \frac{1}{\omega} - n^2 \times (4\omega e^{-\omega} + 2a \left(\frac{2}{\omega} + e^{-\omega} \left(-\frac{2}{\omega} + 2 + 2\omega \right) \right) + b \left(\frac{4}{\omega} + \left(-\frac{12}{\omega} + 12 + 2\omega \right) e^{-\omega} \right) \right\} \right\},$$

$$\begin{aligned}
h_{40} &= - \epsilon \hbar e^{-\frac{\omega}{4}} \left\{ \frac{\omega^2}{48} + a \left(\frac{1}{4} + \frac{\omega}{96} + \frac{\omega^2}{192} \right) \right. \\
&\quad \left. + b \left(\frac{1}{2\omega^2} + \frac{3}{4\omega} + \frac{25}{64} + \frac{\omega^2}{768} \right) \right\} + |V| \hbar^2 e^{-\omega} \\
&\quad \times \left\{ \frac{\omega^2}{3} + 2a \left(1 + \frac{\omega}{6} + \frac{\omega^2}{6} \right) + b \left(6 + \omega + \frac{\omega^2}{6} \right) \right\}, \\
h_{31} &= - \epsilon \hbar e^{-\frac{\omega}{4}} \left\{ \frac{\omega^2}{12} + a \left(\frac{\omega}{24} + \frac{\omega^2}{48} \right) + b \right. \\
&\quad \times \left(-\frac{2}{\omega^2} + \frac{1}{16} + \frac{\omega}{16} + \frac{\omega^2}{192} \right) - |V| \hbar^2 e^{-\omega} \left\{ \frac{4}{3} \omega^2 \right. \\
&\quad \left. + 2a \left(\frac{2}{3}\omega + \frac{2}{3}\omega^2 \right) + b \left(4\omega + \frac{2}{3}\omega^2 \right) \right\} \\
h_{22} &= - \epsilon \hbar e^{-\frac{\omega}{4}} \left\{ \frac{\omega^2}{8} + a \left(-\frac{1}{2} + \frac{\omega}{16} + \frac{\omega^2}{32} \right) \right. \\
&\quad \left. + b \left(\frac{3}{\omega^2} - \frac{3}{2\omega} - \frac{21}{32} + \frac{3\omega}{32} + \frac{\omega^2}{128} \right) \right\} - |V| \hbar^2 e^{-\omega} \\
&\quad \times \left\{ -2\omega^2 + 2a (2 - \omega - \omega^2) + b (12 - 6\omega - \omega^2) \right\}. \quad (4.11)
\end{aligned}$$

If one uses again the relations $b = -\frac{1}{2} a^2$ and $a = -\frac{\omega^2}{2}$, one finds that the coefficients (4.11) differ from the set (4.10) by terms of order ω^3 . Both sets of coefficients are correct to order ω^2 .

For the set (4.11) contributions arising from the terms $\hat{X}^n \hat{P}^6$, $\hat{X}^n \hat{P}^8$ etc. are not included while terms arising from $\hat{X}^n \hat{P}^4$, $\hat{X}^n \hat{P}^2$ and $\hat{X}^n$ are given correctly.

For the boson representation in the strong coupling region, we make the replacement

$$\begin{aligned}
\hat{X} &\rightarrow \frac{\pi}{2} - \hat{X}', \\
\hat{P} &\rightarrow -\hat{P}',
\end{aligned} \quad (4.12)$$

so that expansion of the Hamiltonian (4.1) in $\hat{X}'$ will correspond to an expansion in $\hat{X}$ about the equilibrium point $X_0 = \frac{\pi}{2}$.

For the resulting operators

$$\begin{aligned}\hat{J}_x &= n \{ \cos \hat{X}', \emptyset(\hat{P}'^2) \} , \\ \hat{J}_y &= \hat{P}' , \\ \hat{J}_z &= -n \{ \sin \hat{X}', \emptyset(\hat{P}'^2) \} ,\end{aligned}\quad (4.13)$$

one can check directly that the commutators (3.2) - (3.4) are satisfied with the same function $\emptyset(P^2)$ as before. The considerations concerning the Casimir operator (Eq. (3.12)) and the following are also unaffected by the replacement (4.12).

We then obtain, to order P'^4 , the collective Hamiltonian

$$\begin{aligned}\hat{H} &= [-2\epsilon n \sin \hat{X}' - 4|V|n^2 \cos \hat{X}'^2] + [-\epsilon n a \\ &\times \{ \sin \hat{X}', \hat{P}'^2 \} - |V|(-\hat{P}'^2 + 2a n^2) \\ &\times \{ \cos \hat{X}', \{ \cos \hat{X}', \hat{P}'^2 \} \}] + [-\epsilon n b \{ \sin \hat{X}', \hat{P}'^4 \} \\ &- 2b n^2 |V| \{ \cos \hat{X}', \{ \cos \hat{X}', \hat{P}'^4 \} \} \\ &- \{ \cos \hat{X}', \hat{P}'^2 \}^2].\end{aligned}\quad (4.14)$$

Introduction of the boson operators according to

$$\begin{aligned}\hat{X}' &= \left(\frac{\omega}{2} \right)^{\frac{1}{2}} (B^\dagger + B) , \\ \hat{P}' &= i \left(\frac{1}{2\omega} \right)^{\frac{1}{2}} (B^\dagger - B) ,\end{aligned}\quad (4.15)$$

yields the exactly normal ordered Hamiltonian

$$H = \sum_{m,n} g_{m,n} (B^{\dagger m} B^n + B^{\dagger n} B^m) \quad (4.16)$$

The coefficients (to order $m+n=5$) are

$$\begin{aligned} g_{00} &= -\frac{|V|}{2} \left[-\frac{1}{2\omega} + n^2 (2(1 + e^{-\omega}) + 2a (1 + \frac{1}{\omega}) \right. \\ &\quad \left. \times (1 + e^{-\omega}) + b((1 + \frac{2}{\omega}) + e^{-\omega} (1 + \frac{6}{\omega})) \right] , \\ g_{10} &= -\epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[1 + \frac{a}{2\omega} (1 + \frac{\omega}{2}) + \frac{b}{4\omega^2} (3 + 3\omega + \frac{\omega^2}{4}) \right] , \\ g_{20} &= -|V| \left[\frac{1}{2\omega} - n^2 (2\omega e^{-\omega} + 2a (\frac{1}{\omega} \right. \\ &\quad \left. + (\frac{1}{\omega} + 1 + \omega) e^{-\omega} + b(\frac{2}{\omega} + (\frac{6}{\omega} + 6 + \omega) e^{-\omega})) \right] , \\ g_{11} &= -\frac{|V|}{2} \left[-\frac{1}{\omega} + n^2 (-4\omega e^{-\omega} + 4a (\frac{1}{\omega} + (\frac{1}{\omega} \right. \\ &\quad \left. + 1 + \omega) e^{-\omega} + b (\frac{4}{\omega} + (\frac{12}{\omega} - 12 - 2\omega) e^{-\omega})) \right] , \\ g_{30} &= -\epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[-\frac{\omega}{12} - \frac{a}{2\omega} (1 + \frac{\omega}{12} \right. \\ &\quad \left. + \frac{\omega^2}{24}) - \frac{b}{4\omega^2} (6 + \frac{13}{4}\omega + \frac{1}{4}\omega^2 + \frac{1}{48}\omega^3) \right] , \\ g_{21} &= -\epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[-\frac{\omega}{4} + \frac{a}{2\omega} (1 - \frac{\omega}{4} - \frac{\omega^2}{8}) \right. \\ &\quad \left. + \frac{b}{4\omega^2} (6 + \frac{a}{4}\omega - \frac{3}{4}\omega^2 - \frac{1}{16}\omega^3) \right] , \\ g_{40} &= -|V| n^2 e^{-\omega} \left[\frac{\omega^2}{3} + 2a (1 + \frac{1}{6}\omega + \frac{1}{6}\omega^2) \right. \\ &\quad \left. + b(6 + \omega + \frac{\omega^2}{6}) \right] , \end{aligned}$$

$$\begin{aligned}
g_{31} &= - |V|n^2 e^{-\omega} \left[\frac{4}{3} \omega^2 + 2a \left(\frac{2}{3} \omega + \frac{2}{3} \omega^2 \right) + b(4\omega + \frac{2}{3} \omega^2) \right], \\
g_{22} &= - |V|n^2 e^{-\omega} [2\omega^2 + 2a (-2 + \omega + \omega^2) + b(-12 + 6\omega + \omega^2)], \\
g_{50} &= - \epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[\frac{\omega^2}{480} + \frac{a}{2\omega} \left(\frac{\omega}{12} + \frac{\omega^2}{480} + \frac{\omega^2}{960} \right) \right. \\
&\quad \left. + \frac{b}{4\omega^2} \left(1 + \frac{\omega}{2} + \frac{41}{160} \omega^2 + \frac{1}{160} \omega^3 + \frac{1}{1920} \omega^4 \right) \right], \\
g_{41} &= - \epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[\frac{\omega^2}{96} + \frac{a}{2\omega} \left(\frac{\omega}{12} + \frac{\omega^2}{96} + \frac{\omega^3}{192} \right) \right. \\
&\quad \left. - \frac{b}{4\omega^2} \left(3 - \frac{\omega}{2} - \frac{9}{32} \omega^2 - \frac{1}{32} \omega^3 - \frac{1}{384} \omega^4 \right) \right], \\
g_{32} &= - \epsilon n (2\omega)^{\frac{1}{2}} e^{-\omega/4} \left[\frac{\omega^2}{48} - \frac{a}{2\omega} \left(\frac{\omega}{6} - \frac{\omega^2}{48} - \frac{\omega^3}{96} \right) \right. \\
&\quad \left. + \frac{b}{4\omega^2} \left(2 - \omega - \frac{7}{16} \omega^2 + \frac{1}{16} \omega^3 + \frac{1}{192} \omega^4 \right) \right]. \tag{4.17}
\end{aligned}$$

To fix the frequency we use the condition

$$g_{20} = 0,$$

which gives approximately

$$\omega \approx \frac{1}{2n} = (-2a)^{\frac{1}{2}},$$

which is also the value obtained from Eq. (4.5) in the weak coupling situation.

V. Results and Discussion

We have compared the energies of the low-lying states obtained from the boson expansions (3.9) and (4.13) with the results of an exact diagonalization for the particle numbers $N = 14$ and $N = 20$.

For the weak coupling situation the procedure is straightforward. We construct a set of (normalized) boson basis states

$$|r\rangle = \frac{1}{(r!)^{1/2}} (B^\dagger)^r |vac\rangle, \quad r = 0, 1, \dots, N, \quad (5.1)$$

evaluate the matrix elements

$$\begin{aligned} \langle r | B^{\dagger m} B^n | r' \rangle &= \delta_{r', r+n-m} \\ &\times \frac{r!}{(r-m)!} \left[\frac{(r+n-m)!}{r!} \right]^{1/2} \end{aligned}$$

and diagonalize the Hamiltonian (4.9), (4.10) or (4.9), (4.11).

The resulting ground state energies are given in Table I, the excitation energies of the first and second excited states in Figs. (1) - (4). The ground state energies are well reproduced (with a deviation of less than 5%) for values of $\delta \leq 2$. An increasing deviation is observed if one goes too far beyond the transition point to the "deformed" solution. The excitation energies of the first excited state shows the correct decrease as a function of δ , but the boson expansions are not able to reproduce the very small

splitting of the doublet beyond $\delta = 2$. By comparison one finds for the excitation energies of the second excited states the correct trend even beyond $\delta = 2$. They show, however, some difficulties in the region of the phase transition. In general we can remark that the quality of the boson expansion improves with the particle number N , as expected from the discussion of Sec. 3.

The normal ordered version gives, in each instance more accurate results than the expanded version. Different choices of ω (as for instance the relation $\omega = (-2a)^{\frac{1}{2}}$ versus determination from the ϵ dependent part of h_{20} in Eq. (4.11)) lead only to minor variations of the results.

In Figs. (5) and (6) we show the percentage deviation from exact points of some of the individual transition operator matrix elements for the low lying states.

For the diagonalization of the Hamiltonian with the strong coupling boson expansion we have to keep in mind, that the strong coupling boson describes the doublets and not the individual levels. For this reason the size of the boson space is chosen to be $\frac{1}{2}N + 1$. We have checked that variation of the dimension of the boson space does not affect the results beyond the value $\delta = 2$. For comparison with the exact results we use:

$(E_0 + E_1)_{\text{exact}}$ is to be compared with $(E_0)_{\text{boson}}$.

$(E_3 + E_2 - E_1 - E_0)_{\text{exact}}$ is to be compared with $(E_1 - E_0)_{\text{boson}}$.

The results for the ground state energy and the first excited doublet are given in Tables II and III, respectively. We find excellent agreement as soon as the doublet structure is present and note that the boson expansion reproduces the center of gravity of the developing doublets even for the values of $\delta \approx 1 - 5$.

The results presented here do correspond favourably in all respects with another new boson expansion based on corrections to the Gaussian overlap approximation.^(10,11) There remains the question of investigating a boson expansion for an intermediate situation. As, however, the two extreme cases already span the whole range of the interaction strength successfully, we did not consider it worthwhile for the simple model investigated here. The results already demonstrate that a choice of the boson expansion on the basis of the dynamics of the system as expressed through the transition operators should be a worthwhile endeavor for a more realistic fermion system.

Acknowledgements.

The authors wish to express their gratitude to the Zentrale Recheninstitut of the University Frankfurt, Germany, where part of the numerical calculations were carried out.

References

1. S.T. Beliaev and V. G. Zelevinsky, Nucl. Phys. 39, 582 (1962).
2. T. Marumori, M. Yamamura and A. Tokunaga, Progr. Theor. Physics (Kyoto) 31, 1009 (1964); M. Yamamura, A. Tokunaga, and T. Marumori ibid. 53, 336 (1967).
3. B. Sorensen, Nucl. Phys. A97, 1 (1967); Progr. Theoret. Phys. (Kyoto) 39, 1468 (1968); Nucl. Phys. A119, 65 (1968); R. Broglia and B. Sorensen, Nucl. Phys. A110, 241 (1969); M. Kleber, Phys. Letters 30B, 588 (1969); S. Y. Li, R. M. Dreizler and A. Klein, Phys. Letters 32B, 169 (1970); Phys. Rev. C4, 1571 (1971); S. Y. Li, A. Klein and R. M. Dreizler, J. Math. Phys. 11, 975 (1970); E. R. Marshalek, Nucl. Phys. A161, 401 (1971).
4. H. J. Lipkin, N. Meshkov and A. J. Glick, Nucl. Phys. 62, 188 (1965); N. Meshkov, A. J. Glick and H. J. Lipkin, Nucl. Phys. 62, 199 (1965); A. J. Glick, H. J. Lipkin and N. Meshkov, Nucl. Phys. 62, 199 (1965).
5. D. M. Brink, A. F. R. de Toledo Piza and A. K. Kerman, Phys. Letters 19, 413 (1965).
6. T. K. Das, R. M. Dreizler, and A. Klein, Phys. Rev. C, 2, 632 (1970).
7. A. Klein, Theories of Collective Motion in Dynamical Structure of Nuclear States, Edited by D.J. Rowe, et al. U. of Toronto Press (1972).

8. D. Agassi, H. J. Lipkin and N. Meshkov, Nucl. Phys. 62, 321 (1966); J. C. Parikh and D. J. Rowe, Phys. Rev. 175, 1293 (1968); A. Faessler and A. Pastino, Z. Phys. 220, 88 (1969); K. Bleuler, A. Frederick and D. Schutte, Nucl. Phys. A126, 628 (1969); R. M. Dreizler, A. Klein, F. R. Krejs and G. J. Dreiss, Nucl. Phys. A166, 624 (1971).
9. S. C. Pang, A. Klein and R. M. Drèizler, Ann. Phys. 49, 477 (1968).
10. J. da Providencia, J. N. Urbano, and L. S. Ferreira, Nucl. Phys. A170, 129 (1971).
11. A. J. Alves and E. M. Silva, Nucl. Phys. A181, 669 (1972).

Figure Captions

- Fig. 1. Excitation energy of the first excited state ($E_1 - E_0$) in units of the single particle energy ϵ plotted vs. $\delta = (NV/\epsilon)$, the dimensionless measure of the interaction strength for $N = 14$ particles. The three curves compare the exact result with two boson approximations as described in the text.
- Fig. 2. Excitation energy of the second excited state, where otherwise the caption for Fig. 1. applies.
- Fig. 3. The analogue of Fig. 1 for $N = 20$.
- Fig. 4. The analogue of Fig. 2 for $N = 20$.
- Fig. 5. The absolute percentage deviation $|\Delta J/J|$ for $J_z(0,0)$ ($\circ$), $J_z(02)$ (Δ), $J_z(11)$ (x), $J_x(01) = J_x(10)$ ($\bullet$) for $N = 20$ and the boson expansion of Eq. (4.10). Note the difference in scale to the left and to the right of $\delta = 1.4$.
- Fig. 6. The same quantities as in Fig. 5 for the boson expansion of Eq. (4.11).

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.

Table I: Groundstate Energy ($\langle E_0 + N/2 \rangle$ in units of ϵ) with the boson expansion near the weak coupling limit.

δ	N = 14			N = 20		
	(1)	(2)	(3)	(1)	(2)	(3)
.0	-.000	-.003	-.002	-.000	-.002	-.001
.2	-.009	-.011	-.011	-.010	-.010	-.010
.4	-.038	-.038	-.039	-.039	-.039	-.039
.6	-.088	-.086	-.088	-.091	-.090	-.091
.8	-.163	-.157	-.163	-.172	-.168	-.171
1.0	-.270	-.234	-.268	-.292	-.280	-.290
1.2	-.418	-.379	-.410	-.472	-.439	-.463
1.4	-.621	-.537	-.598	-.748	-.657	-.715
1.6	-.891	-.727	-.835	-1.155	-.946	-1.064
1.8	-1.232	-.948	-1.120	-1.691	-1.304	-1.503
2.0	-1.636	-1.197	-1.447	-2.324	-1.720	-2.012
2.5	-2.843	-1.914	-2.394	-4.158	-2.921	-3.464
3.0	-4.212	-2.719	-3.448	-6.190	-4.247	-5.047
4.0	-7.171	-4.457	-5.699	10.531	-7.060	-8.380
5.0	-10.268	-6.276	-8.032	-15.056	-9.969	-11.815

(1) exact solution.

(2) expanded Hamiltonian (4.9), (4.10) with $\omega = (-2a)^{\frac{1}{2}}$.

(3) normal ordered Hamiltonian (4.9), (4.11) with ω determined by setting the coefficient of $\epsilon (B^{\dagger 2} + B^2)$ equal to zero.

Table II. Groundstate energies with the boson expansion (4.16). compared with $\frac{1}{2} (E_0 + E_1)$ calculated exactly. The dimension of the boson space is $n_B = \frac{1}{2}N + 1$ for each case.

δ	N = 14		N = 20		Units
	1	2	1	2	
∞	-46.151	-46.152	-95.906	-95.907	$ V $
100	-329.69	-329.70	-479.58	-479.59	ϵ
50	-164.90	-164.91	-239.87	-239.87	
25	-82.569	-82.571	-120.10	-120.10	
20	-66.125	-66.127	-96.175	-96.176	
10	-33.356	-33.357	-48.492	-48.492	
5	-17.266	-17.265	-25.051	-25.054	
2	-8.5766	-8.5694	-12.309	-12.289	
1	-6.9460	-6.9799	-10.001	-10.106	ϵ

1. $\frac{1}{2} (E_0 + E_1)$ from exact diagonalization.
2. E_0 boson expansion (to fourth order in the boson variable).

Table III. Excitation energy for the first doublet according to the boson expansion (4.16) compared with $\frac{1}{2} (\Delta E_3 + \Delta E_2 - \Delta E_1)$ calculated exactly. The dimension of the boson space is $n_B = \frac{1}{2} N + 1$.

δ	N = 14		N = 20		Units
	1	2	1	2	
∞	18.124	18.132	26.652	26.649	$ V $
100	129.54	129.50	133.25	133.24	ϵ
50	64.748	64.729	66.609	66.602	↑
25	32.332	32.321	33.272	33.268	
20	25.841	25.831	26.591	26.595	
10	12.814	12.802	13.217	13.214	
5	6.1813	6.1292	6.4375	6.4236	
2	2.0724	1.6874	2.0237	1.5323	↓
1	1.7415	1.3000	1.5645	1.1750	

1. $\frac{1}{2} (\Delta E_3 + \Delta E_2 - \Delta E_1)$ from exact diagonalization.
2. $(E_1 - E_0)$ from boson expansion (to fourth order in the boson variable).