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EXACT EVALUATION OF THE LOWEST ORDER VACUUM POLARIZATION CONTRIBUTIONS TO THE MAGNETIC MOMENT OF THE MUON

Glen W. Erickson and Henry H. T. Liu

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Exact Evaluation of
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Glen W. Erickson and Henry H. T. Liu

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ABSTRACT

The contribution of the lowest order vacuum polarization diagram to the anomalous magnetic moment of a lepton is given in closed form for all values of the mass of the virtual pair. The calculation independently verifies Elend's calculation, as corrected by Kroll, and the result, in terms of Spence functions, is simpler than that given by Elend. Attention is directed to some useful but not well-known Spence function identities.

Introduction

The magnetic moment, $\frac{e\hbar}{2Mc} g$, of a spin- $\frac{1}{2}$ particle with charge e and mass M is well-known to have, besides the g factor of 2 according to the Dirac equation, an anomalous moment given by

$$\frac{g}{2} - 1 = \delta_1 + \delta_2$$

where (1)

$$\delta_1 = \frac{\alpha}{2\pi} - 0.328 \frac{\alpha^2}{\pi^2} + 0.13 \frac{\alpha^3}{\pi^3} + \dots \quad (1)$$

is the radiative correction due to interactions of the particle with the quantized electromagnetic field, and δ_2 is the correction due to interactions involving other particles. For muons, the lowest order contribution to δ_2 is the vacuum polarization diagram

Figure I

in which the virtual pair is an electron-positron pair. This contribution in the integral representation

$$\delta_2\left(\frac{m}{M}\right) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx \int_0^1 dy \frac{x^2(1-x)y^2(1-y^2/3)}{x^2(1-y^2) + 4(m/M)^2(1-x)}, \quad (2)$$

where m is the electron mass, may be obtained (2) by modifying the calculation of the corresponding term in δ_1 , for which the virtual pair is a muon pair, with $m = M$.

In this paper, we evaluate the integral $\delta_2(\frac{m}{M})$ exactly (in terms of Spence functions) in a form valid for all positive finite values of the mass ratio $\frac{m}{M}$, not just for the limit $\frac{m}{M} \ll 1$ giving the dominant contribution for the muon moment. The result, therefore, can not only be used to obtain a more accurate value for the muon moment, but is also applicable to similar problems with different values of the mass ratio. Thus, for example, the contribution of the muon pair vacuum polarization to the magnetic moment of the electron is found by taking $\frac{m}{M} \gg 1$. For $\frac{m}{M} = 1$, one finds the lepton pair vacuum polarization contribution to the magnetic moment of the same lepton.

Results of Calculation

For the given case of the muon moment, we have $\frac{m}{M} \ll 1$, and the integral diverges logarithmically as $\frac{m}{M}$ goes to zero. The two leading terms in the corresponding asymptotic expansion, previously calculated by several authors⁽²⁾, can be extracted from the integral (2) by standard techniques, leaving some integrals of orders $\frac{m}{M}$ and higher. By integrating by parts in the y and x variables successively, however, we may extract the two leading terms in their entirety; the higher order remainder is then contained in a single integral of much simpler form. Thus, we obtain

$$\begin{aligned}
 \left(\frac{\alpha}{\pi}\right)^2 \delta_2 &= \int_0^1 dx \int_0^1 dy \frac{x^2(1-x)y^2(1-y^2/3)}{x^2(1-y^2) + 4(m/M)^2(1-x)} \\
 &= \frac{1}{3} \ln\left(\frac{M}{2m}\right) - \frac{5}{12} + \frac{1}{2} \int_0^1 dx \int_0^1 dy (1-x)(1-y^2) \ln\left(\frac{x^2 + 4(m/M)^2(1-x)}{x^2} - y^2\right) \\
 &= \frac{1}{3} \ln\left(\frac{M}{m}\right) - \frac{25}{36} + \epsilon_2' \quad ,
 \end{aligned}$$

where

$$\delta_2' \equiv \left(\frac{m}{M}\right)^2 \int_0^1 dx \int_0^1 dy \frac{(1-y^2)(2-x)^2}{x^2(1-y^2) + 4(m/M)^2(1-x)} \quad (4)$$

The next term in the expansion can now be evaluated easily by the following usual technique. For $\frac{m}{M} \ll 1$, the main contribution to δ_2' comes from the neighborhood of $x \approx 0$ since the integral diverges there if we set $\frac{m}{M} = 0$. Hence, we approximate δ_2' by

$$\begin{aligned} \delta_2'(x \approx 0) &= 4 \left(\frac{m}{M}\right)^2 \int_0^1 dx \int_0^1 dy \frac{1-y^2}{x^2(1-y^2) + 4(m/M)^2} \\ &= 2 \left(\frac{m}{M}\right) \int_0^1 dy (1-y^2)^{\frac{1}{2}} \tan^{-1} \left[\frac{(1-y^2)^{\frac{1}{2}} M}{2m} \right] \\ &= 2 \left(\frac{m}{M}\right) \int_0^1 dy (1-y^2)^{\frac{1}{2}} \left[\frac{\pi}{2} - \frac{2m}{(1-y^2)^{\frac{1}{2}} M} + \dots \right] \\ &\approx \frac{\pi^2}{4} \left(\frac{m}{M}\right) \end{aligned} \quad (5)$$

The remainder $\delta_2' - \delta_2'(x \approx 0)$ contains terms of higher order than $\frac{m}{M}$.

We shall, however, evaluate δ_2' exactly. We first evaluate the integral with the assumption $\frac{m}{M} > 1$ (see Appendix II) and then analytically continue our result to the region of $\frac{m}{M} \leq 1$. The resulting expression, valid for all positive finite values of the mass ratio $\frac{m}{M}$, is

$$\begin{aligned} \delta_2 &= \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{1}{3} \ln\left(\frac{M}{m}\right) - \frac{25}{36} + \left(\frac{m}{M}\right)^2 \left[4 + 3 \ln\left(\frac{m}{M}\right) \right] \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{m}{M} - 5 \left(\frac{m}{M}\right)^3 \right) \left[\frac{\pi^2}{2} + \text{Li}_2\left(1 - \frac{m}{M}\right) - \text{Li}_2\left(1 + \frac{m}{M}\right) \right] + \left(\frac{m}{M}\right)^4 \left[\frac{\pi^2}{6} - \text{Li}_2\left(1 - \frac{M^2}{m^2}\right) \right] \right\} \quad (6) \end{aligned}$$

where $\text{Li}_2(z)$ is the dilogarithm or Spence function^(3,4).

From equation (6) is it now tedious but fairly straightforward to obtain the various limiting forms of δ_2 by using equations (I-2, (I-3), (I-7), and (I-9), of Appendix I:

$$\delta_2\left(\frac{m}{M} \ll 1\right) = \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{1}{3} \ln\left(\frac{M}{m}\right) - \frac{25}{36} + \frac{\pi^2}{4} \left(\frac{m}{M}\right) + \left[4 \ln\left(\frac{m}{M}\right) + 3 \right] \left(\frac{m}{M}\right)^2 + \dots \right\} \quad (7a)$$

$$\delta_2\left(\frac{m}{M} \gg 1\right) = \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{1}{45} \left(\frac{M}{m}\right)^2 + \frac{1}{70} \left(\frac{M}{m}\right)^4 \left[\ln\left(\frac{M}{m}\right) + \frac{9}{280} \right] + \dots \right\} \quad (7b)$$

$$\delta_2\left(\frac{m}{M} = 1\right) = \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{119}{36} - \frac{\pi^2}{3} \right\} \quad (7c)$$

The results (7), of course, can also be derived directly from the original integral (2) or (3) by standard techniques similar to that used for equation (5).

Comparison with other Calculations

The integral δ_2 was first calculated exactly by H. H. Elend⁽⁵⁾, but an error in this calculation was later noted⁽⁶⁾. His corrected result, $\delta_2(\text{Elend})$

$$\begin{aligned} &= \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{1}{3} \ln\left(\frac{M}{m}\right) - \frac{25}{36} + \left(\frac{m}{M}\right)^2 [4 + 3 \ln\left(\frac{m}{M}\right)] \right. \\ &+ \frac{1}{2} \left[\frac{m}{M} - 5 \left(\frac{m}{M}\right)^3 \right] \left[\frac{\pi^2}{4} + \ln\left(\frac{m}{M}\right) \ln\left(\frac{M+m}{M-m}\right) + \text{Li}_2\left(\frac{2M}{M-m}\right) - \text{Li}_2\left(\frac{2m}{m-M}\right) - 2\text{Li}_2\left(\frac{m}{M}\right) + 2\text{Li}_2\left(-\frac{m}{M}\right) \right] \\ &\left. + 2 \left(\frac{m}{M}\right)^4 \left[\frac{\pi^2}{6} - \ln^2 2 + \frac{1}{4} \ln^2\left(\frac{4m^2}{M^2}\right) + \frac{1}{2} \ln\left(\frac{M^2}{m^2}\right) \ln\left(4 - \frac{4m^2}{M^2}\right) - \frac{1}{2} \text{Li}_2\left(\frac{m^2}{M^2}\right) \right] \right\} \quad (8) \end{aligned}$$

(somewhat simplified here) contains almost twice as many terms as our equation (6), but may be reduced to the same form by the use of various identities. It is easy to show that the six terms multiplying $\frac{1}{2} \left[\frac{m}{M} - 5 \left(\frac{m}{M}\right)^3 \right]$,

for example, can be put in the form

$$\begin{aligned}
 (\text{Elend's six terms}) &= \frac{3\pi^2}{4} - \text{Li}_2\left(\frac{M-m}{M+m}\right) + \text{Li}_2\left(\frac{m-M}{m+M}\right) + 2\text{Li}_2\left(1 - \frac{m}{M}\right) - 2\text{Li}_2\left(1 + \frac{m}{M}\right) \\
 &= \frac{\pi^2}{2} + \text{Li}_2\left(1 - \frac{m}{M}\right) - \text{Li}_2\left(1 + \frac{m}{M}\right)
 \end{aligned} \tag{9}$$

which agrees with the corresponding terms in equation (6). The first equality in (9) follows from (I-7) and (I-8) of Appendix I, and the final form is obtained by making use of (I-11) with $x = -y = \frac{m}{M}$. Similarly, it follows from equation (I-9b) and standard logarithm identities that the five terms multiplying $\left(\frac{m}{M}\right)^4$ in equation (8) reduce to our corresponding two terms in equation (6).

From a very different approach, T. Kinoshita⁽⁷⁾, by algebraic manipulations of lower order cut-off dependent results for g and the charge renormalization constant Z_3 , evaluated the leading term of Equation (7a) and also the leading terms of the α^3 part of δ_2 .

The $\frac{m}{M} = 1$ contribution of Figure I, as well as all other diagrams of order α^2 , was evaluated first by Karplus and Kroll⁽¹⁾. Checks were performed by Sommerfield⁽¹⁾, using the mass-operator formalism of Schwinger, and by Petermann⁽¹⁾, following the same method as Karplus and Kroll. Their result was also obtained by Terent'ev⁽¹⁾, using dispersion relations techniques.

The contribution of structure-less meson pairs in Fig. I is given by equation (2) with $(1 - \frac{y^2}{3})$ replaced by $\frac{2y^2}{9}$, but is small compared with the contribution of hadronic resonances⁽⁸⁾. Brodsky and deRafael⁽⁹⁾ give references to these and other contributions to the muon moment.

Our work was undertaken in order to resolve the discrepancy between Elend's uncorrected result⁽⁵⁾ and a previous direct evaluation by one of us (GWE, unpublished) of the coefficient of $\frac{m}{M}$ in Equation (7a).

Elend's correction⁽⁶⁾ went unnoticed by us until after we had confirmed our first result by the calculation in Equation (5) and had carried out an exact evaluation (Appendix II) of δ_2 . Comparisons of our results with Elend's and discovery of the two-variable identities (I-8) and (I-9) in Lewin⁽³⁾ finally yielded the simpler form (6) for the exact result. This (relatively) simple form and our independent derivation of it are the main contributions^(*) of this report.

(*) We also consider the report pedagogically useful in pointing out the advantages of describing functions by integral representations such as Equation (2) and Equation (4) in which the analytic properties and limiting behavior are easily seen and may be directly evaluated term-by-term as in Equation (5), in contrast to so-called "closed forms", such as Equation (6), which require painstaking work to obtain correctly and yet require still more work to find either numerical values, analytic properties or limiting behavior as in Equation (7).

APPENDIX I

The Dilogarithm or Spence Function:Definition and Useful Identities^(3,4)

For all real x , we define

$$\text{Li}_2(x) = \int_0^x \frac{-\ln|1-y|}{y} dy, \quad (\text{I-1a})$$

while for complex Z , we define a function analytic everywhere in a complex plane cut from $+1$ to infinity along the real axis:

$$\text{Li}_2(Z) = \int_0^Z \frac{-\ln(1-\zeta)}{\zeta} d\zeta \quad (\text{I-1b})$$

where

$$\ln(1-\zeta) = \ln|1-\zeta| + i \arg(1-\zeta), \quad |\arg(1-\zeta)| < \pi,$$

and the path of integration is the straight line from the origin to the point Z . In order that (I-1a) and (I-1b) agree, $\text{Li}_2(Z)$ must be defined for real values of Z greater than unity as the limiting value of the arithmetic mean of values immediately above and below the real axis. We note that this definition is the same as that of the usual references⁽⁴⁾ but differs from that of Lewin⁽³⁾, who uses only (I-1b) with $-\pi < \arg(1-\zeta) \leq \pi$ and thus has $\text{Im Li}_2(x) = i\pi \ln x$ for $x > 1$.

For values of Z of modulus not greater than unity the function possesses the absolutely and uniformly convergent expansion

$$\text{Li}_2(Z) = \sum_{n=1}^{\infty} \frac{Z^n}{n^2}. \quad (\text{I-2})$$

The special values

$$\text{Li}_2(1) = \frac{\pi^2}{6}, \quad \text{Li}_2(-1) = -\frac{\pi^2}{12}, \quad \text{Li}_2(2) = \frac{\pi^2}{4} \quad (\text{I-3})$$

(and others) can be derived from the basic identities:

$$\text{Li}_2(Z) + \text{Li}_2(1-Z) + \ln(Z) \ln(1-Z) = \frac{\pi^2}{6} \quad (\text{I-4})$$

$$\text{Li}_2(Z) + \text{Li}_2\left(\frac{1}{Z}\right) + \frac{1}{2} \ln^2(-Z) = -\frac{\pi^2}{6} \quad (\text{I-5})$$

$$\text{Li}_2(Z) + \text{Li}_2(-Z) = \frac{1}{2} \text{Li}_2(Z^2) \quad (\text{I-6})$$

For real x , we have

$$\text{Li}_2(x) + \text{Li}_2(1-x) + \ln|x| \cdot \ln|1-x| = \frac{\pi^2}{6} \quad (\text{I-7})$$

$$\text{Li}_2\left(\frac{1}{x}\right) + \text{Li}_2(x) + \frac{1}{2} \ln^2|x| = -\frac{\pi^2}{6} \quad \text{for } x < 0 \quad (\text{I-8a})$$

$$= \frac{\pi^2}{3} \quad \text{for } x > 0 \quad (\text{I-8b})$$

and many others, such as

$$\text{Li}_2(x) - \text{Li}_2\left(\frac{x-1}{x}\right) + \frac{1}{2} \ln|x| \cdot \ln\left|\frac{x-1}{x}\right| = -\frac{\pi^2}{3} \quad \text{for } x < 0 \quad (\text{I-9a})$$

$$= \frac{\pi^2}{6} \quad \text{for } x > 0 \quad (\text{I-9b})$$

which may be obtained by combining (I-7) and (I-8). Many others, such as

$$\text{Li}_2(1-x) + \text{Li}_2(1+x) = \frac{\pi^2}{4} + \frac{1}{2} \text{Li}_2(1-x^2) \quad (\text{I-10})$$

may be found by also using (I-6). Finally, we especially note the two-variable identity:

$$\text{Li}_2\left[\frac{y(1-x)}{x(1-y)}\right] = \text{Li}_2(x) - \text{Li}_2(y) + \text{Li}_2\left(\frac{y}{x}\right) + \text{Li}_2\left(\frac{1-x}{1-y}\right) - \frac{\pi^2}{6} + \ln|x| \ln\left|\frac{1-x}{1-y}\right| \quad (\text{I-11})$$

which is not found in the usually cited references⁽⁴⁾, but is among those given by Lewin⁽⁵⁾, who also discusses their century-old formulations.

APPENDIX II

We evaluate exactly the integral

$$\delta'_2 = \rho^2 \int_0^1 dx \int_0^1 dy \frac{(1-y^2)(2-x)^2}{x^2(1-y^2) + 4\rho^2(1-x)} \quad (\text{II-1})$$

as obtained in Equation (3), where $\rho = \frac{m}{M}$.

If we first assume $\rho > 1$, we may make the change of variables of integration

$$x = \frac{2z}{1+z}, \quad y = (\rho^2-1)^{\frac{1}{2}} \tanh \eta \quad (\text{II-2a})$$

and obtain

$$\begin{aligned} \delta'_2 &= 2\rho^2(\rho^2-1)^{\frac{1}{2}} \int_0^1 \frac{dz}{(1+z)^2} \int_0^{\eta_0} \frac{d\eta}{\cos^2 \eta} \left(1 - \frac{k^2(1-z^2)}{\cos^2 \eta - k^2 z^2} \right) \\ &= \rho^2 + \rho(\rho^2-1) \int_0^{\eta_0} \frac{d\eta}{\cos^3 \eta} \ln \left| \frac{\cos \eta + k}{\cos \eta - k} \right| - 4(\rho^2-1)^{3/2} \int_0^1 \frac{dz}{1+z} \int_0^{\eta_0} \frac{d\eta}{\cos^2 \eta} \frac{1}{\cos^2 \eta - k^2 z^2} \end{aligned} \quad (\text{II-2b})$$

where

$$\eta_0 \equiv \tan^{-1}(\rho^2-1)^{\frac{1}{2}} \text{ and } k \equiv \frac{\sqrt{\rho^2-1}}{\rho} = \cos \eta_0. \quad (\text{II-2c})$$

By writing the last integrand in partial fraction of z , performing some of the simple integrals and collecting terms, we find

$$\begin{aligned} \delta'_2 &= \rho^2 \left[4 + 3 \ln \rho \right] + \frac{1}{2}(\rho-5\rho^3) \int_0^{\eta_0} \frac{d\eta}{\cos \eta} \ln \left| \frac{\cos \eta + k}{\cos \eta - k} \right| \\ &+ 2\rho^4 \left\{ \ln 2 \ln \left| \frac{1-(\rho^2-1)^{\frac{1}{2}} \tanh \eta}{2} \right| \right|_{\eta=\eta_0} + \frac{1}{\rho} \int_0^{\eta_0} \frac{d\eta}{\cos^2 \eta - k^2} \left[\cos \eta \ln \left| \frac{\cos \eta + k}{\cos \eta - k} \right| + k \ln \left| \frac{\cos^2 \eta - k^2}{\cos^2 \eta} \right| \right] \right\}. \end{aligned} \quad (\text{II-3})$$

The spurious infinities in the coefficient of ρ^4 will cancel out in the final result provided one is careful in taking limits in a consistent manner.

The coefficients of $\frac{1}{2}(\rho-5\rho^3)$ and $2\rho^4$ can be put explicitly in the form of Spence functions by yet another change of variable of integration, to $u = \tan(\eta/2)$. Then, after a short computation, we have

$$\delta'_2 = \rho^2 \left[4 + 3 \ln \rho \right]$$

$$+ \frac{1}{2}(\rho - 5\rho^3) \left[\text{Li}_2(\sqrt{\rho^2-1}-\rho) - \text{Li}_2(\rho-\sqrt{\rho^2-1}) + \text{Li}_2\left(\sqrt{\frac{\rho+1}{\rho-1}}\right) - \text{Li}_2\left(\sqrt{\frac{\rho-1}{\rho+1}}\right) \right. \\ \left. + \text{Li}_2\left(\sqrt{\frac{\rho-1}{\rho+1}} \frac{1}{\rho+\sqrt{\rho^2-1}}\right) - \text{Li}_2\left(\sqrt{\frac{\rho+1}{\rho-1}} (\sqrt{\rho^2-1}-\rho)\right) \right] \\ + \rho^4 \left[\frac{\pi^2}{6} + \ln^2|\rho(\rho-\sqrt{\rho^2-1})| + 2\text{Li}_2(1-\rho^2+\rho\sqrt{\rho^2-1}) - 2\text{Li}_2\left(\frac{\sqrt{\rho^2-1}}{\rho}\right) \right] \quad (\text{II-4})$$

$$= \rho^2 \left[4 + 3 \ln \rho \right] + \frac{1}{2}(\rho - 5\rho^3) \left[\frac{\pi^2}{4} + 2\text{Li}_2(1-\rho) - \frac{1}{2} \text{Li}_2(1-\rho^2) \right] + \rho^4 \left[\frac{\pi^2}{6} - \text{Li}_2\left(1-\frac{1}{\rho^2}\right) \right] \quad (\text{II-5})$$

Equation (II-5) is obtained by using (I-9b), (I-8a), and (I-6) of Appendix I for the coefficient of ρ^4 , which is the same as in equation (6). For the coefficient of $\frac{1}{2}(\rho - 5\rho^3)$ in (II-4), we use (I-11) to obtain

$$\text{Li}_2\left(\frac{1}{\rho}\right) - \text{Li}_2\left(\frac{-1}{\rho}\right) = \ln \rho \ln \left| \frac{\rho-1}{\rho+1} \right|, \quad (\text{II-6})$$

then (I-9) to obtain the coefficient in equation (6), and then (I-10) to obtain the coefficient in the (II-5).

The integral δ'_2 has been evaluated here assuming $\rho > 1$. Since δ'_2 is analytic at $\rho = 1$, and each term in (II-5) can be analytically continued across the point $\rho = 1$ to the region $0 < \rho < 1$, our result for δ'_2 , and hence eq. (6) for δ_2 , are valid for all positive finite values of the mass ratio

$$\rho = \frac{m}{M}.$$

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