Uniqueness of R-matrix Parameters
in the Analysis of Low Energy Neutron Cross Sections of Fissile Nuclei.[†]

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ABSTRACT

The existence of different sets of R-matrix parameters describing the same cross sections for a given spin-parity state is studied in the context of low energy neutron reactions in fissile elements. For two-fission channels and well-spaced resonances different sets of fission vectors are found describing essentially the same set of cross section curves.

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1. Introduction

The R-matrix theory^(1,2,3) has been applied widely to the analysis of neutron resonance reactions in fissile elements.⁽⁴⁻⁷⁾ In this context the uniqueness of representation of experimental data has been investigated by Michaudon,⁽⁸⁾ who has considered the role of missed levels. Using "mock cross sections" calculated from prescribed R-matrix parameters Lynn⁽⁹⁾ has pointed out the possible occurrence of quasi-resonances, while Auchampaugh,⁽¹⁰⁾ applying least-squares techniques to such data has found that more than one set of R-matrix parameters gives a good fit to the data. Auchampaugh stated that the phase angles obtained are not unique because the correct minimum of the least-squares fit is obscured by many false minima, all of which give good fits to the data.

A related and perhaps more basic problem is to ascertain whether, given mathematically defined fission, capture, and total cross sections for a prescribed spin-parity state one can predict, within R-matrix theory, the existence of different parameter sets describing the same cross section curves. Using such idealized data, with well-spaced resonances so that perturbation methods apply, the case of two fission channels is discussed here. Numerical examples are also presented showing that in the case of three interfering resonances two sets of fission channel width amplitudes can be obtained, while for the case of four resonances four different sets have been found. For N well-spaced resonances the number of sets of fission vectors, differing from the original, is at least $(N-1)(N-2)/2$. It may be expected that the same type of degeneracy occurs in the general case, to which perturbation methods do not apply.

2. Background

Using the standard notation of Lane and Thomas⁽³⁾ and Vogt,⁽¹¹⁾ the collision matrix elements can be written as

$$U_{cc'} = e^{i\Omega_c + i\Omega_{c'}} [\delta_{cc'} + iT_{cc'}] \quad (1)$$

The transition matrix T contains the contributions of the resonance terms in the form

$$T_{cc'} = \sum_{\lambda\lambda'} A_{\lambda\lambda'} \sqrt{\Gamma_{\lambda c} \Gamma_{\lambda' c'}} \quad (2)$$

where A is the Wigner level matrix, defined through its inverse:

$$A_{\lambda\lambda'}^{-1} = (E_\lambda - E) \delta_{\lambda\lambda'} - i\Gamma_{\lambda\lambda'} \quad (3)$$

and

$$\Gamma_{\lambda\lambda'} = \frac{1}{2} \sum_c \sqrt{\Gamma_{\lambda c} \Gamma_{\lambda' c}} \quad (4)$$

The level matrix approach, assuming a limited number of levels and without need of specifying the number of channels, was used by Vogt⁽⁴⁾ in neutron resonance analysis of fissile elements. An alternative approach, developed by Reich and Moore⁽⁶⁾ consists in a specialization of R-matrix theory, based on the partitioning of the channel matrix and the elimination of the radiative capture channels thus sidestepping the level matrix inversion; the channel matrix can be inverted for a small number of fission channels and an unspecified number of levels. These two approaches (in practice complementary) are in form equivalent, as it is always possible to associate a level matrix to the partitioned channel matrix of the Reich-Moore formulation.

An alternative expansion for T, or U, is obtainable by expressing the level matrix in terms of its eigenvalues^(1,12)

$$A_{\lambda\lambda'} = \sum_k \frac{S_{\lambda k} S_{\lambda'k}}{c_k - E} \quad (5)$$

where

$$c_k = \sum_{\lambda\lambda'} S_{\lambda k} (A_{\lambda\lambda'}^{-1} + \delta_{\lambda\lambda'} E) S_{\lambda'k} \quad (6)$$

are the complex eigenvalues of $A^{-1} + IE$, and S represents the complex orthogonal matrix of the related eigenvectors,⁽¹³⁾ performing the diagonalization (assuming that the eigenvalues of $A^{-1} + IE$ are not degenerate). The matrix T then acquires the form

$$T_{cc'} = \sum_k \frac{\tilde{Y}_{kc} \tilde{Y}_{kc'}}{c_k - E} \quad (7)$$

where the complex vector $\tilde{Y}$, defined by

$$\tilde{Y}_{kc} = \sum_{\lambda} S_{\lambda k} \sqrt{\Gamma_{\lambda c}} \quad , \quad (8)$$

represents a complex width amplitude, and

$$c_k - E \equiv \mu_k - i\nu_k - E \quad (9)$$

can be interpreted as a resonance denominator with resonance energy at μ_k and half width ν_k .

In general the eigenvalues c_k depend on the neutron energy, through the

elements $\Gamma_{\lambda\lambda}$, of (4), and so does the matrix S .⁽¹⁴⁾ For neutron reactions in fissile elements in the resolved resonance region, however, where

$$\Gamma_n \ll \Gamma_f$$

and the effects of the momentum dependence of Γ_n are negligible, one can assume that c_k and $S_{\lambda k}$ are energy independent to a very good approximation. The effects of the energy dependence of the complex poles c_k and of the residues have been discussed elsewhere,¹⁵ and, in the context of this investigation, in Appendix C and Section 5.

The ensuing cross section expansions are expressed as superpositions of Breit-Wigner lines containing interference terms,⁽¹⁶⁾ and they can be written in terms of line shape parameters which occur as linear coefficients of the symmetrical and asymmetrical line-shape functions. Thus for a reaction cross section $\sigma(x)$

$$\sigma^{(x)}(E) = \frac{c}{\sqrt{E}} \sum_J \sum_k \frac{G_k^{(x)} \nu_k + (\mu_k - E) H_k^{(x)}}{(\mu_k - E)^2 + \nu_k^2} \quad (10)$$

while for the total cross section

$$\sigma_n^T(E) = \frac{c}{\sqrt{E}} \sum_J \left\{ \sum_k \frac{(\alpha_k^0 \cos \omega - \beta_k^0 \sin \omega) v_k + (\mu_k - E) (\alpha_k^0 \sin \omega + \beta_k^0 \cos \omega)}{(\mu_k - E)^2 + v_k^2} + \frac{2g_J}{\sqrt{E}} (1 - \cos \omega) \right\} \quad (11)$$

where ω is the potential scattering phase shift, c a conversion constant, g_J the statistical factor, and k runs over the resonances of the same spin-parity state, J .

Through S , these parameters can be related explicitly to the original R -matrix parameters. For the total cross section one has:

$$\alpha_k^0 + i\beta_k^0 = \tilde{\gamma}_{kn}^2 = \sum_{\lambda\lambda'} S_{\lambda k} S_{\lambda' k}^* \sqrt{\Gamma_{\lambda n}^0 \Gamma_{\lambda' n}^0} \quad (12)$$

The expression for reaction line shape parameters can be written concisely in terms of auxiliary hermitean matrices:

$$X_{kk'}^{(x)} = \sum_{c(x)} \tilde{\gamma}_{kc} \tilde{\gamma}_{k'c}^* = \sum_{\lambda\lambda'} S_{\lambda k} S_{\lambda' k'}^* \sum_{c(x)} \gamma_{\lambda c} \gamma_{\lambda' c}^* \quad (13)$$

so that

$$G_k^{(x)} + iH_k^{(x)} = i \sum_{k'} \frac{X_{kk'}^{(n)} X_{kk'}^{(x)}}{c_{k'}^* - c_k} \quad (14)$$

where $*$ denotes complex conjugate, and

$$X_{kk'}^{(n)} = \tilde{\gamma}_{kn} \tilde{\gamma}_{k'n}^* \quad (15)$$

From the orthogonality of S and the hermiticity of X one verifies that

$$\sum_k \beta_k^0 = 0, \quad \sum_k H_k^{(x)} = 0 \quad (16)$$

and, using (6) and (13) that:

$$\sum_{(x)} (G_k^{(x)} + iH_k^{(x)}) = \alpha_k^0 + i\beta_k^0. \quad (17)$$

These parameters can again be considered as energy-independent, in the same sense as c_k , when the energy dependence of the $\Gamma_{\lambda\lambda}$ can be neglected.⁽¹⁷⁾ They can be utilized in data analysis⁽¹⁶⁾ where the simplicity of the cross section formulae allows to treat many levels at the same time, without specializing assumptions on the number of channels, and to evaluate broadening and resolution effects by standard techniques. The advantage of this expansion for a study of the uniqueness of R-matrix parameters is apparent: once the poles of the collision matrix are prescribed, the cross sections are uniquely defined by the coefficients $G_k^{(x)}$, $H_k^{(x)}$, or α_k^0 , β_k^0 , which enter linearly in the cross section formulae. On the other hand, as seen readily from the definition:

$$\sigma^{(x)}(E) \sim \sum_{c(x)} |U_{nc}|^2 \quad (18)$$

the R-matrix reduced reaction widths contribute nonlinearly to the cross section expansions, so that it is expectable that the same cross section values can be constructed from different sets of R-matrix parameters, while keeping invariant the poles of the collision matrix.

3. Multiplicity of R-matrix parameters

The problem, whether more than one set of R-matrix parameters represents the same cross section curves, can be treated by searching for different R-matrix parameter sets, giving the same eigenvalues of A^{-1} and the same line shape parameters (which are thus introduced as constants for the problem considered).

While the reduced complex neutron width $\tilde{\gamma}_{kn}$, giving the total cross section parameters [Eq. (12)], is kept constant, the matrices $X^{(x)}$ will in principle change when changing from one set of R-matrix parameters to another, but they will still satisfy Eq. (14), as a constraint, for the same values of $G^{(x)}$ and $H^{(x)}$.

The analysis of the general problem is laborious and is not reported here.

For the special case in which the spacings between levels are large compared with the off-diagonal elements of A^{-1} , a perturbation approximation (discussed in Appendix A) can be used to simplify the previous relations. Within perturbation, it is shown that for more than one fission channel the same line shape parameters may correspond to different sets of fission width amplitudes. The degree of this "degeneracy" is related to the number of resonances and to the number of channels. Because the condition

$$|\epsilon_{\lambda\lambda'}| = |\Gamma_{\lambda\lambda'}| / |E_{\lambda} - E_{\lambda'}| \ll 1, \quad (\lambda \neq \lambda') \quad (19)$$

which governs the perturbation approximation, is verified for instance for the low energy resonances ¹⁶⁾ of ²³⁹Pu, the results derived here, although derived from first-order perturbation, may reflect realistic cases.

As discussed in Appendix A, the line-shape parameters of Eq. (12) and (14) in this case can be expanded in successive powers of $\epsilon_{\lambda\lambda'}$. For the same spin-parity state, and utilizing the results of Appendix C ($\Gamma_{\lambda n} = \bar{\Gamma}_{\lambda n}$ of Eq. (C-5)), they reduce in first order to:

$$G_k^0 + iH_k^0 = \Gamma_{kn}^0 + 2i \sum_{\lambda \neq k} \Gamma_{\lambda k} \sqrt{\Gamma_{\lambda n}^0 \Gamma_{kn}^0} / (E_\lambda - E_k) \quad (20)$$

$$G_k^F + iH_k^F = \Gamma_{kn}^0 \left(\Gamma_{kf} / 2\Gamma_{kk} \right) + 2i \sum_{\lambda \neq k} \Gamma_{\lambda k}^F \sqrt{\Gamma_{\lambda n}^0 \Gamma_{kn}^0} / (E_\lambda - E_k) \quad (21)$$

$$G_k^Y = \Gamma_{kn}^0 \left(\Gamma_{kY} / 2\Gamma_{kk} \right) \quad (22)$$

$$H_k^Y = 0 \quad (23)$$

where

$$\Gamma_{\lambda k} = \Gamma_{\lambda k}^F + \sqrt{\Gamma_{kn} \Gamma_{\lambda n}} / 2 \quad (24)$$

$$\Gamma_{kn}^0 = 2g \Gamma_{kn} \sqrt{E} \quad (25)$$

and

$$2\Gamma_{kk} = \Gamma_{kn} + \Gamma_{kY} + \Gamma_{kf} \quad (26)$$

To this order of approximation, the condition that the symmetrical parameters $G^{(x)}$, $\theta^{(o)}$, remain constant means that the partial reaction widths,

$$\Gamma_{\lambda n}, \Gamma_{\lambda Y}, \Gamma_{\lambda f}$$

are invariant. Similarly, because to this order

$$\mu_k \approx E_k,$$

the resonance energies also do not vary.

The only quantities thus subject to changes are the orientations in channel space of the fission width amplitudes, $\gamma_{\lambda c}$, which induce changes in the off-diagonal elements of the matrix Γ^F :

$$\Gamma_{\lambda\lambda'}^F = \frac{1}{2} \sum_{c(F)} \gamma_{\lambda c} \gamma_{\lambda' c} = \frac{1}{2} \sqrt{\Gamma_{\lambda f} \Gamma_{\lambda' f}} \cos(\theta_{\lambda} - \theta_{\lambda'}) . \quad (27)$$

Here, and in the following for brevity $\gamma_{\lambda c}$, $\gamma_{\lambda n}$ will denote

$$\gamma_{\lambda c} = \sqrt{\Gamma_{\lambda c}} , \quad \gamma_{\lambda n} = \sqrt{\Gamma_{\lambda n}^0} . \quad (28)$$

Rigid body rotations, and mirror reflections in channel space, which do not alter the values of $\cos(\theta_{\lambda} - \theta_{\lambda'})$, are not considered here. Rotations in channel space may occur, however, which transform Γ^F into a matrix $\bar{\Gamma}^F$, sharing with Γ^F the diagonal elements, but differing in the off-diagonal ones.

The condition, that the asymmetrical parameters are constant, imposes constraints on $\bar{\Gamma}^F$. Using Eq. (21) one has:

$$\frac{H_k^F}{2\gamma_{kn}} = \sum_{\lambda \neq k} \bar{\Gamma}_{\lambda k}^F \gamma_{\lambda n} / (E_{\lambda} - E_k) = \sum_{\lambda \neq k} \Gamma_{\lambda k}^F \gamma_{\lambda n} / (E_{\lambda} - E_k) \quad (29)$$

for $k = 1, N$.

It is apparent that for $N = 2$ there is no other set of parameters, but the original, to satisfy Eq.(29), as there exists only one element, Γ_{12}^F , defining H_1 and H_2 . Hence the lowest number of resonances for which this "degeneracy" may occur is $N = 3$.

The answer to the problem of defining the possible $\bar{\Gamma}^F$ which satisfy Eq.(29), is directly connected with the number of fission channels describing the nuclear reactions. As discussed in the following section, the lowest number of fission channels inducing degeneracy is $N_c = 2$.

4. Properties of the matrix Γ^F

As discussed by Vogt⁽⁴⁾, for a set of N interfering levels at most N fission channels are sufficient to describe the cross sections, that is to define the matrix Γ^F . If the number of channels, N_c , is less than N , some of the elements of Γ^F depend on the others.

These considerations can be illustrated by writing the fission width amplitudes, $\gamma_{\lambda c}$, as elements of an $N \times N_c$ rectangular matrix, $C_{\lambda c}$, so that Γ^F can be regarded as a matrix product:

$$\Gamma^F = C \tilde{C} \quad (30)$$

(where $\tilde{C}$ is the transposed of C). If $N_c < N$, Γ^F has rank N_c ; the vanishing of the determinant of Γ^F and, for $N_c < N-1$, of the minors of order $N_c + 1$, leads to relations among the elements of Γ^F . If $N_c > N$ the rank of Γ^F is N , that is, the excess channels ($N_c - N$) are superfluous, and⁽⁴⁾ only N linear combinations of the N_c components $\gamma_{\lambda c}$ suffice to describe the cross sections.

The matrix $\bar{\Gamma}^F$, being obtained from rotations in channel space, as discussed in the previous section, is of the same rank as the matrix Γ^F . This is the basic condition which permits to solve Eq. (29).

For the particular case of one fission channel, ($N_c = 1$) all minors of order 2 vanish, and one has

$$(\Gamma_{\lambda\lambda'}^F)^2 = \Gamma_{\lambda\lambda}^F \Gamma_{\lambda'\lambda'}^F = (\bar{\Gamma}_{\lambda'\lambda}^F)^2 \quad (31)$$

for $\lambda, \lambda' = 1, N$. Since sign changes in $\Gamma_{\lambda\lambda}^F$ would induce changes in the values of H_λ and β_λ^0 , this case has no other solution, but

$$\bar{\Gamma}^F = \Gamma^F$$

The lowest number of fission channels for which degeneracy may occur is therefore $N_c = 2$. In this case one can write

$$C = \begin{pmatrix} \gamma_{1f} \cos \theta_1 & \gamma_{1f} \sin \theta_1 \\ \gamma_{2f} \cos \theta_2 & \gamma_{2f} \sin \theta_2 \\ . & . \\ . & . \\ \gamma_{Nf} & 0 \end{pmatrix} \quad (32)$$

in a frame of reference in which γ_N coincides with the polar axis. There are in this case $N-1$ independent angles θ , or, equivalently, $N-1$ independent elements of Γ^F , so that $(N-1)(N-2)/2$ elements of Γ^F are functions of the other $N-1$, and Γ^F has rank 2.

Thus, in the two-channel case, the problem can be solved by associating to Eq. (29) the conditions that the $(N-1)(N-2)/2$ independent minors of rank 3 of Γ^F vanish. From the practical point of view, it is convenient to use first Eq. (29) in order to express $N-1$ elements of Γ^F , in terms of the other $(N-1)(N-2)/2$, of the asymmetrical parameters H , and of Γ_n^O , Γ_f , and then to solve the $(N-1)(N-2)/2$ homogeneous equations for the minors. The result is a system of coupled cubic equations for $(N-1)(N-2)/2$ elements of Γ^F . Although complicated, these equations are preferable to those obtainable using first the conditions on the minors and solving Eq. (29) for $N-1$ elements of Γ^F . The equations in this case contain the unknowns in irrational form.

The analysis of Eq. (29) is facilitated by defining first a matrix M ,

$$M_{\lambda k} = (\bar{\Gamma}_{\lambda k}^F - \Gamma_{\lambda k}^F) / (E_\lambda - E_k) \quad (33)$$

which satisfies the conditions:

$$\sum_{\lambda=1}^N M_{\lambda k} \gamma_{\lambda n} = 0 \quad (k=1, N-1) \quad (34)$$

where the N-th condition is an identity, because of the constraint given by Eq. (16).

It is apparent that M is real, skewsymmetric, and related to the neutron width amplitudes $\gamma_{\lambda n}$. This matrix can be constructed from the vector

$$\vec{\gamma}_n = \begin{pmatrix} \gamma_{1n} \\ \gamma_{2n} \\ \vdots \\ \gamma_{Nn} \end{pmatrix} \quad (35)$$

as shown in Appendix B, through a real orthogonal transformation $\mathcal{O}$, operating on $\vec{\gamma}_n$. As shown in Appendix B, one can construct from $\mathcal{O}$, $(N-1)(N-2)/2$ independent matrices $M^{(jj')}$

$$M_{\lambda\lambda'}^{(jj')} = \mathcal{O}_{\lambda j} \mathcal{O}_{\lambda' j'} - \mathcal{O}_{\lambda j'} \mathcal{O}_{\lambda' j} \quad (36)$$

which satisfy Eq. (34) and are real and skewsymmetric, by combining in all possible ways the column indices, $j \neq j'$, for $j \neq N$, $j' \neq N$. The general solution of Eq. (34) can be written as a linear combination of the $(N-1)(N-2)/2$ matrices $M^{(jj')}$

$$M = \sum_{jj'} \mathcal{O}_{jj'} M^{(jj')} \quad (37)$$

The unknown coefficients $\mathcal{O}_{jj'}$ can be determined now from the conditions that $\bar{\Gamma}^F$ has rank N_c . In terms of M, $\bar{\Gamma}^F$ can be written as

$$\bar{\Gamma}_{\lambda\lambda'}^F = \Gamma_{\lambda\lambda'}^F + (E_\lambda - E_{\lambda'}) \sum_{jj'} \mathcal{O}_{jj'} M_{\lambda\lambda'}^{jj'} \quad (38)$$

For $N_C=2$ the vanishing of the minors of order 3 provide $(N-1)(N-2)/2$ coupled cubic equations for just as many unknowns.

This procedure makes it apparent that for two fission channels there exist at least $(N-1)(N-2)/2$ different sets of angles in channel space beyond the original, reproducing the same cross-section curves. Exceptional numerical situation may occur in which the real solutions of the cubic equations are unique. Using the relations (38) the system of equations to be solved appears less complicated than the one based on eliminating variables through Eq. (29), because these equations are homogeneous.

The special cases of $N=3$ and $N=4$ will be illustrated in Section 5.

5. Numerical examples

The "degeneracy" discussed in this paper has been encountered by G. Auchampaugh⁽¹⁰⁾ in a different context. From numerical experiments made with a least-square fitting program, using the Reich-Moore formalism to fit idealized cross section data, Auchampaugh reported the occurrence of "false solutions", that is of sets of fission width amplitudes differing from those which had been used to originate the "idealized data.". The author interprets this fact as an effect of the least-squares technique, i.e., devoid of physical content. The cases treated by Auchampaugh fall into the "first-order perturbation" class, and some of them have been considered here as examples to illustrate the previous considerations.

Independently, in the process of extracting R-matrix parameters from multilevel line shape parameters, with a technique reported earlier,⁽¹⁸⁾ the same type of degeneracy has been experienced. Thus, investigation of cases for which the perturbation approximation is valid is of some physical interest. It appears, in fact, that the perturbation approximation can cast some light on the general case, when the resonances are more closely spaced, and provide a guideline for the

procedures to use in that case. It appears unlikely that, apart from singular numerical situations, the general case will be less degenerate than the cases treated by perturbation. Numerical examples of three and four resonances with two-fission channels are presented in Tables I to IV.

This study of the degeneracy of the R-matrix parameters is based on the assumption that the line-shape parameters given in eqs. (12) and (14) can be considered as energy independent, and thus be used as constants for the problem. In Appendix C it is shown that this assumption is legitimate for fissile element cross sections in the low-energy region, as long as $\Gamma_{\lambda n}^0 \ll \Gamma_{\lambda f}^0$. For the numerical cases presented here this assumption has been tested also by comparing the cross section values calculated with the exact R-matrix formalism, with values obtained by the multilevel expansion of eqs. (10) and (11),⁽¹⁹⁾ as discussed in Appendix C.

The method employed in obtaining the "degenerate solutions" is based on the first order perturbation approximation, discussed in Appendix A. The convergence of this approximation depends on the values of the perturbation parameters $\epsilon_{\lambda\lambda}$, as defined in eq. (19), which varies from one case to another. When valid, this approximation simplifies the calculation, restricting the unknowns of the problem to the fission vectors angles, as opposed to the general case, in which in principle all pertinent R-matrix parameters would enter as unknowns.

Within perturbation, different angle sets are obtained, as roots of the system of equations (29), which reproduce exactly the first order perturbation values of the multilevel parameters.

For three resonances with two fission channels one can solve the equation

$$\det (\Gamma^F) = 0 , \quad (39)$$

where two of the off-diagonal elements of Γ^F (e.g., Γ_{12}^F , Γ_{23}^F) are expressed in terms of one (e.g., Γ_{13}^F), through Eq. (29). Omitting for simplicity the superscript, one has explicitly

$$\Gamma_{12} = (H_1 - \rho_{13} \Gamma_{13}) / \rho_{12} \quad (40)$$

$$\Gamma_{23} = (H_3 - \rho_{31} \Gamma_{13}) / \rho_{32} \quad (41)$$

where

$$\rho_{\lambda\lambda'} = \gamma_{\lambda n} \gamma_{\lambda' n} / (E_{\lambda'} - E_{\lambda}) \quad (\lambda \neq \lambda') \quad (42)$$

so that Eq. (39) becomes a cubic of the form

$$x^3 + ax^2 + bx + c = 0, \quad (43)$$

in which the unknown is $x = \cos \theta_1$, (with reference to γ_{3f}). It is perhaps relevant to note that the coefficients a, b, c, depend exclusively on the products

$$\Gamma_{\lambda n}^O \Gamma_{\lambda\lambda}^F \approx G_{\lambda}^F \gamma_{\lambda}, \quad (44)$$

on the asymmetries, H_{λ} , and on the spacings

$$E_{\lambda'} - E_{\lambda} \approx \mu_{\lambda'} - \mu_{\lambda}. \quad (45)$$

Thus, Eq. (43) encompasses all problems sharing the same sets of values (44), the same asymmetries and the same spacings. In particular, as shown in Tables I, II, and III, the values of $\cos \theta_{\lambda}$ are more sensitive to changes of $\Gamma_{\lambda n}^O \Gamma_{\lambda\lambda}^F$, than they are to the actual values of the spacings. Thus the solutions of Eq. (43) for

each specific case describe the totality of the cases which have in common the same sets of quantities specified above. Cases where

$$|x| > 1 \quad (46)$$

could be interpreted as providing values of $\cos\theta$, for a larger $\Gamma_{\lambda f}$, lower value of $\Gamma_{\lambda n}^0$, and same fission cross section asymmetries. This is however a type of degeneracy which is not of interest in this case, where $\Gamma_{\lambda f}$ is kept constant. Thus solutions not satisfying the condition

$$|x| \leq 1 \quad (47)$$

are here considered "spurious" and discarded. Cubic equations with only one real solution can, of course, occur. This case, appearing as a non degenerate case, is not discussed here.

The case of three solutions, of which one spurious and two identical is a case of uniqueness.

The same cases treated with the matrix M reduce to the solution of a quadratic equation. For $N_1 = 3$ there is only one matrix $M^{jj'}$ (as shown in Appendix B), that is

$$M = \alpha M^{(1,2)}, \quad (48)$$

where $\alpha = Q_{12}$. Thus the matrix $\bar{\Gamma}^F$ defined in Eq. (38) contains only the unknown constant α . In this case Eq. (44) reduces to a cubic equation of the form

$$\alpha^3 + A\alpha^2 + B\alpha = 0. \quad (49)$$

The solution $\alpha = 0$ obviously corresponds to the known solution, $\bar{\Gamma}^F$. If the roots of the quadratic in Eq. (49) are real, the values of α , subject again

to a limiting condition similar to (47) give additional solutions to the problem. Cases of complex solutions represent, as for the cubic, a "non degenerate" situation.

Numerical examples for two fission channels are presented in Tables I to IV. In each case, the initial R-matrix parameters

$$E_{\lambda}, \Gamma_{\lambda n}^0, \Gamma_{\lambda \gamma}, \Gamma_{\lambda f}, \theta_{\lambda}^0$$

are given together with the line shape parameters $G_{\lambda}^F, H_{\lambda}^F$ obtained in first order perturbation.

To ascertain the accuracy of the results three different numerical methods have been used in treating the cases of three resonances:

- (1) by solution of the cubic equation [see Eq. (43)];
- (2) by construction of the matrix M [see Eq. (49)];
- (3) by numerical searching routines for $\cos\theta_1$ and $\cos\theta_2$.

Four cases of degeneracy are illustrated in Table I. In each case the second set of angles $\theta_{\lambda}^{(1)}$ is not related in any obvious way to the original set of angles $\theta_{\lambda}^{(0)}$. Cases 1 to 4 share the values of $E_{\lambda}, \Gamma_{\lambda \gamma}$, and $\Gamma_{\lambda f}$; cases 1 and 2 have in common also the values of $\Gamma_{\lambda n}^0$ and differ in the initial values $\theta_{\lambda}^{(0)}$. Consequently the calculated values of G_{λ}^F are also in common while the values of H_{λ}^F differ in the two cases, leading to different values for the second set $\theta_{\lambda}^{(1)}$. The cases 3 and 4 having again $\Gamma_{\lambda n}^0$ in common follow the same pattern. Comparing Case 1 with Case 3, or Case 2 with Case 4, the effect of a change in $\Gamma_{\lambda n}^0$ on the "second solution $\theta_{\lambda}^{(1)}$ " is made apparent. In each of the cases considered in Table I, the cubic equation has three real solutions, one reproducing the original $\theta_{\lambda}^{(0)}$, one giving $\theta_{\lambda}^{(1)}$, and a third one which is "spurious"; at the same time, the quadratic equation has two real solutions, completely equivalent to the second solution of the cubic, and to the "spurious" solution of the cubic.

From Table I it is apparent that as an effect of the changes in orientation of the $\gamma_{\lambda f}$ in each case the new fission matrix $\overline{\Gamma}^F$ differs from the original matrix Γ^F .

Table II shows an example of "uniqueness." Using the same set of values E_λ , $\Gamma_{\lambda\gamma}$, $\Gamma_{\lambda f}$, as in Table I, but different $\theta_\lambda^{(0)}$ and $\Gamma_{\lambda n}^{(0)}$ the cubic equations in each case gave three real solutions of which one "spurious" and two identical.

Table III reproduces in case 6 one of the sets of Auchampaugh's.⁽¹⁰⁾ The results match, within the numerical accuracy, those of ref. (10). The same R-matrix parameters of this case have also been used with drastically different values of E_λ and reduced spacings. The case 7 of Table III shows that, while the H_λ are different from those of case 6, very little change occurs in the second angle set. The reason for the change in H_λ , from one case to the other, is here due to the sizeable change in the level spacings.

The case of four interfering resonances leads to equations that are complicated. For this case a different method⁽²⁰⁾ was used, numerically less accurate, but picturesque enough to be reported. After elimination of $\cos\theta_3$, curves of equal H^F (or equal β^0) are drawn in the phase plane θ_1, θ_2 . Intersections of the two curves represent the joint solutions, θ_1, θ_2 , to the problem. The four resonances of Table IV (case 8) allow four intersections, one of which reproduces the initial angle set, as shown in Fig. 1. As an illustration the iso- β curves for case 6 are shown in Fig. 2. The graphic solution in this case indicates also the "mirror-images" of the original and of the second angle set.

For all cases presented here the convergence of the perturbation approximation was tested by comparing the first order perturbation parameters G_λ , H_λ , shown in Tables I to IV, to the "exact" multilevel parameters derived successively from initial and degenerate angles sets. As discussed in Appendix C, the only approximation implied in this calculation⁽¹⁹⁾

is the assumption that the neutron widths, because small compared to the other partial widths, can be taken as energy independent.

The comparison between "exact" parameters and first order parameters shows, for each angle set, deviations of the second order or higher, which vary from case to case, depending on the values of the parameters $\epsilon_{\lambda\lambda}$. As measured by the maximum deviation found numerically, these differences range from .5% in case 1 to 1.5 % in case 8, where the largest ϵ has the values .24. The "exact" multilevel parameters pertaining to the initial and to the degenerate angle sets have also been compared among each other. As expectable, differences of the second order, or higher, occur, measuring the accuracy of the "degenerate solutions". That is, the degenerate angle set, which was determined numerically so as to reproduce exactly the first order parameters, reproduces the (initial) "exact" parameters well within the limits of the perturbation approximation. These deviations point to the fact that, in using a higher order of approximation (in which the $\cos \theta_\lambda$ appear non-linearly) other R-matrix parameters ought to be allowed to vary, if degenerate solution were to be investigated. From the point of view of data analysis, as long as the first order perturbation approximation is adequate ($|\epsilon_{\lambda\lambda}| \ll 1$) the degenerate angle sets, obtained with the numerical technique described in this context, generate curves which are in practice indistinguishable from the initial ones, as numerically they present deviations ranging from the fourth to the sixth significant figure.

6. Conclusions

The existence of multiple sets of the fission amplitudes for two fission channels is shown here for cases of well-spaced resonances using a perturbation expansion and the assumption that $\Gamma_{\lambda n} \approx \Gamma_{\lambda n}^0 \sqrt{E_\lambda/2g}$. The validity of this

this assumption is discussed in Appendix C. The lowest number of resonances for which these considerations apply is $N = 3$. The numerical solution of problems with more than four resonances are complicated, but it appears that for two-fission channels there exist in general at least $1 + (N-1)(N-2)/2$ different sets of fission vectors describing the same cross section curves. (The cases of "uniqueness" are characterized by relations among the angles in channel space, which amount to elimination of one (or more) variables, thus reducing the multiplicity of solutions.) The cases treated here correspond to rotations in channel space which rearrange the orientation of the individual fission amplitudes leaving unchanged the line-shape parameters describing the cross section curves.

The implication of these results for data analysis is straightforward: from the set of cross sections describing a "complete experiment" (that is correlated measurement of fission, scattering, capture, and total cross sections) it is not in general possible to resolve uniquely the channel components of the fission width, while the resonance energies and reaction widths, at least for well-spaced levels, are uniquely definable. It is therefore not surprising that computer programs for the least-squares analysis in terms of R-matrix parameters reproduce this occurrence.⁽¹⁰⁾

For closely spaced resonances, where perturbation does not apply, it may be surmised that further ambiguities (e.g., in the partial widths) may occur which are not discussed here.

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APPENDIX A: PERTURBATION APPROXIMATION

A perturbation approach for the inversion of the level matrix has been introduced by Wigner⁽¹⁾ and Thomas.⁽¹²⁾ Assuming the diagonal elements of A to be the dominant terms of A, this procedure gives an expansion of the collision matrix in ascending powers of the parameters $\epsilon_\lambda = 1/(E_\lambda - E - i\Gamma_{\lambda\lambda})$. An alternate approach^{15/} is based on the perturbation parameters

$$\epsilon_{\lambda\lambda'} = \Gamma_{\lambda\lambda'} / (E_{\lambda'} - E_\lambda) \quad (\lambda \neq \lambda') \quad (A-1)$$

which are used to expand the eigenvalues of A^{-1} and the matrix S for well-spaced levels, whenever

$$|\epsilon_{\lambda\lambda'}| \ll 1. \quad (A-2)$$

This method has been used in deriving eigenvalues and line shape parameters from existing R-matrix parameters^(15,16,21) and has also been reported by Garrison.⁽²²⁾ In this Appendix a more concise derivation of the same expansion is given which is more suited for computer use. The explicit energy dependence of A^{-1} is removed by introducing the matrix

$$B = A^{-1} + IE. \quad (A-3)$$

Indicating with D the diagonal matrix with elements c_λ , Eq. (6) can be

written as

$$\tilde{S} B S = D \quad (A-4)$$

Expanding B, S, and D in successive order of perturbation terms one has:

$$B_{\lambda k} = E_{\lambda} \delta_{\lambda k} - i\Gamma_{\lambda k} \quad (A-5)$$

$$S_{\lambda k} = S_{\lambda k}^{(0)} + S_{\lambda k}^{(1)} + S_{\lambda k}^{(2)} + \dots \quad (A-6)$$

$$D_{\lambda} = D_{\lambda}^{(0)} + D_{\lambda}^{(1)} + D_{\lambda}^{(2)} + \dots \quad (A-7)$$

The unknown terms of S and D can be determined, using (A-5), the orthogonality condition

$$S \tilde{S} = I \quad (A-8)$$

and comparing in (A-4) terms of the same order of magnitude. One obtains in the zero order:

$$D_{\lambda}^{(0)} = E_{\lambda} \quad (A-9)$$

and

$$S_{\lambda k}^{(0)} = \delta_{\lambda k} \quad (A-10)$$

In first order one obtains

$$D_{\lambda}^{(1)} = -i\Gamma_{\lambda\lambda} \quad (A-11)$$

and

$$\begin{aligned} S_{\lambda k}^{(1)} &= i\Gamma_{\lambda k} / (E_{\lambda} - E_k) \quad \text{for } \lambda \neq k \\ &= 0 \quad \text{for } \lambda = k \end{aligned} \quad (\text{A-12})$$

In the second order mixed terms occur, and the result is

$$D_{\lambda}^{(2)} = \sum_{k \neq \lambda} \Gamma_{\lambda k}^2 / (E_k - E_{\lambda}) \quad (\text{A-13})$$

$$S_{\lambda k}^{(2)} = - \frac{1}{E_{\lambda} - E_k} \left[\frac{\Gamma_{\lambda\lambda} - \Gamma_{kk}}{E_{\lambda} - E_k} \Gamma_{\lambda k} + \sum_{\substack{j \neq \lambda \\ j \neq k}} \frac{\Gamma_{\lambda j} \Gamma_{jk}}{E_j - E_k} \right] \quad \text{for } \lambda \neq k \quad (\text{A-14})$$

and

$$S_{\lambda\lambda}^{(2)} = \frac{1}{2} \sum_{k \neq \lambda} \frac{\Gamma_{\lambda k}^2}{(E_{\lambda} - E_k)^2} \quad (\text{A-15})$$

Finally, the next order correction to the eigenvalue and to S become

$$D_{\lambda}^{(3)} = - i \sum_{k \neq \lambda} \Gamma_{k\lambda} S_{k\lambda}^{(2)} \quad (\text{A-16})$$

$$S_{\lambda k}^{(3)} = \frac{1}{E_{\lambda} - E_k} \left[i \sum_j \Gamma_{\lambda j} S_{jk}^{(2)} + S_{\lambda k}^{(1)} D_k^{(2)} + S_{\lambda k}^{(2)} D_k^{(1)} \right] \quad (\text{A-17})$$

and so on, by further successive approximations. To the second order the real and imaginary parts of the eigenvalues become

$$\mu_k = E_k + \sum_{\lambda \neq k} \frac{\Gamma_{\lambda k}^2}{E_\lambda - E_k} \quad (A-18)$$

$$v_k = \Gamma_{kk} + \sum_{\lambda \neq k} S_{\lambda k}^{(2)} \Gamma_{\lambda k} \quad (A-19)$$

where the invariance of the trace of A^{-1} is readily verified.

Thus if the parameters $\epsilon_{\lambda\lambda}$ are kept of the same order, the relative correction to the resonance energy tends to become less appreciable the larger the resonance energy, while the relative change in the half width does not necessarily decrease.

A convenient notation for the complex orthogonal matrix $S^{(13)}$ is also:

$$S = Q e^{i\eta} \quad (A-20)$$

where Q is real orthogonal and η real skew-symmetric. These matrices can be derived with the same type of expansion as in (A-7), using the fact that in D and S even order terms are real and odd order terms are pure imaginary.

Writing

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + \dots, \quad (A-21)$$

and expanding

$$e^{i\eta} = I + i\eta - \frac{1}{2} \eta^2 - \frac{i}{6} \eta^3 \dots \quad (A-22)$$

one may write:

$$\begin{aligned} S = Q e^{i\eta} = & Q^{(0)} + Q^{(1)} + Q^{(0)} i\eta + \left[(Q^{(2)} - \frac{1}{2} Q^{(0)} \eta^2) + i Q^{(1)} \eta \right] \\ & + \left[(Q^{(3)} - \frac{1}{2} Q^{(1)} \eta^2) + i (Q^{(2)} \eta - \frac{1}{6} Q^{(0)} \eta^3) \right] + \dots \quad (A-23) \end{aligned}$$

Comparing term by term this expansion with (A-7), one obtains successively

$$Q_{\lambda k}^{(0)} = \delta_{\lambda k} \quad (A-24)$$

$$\eta_{\lambda k} = (1 - \delta_{\lambda k}) \Gamma_{\lambda k} / (E_{\lambda} - E_k) \quad (A-25)$$

$$Q_{\lambda k}^{(1)} = 0 \quad (A-26)$$

$$(Q^{(2)} - \frac{1}{2} \Gamma^2)_{\lambda k} = S_{\lambda k}^{(2)} \quad (A-27)$$

and so on. Thus, $\eta_{\lambda k}$ is identical to the expansion parameter $\epsilon_{\lambda k}$ defined in Eq. (A-3); while the first correction term to $Q^{(0)} = I$ is of the second order.

One may also verify that $Q^{(2)}$ is a skew-symmetric matrix, as expectable, since Q is a real orthogonal matrix.

Using the previous results to the second order, the matrix $X^{(x)}$ of Eq. (13) becomes:

$$\begin{aligned} X_{kk'}^{(x)} &= 2 \Gamma_{kk'}^{(x)} + 2 \sum_{\lambda} \left(S_{\lambda k}^{(1)} \Gamma_{\lambda k'}^{(x)} - S_{\lambda k'}^{(1)} \Gamma_{\lambda k}^{(x)} \right) \\ &+ 2 \sum_{\lambda} \left(S_{\lambda k}^{(2)} \Gamma_{\lambda k'}^{(x)} + S_{\lambda k'}^{(2)} \Gamma_{\lambda k}^{(x)} \right) - 2 \sum_{\lambda \lambda'} S_{\lambda k}^{(1)} S_{\lambda' k'}^{(1)} \Gamma_{\lambda \lambda'}^{(x)} \end{aligned} \quad (A-28)$$

where

$$2 \Gamma_{\lambda \lambda'}^{(x)} = \sum_{c(x)} \gamma_{\lambda c} \gamma_{\lambda' c} \quad (A-29)$$

Separating in $X^{(x)}$ the terms of increasing order, one has

$$X_{kk'}^{(x)(0)} = 2 \Gamma_{kk'}^{(x)} \quad (A-30)$$

$$X_{kk'}^{(x)(1)} = 2i \sum_{\lambda \neq k} \frac{\Gamma_{\lambda k} \Gamma_{\lambda k'}^{(x)}}{E_{\lambda} - E_k} - 2i \sum_{\lambda \neq k'} \frac{\Gamma_{\lambda k'} \Gamma_{\lambda k}^{(x)}}{E_{\lambda} - E_{k'}} \quad (A-31)$$

$$\begin{aligned} X_{kk'}^{(x)(2)} &= 2 \sum_{\lambda} \left(S_{\lambda k}^{(2)} \Gamma_{\lambda k'}^{(x)} + S_{\lambda k'}^{(2)} \Gamma_{\lambda k}^{(x)} \right) \\ &+ 2 \sum_{\lambda \lambda'} S_{\lambda k}^{(1)} S_{\lambda' k'}^{(1)} \Gamma_{\lambda \lambda'}^{(x)} \end{aligned} \quad (A-32)$$

where the hermiticity of $X^{(x)}$ is apparent.

From the matrices $X^{(x)}$, $X^{(n)}$, the parameters of Eq. (14) are derived to the second order:

$$\begin{aligned} G_k^{(x)} + iH_k^{(x)} &= \frac{1}{2v_k} \left\{ X_{kk}^{n(0)} \left(X_{kk}^{(x)(0)} + X_{kk}^{(x)(2)} \right) + X_{kk}^{n(2)} X_{kk}^{(x)(0)} \right\} \\ &+ i \sum_{k' \neq k} \frac{1}{E_{k'} - E_k} \left\{ X_{kk'}^{n(0)} \left(X_{kk'}^{(x)(0)} + X_{kk'}^{(x)(1)} \right) + X_{kk'}^{n(1)} X_{kk'}^{(x)(0)} \right\} \\ &+ \sum_{k' \neq k} \frac{\Gamma_{kk} + \Gamma_{k'k'}}{(E_{k'} - E_k)^2} X_{kk'}^{n(0)} X_{kk'}^{(x)(0)}, \end{aligned} \quad (A-33)$$

where explicitly

$$H_k^{(x)} = 2i \sum_{k' \neq k} \frac{\sqrt{\Gamma_{kn}^0 \Gamma_{k'n}^0} \Gamma_{kk'}^{(x)}}{E_{k'} - E_k} \quad (A-34)$$

and

$$\sum_k H_k^{(x)} = 0 \quad (A-35)$$

As noted before, the symmetrical parameter contains even order terms and the asymmetrical parameters contains odd order terms, so that the second order approximation for H_k coincides with the first order approximation. For radiative capture, where

$$x_{\lambda\lambda'}^{(y)(0)} = 2 \Gamma_{\lambda\lambda'}^y = \Gamma_{\lambda y} \delta_{\lambda\lambda'}, \quad (A-36)$$

the expansion (A-33) to the same order is real:

$$H_k^y = 0. \quad (A-37)$$

The total cross section parameters of Eq. (12) are similarly obtained as

$$\begin{aligned} \alpha_k^0 + i\beta_k^0 &= \tilde{\gamma}_{kn}^2 = i_{kn}^0 + \sum_{\lambda\lambda'} S_{\lambda k}^{(1)} S_{\lambda' k}^{(1)} \sqrt{\Gamma_{\lambda n}^0 \Gamma_{\lambda' n}^0} \\ &+ 2 \sum_{\lambda} S_{\lambda k}^{(2)} \sqrt{\Gamma_{kn}^0 \Gamma_{\lambda n}^0} \\ &+ 2 \sum_{\lambda} S_{\lambda k}^{(1)} \sqrt{\Gamma_{kn}^0 \Gamma_{\lambda n}^0} \end{aligned} \quad (A-38)$$

where

$$\beta_k^0 = 2i \sum_{\lambda \neq k} \frac{\Gamma_{\lambda k}}{E_{\lambda} - E_k} \sqrt{\Gamma_{\lambda n}^0 \Gamma_{kn}^0} \quad (A-39)$$

is of the first order. It should be noted that, to this order, the asymmetry of the scattering cross section does not vanish, although small compared to β^0 :

$$H_k^{(n)} = \sum_{\lambda \neq k} \frac{\sqrt{\Gamma_{\lambda n} \Gamma_{kn}}}{E_{\lambda} - E_k} \sqrt{\Gamma_{kn}^0 \Gamma_{\lambda n}^0} \quad (A-40)$$

For fissile elements in the resolved energy region, where in practical cases

$$\Gamma_{\lambda n} \ll \Gamma_{\lambda f} \quad (A-41)$$

the matrices $X^{(n)(1)}$, $X^{(n)(2)}$ in (A-33) and the second order terms in (A-38) are smaller than the other terms, so that a first order approximation is often sufficient, as long as $|\epsilon_{\lambda\lambda}| \ll 1$. The first order terms of (A-33) and (A-38) were used in the analysis presented in this paper (Eq. (20), (21), (22)). In deriving (A-33) the "resonance denominator"

$$\frac{1}{c_{k'}^* - c_k} \approx \frac{1}{E_{k'} - E_k + i(\Gamma_{kk} + \Gamma_{k'k'})} \quad (A-42)$$

was also expanded for consistency:

$$\frac{1}{c_{k'}^* - c_k} \approx \frac{1}{E_{k'} - E_k} - i \frac{\Gamma_{kk} + \Gamma_{k'k'}}{(E_{k'} - E_k)^2} + \dots \quad (A-43)$$

Results obtained retaining the form (A-42) in the previous expansions have also been given by Harris.⁽²³⁾

APPENDIX B: CONSTRUCTION OF THE MATRIX M

Examples for Three, Four and Five Interfering Resonances

Starting from the vector

$$\vec{y}_n = \begin{pmatrix} y_{1n} \\ \vdots \\ y_{Nn} \end{pmatrix} \quad (B-1)$$

as defined in Eq. (35), a real orthogonal matrix $\mathcal{O}$ can be constructed by standard procedures,⁽¹³⁾ which has the following properties:

$$\sum_{\lambda} \mathcal{O}_{\lambda j} y_{\lambda n} = \rho_N \delta_{jN} \quad (B-2)$$

with

$$\rho_N^2 = \sum_{k=1}^N \Gamma_{kn}^0 \quad ; \quad (B-3)$$

any column of $\mathcal{O}$ can be taken as preferred, the results differing only by a permutation of the column indices. One constructs $\mathcal{O}$ so that

$$\sum_{\lambda} \mathcal{O}_{\lambda k} y_{\lambda} = 0 \quad \text{for } k < N \quad (B-4)$$

where now y_{λ} stands for $y_{\lambda n}$.

Let $\vec{r}_k$ represent the k-th vector column, so that

$$(\vec{r}_k \cdot \vec{y}_n) = 0 \quad (B-5)$$

and

$$(\vec{r}_k \cdot \vec{r}_{k'}) = \delta_{kk'} \quad (B-6)$$

For $k = 1$ one chooses 2 components, obtaining

$$\vec{r}_1 = \begin{pmatrix} \gamma_2/\rho_2 \\ -\gamma_1/\rho_2 \\ 0 \end{pmatrix} \quad (B-7)$$

For column-2 one chooses three components, giving

$$\vec{r}_2 = \begin{pmatrix} \gamma_1\gamma_3/\rho_2\rho_3 \\ \gamma_2\gamma_3/\rho_2\rho_3 \\ -\rho_2/\rho_3 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \quad (B-8)$$

where

$$\rho_k^2 = \sum_{\lambda=1}^k \Gamma_{\lambda n}^0 \quad (B-9)$$

proceeding successively, in a manner analogous to the method of obtaining an orthogonal basis. The $(N-1)$ -th column thus has all the element different from zero and the N -th column is given by

$$\vec{r}_N = \begin{pmatrix} \gamma_1/\rho_N \\ \gamma_2/\rho_N \\ \vdots \\ \gamma_N/\rho_N \end{pmatrix} \quad (B-10)$$

For $N = 3$ one has explicitly

$$G^{(3)} = \begin{pmatrix} \gamma_2/\rho_2 & \gamma_1\gamma_3/\rho_2\rho_3 & \gamma_1/\rho_3 \\ -\gamma_1/\rho_2 & \gamma_2\gamma_3/\rho_2\rho_3 & \gamma_2/\rho_3 \\ 0 & -\rho_2/\rho_3 & \gamma_3/\rho_3 \end{pmatrix} \quad (B-11)$$

for $N = 4$

$$Q^{(4)} = \begin{pmatrix} \gamma_2/\rho_2 & \gamma_1\gamma_3/\rho_2\rho_3 & \gamma_1\gamma_4/\rho_3\rho_4 & \gamma_1/\rho_4 \\ -\gamma_1/\rho_2 & \gamma_2\gamma_3/\rho_2\rho_3 & \gamma_2\gamma_4/\rho_3\rho_4 & \gamma_2/\rho_4 \\ 0 & -\rho_2/\rho_3 & \gamma_3\gamma_4/\rho_3\rho_4 & \gamma_3/\rho_4 \\ 0 & 0 & -\rho_3/\rho_4 & \gamma_4/\rho_4 \end{pmatrix} \quad (B-12)$$

for $N = 5$

$$Q^{(5)} = \begin{pmatrix} \gamma_2/\rho_2 & \gamma_1\gamma_3/\rho_2\rho_3 & \gamma_1\gamma_4/\rho_3\rho_4 & \gamma_1\gamma_5/\rho_4\rho_5 & \gamma_1/\rho_5 \\ -\gamma_1/\rho_2 & \gamma_2\gamma_3/\rho_2\rho_3 & \cdot & \cdot & \cdot \\ 0 & -\rho_2/\rho_3 & \cdot & \cdot & \cdot \\ 0 & 0 & -\rho_3/\rho_4 & \gamma_4\gamma_5/\rho_4\rho_5 & \gamma_4/\rho_5 \\ 0 & 0 & 0 & -\rho_4/\rho_5 & \gamma_5/\rho_5 \end{pmatrix} \quad (B-13)$$

For $j \neq j'$ and $j < N$, $j' < N$ one can construct readily $(N-1)(N-2)/2$ skew-symmetric matrices of the form:

$$M_{\lambda\lambda'}^{(jj')} = Q_{\lambda j} Q_{\lambda' j'} - Q_{\lambda j'} Q_{\lambda' j} \quad (B-14)$$

For $N = 3$, $(N-1)(N-2)/2 = 1$: there is only one matrix

$$M^{(1,2)} = \begin{pmatrix} 0 & \gamma_3/\rho_3 & -\gamma_2/\rho_3 \\ -\gamma_3/\rho_3 & 0 & \gamma_1/\rho_3 \\ \gamma_2/\rho_3 & -\gamma_1/\rho_3 & 0 \end{pmatrix} \quad (B-15)$$

For $N = 4$, $(N-1)(N-2)/2 = 3$, and one has

$$M^{(1,2)} = \begin{pmatrix} 0 & \gamma_3/\rho_3 & -\gamma_2/\rho_3 & 0 \\ -\gamma_3/\rho_3 & 0 & \gamma_1/\rho_3 & 0 \\ \gamma_2/\rho_3 & -\gamma_1/\rho_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (B-16)$$

$$M^{(1,3)} = \begin{pmatrix} 0 & \gamma_4 \rho_2 / \rho_3 \rho_4 & \gamma_2 \gamma_3 \gamma_4 / \rho_2 \rho_3 \rho_4 & -\gamma_2 \rho_3 / \rho_2 \rho_4 \\ -\rho_2 \gamma_4 / \rho_3 \rho_4 & 0 & -\gamma_1 \gamma_3 \gamma_4 / \rho_2 \rho_3 \rho_4 & \gamma_1 \rho_3 / \rho_2 \rho_4 \\ -\gamma_2 \gamma_3 \gamma_4 / \rho_2 \rho_3 \rho_4 & \gamma_1 \gamma_3 \gamma_4 / \rho_2 \rho_3 \rho_4 & 0 & 0 \\ \gamma_2 \rho_3 / \rho_2 \rho_4 & -\gamma_1 \rho_3 / \rho_2 \rho_4 & 0 & 0 \end{pmatrix} \quad (B-17)$$

$$M^{(2,3)} = \begin{pmatrix} 0 & 0 & \gamma_1 \gamma_4 / \rho_2 \rho_4 & -\gamma_1 \gamma_3 / \rho_2 \rho_4 \\ 0 & 0 & \gamma_2 \gamma_4 / \rho_2 \rho_4 & -\gamma_2 \gamma_3 / \rho_2 \rho_4 \\ -\gamma_1 \gamma_4 / \rho_2 \rho_4 & -\gamma_2 \gamma_4 / \rho_2 \rho_4 & 0 & \rho_2 / \rho_4 \\ \gamma_1 \gamma_3 / \rho_2 \rho_4 & \gamma_2 \gamma_3 / \rho_2 \rho_4 & -\rho_2 / \rho_4 & 0 \end{pmatrix} \quad (B-18)$$

For $N = 5$ there are 6 independent matrices $M^{jj'}$ which can be derived in a similar manner from $\mathcal{O}^{(5)}$.

It is obvious that

$$\sum_{\lambda} M_{k\lambda}^{(jj')} \gamma_{\lambda n} = 0. \quad (B-19)$$

It may be added that the matrices $M^{(jj')}$ for different pairs (jj') do not commute. For a given jj' pair they satisfy the recurrence relation

$$M^3 = -M, \quad (B-20)$$

so that

$$e^{i\alpha M} = I + (i \operatorname{Sh} \alpha) M - (\operatorname{Ch} \alpha - 1) M^2 \quad (B-21)$$

also satisfies eq. (B-19).

APPENDIX C:

EFFECTS OF THE ENERGY DEPENDENCE OF NEUTRON WIDTH IN THE LOW ENERGY RESONANCES OF FISSILE ELEMENTS

The condition:

$$\Gamma_{\lambda n}^0 \ll \Gamma_{\lambda f} \quad (C-1)$$

holds for most of the neutron resonances in the low energy region, so that the energy dependence of the neutron widths can be treated as a perturbation of an energy-independent level matrix, A^0 . The exact inverse level matrix is split into two terms

$$A^{-1} = (A^0)^{-1} + i \Gamma^{(1)} \quad (C-2)$$

where

$$(A^0)^{-1}_{\lambda\lambda'} = (E_\lambda - E - i\Gamma_{\lambda f}/2)\delta_{\lambda\lambda'} - i\Gamma_{\lambda\lambda'} - i\sqrt{\Gamma_{\lambda n}\bar{\Gamma}_{\lambda'n}}/2 \quad (C-3)$$

and the momentum dependent matrix $\Gamma^{(1)}$ is defined as

$$\Gamma^{(1)}_{\lambda\lambda'} = \sqrt{\Gamma_{\lambda n}\bar{\Gamma}_{\lambda'n}} (k/\sqrt{k_\lambda k_{\lambda'}} - 1)/2 \quad (C-4)$$

Here

$$\bar{\Gamma}_{\lambda n} = \Gamma_{\lambda n}^0 k_\lambda/2g, \quad k_\lambda = \sqrt{E_\lambda} \quad (C-5)$$

and $k = \sqrt{E}$. The elements of $\Gamma^{(1)}$ are slowly varying functions of E , and small compared with the elements of Γ^F in the energy region of the resonances in consideration, as long as the condition (C-1) holds.

One may note that at the energy

$$E^* = E_\lambda + .4g \Gamma_{\lambda f} k_\lambda / \Gamma_{\lambda n}^0 + 4g^2 (\Gamma_{\lambda f} / \Gamma_{\lambda n}^0)^2 10^{-2} \quad (C-6)$$

the diagonal elements of $\Gamma^{(1)}$ become of the order of $\Gamma^R/10$. As long as the condition (C-1) holds, E^* falls far from E_λ , where the contribution of the resonance line to the cross section becomes very small. As an example, for values of $\Gamma_{\lambda n}^0 = .8$ mV, $E_\lambda = 20$ eV of Table I, and with $g = .5$, $E^* \approx 200$ eV. Under these conditions it is legitimate to treat $\Gamma^{(1)}$ as a perturbation to A^0 , in the energy range dominated by the resonances in consideration, and for which (C-1) is valid. Cases for which

$$\Gamma_{\lambda n}^0 \approx \Gamma_{\lambda f} \quad (C-7)$$

occur rarely in the low energy region (e.g. in Pu^{239}) and are characterized by low $\Gamma_{\lambda f}$ values rather than by large $\Gamma_{\lambda n}^0$. Although smaller than those of the surrounding resonances, their contribution can be described by adding to the resonance terms a smoothly varying background, as shown below.

Indicating with S^0 and c^0 the eigenvectors and eigenvalues of A^0 , and with S and c those of A , one has:

$$(A^0)^{-1} S^0 = S^0 c^0 \quad (C-8)$$

$$A^{-1} S = S c \quad (C-9)$$

Following standard procedures, S can be related to S^0 through an expansion of the form

$$S_{\lambda k} = \sum_m Q_{km} S_{\lambda m}^0 \quad (C-10)$$

where the matrix Q is to be determined. Denoting with

$$c_k = c_k^0 + \Delta c_k^{(1)} \quad (C-11)$$

and using the orthogonality of S^0 , one obtains from Eq. (C-9) the system of equations for the elements of Q :

$$Q_{kj} (c_j^0 - c_k^0) = Q_{kj} \Delta c_k + i \sum_m Q_{km} \rho_{jm}. \quad (C-12)$$

The matrix elements

$$\rho_{jm} = \sum_{\lambda\lambda'} S^0_{\lambda j} \Gamma^{(1)}_{\lambda\lambda'} S^0_{\lambda'm} \quad (C-13)$$

are linear functions of k and can be treated as perturbation parameters under the conditions discussed above, so that Q and Δc_k can be expanded in ascending powers of the parameters ρ_{jm} , in a manner similar to the one used in Appendix A.

In zero order one has immediately:

$$Q_{kj}^{(0)} (c_j^0 - c_k^0) = 0 \quad (C-14)$$

so that:

$$Q_{kj}^{(0)} = Q_{kk}^{(0)} \delta_{kj} \quad (C-15)$$

To the first order:

$$\Delta c_k^{(1)} = -i \rho_{kk} \quad (C-16)$$

$$Q_{kj}^{(1)} (1 - \delta_{kj}) = i \rho_{jk} Q_{kk}^{(0)} / (c_j^0 - c_k^0) \quad (C-17)$$

and using the orthogonality of S :

$$Q_{kk}^{(0)} = 1. \quad (C-18)$$

In first order, therefore, the eigenvalues and eigenvectors of A become slowly varying function of the momentum, the k dependent part remaining small under the conditions discussed above:

$$S_{\lambda k} \approx S_{\lambda k}^{\circ} + i \sum_{j \neq k} S_{\lambda j}^{\circ} \sum_{\lambda' \lambda''} S_{\lambda' j}^{\circ} \Gamma_{\lambda' \lambda''}^{(1)} S_{\lambda'' k}^{\circ} / (c_j^{\circ} - c_k^{\circ}) \quad (C-19)$$

$$c_k = c_k^{\circ} - i \sum_{\lambda \lambda'} S_{\lambda k}^{\circ} \Gamma_{\lambda \lambda'}^{(1)} S_{\lambda' k}^{\circ} \quad (C-20)$$

The complex partial width amplitudes, defined in eq. (8) can similarly be written as linear functions of k,

$$\tilde{\gamma}_{kc} = \tilde{\gamma}_{kc}^{\circ} + i \sum_{\lambda} \sum_{j \neq k} S_{\lambda j}^{\circ} \gamma_{\lambda c} \sum_{\lambda' \lambda''} S_{\lambda' j}^{\circ} \Gamma_{\lambda' \lambda''}^{(1)} S_{\lambda'' k}^{\circ} / (c_j^{\circ} - c_k^{\circ}) \quad (C-21)$$

where $\tilde{\gamma}_{kc}^{\circ}$ is energy independent. These relations are more general than those obtained neglecting the off-diagonal contributions of $\Gamma_{\lambda n}$ in $(A^{\circ})^{-1}$ ⁽¹⁵⁾, or neglecting altogether $\Gamma_{\lambda n}$ in $(A^{\circ})^{-1}$ ⁽²³⁾. The evaluation of the line shape parameters, using the relations given in eq. (14) has the disadvantage of introducing the momentum dependence of c_j in the denominators. This can be avoided by redefining through (C-20), the complex roots of c_j in the momentum plane. One may write:

$$c_j - E = (k_j^{(1)} - k) (k_j^{(2)} + k) \quad (C-22)$$

where:

$$k_j^{(1)} k_j^{(2)} = c_j^{\circ} + i \sum_{\lambda \lambda'} S_{\lambda j}^{\circ} S_{\lambda' j}^{\circ} \sqrt{\Gamma_{\lambda n} \Gamma_{\lambda' n}} / 2 \quad (C-23)$$

$$k_j^{(1)} - k_j^{(2)} = -i \sum_{\lambda \lambda'} S_{\lambda j}^{\circ} S_{\lambda' j}^{\circ} \sqrt{\Gamma_{\lambda n} \Gamma_{\lambda' n}} / 2 \sqrt{k_{\lambda} k_{\lambda'}} \quad (C-23')$$

When (C-1) holds the corrective terms in (C-23) and (C-23') are always small and can be disregarded, except in cases of closely spaced resonances, when the level matrix eigenvalues c_j^0 are degenerate. In this case, the correction shown in (C-20) may remove such a degeneracy. (19)

For cases in which $\Gamma_{\lambda f} \approx \Gamma_{\lambda n}^0$ the k dependence of c_j can be removed in the cross section expansions by deriving relations in form similar to those of the eqs. (10) and (11), where however the relation of the line shape parameters to the R-matrix parameters is algebraically more complicated than the one shown in eqs. (12) and (14).

In the notations of eq. (7) one has:

$$T_{nc} = \sum_j \left(\frac{1}{k_j^{(1)} - k} + \frac{1}{k_j^{(2)} + k} \right) \frac{\tilde{Y}_{jn}}{k_j^{(1)} + k_j^{(2)}} \quad (C-24)$$

where $\tilde{Y}_{jn}$, $\tilde{Y}_{jc}$ are momentum dependent, as in (C-21).

In evaluating the generic reaction cross section $\sigma(x)$, one can further separate resonance contributions to obtain:

$$\begin{aligned} \sum_{c(x)} T_{nc} T_{nc}^* &= \sum_{jj'} \frac{X_{jj'}^n X_{jj'}^{(x)}}{(k_j^{(1)} + k_j^{(2)})(k_{j'}^{(1)*} + k_{j'}^{(2)*})} \\ &\left\{ \frac{(k_j^{(1)} + k)}{\eta_j - E} \left(\frac{1}{k_{j'}^{(1)*} - k_j^{(1)}} + \frac{1}{k_{j'}^{(2)*} + k_j^{(1)}} \right) \right. \\ &+ \frac{k_{j'}^{(1)*} + k}{\eta_{j'}^* - E} \left(\frac{1}{k_j^{(2)} + k_{j'}^{(1)*}} + \frac{1}{k_j^{(1)} - k_{j'}^{(1)*}} \right) \\ &\left. + B_{jj'}(k) \right\} \quad (C-25) \end{aligned}$$

where the momentum dependent matrices X^n , $X^{(x)}$ are defined as in eq. (13) through $\tilde{\gamma}_n$ and $\tilde{\gamma}_c$, and

$$\eta_j = (k_j^{(1)})^2 \quad (C-26)$$

The matrix

$$\begin{aligned} B_{jj'}(k) = & \frac{1}{k_j^{(2)} + k} \frac{1}{k_j^{(2)} + k_{j'}^{(1)*}} \\ & + \frac{1}{k_j^{(2)*} + k} \frac{1}{k_{j'}^{(2)*} + k_j^{(1)}} \\ & + \frac{1}{k_j^{(2)} + k} \frac{1}{k_{j'}^{(2)*} + k} \end{aligned} \quad (C-27)$$

is a slowly varying function of k . The resonance denominators are here expressed in terms of the energy independent complex "resonance energies" η_j , while the coefficients of the resonance terms are slowly varying function of E , for all values of E , except near a threshold. In these notations one can write

$$G_j^{(x)} + iH_j^{(x)} = \bar{G}_j^{(x)} + i\bar{H}_j^{(x)} + g_j^{(1)}k + g_j^{(2)}k^2 + \dots \quad (C-28)$$

where the barred values are energy independent. Because the corrective terms to $\tilde{\gamma}_k$, as in (C-21) are usually small, even when the condition (C-1) is not satisfied, the terms with powers larger than the second in (C-28) can be neglected, and the contribution of the j -th resonance can be written as

$$\frac{G_j^{(x)} + iH_j^{(x)}}{\eta_j - E} \approx \frac{\bar{G}_j^{(x)} + i\bar{H}_j^{(x)} - g_j^{(2)}}{\eta_j - E} - \frac{g_j^{(1)}k_j^{(1)}}{k_j^{(1)} + k} + g_j^{(2)} \quad (C-29)$$

In this manner the cross section expansion can still be considered of the same form as the one of eqs. (10) and (11) with resonance energies and half-widths and multilevel parameters which are modified with regard to those of Section II, but are energy independent. The corrective terms appearing in (C-29), together with the smoothly varying background terms defined in (C-27) represent, to this order of approximation, the effects of the momentum dependence of the neutron widths. In presence of many resonances or of a theoretically infinite level matrix these terms can be included in the general background terms arising from the "distant resonances", that is from those which were not included in the finite (truncated) level matrix.

In actual numerical calculations the previous considerations can be replaced by a direct comparison of the cross section values calculated with the exact R-matrix theory (either level matrix or channel matrix inversion) and with the multilevel expansion.

Representative values for the total cross section of Case 8 (Table IV) are shown in Table V. Column (A) gives the results obtained from the exact level matrix inversion, and column (B) gives values computed from the multilevel expansion. The differences between the two sets of values appear in the sixth or seventh significant figure (where computer induced truncation errors may affect the result), the accuracy being higher near the peak than in the valleys. Similar accuracy is obtained for the other cases of Tables I to III. As expectable, the accuracy in the valleys between the resonance peaks decreases for larger values of the ratio $\Gamma_{\lambda n}^0 / \Gamma_{\lambda f}$ (e.g., case 1 to 4 of Table I). While the agreement at the resonance peaks is still as good as that shown in Table V, the differences become more noticeable in the wings of the lines (deviations occur in the fourth or fifth significant figure). At these energies, however, the cross section values are of the order of one percent or less of their peak values, so

that the absolute deviations from a case to another remain of the same order.

The cross section curves calculated with the two approaches would be indistinguishable in practice, if they were to represent measured data. Thus, in the limits of validity of the condition (C-1), that is for low energy fissile element cross sections, the multilevel expansion appears physically equivalent to the exact R-matrix representation, although the two approaches are not mathematically identical.

Table VI was obtained through computer program LEMA-CHAMA⁽¹⁹⁾ which simultaneously evaluates the energy dependent cross section curves from the inversion of the exact level matrix (or, by option, of the channel matrix), that is, including at each energy step the momentum dependence of Γ_n , and from the multilevel expansion based on the diagonalization of the level matrix. In the latter calculation the approximation

$$\Gamma_{\lambda n} \approx \bar{\Gamma}_{\lambda n}$$

is implied.

Table I

Three Interfering Resonances with Two Fission Channels (Cases 1 to 4)

Parameters common to cases 1 to 4

λ	$E_{\lambda}^{(1)}$	$\Gamma_{\lambda\gamma}^{(1)}$	$\Gamma_{\lambda f}^{(1)}$
1	20.0	.040	.080
2	22.5	.040	.070
3	24.0	.040	.060

Parameters common to:

Cases 1 and 2			Cases 3 and 4		
λ	$\Gamma_{\lambda n}^{(2)}$	$G_{\lambda}^f (2)$	$\Gamma_{\lambda n}^{(2)}$	$G_{\lambda}^f (2)$	
1	.8 10^{-3}	.5180 10^{-3}	1.6 10^{-3}	1.066 10^{-3}	
2	.6 10^{-3}	.3722 10^{-3}	1.0 10^{-3}	.6101 10^{-3}	
3	.6 10^{-3}	.3497 10^{-3}	1.0 10^{-3}	.5720 10^{-3}	

	Case 1			Case 2		
λ	θ_{λ}^o (3)	H_{λ}^f (2)	$\theta_{\lambda}^{(1)}$ (4)	$\theta_{\lambda}^{(o)}$ (3)	H_{λ}^f (2)	$\theta_{\lambda}^{(1)}$ (4)
1	104.56	.1217 10^{-4}	134.88	5.905	.3030 10^{-4}	34.48
2	147.47	-.3704 10^{-4}	129.26	33.59	.0323 10^{-4}	24.24
3	0.0	.2487 10^{-4}	0.0	0.0	-.3353 10^{-4}	0.

	Case 3			Case 4		
λ	$\theta_{\lambda}^{(o)} (2)$	$H_{\lambda}^f (2)$	$\theta_{\lambda}^{(1)} (4)$	$\theta_{\lambda}^{(o)} (3)$	$H_{\lambda}^f (2)$	$\theta_{\lambda}^{(1)} (4)$
1	104.56	.2222 10^{-4}	134.60	5.905	.5532 10^{-4}	34.17
2	147.47	-.6416 10^{-4}	127.92	33.59	.0246 10^{-4}	23.39
3	0.0	.4194 10^{-4}	0.0	0.0	-.5778 10^{-4}	0.0

(1) in eV

(2) in $(\text{eV})^{\frac{1}{2}}$

(3) original angle with reference to γ_{3f}

(4) second solution with reference to γ_{3f}

} in degrees

Table II

Three Interfering Resonances with Two Fission Channels

Case 5: Example of Uniqueness

(The values of E_λ , $\Gamma_{\lambda\gamma}$, $\Gamma_{\lambda f}$ are those of Table I)

λ	$\Gamma_{\lambda n}^o$ (1)	θ_λ^o (2)	G_λ^F (2)	H_λ^F (1)
1	16. 10^{-3}	163.0	8.217 10^{-3}	2.266 10^{-4}
2	6. 10^{-3}	166.1	3.380 10^{-3}	-3.956 10^{-4}
3	1. 10^{-3}	0.0	.586 10^{-3}	1.690 10^{-4}

(1) in $(\text{eV})^{\frac{1}{2}}$

(2) angle with reference to γ_{3f} , in degrees

TABLE III

Three interfering resonances with two fission channels.

Effect of level spacing on the angles θ_λ

Parameters common to cases 7 and 8

λ	$\Gamma_{\lambda n}^o$ (2)	$\Gamma_{\lambda \gamma}^{(1)}$	$\Gamma_{\lambda f}^{(1)}$	$\theta_\lambda^{(o)} (3)$
1	3.66×10^{-3}	.040	0.3095	-8.46
2	$.340 \times 10^{-3}$	.040	0.5156	230.0
3	1.86×10^{-3}	.040	0.9761	0.0

Case 7

λ	E_λ (1)	G_λ^F (2)	H_λ^F (2)	θ_λ (1) (4)
1	141.7	3.056×10^{-3}	0.728×10^{-4}	82.38
2	149.1	0.314×10^{-3}	-0.270×10^{-4}	69.60
3	155.3	1.765×10^{-3}	-0.458×10^{-4}	0.0

Case 8

λ	E_λ (1)	G_λ^F (2)	H_λ^F (2)	θ_λ (1) (4)
1	14.00	3.242×10^{-3}	2.379×10^{-4}	84.53
2	16.00	0.315×10^{-3}	-0.649×10^{-4}	60.53
3	18.00	1.770×10^{-3}	-1.730×10^{-4}	0.0

(1) in eV

(2) in $(\text{eV})^{\frac{1}{2}}$

(3) original angle with reference to γ_{3f} (in degrees)

(4) second solution with reference to γ_{3f} (in degrees)

TABLE IV

Four interfering resonances with two fission channels

λ	$E_{\lambda}^{(1)}$	$\Gamma_{\lambda n}^{(2)}$	$\Gamma_{\lambda \gamma}^{(1)}$	$\Gamma_{\lambda f}^{(1)}$	$\theta_{\lambda}^{(o) (3)}$	$G_{\lambda}^f (2)$	$H_{\lambda}^f (2)$	$\theta_{\lambda}^{(1) (4)}$	$\theta_{\lambda}^{(2) (4)}$	$\theta_{\lambda}^{(3) (4)}$
1	150.0	$.2 \cdot 10^{-3}$	.040	.350	115.98	$1.789 \cdot 10^{-4}$	$.2746 \cdot 10^{-4}$	104.15	38.26	50.67
2	155.0	$.4 \cdot 10^{-3}$	.040	.400	93.19	$3.616 \cdot 10^{-4}$	$.2695 \cdot 10^{-4}$	66.90	83.80	110.09
3	159.0	$.2 \cdot 10^{-3}$	.040	1.200	102.93	$1.934 \cdot 10^{-4}$	$-.7841 \cdot 10^{-4}$	117.66	122.18	107.45
4	164.0	$.6 \cdot 10^{-3}$	.040	.900	0.0	$5.721 \cdot 10^{-4}$	$.2400 \cdot 10^{-4}$	0.0	0.0	0.0

(1) in eV

(2) in $(\text{eV})^{\frac{1}{2}}$ (3) original angles in reference to γ_{4f} (in degrees)(4) solutions in reference to γ_{4f} (in degrees)

Table V
Comparison of Exact R-Matrix Cross Section Calculation
with Multilevel Expansion

Energy (eV)	Total Cross section in barns	
	A	B
145.000	9.81450	9.81459
146.000	9.90783	9.90794
147.000	10.08814	10.08827
148.000	10.52872	10.52888
149.000	12.40207	12.40229
150.000	63.96998	63.97003
151.000	10.81220	10.81218
152.000	9.97801	9.97810
153.000	10.50062	10.50078
154.000	13.78116	13.78145
155.000	104.29809	104.29811
156.000	13.16931	13.16911
157.000	10.16510	10.16507
158.000	11.75714	11.75722
159.000	26.74377	26.74382
160.000	17.99287	17.99285
161.000	14.55518	14.55521
162.000	15.02620	15.02631
163.000	22.75375	22.75400
164.000	75.80992	75.80995
165.000	22.61266	22.61231

A: Exact R-Matrix Calculation
B: Multilevel Expansion

Fig. 1. Iso- β Curves for Four Interfering Resonances (Case 8)

Fig. 2. Iso- β Curves for Three Interfering Resonances (Case 6)