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A SPECTRAL THEORY FOR THE STATIONARY TRANSPORT OPERATOR IN SLAB GEOMETRY

by

Erwin H. Bareiss

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ANL-6940
Mathematics and Computers
(TID-4500, 37th Ed.)
AEC Research and
Development Report

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9700 South Cass Avenue
Argonne, Illinois 60440

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Applied Mathematics Division

December 1964

Operated by The University of Chicago
under
Contract W-31-109-eng-38
with the
U. S. Atomic Energy Commission

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ABSTRACT

Patterned after the operational calculus of functional analysis, a spectral theory is constructed over a general function space for stationary transport operators in plane geometry as they occur in neutron theory, radiative transfer, and other areas. As an application, the general solution of the corresponding transport equations is given in closed form. It is shown that the spectrum of the isotropic transport operator consists of the union of a discrete spectrum of exactly two points, either real or pure imaginary, and a continuous spectrum which extends from ± 1 to $\pm \infty$ of the real axis. If the discrete spectrum of the anisotropic operator consists of two points only, as is the case in practical applications, the spectrum has the same properties as in the isotropic case. Proof and additional properties of the spectrum are derived under the assumption that the scattering functions have the physically important form $f(\Omega, \Omega') = f(\Omega \cdot \Omega') > 0$.

I. INTRODUCTION

In this paper, we construct a spectral theory for a simple form of the transport operator, which is defined here by the following equation:

$$\Omega \cdot \nabla \phi + \sigma \phi = \frac{c}{4\pi} \sigma \int_{\{\Omega\}} f(r, \Omega \cdot \Omega') \phi(r, \Omega') d\Omega' + Q(r, \Omega), \quad (1.1)$$

where

$\phi = \phi(r, \Omega)$ is the magnitude of the directional flux,

r is the position vector,

σ is the total macroscopic cross section,

c is the net number of neutrons produced per collision

$$[= (\sigma_s + \nu \sigma_f) / \sigma],$$

f is the scattering function,

and

Q is the rate of production of neutrons or photons by sources.

The scattering function is so normalized that

$$\int_{\{\Omega\}} f(\Omega \cdot \Omega') d\Omega' = 1.$$

There are applications of this transport equation (besides in neutron physics) in investigations concerning dispersion of light in the atmosphere, passage of γ -rays through a dispersive medium, transport of radiation in stellar atmospheres, cosmic rays, and other areas.

The reason for investigating the mathematical structure of the transport operator was to explain some of the difficulties which arose in actual computations, and to find better numerical methods and more efficient approximations.

Dividing Eq. (1.1) by σ , one gets the following representation

$$\mathcal{L}\phi = \frac{c}{4\pi} \mathcal{S}\phi + F(r, \Omega), \quad (1.2)$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\psi = \frac{1}{\sigma} \Omega \cdot \nabla \psi + \psi, \quad (1.3)$$

and the operator $\mathcal{S}$ by

$$\mathcal{S}\psi = \int_{\{\Omega\}} f(r, \Omega \cdot \Omega') \psi(r, \Omega') d\Omega', \quad (1.4)$$

which is an integral operator. $F(r, \Omega)$ is $Q(r, \Omega)/\sigma$.

In 1952, Uhlenbeck and Wang⁽¹³⁾ investigated the linearized Boltzmann integral equation for a rarefied gas,

$$\left(\frac{m}{2KT}\right)^{1/2} \frac{\partial \phi}{\partial t} + p \cdot \nabla \phi = n \mathcal{J} \phi. \quad (1.5)$$

If we write this equation as

$$\mathcal{D}\phi = \mathcal{J}\phi, \quad (1.6)$$

the left-hand side defines a differential operator $\mathcal{D}$, and the right-hand side the collision operator $\mathcal{J}$. The method of Uhlenbeck and Wang for solving (1.6) is to expand ϕ in terms of the complete set of eigenfunctions ϕ_i of the collision operator; i.e.,

$$\phi = \sum \alpha_i \phi_i,$$

where

$$\mathcal{J}\phi_i = \lambda_i \phi_i.$$

Then

$$\mathcal{D}\phi = \sum \alpha_i \mathcal{D}\phi_i = \sum \alpha_i \lambda_i \phi_i,$$

and the problem is transformed to a system of ordinary differential equations.

One is tempted to try the same approach in Eq. (1.2). If one assumes $f = 1$ (isotropic scattering), one sees immediately that there exists only one eigenfunction, a constant. Even when we include the absorption term in the collision operator, we have only

$$(\mathcal{J} - I)\phi = \lambda\phi.$$

There are, however, two approaches which can be applied:

a) We assume homogeneous boundary conditions and the existence of $\mathcal{L}^{-1}$. Then we may write (1.2) in the equivalent form,

$$\phi = \lambda \mathcal{L}^{-1} \mathcal{J}\phi + \mathcal{L}^{-1} F,$$

where $\lambda = c/(4\pi)$. The corresponding characteristic value problem is then given by

$$\phi = \lambda \mathcal{L}^{-1} \mathcal{J}\phi. \quad (1.7)$$

This equation is a linear integral equation, and the method may be called the "Integral Equation Approach." Some mathematical aspects of its theory have been worked out by V. S. Vladimirov.⁽¹¹⁾

b) Contrasted to the "Integral Equation Approach" is the "Differential Equation Approach." This second approach is carried out below for the case of plane geometry, taking methods of functional analysis as tools

of investigation. The final results include those obtained by Case⁽³⁾ and Wigner.⁽¹⁴⁾ The method used here is mathematically rigorous. The case of arbitrary (three-dimensional) geometry was treated by the author in Ref. 2. The results turned out to be very unexpected.

We now specialize the transport Eq. (1.1) to one of the simplest but still practically important forms, namely to the one-velocity normalized ($\sigma=1$) stationary, isotropic transport equation for plane geometry and axially symmetric vector flux ϕ ; that is,

$$\mu \frac{\partial \phi}{\partial x} + \phi = \frac{\epsilon}{2} \int_{-1}^{+1} \phi(x, \mu') d\mu' + Q(x, \mu) \quad -1 \leq \mu \leq 1 \quad (1.8)$$

$$-\infty < x < \infty.$$

The axis of symmetry is taken in the x direction, and $\mu (= \Omega_x)$ is the direction cosine of the vector flux.

In the following sections, we define mathematically an operator $T_\epsilon = D + L_\epsilon$ corresponding to the homogeneous form of Eq. (1.8), and depending on a parameter ϵ . We investigate the properties of L_ϵ and show that by a simple extension of the limit operator

$$\lim_{\epsilon \rightarrow +0} T_\epsilon \phi = T \phi, \quad (1.9)$$

the general physical problem of Eq. (1.8) can be solved in closed form. Finally, the restriction to the case of isotropic scattering is removed, and anisotropic scattering is admitted.

The procedure we follow is similar to that used by Friedman (Ref. 6, Chapter 5) for the spectral theory of partial differential operators that are the sum of two commutative operators. Since the differential operator D is amply discussed in the literature, but only formally defined until the actual problem to be solved is specified, we necessarily start with, and concentrate our investigation on, the operator L_ϵ . Our problem therefore consists of constructing a spectral theory for the operator L_ϵ . In other words, we are confronted with the eigenvalue problem,

$$L_\epsilon \phi = \lambda \phi. \quad (1.10)$$

Our tool will be an operational calculus modeled after the theorems of functional analysis (Taylor, Ref. 10, Section 5.6; or Dunford and Schwartz, Ref. 5). The results of this paper are formally the same as in functional analysis and therefore easy to remember but are conceptually different; i.e., the formulas carry a different meaning.

II. THE OPERATORS L_ϵ AND L_0

For reasons that will be apparent later, we define a vector space S_0 as follows:

Definition of S_0

Given $-\infty < x_0 < x_1 < \infty$ and the real sets

$$E_1 = \{x : x_0 \leq x \leq x_1\}$$

and

$$M_0 = \{\mu : 0 < \mu^2 \leq 1\}.$$

Then S_0 is the set of all complex-valued Borel functions ψ over $E_1 \times M_0$ with the following three properties:

$$(a) \int_{M_0} \psi(x, \mu) d\mu \text{ exists for all } x \in E_1.$$

$$(b) v(x) = \int_{M_0} \frac{\mu \psi(x, \mu)}{1 - \lambda \mu} d\mu \text{ exists for all } x \in E_1 \text{ and } \lambda$$

in the complex plane. The integral is understood to be taken in the sense of Cauchy's principal value for real λ .

$$(c) \int_{x_0}^x v(\xi) e^{(x-\xi)\lambda} d\xi \text{ exists for all } x \in E_1 \text{ and all complex } \lambda.$$

Addition of elements in S_0 is the usual addition of functions, scalar multiplication is the usual multiplication of functions by a real or complex scalar.

A typical element in S_0 is

$$\psi(x, \mu) = \frac{e^{-x}}{|\mu_0 - \mu|^\alpha}, \quad (-1 < \mu_0 < 1; 0 < \alpha < 1).$$

To give a clearer picture of the type of function in S_0 , we present the following example:

A proper subspace of S_0 is the set of all functions ψ such that

$$\psi(x, \mu) = f(x)\phi(\mu),$$

where f is integrable over E_1 and Φ is integrable over M_0 . Furthermore, let

$$\Phi \in L_p(M_0) \quad (p > 1)$$

be such that in the neighborhood of $\mu = +1$ (and $\mu = -1$)

$$\Phi(\mu) = C(1 \pm \mu)^{-\alpha} + \phi(\mu), \quad (0 < \alpha < 1)$$

where C is a constant and $\phi(\mu)$ vanishes at $\mu = +1$ ($\mu = -1$) and satisfies a uniform Hölder condition of positive order γ ; i.e.,

$$\|\phi(\mu) - \phi(\mu_0)\| < K|\mu - \mu_0|^\gamma.$$

This subspace is similar to the function space often used in the theory of singular integral equations with Cauchy kernel.

The space S_0 is not complete. This is demonstrated by the following example. Let

$$\psi_n = \begin{cases} 1/\mu^2 & |\mu| > 1/n \\ 1/n^2 & |\mu| \leq 1/n \end{cases}$$

Definition of $\oint$

$$\oint \psi(x, \mu) d\mu = \int_{-1}^{-\epsilon} \psi(x, \mu) d\mu + \int_{\epsilon}^1 \psi(x, \mu) d\mu. \quad (2.1)$$

Definition of Convergence

A sequence $\{\psi_n\} \subset S_0$ converges to $\psi \in S_0$; i.e.,

$$\lim_{n \rightarrow \infty} \psi_n(x, \mu) = \psi(x, \mu) \quad (2.2)$$

if

$$\lim_{n \rightarrow \infty} \oint \psi_n(x, \mu) d\mu = \oint \psi(x, \mu) d\mu \quad (2.3a)$$

for every ϵ such that $0 < \epsilon < 1$ and

$$\lim_{n \rightarrow \infty} |\psi_n(x, \mu) - \psi(x, \mu)| = 0 \quad (2.3b)$$

pointwise in x and μ .

Definition of L_ϵ

Given a real $\epsilon > 0$ and a positive c , the linear operator L_ϵ on S_0 is defined by

$$(L_\epsilon \psi)(x, \mu) = \begin{cases} \frac{\psi(x, \mu) - \frac{c}{2} \oint \psi(x, \mu') d\mu'}{\mu} & \text{for } \epsilon \leq |\mu| < 1; \\ \frac{\psi(x, \mu) - \frac{c}{2} \oint \psi(x, \mu') d\mu'}{\epsilon \operatorname{sign} \mu} & \text{for } 0 < |\mu| \leq \epsilon. \end{cases} \quad (2.4)$$

It can be verified that $L_\epsilon \psi \in S_0$ for every $\psi \in S_0$.

Lemma 2.1

The linear operator L_ϵ on S_0 is continuous in the following sense that $\psi_n, \psi \in S_0$ and

$$\lim_{n \rightarrow \infty} \psi_n(x, \mu) = \psi(x, \mu) \quad (2.5)$$

in the sense of (2.2), imply

$$\lim_{n \rightarrow \infty} (L_\epsilon \psi_n)(x, \mu) = (L_\epsilon \psi)(x, \mu) \quad (2.6)$$

for

$$(x, \mu) \in E_1 \times M_0.$$

In fact,

$$\lim_{n \rightarrow \infty} \left| \oint \psi_n(x, \mu) d\mu - \oint \psi(x, \mu) d\mu \right| \leq \lim_{n \rightarrow \infty} \oint |\psi_n(x, \mu) - \psi(x, \mu)| d\mu = 0,$$

from which Lemma 2.1 follows easily.

Definition of L_0 :

The linear operator L_0 on S_0 is defined by

$$(L_0 \psi)(x, \mu) = \frac{\psi(x, \mu) - \frac{c}{2} \int_{M_0} \psi(x, \mu') d\mu'}{\mu} \quad (2.7)$$

for all $\psi \in S_0$, where c is a positive constant.

It can be verified that $L_0 \psi \in S_0$ for every $\psi \in S_0$.

Theorem 2.1

For any sequence $\{\epsilon_n\}$, where $\epsilon \downarrow 0$ and any given $\psi \in S_0$, we have

$$\lim_{n \rightarrow \infty} (L_{\epsilon_n} \psi)(x, \mu) = (L_0 \psi)(x, \mu). \quad (\text{Pointwise convergence}) \quad (2.8)$$

Proof:

Under the hypothesis, we have

$$\lim_{n \rightarrow \infty} \int_{E \in \epsilon_n} \psi d\mu = \lim_{\epsilon \rightarrow 0} \int \psi d\mu = \int_{M_0} \psi d\mu.$$

It follows then from the definitions of L_ϵ and L_0 , that for $x \in E_1$ and any $\mu \in M_0$,

$$\lim_{n \rightarrow \infty} |[(L_{\epsilon_n} - L_0) \psi](x, \mu)| = 0.$$

III. THE SPECTRUM OF L_ϵ

Definitions

Let L be a continuous linear operator on S_0 . Then the inverse of the operator $L - \lambda I$ is represented by the symbol

$$R_\lambda = (L - \lambda I)^{-1}.$$

The resolvent set $\rho(L)$ is the set of all λ 's such that R_λ exists and is continuous.

The complement of $\rho(L)$ is the spectrum $\sigma(L)$. In other words, $\lambda \in \sigma(L) \Leftrightarrow L - \lambda I$ has no continuous inverse.

The point spectrum $\sigma_p(L)$ is defined as the set of those λ 's for which there exists a $\phi \in S_0$ ($\phi \neq 0$) such that $L\phi = \lambda\phi$; i.e.,

$$\lambda \in \sigma_p(L) \Leftrightarrow \left\{ \phi \in S_0, \phi \neq 0, \text{ s.th. } L\phi = \lambda\phi. \right.$$

The λ 's in σ_p are called eigenvalues of L .

The continuous spectrum $\sigma_c(L)$ is defined as the complement of $\sigma_p(L)$ with respect to $\sigma(L)$; i.e.,

$$\sigma_c(L) = \sigma(L) - \sigma_p(L).$$

For the pair of operators L_ϵ and L_0 introduced in the previous section, we use R_λ^ϵ , $\rho(L_\epsilon)$, $\sigma(L_\epsilon)$, etc., and R_λ^0 , $\rho(L_0)$, $\sigma(L_0)$, etc., respectively.

Theorem 3.1

For any $\epsilon > 0$, the point spectrum $\sigma_p(L_\epsilon)$ consists of two points λ_0^ϵ and $-\lambda_0^\epsilon$, which are either real or pure imaginary. If they are real, then $0 \leq \lambda_0^\epsilon < 1$.

The continuous spectrum $\sigma_c(L_\epsilon)$ consists of the real intervals,

$$-\frac{1}{\epsilon} \leq \lambda \leq -1 \text{ and } 1 \leq \lambda \leq \frac{1}{\epsilon}.$$

Proof. It suffices to prove the following statements:

- (a) If λ is neither in the interval $-1/\epsilon \leq \lambda \leq -1$ nor in $1 \leq \lambda \leq 1/\epsilon$, then $\lambda \in \sigma_p(L_\epsilon)$ if and only if $\lambda = \pm \lambda_0^\epsilon$, where $\pm \lambda_0^\epsilon$ are the only solutions of (3.6) below.
- (b) If λ is neither in $-1/\epsilon \leq \lambda \leq -1$ nor in $1 \leq \lambda \leq 1/\epsilon$, and if $\lambda \neq \pm \lambda_0^\epsilon$, then $\lambda \in \rho(L_\epsilon)$.
- (c) If $-1/\epsilon \leq \lambda \leq -1$ or $1 \leq \lambda \leq 1/\epsilon$, then $\lambda \in \sigma_c(L_\epsilon)$.

Proof of Statement (a). We determine the point spectrum of L_ϵ . Under the hypothesis of statement (a), we assume $\lambda \in \sigma_p(L_\epsilon)$ and let $\phi \in S_0$, $\phi \neq 0$, be such that

$$(L_\epsilon - \lambda)\phi = 0. \quad (3.1)$$

We have explicitly

$$[(L_\epsilon - \lambda)\phi](x, \mu) = \begin{cases} \frac{(1 - \lambda\mu)\phi(x, \mu) - \frac{c}{2} \oint \phi(x, \mu') d\mu'}{\mu} & \text{for } \epsilon \leq |\mu| \leq 1; \\ \frac{(1 - \lambda \epsilon \operatorname{sign} \mu)\phi(x, \mu) - \frac{c}{2} \oint \phi(x, \mu') d\mu'}{\epsilon \operatorname{sign} \mu} & \text{for } 0 < |\mu| \leq \epsilon \end{cases} \quad (3.2)$$

If we let

$$J_\epsilon(x) = \oint \phi(x, \mu') d\mu', \quad (3.3)$$

it follows that

$$\phi^\epsilon(x, \mu) = \begin{cases} \frac{c}{2} \frac{J_\epsilon(x)}{1 - \lambda\mu} & (\epsilon \leq |\mu| \leq 1); \\ \frac{c}{2} \frac{J_\epsilon(x)}{1 - \lambda \epsilon \operatorname{sign} \mu} & (0 < |\mu| \leq \epsilon). \end{cases} \quad (3.4)$$

Applying the operation $\oint \cdot d\mu$ on both sides of (3.4) yields

$$\left[1 - \frac{c}{2} \oint \frac{d\mu}{1 - \lambda\mu} \right] J_\epsilon(x) = 0. \quad (3.5)$$

$J_\epsilon(x)$ cannot vanish identically, since otherwise, by (3.4), $\phi^\epsilon = 0$. Hence, every $\lambda \in \sigma_p(L_\epsilon)$ necessarily satisfies the characteristic equation,

$$1 - \frac{c}{2} \frac{d\mu}{1 - \lambda\mu} = 0. \quad (3.6)$$

Conversely, for every λ satisfying (3.6), the function ϕ defined by (3.4) is in S_0 and will yield $(L_\epsilon - \lambda)\phi = 0$. According to the Lemma of Appendix A, the characteristic Eq. (3.6) has exactly two solutions $\pm\lambda_0^\epsilon$. Hence, $\sigma_p(L_\epsilon)$ is precisely the set $\{\lambda_0^\epsilon, -\lambda_0^\epsilon\}$, where

$$\left. \begin{aligned} 0 < \lambda_0^\epsilon < 1 & \quad \text{for } 0 < c < 1/(1 - \epsilon); \\ \lambda_0^\epsilon = 0 & \quad \text{for } c = 1/(1 - \epsilon); \\ \lambda_0^\epsilon = i |\lambda_0^\epsilon| \text{ (pure imaginary)} & \quad \text{for } c > 1/(1 - \epsilon). \end{aligned} \right\} \quad (3.7)$$

We define two eigenfunctions of L_ϵ corresponding to the two eigenvalues $\pm\lambda_0^\epsilon$ as

$$\phi_{\pm\lambda_0^\epsilon}^\epsilon(\mu) = \begin{cases} \frac{c}{2} \frac{1}{1 + \lambda_0^\epsilon \mu} & \epsilon \leq |\mu| \leq 1; \\ \frac{c}{2} \frac{1}{1 \mp \lambda_0^\epsilon \text{ sign } \mu} & 0 < |\mu| < \epsilon. \end{cases} \quad (3.8)$$

by (3.4), we see that every other eigenfunction differs from (3.8) only by a function, $a(x)$.

Proof of Statement (b). Next we determine $\rho(L_\epsilon)$. Let ψ be any element of S_0 . Then for each $\lambda \in \rho(L_\epsilon)$ there must exist $\phi \in S_0$ such that

$$(L_\epsilon - \lambda)\phi = \psi, \quad (3.9)$$

and R_λ^ϵ is continuous. We first determine the set of λ 's where R_λ^ϵ exists. Then we show that R_λ^ϵ is continuous on S_0 .

By simple algebraic manipulations, one finds from (3.9) the explicit form of $R_\lambda^\epsilon \psi$; viz.,

$$(R_{\lambda}^{\epsilon} \psi)(x, \mu) = \phi(x, \mu) = \begin{cases} \frac{\mu \psi(x, \mu) + \frac{c}{2} J(x)}{1 - \lambda \mu} & (\epsilon \leq |\mu| \leq 1), \\ \frac{\epsilon \operatorname{sign} \mu \psi(x, \mu) + \frac{c}{2} J(x)}{1 - \lambda \epsilon \operatorname{sign} \mu} & (0 < |\mu| \leq \epsilon), \end{cases} \quad (3.10)$$

where $J(x)$ is defined as in (3.3). Applying the operation $\oint \cdot d\mu$ to (3.10) yields, after some elementary computation,

$$J(x) = \frac{\oint \frac{\mu \psi(x, \mu) d\mu}{1 - \lambda \mu}}{1 - \frac{c}{2} \oint \frac{d\mu}{1 - \lambda \mu}} = J(x; \lambda), \quad (3.11)$$

where the right-hand side indicates that J also depends on λ . The denominator in (3.11) vanishes only on $\sigma_p(L_{\epsilon})$, i.e., for the eigenvalues $\lambda = \pm \lambda_{\epsilon}$, and possibly on two points in $1 \leq |\lambda| \leq 1/\epsilon$ that are excluded by hypothesis. The numerator exists in the entire λ -plane because $\psi \in S_0$. Substituting (3.11) into (3.10) and observing that under the hypothesis of statement b, $1 - \lambda \mu \neq 0$ for $\epsilon \leq |\mu| \leq 1$, we have shown the existence of $(R_{\lambda}^{\epsilon} \psi)(x, \mu)$ or all λ 's except for those that are elements of $-1/\epsilon \leq \lambda \leq -1$ or $1 \leq \lambda \leq 1/\epsilon$ or $\sigma_p(L_{\epsilon})$. One can verify that these $R_{\lambda}^{\epsilon} \psi$ are elements of S_0 .

We show that R_{λ}^{ϵ} is continuous on S_0 . Let $\{\psi\} \subset S_0$ be a sequence such that

$$\lim_{n \rightarrow \infty} \psi_n(x, \mu) = \psi(x, \mu) \quad (\psi \in S_0). \quad (3.12)$$

We have to show that

$$\lim_{n \rightarrow \infty} (R_{\lambda}^{\epsilon} \psi_n)(x, \mu) = (R_{\lambda}^{\epsilon} \psi)(x, \mu). \quad (3.13)$$

From (3.10), we see immediately that this is true when $J(x)$ [which is a functional of ψ] is continuous. From (3.11), we see then that the problem can be reduced to the proof of

$$\lim_{n \rightarrow \infty} \oint \frac{\mu \psi_n(x, \mu) d\mu}{1 - \lambda \mu} = \oint \frac{\mu \psi(x, \mu) d\mu}{1 - \lambda \mu}. \quad (3.14)$$

Equation (3.14) is proven by

$$\lim_{n \rightarrow \infty} \left| \oint \frac{\mu [\psi_n(x, \mu) - \psi(x, \mu)]}{1 - \lambda \mu} \right| \leq \lim_{n \rightarrow \infty} \oint |\psi_n(x, \mu) - \psi(x, \mu)| \left| \frac{\mu}{1 - \lambda \mu} \right| d\mu = 0.$$

Hence R_λ^ϵ is continuous for all λ 's admitted under the hypothesis.

Proof of Statement c. To conclude the proof of Theorem 3.1, it remains to be shown that R_λ^ϵ is not continuous for $-1/\epsilon \leq \lambda \leq -1$ or $1 \leq \lambda \leq 1/\epsilon$; i.e. (3.13) does not hold for those λ 's. Take any real λ such that $1 \leq |\lambda| \leq 1/\epsilon$. Then for μ such that $\mu \rightarrow 1/\lambda$, we have for the first term on the right-hand side of (3.10)

$$\lim_{n \rightarrow \infty} \left| \lim_{\mu \rightarrow 1/\lambda} \frac{\psi_n(x, \mu) - \psi(x, \mu)}{1 - \lambda \mu} \right| \rightarrow \infty. \quad (3.16)$$

A similar argument holds for the second term on the right-hand side of (3.10), which concludes the proof of Theorem 3.1.

Theorem 3.2

$$\sigma_p(L_0) = \{\pm \lambda_0^0\}; \quad (3.17)$$

$$\sigma_c(L_0) = \{\lambda: -\infty < \lambda \leq -1 \text{ or } 1 \leq \lambda < \infty\}. \quad (3.18)$$

Proof. We proceed exactly as in the case of the operator L_ϵ , but substitute L_0 for L_ϵ . The details are left to the reader. The following remarks are of interest:

The characteristic equation of L_ϵ [defined in (3.6)] is in the limit $\epsilon \rightarrow +0$ exactly the characteristic equation for L_0 ; viz.,

$$1 - \frac{c}{2} \int_{M_0} \frac{d\mu}{1 - \lambda \mu} = 0. \quad (3.19)$$

The roots of this equation are denoted by $\pm \lambda_0$ and are equal to $\pm \lim_{\epsilon \rightarrow +0} \lambda_0^\epsilon$; i.e., $\lambda_0 = \lambda_0^0$.

Hence,

$$\lim_{\epsilon \rightarrow +0} \sigma_p(L_\epsilon) = \sigma_p(L_0). \quad (3.20)$$

For the continuous spectrum, we have

$$\lim_{\epsilon \rightarrow +0} \sigma_c(L_\epsilon) = \{\lambda: 1 \leq \lambda^2 < \infty\}. \quad (3.21)$$

The inverse of $L_0 - \lambda$ is given by

$$(\mathcal{R}_\lambda^0 \psi)(x, \mu) = \frac{\mu \psi(x, \mu) + \frac{c}{2} \frac{\int_{M_0} \frac{\mu \psi(x, \mu) d\mu}{1 - \lambda \mu}}{1 - \frac{c}{2} \int_{M_0} \frac{d\mu}{1 - \lambda \mu}}}{1 - \lambda \mu} (\psi \in S_0); \quad (3.22)$$

and by repeating the steps that lead to $\sigma_c(L_\epsilon)$, one finds that

$$\sigma_c(L_0) = \{\lambda: 1 \leq \lambda^2 < \infty\}. \quad (3.23)$$

Comparing (3.23) with (3.21) yields

$$\lim_{\epsilon \rightarrow +0} \sigma_c(L_\epsilon) = \sigma_c(L_0).$$

Theorem 3.3

$$\lim_{\epsilon \rightarrow +0} \mathcal{R}_\lambda^\epsilon \phi = \mathcal{R}_\lambda^0 \phi$$

for

$$\phi \in S_0 \text{ and } \lambda \in \rho(L_0).$$

The proof of this theorem follows by comparing (3.22) with the limit $\epsilon \rightarrow +0$ of (3.10).

IV. RESOLUTION OF THE IDENTITY FOR L_ϵ

We show first that $(R_\lambda^\epsilon \psi)(x, \mu)$ for $(x, \mu) \in E_1 \times M_0$ is analytic in λ on $\rho(L_\epsilon)$. If $\lambda, \lambda' \in \rho(L_\epsilon)$, $\lambda \neq \lambda'$, it follows from

$$(L_\epsilon - \lambda' I) - (L_\epsilon - \lambda I) = (\lambda - \lambda') I$$

that

$$(R_\lambda^\epsilon \psi - R_{\lambda'}^\epsilon \psi)(x, \mu) = (\lambda - \lambda')(R_\lambda^\epsilon R_{\lambda'}^\epsilon \psi)(x, \mu). \quad (4.1)$$

Then

$$\lim_{\lambda' \rightarrow \lambda} \frac{R_\lambda^\epsilon \psi - R_{\lambda'}^\epsilon \psi}{\lambda - \lambda'}(x, \mu) = [(R_\lambda^\epsilon)^2 \psi](x, \mu) \quad (4.2)$$

exists and $(R_\lambda^\epsilon)^2 \psi \in S_0$. Hence $(R_\lambda^\epsilon \psi)(x, \mu)$ is analytic on $\rho(L_\epsilon)$. The analyticity can also be obtained directly from (3.10).

As a further preliminary remark, we conclude from Eqs. (3.10) and (3.11) that

$$(R_\lambda^\epsilon \psi)(x, \mu) = -\frac{\psi(x, \mu)}{\lambda} + o\left(\frac{1}{\lambda^2}\right) \text{ for } |\lambda| \gg \frac{1}{\epsilon} > 1. \quad (4.3)$$

The simple details for the derivation of (4.3) are left to the reader.

Our next goal is to find the resolution of the identity.

Since $(R_\lambda^\epsilon \psi)(x, \mu)$ is an analytic function in λ on $\rho(L_\epsilon)$ for any bounded $\psi(x, \mu)$, Cauchy's Fundamental Theorem can be applied:

$$\oint_{+B(D)} (R_\lambda^\epsilon \psi)(x, \mu) d\lambda = 0 \quad (4.4)$$

for any Cauchy domain $D[\rho(L_\epsilon)]$.

The notation in this contour integral will now be explained.

Definition. A set D in the complex plane is called a Cauchy domain if (1) it is open; (2) it has a finite number of components, the closures of any two of which are disjoint; and (3) the boundary of D is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect.

If C is one of the curves forming part of the boundary of D , the positive orientation of C is clockwise or counterclockwise according to whether the points of D near a point of C are outside or inside C . The positively oriented boundary of D is denoted by $+B(D)$; with the reverse orientation it is denoted by $-B(D)$. (Taylor, Ref. 10, page 288.)

We choose as Cauchy domain the interior of a circle C_0 of radius $r > 1/\epsilon + \epsilon$, minus the interior of four closed rectifiable Jordan curves that contain the spectrum of L_ϵ and their closures, as illustrated in Fig. 1 for $c > 1$.

Fig. 1
Contour of
Cauchy Domain

At this moment,

$$B(D) = \bigcup_{\ell=0}^4 C_\ell. \quad (4.5)$$

Before deforming $B(D)$, we note that

$$\lim_{\epsilon \rightarrow 0} \oint_{C_0} (R_\lambda^\epsilon \psi)(x, \mu) d\lambda = -2\pi i \psi(x, \mu). \quad (4.6)$$

The proof of (4.6) is obvious. We introduce polar coordinates such that C_0 is given by

$$C_0 : \lambda = r e^{i\omega}, r > \frac{1}{\epsilon} + \epsilon, 0 \leq \omega < 2\pi. \quad (4.7)$$

Then, for $r \rightarrow \infty$ and keeping ϵ fixed, by (4.3),

$$\lim_{r \rightarrow \infty} \oint_{C_0} R_{\lambda}^{\epsilon} \psi d\lambda = - \lim_{r \rightarrow \infty} i \int_{\omega=0}^{2\pi} \frac{\psi}{re^{i\omega}} re^{i\omega} d\omega = -2\pi i \psi \quad (4.8)$$

pointwise in $(x, \mu) \in E_1 \times M_0$. However, when the limit is taken in (4.8), the continuous spectrum $\sigma_c(L_{\epsilon})$ can also be expanded to any point $\pm 1/\epsilon$ ($\epsilon > 0$). It is prudent to investigate the behavior of the integral $\int R_{\lambda}^{\epsilon} \psi d\lambda$ in the neighborhood of $\omega = 0$ and $\omega = \pi$, as indicated by C_5 and C_6 in Fig. 1. We assume the end points C_5 to have fixed imaginary parts $\pm is$. Substituting in (3.10) the bound ϵ for μ , and $re^{i\omega}$ for λ , we obtain (after integrating with respect to ω)

$$\int_{C_5} R_{\lambda}^{\epsilon} \psi d\lambda = i \int_{\omega^+}^{\omega^-} \frac{\epsilon \psi + \frac{c}{2} J}{1 - \epsilon re^{i\omega}} re^{i\omega} d\omega \quad (4.9)$$

pointwise in

$$(x, \mu) \in E_1 \times M_0,$$

where

$$\omega^{\pm} = \sin^{-1} \pm s/r \approx \pm s/r.$$

Since $\psi(x, \mu)$ and $J(x; \lambda)$ are bounded over C_5 for all r , say $|\psi| \leq \hat{\psi}$ and $|J| \leq \hat{J}$, as can be verified from (3.11),

$$\lim_{r \rightarrow \infty} \left| \int_{C_5} (R_{\lambda}^{\epsilon} \psi)(x, \mu) d\lambda \right| \leq \lim_{r \rightarrow \infty} 2 \frac{s}{r} \cdot \frac{\epsilon \hat{\psi} + \frac{c}{2} \hat{J}}{|1 - \epsilon re^{is/r}|} = 0. \quad (4.10)$$

Hence, the contribution of the integral (4.9) over the segments C_5 of constant length $2s$ to the integral of (4.8) vanishes uniformly for $\|\mu\| > \epsilon > 0$ as $r \rightarrow \infty$. A similar statement can be made concerning C_6 of Fig. 1. Combining (4.4), (4.5), and (4.6) yields the following resolution of the identity of the operator L_{ϵ} :

$$\psi(x, \mu) = + \frac{1}{2\pi i} \int_{\bigcup_{\ell=1}^4 C_{\ell}} (R_{\lambda}^{\epsilon} \psi)(x, \mu) d\lambda, \quad (4.11)$$

pointwise in $\psi \in S_0$. This result has the same appearance as the results of functional analysis in Banach spaces.

We are now prepared to analyze the operator L_0 .

V. SPECTRAL RESOLUTION FOR THE OPERATOR L_0

The operator L_0 is defined by (2.7). By Theorem 2.1, we have

$$\lim_{\epsilon \rightarrow +0} (L_\epsilon \psi)(x, \mu) = (L_0 \psi)(x, \mu). \quad (5.1)$$

Referring to the notations of the previous section, we deform the contours C_3 and C_4 around the continuous spectrum so that its outermost parts coincide with C_5 and C_6 (see Fig. 1) and therefore cancel each other. The boundary $B(D)$ of (4.4) for this modified contour consists now of the union of $B_1 = C_1$, $B_2 = C_2$, $B_3 = C_3 - C_5$, $B_4 = C_4 - C_6$, and $C_0 - C_5 - C_6$. As we let the radius of C_0 increase, $r \rightarrow \infty$, we again obtain a resolution of the identity in the same way as in (4.11); viz.,

$$\psi(x, \mu) = +\frac{1}{2\pi i} \int_{\bigcup_{i=1}^4 B_i} (R_\lambda^\epsilon \psi)(x, \mu) d\lambda, \quad (5.2)$$

since the contributions of C_5 and C_6 are zero in the limit, as demonstrated in the previous section. It is now observed that the open contours B_3 and B_4 extend to infinity. Therefore we can reduce ϵ to the limit $+0$ such that the continuous spectrum $\sigma_c(L_\epsilon)$ becomes, in the limit,

$$\{\lambda: 1 \leq \lambda^2 < \infty\} = \sigma_c(L_0).$$

Thus Eq. (5.2) also exhibits the resolution of the identity for the operator L_0 because of Eq. (5.1) and Theorems 3.2 and 3.3; i.e.,

$$\psi(x, \mu) = \frac{1}{2\pi i} \int_{\bigcup_{i=1}^4 B_i} R_\lambda^0 \psi(x, \mu) d\lambda \quad \psi \in S_0. \quad (5.3)$$

It should be emphasized that in arriving at (5.3) we had to make two limiting processes in a well-defined order: first $r \rightarrow \infty$ to obtain (5.2), and then $\epsilon \rightarrow +0$ to obtain (5.3). It should now be clear why the auxiliary operator L_ϵ was introduced.

The analogy with functional analysis can be carried further. We define for any $\psi \in S_0$, pointwise in $(x, \mu) \in E_1 \times M_0$,

$$P_{\sigma_i} = -\frac{1}{2\pi i} \oint_{B(D_i)} R_{\lambda}^0 d\lambda, \quad (5.4)$$

where D_i is a Cauchy domain such that $\sigma_i \subset D_i$, $\sigma_k \cap D_i = 0$ for $i \neq k$, and σ_i is a part of $\sigma(L)$ such that $\bigcup_{(i)} \sigma_i = \sigma$, $\sigma_i \cap \sigma_k = 0$ ($i \neq k$). The operator

P_{σ_i} is called a projection operator, since

$$P_{\sigma_i} P_{\sigma_i} = P_{\sigma_i}. \quad (5.5)$$

The proof is simple. Observe that the order of integration below is interchangeable. Take two Cauchy domains D_i and D'_i such that $D'_i \subset D_i$ and $B(D_i) \cap B(D'_i) = 0$ then,

$$\begin{aligned} P_{\sigma_i} &= -\frac{1}{2\pi i} \oint_{B(D_i)} R_{\lambda}^0 d\lambda = -\frac{1}{2\pi i} \oint_{B(D'_i)} R_{\lambda}^0 d\lambda; \\ P_{\sigma_i} P_{\sigma_i} &= \left(\frac{1}{2\pi i}\right)^2 \oint_{B(D_i)} \oint_{B(D'_i)} R_{\lambda}^0 R_{\lambda'}^0 d\lambda d\lambda' \\ &= \left(\frac{1}{2\pi i}\right)^2 \oint_{B(D_i)} \oint_{B(D'_i)} \frac{R_{\lambda}^0 - R_{\lambda'}^0}{\lambda - \lambda'} d\lambda d\lambda' \\ &= \left(\frac{1}{2\pi i}\right)^2 \left\{ \oint_{B(D_i)} R_{\lambda}^0 \oint_{B(D'_i)} \frac{d\lambda'}{\lambda - \lambda'} d\lambda \right. \\ &\quad \left. - \oint_{B(D'_i)} R_{\lambda'}^0 \oint_{B(D_i)} \frac{d\lambda}{\lambda - \lambda'} d\lambda' \right\} \\ &= \left(\frac{1}{2\pi i}\right)^2 \left\{ 0 - 2\pi i \oint_{B(D'_i)} R_{\lambda'}^0 d\lambda' \right\} = P_{\sigma_i} \end{aligned}$$

Hence, P_{σ_i} is a projection operator, indeed. After the detailed investigations in the previous sections, it is clear that the Cauchy domains containing the continuous spectrum of L_0 can be extended to infinity without invalidating (5.5).

In a similar way, we can prove the orthogonality relation,

$$P_{\sigma_i} P_{\sigma_k} = 0 \quad \text{for } i \neq k. \quad (5.6)$$

Consider now that the boundaries B_i ($i = 1, 2, 3, 4$) of Fig. 1 can be defined as

$$B_i = -B(D_i),$$

where the $B(D_i)$ are defined as for Eq. (5.4). Therefore, as in classical spectral theory, we have immediately, from (5.2), the following decomposition theorem.

Decomposition Theorem

$$\psi(x, \mu) = \sum_{i=1}^4 P_{\sigma_i} \psi(x, \mu) \quad (\psi \in S_0). \quad (5.7)$$

In the next section, we specialize the contours of integration.

VI. AN EXPLICIT EXPANSION THEOREM AND ITS CONNECTION TO A PAPER BY K. M. CASE

We derive the explicit expressions of the projection operators P_{σ_i} for the point spectrum $\sigma_p = \{\lambda_0, -\lambda_0\}$. From its definition (5.4) and Eq. (3.22),

$$P_{\lambda_0} \psi(x, \mu) = -\frac{1}{2\pi i} \int_{B(D_{\lambda_0})} \frac{\mu \psi(x, \mu) + \frac{c}{2} J_0(x; \lambda)}{1 - \lambda \mu} d\lambda, \quad (6.1)$$

where

$$J_0(x; \lambda) = \lim_{\epsilon \rightarrow +0} J(x; \lambda) \text{ [Eq. (3.11)]}.$$

This is an analytic function in λ , and we can apply Cauchy's residue theorem. The first term $\mu \psi / (1 - \lambda \mu)$ under the integral is analytic at λ_0 , and therefore its residue is zero. The second term $(c/2) J_0 / (1 - \lambda \mu)$ is defined by letting $\epsilon \rightarrow +0$ in (3.11). Since we have shown that λ_0 for $c \neq 1$ is a simple root of the characteristic equation, it follows that λ_0 is a simple pole for $c \neq 1$. For $c = 1$, $\lambda_0 = 0$ is of multiplicity two. Hence the residue of the second term for $c \neq 1$ is

$$\text{res} \left(-\frac{\frac{c}{2} J_0(x; \lambda)}{1 - \lambda \mu} \right) = \frac{c/2}{1 - \lambda_0 \mu} \cdot \frac{\int_{-1}^{+1} \frac{\mu \psi(x, \mu)}{1 - \lambda_0 \mu} d\mu}{\frac{c}{2} \int_{-1}^{+1} \frac{\mu d\mu}{(1 - \lambda_0 \mu)^2}}. \quad (6.2)$$

We define the eigenfunctions ϕ_{λ_0} and $\phi_{-\lambda_0}$ in analogy to (3.8), as follows:

$$\phi_{\pm \lambda_0}(\mu) = \frac{c/2}{1 \mp \lambda_0 \mu} \quad (\mu \in M_0). \quad (6.3)$$

It is easily verified that these eigenfunctions satisfy the eigenvalue problem,

$$L_0 \phi = \lambda \phi, \quad \lambda \in \sigma_p = \{\lambda_0, -\lambda_0\}. \quad (6.4)$$

Multiplying numerator and denominator of the second fraction on the right-hand side of (6.2) by $c/2$ enables us to introduce ϕ_{λ_0} under the integral. Equation (6.1) therefore takes the final form

$$P_{\lambda_0} \psi(x, \mu) = a^+(x) \phi_{\lambda_0}(\mu), \quad (6.5)$$

where the Fourier coefficient $a^+(x)$ is given by

$$a^+(x) = \frac{\int_{-1}^{+1} \mu \psi(x, \mu) \phi_{\lambda_0}(\mu) d\mu}{\int_{-1}^{+1} \mu [\phi_{\lambda_0}(\mu)]^2 d\mu}. \quad (6.6)$$

A corresponding expression is obtained for $P_{-\lambda_0} \psi$ by replacing $a^+(x)$ and ϕ_{λ_0} by $a^-(x)$ and $\phi_{-\lambda_0}$, respectively, in (6.5) and (6.6).

To obtain an explicit form of the Projection operator P_{σ_c} over the continuous spectrum $\sigma_c = \{\lambda : 1 \leq \lambda^2 < \infty\}$, we let the boundaries B_3 and B_4 of Fig. 1 shrink to the band of width zero and bordering each side of σ_c . This limiting process forces us to put restriction (b) on our function space.

Consider first the positive part of σ_c , say $\sigma_c^+ = \{\lambda : 1 \leq \lambda < \infty\}$. The contour of the integral will be given by the following three parts:

$$\lambda' = \begin{cases} \lambda + is & \infty > \lambda \geq 1; \\ 1 + se^{i\omega} & \pi/2 > \omega > 3\pi/2; \\ \lambda - is & 1 \leq \lambda < \infty. \end{cases} \quad (6.7)$$

The positive direction, as explained in the previous section, will be opposite to that indicated in Fig. 1. Hence,

$$P_{\sigma_c^+} \psi = -\frac{1}{2\pi i} \left\{ \int_{\infty}^1 R_{\lambda'} \psi d\lambda' + \int_{\pi}^{3\pi/2} R_{\lambda'} \psi d\omega + \int_1^{\infty} R_{\lambda'} \psi d\lambda' \right\} \quad (6.8)$$

pointwise in $(x, \mu) \in E_1 \times M_0$.

Assume for the moment that μ in (6.8) is negative; i.e., $-1 \leq \mu < 0$. Then, as in (6.1), the first term of $(R_{\lambda} \psi)(x, \mu)$ is analytic, and the contour integral is zero. We now define, for a given function $F(z)$,

$$\lim_{s \rightarrow 0} F(\lambda \pm is) = F^{\pm}(\lambda) \quad (\lambda \text{ real}). \quad (6.9)$$

Then, the integrals $\int_{\infty}^1$ and $\int_1^{\infty}$ of (6.8) yield

$$\lim_{s \rightarrow 0} \int_0^1 + \int_1^\infty = - \int_1^\infty \frac{J_0^+(x, \lambda) - J_0^-(x, \lambda)}{1 - \lambda \mu}, \quad (6.10)$$

where $J_0(x, \lambda)$ is defined as in (6.1) by letting $\epsilon \rightarrow +0$, in (3.11). The remaining term in (6.8) is easily shown to be zero, if

$$\lim_{s \rightarrow 0} J_0(x; 1 + s e^{i\omega}) < \infty. \quad (6.11)$$

This will always be the case since, for $\psi \in S_0$,

$$\left| \int_{-1}^{+1} \frac{\mu \psi(\mu)}{1 - \lambda' \mu} d\mu \right| < \infty \quad (6.12)$$

for any λ' of a complex neighborhood of $\sigma(L_0)$.

If μ is positive, i.e., $0 < \mu \leq 1$, then the contour integral of the first term in $(R_\lambda^0 \psi)(x, \mu)$ does not vanish. The contour integral yields, by Cauchy's integration formula,

$$-\frac{1}{2\pi i} \int_{B(\sigma_c^+)} \frac{\mu \psi(x, \mu)}{1 - \lambda \mu} d\lambda = \frac{\psi(x, \mu)}{2\pi i} \int_{B(\sigma_c^+)} \frac{d\lambda}{\lambda - 1/\mu} = \psi(\mu). \quad (6.13)$$

The second term of R_λ^0 is also singular at $\lambda = 1/\mu$. Hence we exclude this point by two small semicircles of radius s' , as indicated in Fig. 2.

Fig. 2. The Singularity at $\lambda = 1/\mu$ of $B(\sigma_c)$

In the limit, the contribution of these semicircles is

$$\lim_{s' \rightarrow 0} \int \frac{J_0(x, \lambda')}{1 - \lambda' \mu} d\lambda' = J_0^+ \left(x, \frac{1}{\mu} \right) \lim_{s' \rightarrow 0} \int_0^\pi \frac{is' e^{i\omega}}{-\mu s' e^{i\omega}} = \frac{\pi i}{\mu} J_0^+ \left(x; \frac{1}{\mu} \right), \quad (6.14)$$

and

$$\lim_{s' \rightarrow 0} \int \frac{J_0(x; \lambda')}{1 - \lambda' \mu} d\lambda' = J_0^-\left(x, \frac{1}{\mu}\right) \lim_{s' \rightarrow 0} \int_{\pi}^{2\pi} \frac{s' e^{i\omega} d\omega}{-\mu s' e^{i\omega}} = -\frac{\pi i}{\mu} J_0^-\left(x; \frac{1}{\mu}\right). \quad (6.15)$$

The remainder of the line again yields (6.10), but the integral must be taken in the sense of Cauchy's principal value.

Analogous results are obtained for $P_{\sigma_C} \psi$. It is convenient to write just one expression for both parts of the continuous spectrum. Hence, collecting the terms (6.13) to (6.15), and (6.10) yields

$$P_{\sigma_C} \psi(x, \mu) = \psi(x, \mu) + \frac{1}{2\mu} \frac{c}{2} \left[J_0^+\left(x; \frac{1}{\mu}\right) + J_0^-\left(x; \frac{1}{\mu}\right) \right] + \frac{1}{2\pi i} \int_{\{\sigma_C\}} \frac{c/2}{1 - \lambda \mu} [J_0^+(x; \lambda) - J^-(x; \lambda)] d\lambda, \quad (6.16)$$

where

$$\{\sigma_C\} = \{\lambda : 1 \leq \lambda^2 < \infty\}.$$

The integrals must be taken in the sense of Cauchy's principal value.

Equation (6.16) can be brought into a more compact form. To accomplish, this we define

$$Q(\lambda') = 1 - \frac{c}{2} \int_{-1}^{+1} \frac{d\mu}{1 - \lambda' \mu}, \quad (\lambda' \text{ complex}) \quad (6.17)$$

and

$$K(\lambda) = 1 - \frac{c}{2} \int_{-1}^{+1} \frac{d\mu}{1 - \lambda \mu} = 1 - \frac{c}{\lambda} \tanh^{-1}(1/\lambda), \quad \lambda \in \{\lambda : 1 \leq \lambda^2 < \infty\}, \quad (6.18)$$

where the integral is in the sense of Cauchy's principal value. If we let λ' approach the real axis from above (and below) and avoid the pole at $\mu = 1/\lambda$ by an indentation of the contour from -1 to $+1$ below (and above) the real axis, then

$$Q^+(\lambda) = K(\lambda) + \frac{c\pi i}{2\lambda}$$

$$\lambda \in \{\lambda : 1 \leq \lambda^2 < \infty\} \quad (6.19)$$

$$Q^-(\lambda) = K(\lambda) - \frac{c\pi i}{2\lambda}$$

These equations are, of course, a special form of Plemelj's formulas.

Now we observe from Eq. (3.11) that

$$Q(\lambda') J_0(x; \lambda') = \int_{-1}^{+1} \frac{\mu \psi(x, \mu)}{1 - \lambda' \mu} d\mu. \quad (6.20)$$

As $\lambda' \rightarrow \lambda \pm i0$, Eq. (6.20) yields for $\lambda \in \{\lambda : 1 \leq \lambda^2 < \infty\}$, by Plemelj's formula,

$$Q^+(\lambda) J_0^+(x; \lambda) = -\frac{\pi i}{\lambda^2} \psi\left(x, \frac{1}{\lambda}\right) + \int_{-1}^{+1} \frac{\mu \psi(x, \mu) d\mu}{1 - \lambda \mu}, \quad (6.21)$$

and

$$Q^-(\lambda) J_0^-(x; \lambda) = +\frac{\pi i}{\lambda^2} \psi\left(x, \frac{1}{\lambda}\right) + \int_{-1}^{+1} \frac{\mu \psi(x, \mu) d\mu}{1 - \lambda \mu}, \quad (6.22)$$

where the integrals on the right-hand side are in the sense of Cauchy's principal value. The difference of these two equations, under consideration of (6.19), is

$$K(\lambda) [J_0^+(x; \lambda) - J_0^-(x; \lambda)] + \frac{c\pi i}{2\lambda} [J_0^+(x; \lambda) + J_0^-(x; \lambda)] = -\frac{2\pi i}{\lambda^2} \left(x, \frac{1}{\lambda}\right). \quad (6.23)$$

Definition

$$A(x, \lambda) = \frac{1}{2\pi i} [J_0^+(x; \lambda) - J_0^-(x; \lambda)]. \quad (6.24)$$

Substituting first (6.24) into (6.23), then $\frac{1}{\mu}$ for λ , and rearranging terms yields

$$-\frac{1}{\mu^2} K\left(\frac{1}{\mu}\right) A\left(x, \frac{1}{\mu}\right) = \psi(x, \mu) + \frac{1}{2\mu} \frac{c}{2} \left[J_0^+\left(x, \frac{1}{\mu}\right) + J_0^-\left(x, \frac{1}{\mu}\right) \right]. \quad (6.25)$$

We now substitute (6.24) and (6.25) into (6.16) and obtain the projection,

$$P_{\sigma_c} \psi(x, \mu) = -\frac{1}{\mu^2} K\left(\frac{1}{\mu}\right) A\left(x, \frac{1}{\mu}\right) + \int_{\{\sigma_c\}} A(x, \lambda) \frac{c/2}{1 - \lambda\mu} d\lambda, \quad (6.26)$$

where the integral over σ_c is in the sense of Cauchy's principal value. It is of interest to mention that

$$\lim_{\lambda \rightarrow \infty} A(x, \lambda) = \lim_{\mu \rightarrow 0} A\left(x, \frac{1}{\mu}\right) = 0, \quad (6.27)$$

as can be concluded from (6.24).

To obtain our final form of the expansion theorem we define the scalar product for any two functions, $f(\mu)$ and $g(\mu)$ as follows:

$$(f, g) = \int_{-1}^{+1} \mu' f(\mu') g(\mu') d\mu'. \quad (6.28)$$

We note that (f, f) is not positive definite.

The results of this section are summarized in the following expansion theorem.

Expansion Theorem

Any function $\psi(x, \mu) \in S_0$ has the unique resolution

$$\begin{aligned} \psi(x, \mu) = & a^+(x) \phi_{\lambda_0}(\mu) + a^-(x) \phi_{-\lambda_0}(\mu) - \frac{1}{\mu^2} K\left(\frac{1}{\mu}\right) A\left(x, \frac{1}{\mu}\right) \\ & + \int_{\{\sigma_c\}} A(x, \lambda) \phi_{\lambda}(\mu) d\lambda, \end{aligned} \quad (6.29)$$

where

$$\phi_{\lambda}(\mu) = \frac{c}{2} \frac{1}{1 - \lambda\mu}, \quad (\lambda \text{ real or complex})$$

$$a^{\pm}(x) = \frac{(\psi, \phi_{\pm\lambda_0})}{(\phi_{\pm\lambda_0}, \phi_{\pm\lambda_0})}, \quad (\pm\lambda_0 \in \sigma_p)$$

$$A(x, \lambda) = - \frac{K(\lambda) \psi\left(x, \frac{1}{\lambda}\right) + \lambda(\psi, \phi_\lambda)}{\lambda^2 K^2(\lambda) + c^2 \pi^2 / 4}, \quad (\lambda \in \sigma_c)$$

$$K(\lambda) = 1 - \frac{c}{\lambda} \tanh^{-1}(1/\lambda), \quad (\lambda \in \sigma_c)$$

$$(\psi, \phi_\lambda) \equiv \int_{-1}^{+1} \mu \psi(x, \mu) \phi_\lambda(\mu) d\mu,$$

$$\sigma_p = \left\{ \pm \lambda_0 : \frac{c}{\lambda_0} \tanh^{-1} \lambda_0 = 1; \lambda_0 \notin \cup_c \right\},$$

and

$$\sigma_c = \{\lambda : 1 \leq \lambda^2 < \infty\}.$$

The integrals are taken in the sense of Cauchy's principal value when necessary.

The expression for $A(x, \lambda)$ above follows from (6.18), (6.19), (6.21), (6.22), and (6.24).

We show the connection of (6.29) with the expansion formula in a paper by K. M. Case [Ref. 3, Eqs. (23) to (27)]. In (6.29), substitute $-\nu^2 A^{(c)}_\nu$ for $A(x, \lambda)$, where $\nu = 1/\lambda$, and change variables in the integral. Then,

$$\psi = a^+ \phi_{1/\nu_0} + a^- \phi_{1/\nu_0} + [1 - c \mu \tanh^{-1} \mu] A^{(c)}(\mu) + \frac{c}{2} \int_{-1}^{+1} \frac{\nu A^{(c)}(\nu) d\nu}{\nu - \mu} \quad (6.30)$$

is exactly the form obtained by Case, where the index (c) refers to the function $A(\nu)$ of his paper. Furthermore [omitting x in $A(x, \lambda)$, $J_0(x; \lambda)$, and $\psi(x, \mu)$],

$$\begin{aligned} \frac{c}{2} \nu A^{(c)}(\nu) &= - \frac{c}{2} \lambda A(\lambda) \\ &= - \frac{\frac{c}{2}}{2\pi i} \cdot \frac{1}{\nu} \left[J_0^+\left(\frac{1}{\nu}\right) - J_0^-\left(\frac{1}{\nu}\right) \right] \\ &= N^+(\nu) - N^-(\nu), \end{aligned}$$

and

$$N(z) = \frac{1}{2\pi i} \frac{\frac{c}{2} \int_{-1}^{+1} \frac{\mu \psi(\mu)}{z - \mu} d\mu}{1 - \frac{c}{2} z \int_{-1}^{+1} \frac{d\mu}{z - \mu}} = - \frac{1}{2\pi i} \cdot \frac{c}{2} \cdot \frac{1}{z} \cdot J_0\left(\frac{1}{z}\right).$$

$N(z)$ refers to Eqs. (31) and (40) of Ref. 3, but Case uses the function

$$\psi' = \psi - a^+ \phi_1 / \nu_0 - a^- \phi_{-1} / \nu_0$$

instead of ψ . Because of the orthogonality relations (5.6), both functions yield the same result in (6.30). Case used a function space that is contained in S_0 .

VII. THE INVERSE OF THE TRANSPORT OPERATOR

We use the results of the previous sections to obtain a general solution to the problem,

$$T\phi = (D + L_0)\phi = \psi, \quad (\psi \in S_0), \quad (7.1)$$

where $D = \partial/\partial x$ is a differential operator and T is the operator referred to in Eq. (1.9). The last condition (c) in the definition of S_0 is sufficient to ensure that the operators D and L_0 are commutative, since then $\partial/\partial x \int \phi(x, \mu) d\mu = \int (\partial\phi/\partial x) d\mu$. We recall Eq. (5.7) in the form,

$$\psi = P_{\lambda_0} \psi + P_{-\lambda_0} \psi + P_{\sigma_c} \psi, \quad (7.2)$$

which holds pointwise in $(x, \mu) \in E_1 \times M_0$ for any $\psi \in S_0$.

The inverse of the sum of commutative operators is treated in the literature as, for example, by Friedman (Ref. 6, pp. 259-285). We generalize the known results to our operator.

General Inversion Theorem

If the inverse of $(D + \lambda I)$ exists for $\lambda = \pm\lambda_0$ and all points of $B(\sigma_c(L_0))$ and its interior, then the inverse of the operator T defined by Eq. (7.1),

$$(D + L_0)\phi = \psi, \quad (\psi \in S_0)$$

is given by

$$\phi(x, \mu) = (D + \lambda_0)^{-1} a^+(x) \phi_{\lambda_0}(\mu) + (D - \lambda_0)^{-1} a^-(x) \phi_{-\lambda_0}(\mu) \quad (7.3)$$

$$- \frac{1}{2\pi i} \int_{B(\sigma_c)} (D + \lambda)^{-1} (R_{\lambda}^0 \psi)(x, \mu) d\lambda,$$

where $a^+(x), \phi_{\pm\lambda_0}$, and R_{λ}^0 are defined by Eqs. (6.6), (6.3), and (3.22), respectively.

Proof

The validity of this theorem is best shown by verification. To this end, we operate on the left- and right-hand sides of Eq. (7.3) with $D + L_0$. We treat the different parts separately. Equations (7.4) to (7.6)

are considered to hold pointwise in $(x, \mu) \in E_1 \times M_0$. We leave the argument (x, μ) out to give the equations a simpler appearance. Hence,

$$\begin{aligned}
 (D + L_0) (D + \lambda_0)^{-1} a^+ \phi_{\lambda_0} &= [(D + \lambda_0) + (L_0 - \lambda_0)] (D + \lambda_0)^{-1} a^+ \phi_{\lambda_0} \\
 &= a^+ \phi_{\lambda_0} + (D + \lambda_0)^{-1} (L_0 - \lambda_0) a^+ \phi_{\lambda_0} \\
 &= a^+ \phi_{\lambda_0} = P_{\lambda_0} \psi,
 \end{aligned} \tag{7.4}$$

since $DL_0 = L_0D$ and $(L_0 - \lambda_0)\phi_{\lambda_0} = 0$. Similarly,

$$(D + L_0) (D - \lambda_0)^{-1} a^- \phi_{-\lambda_0} = P_{-\lambda_0} \psi. \tag{7.5}$$

Finally,

$$\begin{aligned}
 (D + L_0) \int_{B(\sigma_c)} (D + \lambda)^{-1} R_{\lambda}^0 \psi d\lambda &= \int_{B(\sigma_c)} (D + \lambda) (D + \lambda)^{-1} R_{\lambda}^0 \psi d\lambda \\
 &\quad + \int_{B(\sigma_c)} (D + \lambda)^{-1} (L_0 - \lambda) R_{\lambda}^0 \psi d\lambda \\
 &= \int_{B(\sigma_c)} R_{\lambda}^0 \psi d\lambda + \int_{B(\sigma_c)} (D + \lambda)^{-1} \psi d\lambda \\
 &= -2\pi i P_{\sigma_c} \psi,
 \end{aligned} \tag{7.6}$$

since the second integral vanishes.

We conclude, from (7.4) to (7.6), that

$$(D + L_0) \phi(x, \mu) = P_{\lambda_0} \psi(x, \mu) + P_{-\lambda_0} \psi(x, \mu) + P_{\sigma_c} \psi(x, \mu) = \psi(x, \mu),$$

which confirms the theorem.

To exemplify the inversion theorem, we solve explicitly the following general problem. Given is the transport Eq. (1.8),

$$\mu \frac{\partial \phi}{\partial x} + \phi = \frac{c}{2} \int_{-1}^{+1} \phi(x, \mu') d\mu' + Q(x, \mu), \quad (7.7)$$

where at the boundary $x = x_0$, the flux $\phi(x, \mu)|_{x=x_0} = \phi_0(\mu)$ is known. First we solve

$$\left. \begin{aligned} T\phi &= \psi; & \psi &= \frac{1}{\mu} Q(x, \mu) \in S_0 \\ \phi(0, \mu) &= \phi_0(\mu) \end{aligned} \right\} \quad (7.8)$$

and then extend $T\phi(x, \mu)$ such that $\mu = 0$ is included. The answer for the first part is given by

$$\phi = T^{-1}\psi \quad (7.9)$$

and obtained in the following four steps:

Step 1. We expand $\psi(x, \mu)$ according to the expansion theorem. We obtain the following one-to-one mapping from ψ to the set of Fourier coefficients $a^+(x)$, $a^-(x)$, $A(x, \lambda)$:

$$\psi(x, \mu) \Leftrightarrow \{a^+(x), a^-(x), A(x, \lambda)\}. \quad (7.10)$$

Similarly, we have

$$\phi_0(\mu) \Leftrightarrow \{b_0^+, b_0^-, B_0(\lambda)\}, \quad (7.11)$$

where the coefficients are independent of x .

Step 2. We find the inverse $(D + \lambda)^{-1}$. By definition, we have

$$(D + \lambda)u = \left(\frac{\partial}{\partial x} + \lambda \right) u = v; \quad u(x_0) = u_0. \quad (7.12)$$

As is well known, the inverse is given by

$$u(x) = (D + \lambda)^{-1}v = e^{-(x-x_0)\lambda} \left[u_0 + \int_{x_0}^x v(\xi) e^{(x-\xi)\lambda} d\xi \right]. \quad (7.13)$$

This inverse satisfies the hypothesis of the Inversion Theorem.

Step 3. We insert (7.13) into (7.3) of the inversion theorem. The first term in (7.3) is therefore

$$(D + \lambda_0)^{-1} a^+(x) \phi_{\lambda_0}(\mu) = e^{-(x-x_0)\lambda_0} \left[u_0 + \int_{x_0}^x a^+(\xi) e^{(x-\xi)\lambda_0} d\xi \phi_{\lambda_0}(\mu) \right]. \quad (7.14)$$

For $\phi(x, \mu)$, there exist Fourier coefficients, say

$$\phi(x, \mu) \Leftrightarrow \{b^+(x), b^-(x), B(x, \lambda)\}. \quad (7.15)$$

It follows from (7.14) that, for $x = x_0$,

$$(D + \lambda_0)^{-1} a^+(x) \phi_{\lambda_0}(\mu) = u_0 \quad (x = x_0).$$

Hence,

$$u_0 = b_0^+ \phi_{\lambda_0}(\mu) \quad (b_0^+ = b^+(x_0)).$$

Therefore,

$$b^+(x) = e^{-(x-x_0)\lambda_0} \left[b_0^+ + \int_{x_0}^x a^+(\xi) e^{(x-\xi)\lambda_0} d\xi \right]. \quad (7.16)$$

Similarly, for the second term,

$$b^-(x) = e^{(x-x_0)\lambda_0} \left[b_0^- + \int_{x_0}^x a^-(\xi) e^{-(x-\xi)\lambda_0} d\xi \right], \quad (7.17)$$

and for the third term, as given by the expansion theorem,

$$B(x, \lambda) = e^{-(x-x_0)\lambda} \left[B_0(\lambda) + \int_{x_0}^x A(\xi, \lambda) e^{(x-\xi)\lambda} d\xi \right] \quad (7.18)$$

for $\lambda \in \{\sigma_c\} = \{\lambda: 1 < \lambda^2 \leq \infty\}$.

Step 4 (Final Solution). We have obtained the Fourier coefficient for (x, μ) over the domain,

$$E_1 \times M_0 = \{x, \mu: 0 \leq x^2 < \infty; 0 < \mu^2 \leq 1\}. \quad (7.19)$$

Over this domain, as in (6.29),

$$\left. \begin{aligned} \phi(x, \mu) &= b^+(x) \phi_{\lambda_0}(\mu) + b^-(x) \phi_{-\lambda_0}(\mu) + G(x, \mu); \\ G(x, \mu) &\equiv - \frac{1 - c\mu \tanh^{-1}\mu}{\mu^2} B\left(x, \frac{1}{\mu}\right) + \int_{\{\sigma_c\}} B(x, \lambda) \phi_{\lambda}(\mu) d\lambda. \end{aligned} \right\} \quad (7.20)$$

For $\mu = 0$, Eq. (7.7) reduces to

$$\phi(x, 0) = \frac{c}{2} \int_{-1}^{+1} \phi(x, \mu') d\mu' + Q(x, 0), \quad (7.21)$$

since the derivative of ϕ with respect to x is required to exist.

Hence, we have found in Eqs. (7.16) to (7.21) the general solution to the transport Eq. (7.7) in closed form.

VIII. ANISOTROPIC SCATTERING

One objective in taking the functional analysis approach in this paper was in the expectation that a generalization to the anisotropic case should be relatively easy. This seems to be true. For anisotropic scattering, Eq. (1.8) becomes

$$\mu \frac{\partial \phi}{\partial x} + \phi = c \int_{-1}^{+1} f(\mu, \mu') \phi(x, \mu') d\mu' + Q(x, \mu), \quad (8.1)$$

and the operator L_0 is replaced by L , defined as

$$L\phi = \frac{\phi - c \int_{M_0} f(\mu, \mu') \phi(\mu') d\mu'}{\mu}. \quad (8.2)$$

The eigenvalue problem corresponding to (8.2) is given by

$$\mu(L - \lambda)\phi = (1 - \lambda\mu)\phi - c \int_{M_0} f(\mu, \mu') \phi(\mu') d\mu' = 0. \quad (8.3)$$

We restrict our attention to the physically interesting case where the scattering kernel $f(\mu, \mu')$ depends only on the angle between the directions of the incoming flux, $\phi(\mu')$, and the scattered flux, $\phi(\mu)$. Because $|f(\mu, \mu')| < \infty$, $f(\mu, \mu')$ is square integrable over $-1 \leq \mu, \mu' \leq 1$, and hence in $L_2(-1, 1)$. It is known (Refs. 1, p. 6; 4, p. 232) that $f(\mu, \mu')$ can be expanded as

$$f(\mu, \mu') = \sum_{n=0}^{\infty} \frac{2n+1}{2} \beta_n P_n(\mu) P_n(\mu'); \quad (\beta_0 = 1) \text{ in } L_2; \quad (8.4)$$

where $P_n(\mu)$ are Legendre polynomials. We observe that

$$\int_{-1}^{+1} f(\mu, \mu') d\mu = \int_{-1}^{+1} f(\mu, \mu') d\mu' = 1, \quad (8.5)$$

since

$$\int_{-1}^{+1} P_\ell(\mu) P_m(\mu) d\mu = \frac{2\delta_{\ell m}}{2\ell + 1}. \quad (8.6)$$

Anisotropic scattering for the transport equation has been treated in the literature for scattering functions in the form of a polynomial of

finite degree (see Refs. 7,8). For the behavior of the spectrum of L for a scattering kernel of the form (8.3) such that $f(\mu, \mu') > 0$, Davison gave an account (Ref. 4, p. 242), which we rephrase as follows:

The continuous spectrum is the same for L_0 and L ; i.e.,

$$\sigma_c(L) = \sigma_c(L_0). \quad (8.7)$$

In the case of isotropic scattering, the point spectrum $\sigma_p(L_0)$ has just one pair of elements. In the case of anisotropic scattering, this is no longer necessarily true; examples have been constructed in which $f(\mu, \mu')$ is non-negative and linear in μ, μ' , but if c is large enough, a coefficient β_1 can be found such that $\sigma_p(L)$ has two pairs of elements in the cut plane. Further, for any c , if N is large enough that $\beta_N \neq 0$, then $b_1, b_2, \dots, \beta_N$ can be chosen so that $\sigma_p(L)$ has any prescribed number k of pairs of points ($k \geq 1$), though $f(\mu, \mu')$ remains nonnegative. However, these situations have not yet been encountered in practical applications; all practical problems hitherto considered have been such that $\sigma_p(L)$ has just one pair of elements, as in the case of isotropic scattering. We summarize below the available information on this topic:

When all the β_n ($n \geq 1$) are negligibly small, the situation is identical with isotropic scattering. As the β_n increase, a point is reached when two more elements appear at the ends of $\sigma_c(L)$, i.e., at ± 1 , and begin to move away from $\sigma_c(L)$ along the real axis. When the β_n increase further, two more points appear at the ends of $\sigma_c(L)$, and so on. However, unless c is very large, all these extra points are close to the ends of $\sigma_c(L)$.

Davison obtained these results by physical considerations.

We shall restrict ourselves to the proof of the following statements:

- (a) $\sigma_c(L) = \sigma_c(L_0)$.
- (b) $\lambda \in \sigma_p(L) \Rightarrow -\lambda \in \sigma_p(L)$ and $\bar{\lambda} \in \sigma_p(L)$. If $\phi_\lambda(\mu)$ is an eigenfunction of L , then $\phi_\lambda(-\mu) = \phi_{-\lambda}(\mu)$ is an eigenfunction of $-L$.
- (c) If there are exactly two elements in $\sigma_p(L)$, they are either real or pure imaginary.
- (d) If $c < 1$ and $f(\mu, \mu') > 0$, there exists a pair of real elements $\pm\lambda \in \sigma_p(L)$ such that $|\lambda| < 1$.
- (e) If $\sigma_p(L)$ contains pure imaginary elements and $c\beta_n < 1$ ($n = 1, 2, \dots$), then $c > 1$.

To prove the statement (a), we consider

$$(L - \lambda)\phi = \psi \quad \phi, \psi \in S_0.$$

Then, applying the same reasoning as in the isotropic case, we find

$$\phi = (L - \lambda)^{-1}\psi = \frac{\mu\psi + c \int_{M_0} f(\mu, \mu')\phi}{1 - \lambda\mu},$$

which is unbounded if and only if $1 \leq \lambda^2 < \infty$. Hence,

$$\sigma_c(L) = \sigma_c(L_0). \quad \text{q.e.d.} \quad (8.8)$$

We prove statement (b). The first part is reformulated in the following lemma.

Lemma 8.1

If λ is an eigenvalue of (8.3), then $-\lambda$ and $\bar{\lambda}$ are also eigenvalues of (8.3).

Proof. The scattering function (8.4) also has the property

$$f(\mu, \mu') = f(-\mu, -\mu'). \quad (8.9)$$

By changing variables $\mu \rightarrow -\mu$, $\mu' \rightarrow -\mu'$, and letting $\lambda \rightarrow -\lambda$ in (8.3), we see that this equation remains unchanged because of (8.9); explicitly,

$$-\mu(L + \lambda)\phi = (1 - \lambda\mu)\phi(-\mu) - c \int_{M_0} f(-\mu, -\mu')\phi(-\mu')d\mu' = 0. \quad (8.10)$$

Furthermore, if $\phi_\lambda(\mu)$ is an eigenfunction for the eigenvalue λ , then $\phi_\lambda(-\mu) = \phi_{-\lambda}(\mu)$ is an eigenfunction for the eigenvalue $-\lambda$. Similarly, one deduces from (8.3) that if λ is a complex eigenvalue, then $\bar{\lambda}$ is also an eigenvalue; i.e.,

$$(L - \lambda)\phi = 0 \Rightarrow (L - \bar{\lambda})\bar{\phi} = 0. \quad (8.11)$$

Statement (c) follows from Lemma 8.1. If there are exactly two discrete eigenvalues, say λ_1 and λ_2 , we conclude that they must be either real or pure imaginary to satisfy $\lambda_2 = -\lambda_1 = \pm\bar{\lambda}_1$.

The following remarks will prove helpful. Let

$$J(\mu) = \int_{-1}^{+1} f(\mu, \mu')\phi(\mu')d\mu'. \quad (8.12)$$

As in the case of isotropic scattering, we obtain, from (8.3),

$$\phi = \frac{cJ(\mu)}{1 - \lambda\mu}. \quad (8.13)$$

Introducing (8.13) into (8.12) leads to the characteristic equation for anisotropic scattering,

$$J(\mu) = c \int_{-1}^{+1} \frac{f(\mu, \mu')}{1 - \lambda\mu'} J(\mu') d\mu'. \quad (8.14)$$

We integrate (8.12) and (8.14) over μ , under consideration of (8.5) and the assumption that the order of integration can be interchanged (i.e., the functions have continuous derivatives). This yields

$$\int_{-1}^{+1} J(\mu) d\mu = \int_{-1}^{+1} \phi(\mu') d\mu' = 1; \quad (8.15)$$

$$\int_{-1}^{+1} J(\mu) d\mu = c \int_{-1}^{+1} \frac{J(\mu')}{1 - \lambda\mu'} d\mu' = 1. \quad (8.16)$$

We have normalized $\phi(\mu)$ so that (8.15) yields one.

In the case of isotropic scattering, $J(\mu) = 1/2$, both (8.14) and (8.16) reduce to the characteristic Eq. (A.1).

$$1 - \frac{c}{2} \int_{-1}^{+1} \frac{d\mu'}{1 - \lambda\mu'} = 0. \quad (8.17)$$

After these remarks, we prove statement (d).

Condition (8.4) implies

$$f(\mu, \mu') = f(\mu', \mu). \quad (8.18)$$

Hence, if we let

$$\left. \begin{aligned} U(\mu) &= \frac{J(\mu)}{\sqrt{1 - \lambda\mu}}, \\ K_{\lambda}(\mu, \nu) &= \frac{f(\mu, \nu)}{\sqrt{1 - \lambda\mu} \sqrt{1 - \lambda\nu}} \end{aligned} \right\} \quad (8.19)$$

and λ be real such that $|\lambda| < 1$, Eq. (8.14) becomes a Fredholm equation with symmetric kernel,

$$U(\mu) = c \int_{-1}^{+1} K_{\lambda}(\mu, \nu) U(\nu) d\nu, \quad (8.20)$$

and we can apply Rellich's perturbation theory for any sequence

$$\{K_{\lambda_n}\} \rightarrow K_{\lambda}.$$

It follows then that the corresponding sequence

$$\{c_{\lambda_n}\} \rightarrow c_{\lambda} = c,$$

and therefore c is an analytic function $c(\lambda)$ of λ in $(-1, 1)$.

We assume now that $f(\mu, \mu') > 0$. Then, for $0 \leq \lambda < 1$, the kernel of (8.14) is positive, and by Jentsch's theorem there exists a simple positive c of minimal absolute value such that (8.14) is satisfied. For $\lambda = 0$, Eq. (8.16) yields $c = 1$, and for $\lambda \rightarrow 1$, we obtain $c \rightarrow +0$. Since we proved that c is a continuous function of λ , we have shown that there exists for each $0 < c < 1$ a real eigenvalue, $0 < \lambda < 1$. Similarly, one shows that for each $0 < c < 1$ there exists an eigenvalue, $-1 < \lambda < 0$, which is in agreement with Lemma 8.1. This concludes the proof of statement (d). In addition, one asserts the existence of $J(\mu)$ from (8.20) and (8.19).

Before proving the statement (e), we describe the characteristic equation in a different form.

Substitute the right-hand side of (8.4) into (8.12). This yields

$$J(\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \beta_n P_n(\mu) \int_{-1}^{+1} P_n(\mu') \phi(\mu') d\mu'. \quad (8.21)$$

We note that

$$\int_{-1}^{+1} P_n(\mu') \phi(\mu') d\mu' = \alpha_n \quad (8.22)$$

are the Fourier coefficients of $\phi(\mu)$ such that

$$\phi(\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \alpha_n P_n(\mu). \quad (8.23)$$

From the right-hand side of (8.15), it follows that

$$\alpha_0 = 1. \quad (8.24)$$

For later reference, we note that (8.21) becomes (because $\alpha_0 = \beta_0 = 1$)

$$J(\mu) = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} (2n+1) \alpha_n \beta_n P_n(\mu) \right]. \quad (8.25)$$

We insert (8.4) and (8.23) into (8.3) and obtain, by virtue of (8.6),

$$\mu(L - \lambda)\phi = \sum_{n=0}^{\infty} \frac{2n+1}{2} \{(1 - \lambda\mu)\alpha_n - c\alpha_n\beta_n\} P_n(\mu) = 0. \quad (8.26)$$

Remembering the identity

$$(2n+1)\mu P_n(\mu) = (n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)$$

and (8.6), we can, by standard methods, transform (8.26) into the following infinite system of equations in α_n :

$$\left. \begin{aligned} -\frac{1}{\lambda}(1 - c\beta_0)\alpha_0 + \alpha_1 &= 0 \quad \text{where } (\alpha_0 = \beta_0 = 1); \\ n\alpha_{n-1} - \frac{2n+1}{\lambda}(1 - c\beta_n)\alpha_n + (n+1)\alpha_{n+1} &= 0 \quad (n = 1, 2, 3, \dots). \end{aligned} \right\} \quad (8.27)$$

This is also a linear difference equation for the α_n .

For later reference, we note

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= \frac{1-c}{\lambda}, \\ \alpha_2 &= \frac{1}{2} \left[\frac{3(1-c)(1-c\beta_1)}{\lambda^2} - 1 \right], \\ \alpha_3 &= \frac{1}{6\lambda} \left\{ \frac{15(1-c)(1-c\beta_1)(1-c\beta_2)}{\lambda^2} - [4(1-c) + 5(1-c\beta_3)] \right\}, \end{aligned} \quad (8.28)$$

and so on.

The characteristic equation of the difference Eq. (8.27) for $n \rightarrow \infty$ is

$$x^2 - \frac{2}{\lambda}x + 1 = 0,$$

which shows that absolute convergence of the α_n cannot be proved, since

$$\lim_{n \rightarrow \infty} \max |\alpha_n| > 1 \text{ for } \lambda \neq 1.$$

Our aim is to obtain a relationship between λ , c , and $\{\beta_n\}$ where the α_n are eliminated. To this end, we reformulate (8.27) by defining

$$\rho_n = -\lambda \frac{\alpha_{n+1}}{\alpha_n} \quad (8.29)$$

and introducing (8.29) into (8.27). After some rearranging, we obtain

$$\rho_{n-1} = \frac{-\frac{n}{n+1} \lambda^2}{\frac{2n+1}{n+1} (1 - c\beta_n) + \rho_n} \quad (n = 1, 2, 3, \dots). \quad (8.30)$$

From (8.28) and (8.29),

$$c = 1 + \rho_0. \quad (8.31)$$

Substituting (8.30) into (8.31) yields the following expansion into continued fractions:

$$\begin{aligned} c &= 1 + \rho_0 \\ &= 1 - \frac{\frac{1}{2} \lambda^2}{\frac{3}{2} (1 - c\beta_1) + \rho_1} \\ &= 1 - \frac{\frac{1}{2} \lambda^2}{\frac{3}{2} (1 - c\beta_1) - \frac{\frac{2}{3} \lambda^2}{\frac{5}{3} (1 - c\beta_2) - \frac{\frac{3}{4} \lambda^2}{\frac{7}{4} (1 - c\beta_3) - \dots}} \end{aligned} \quad (8.32)$$

and when $\sum_{i=0}^{\infty} \beta_i^2 < \infty$, it can be shown by standard methods that (8.32) converges in the entire complex plane, except for the cut $1 \leq \lambda^2 < \infty$, i.e., on $\sigma_c(L)$. We

give a short outline for the proof of this statement in Appendix B. For isotropic scattering we have

$$\beta_i = 0, \quad (i = 1, 2, 3, \dots) \quad (8.33)$$

and the right-hand side of (8.32) reduces, after an equivalence transformation (see Appendix B), to the continued-fraction expansion of

$$\frac{\lambda}{\tanh^{-1}\lambda} = 1 - \frac{\lambda^2}{3 - \frac{4\lambda^2}{5 - \frac{9\lambda^2}{7 - \dots}}} \quad (8.34)$$

as given in Wall, Ref. 12, p. 342. Combining the left-hand sides of (8.32) and (8.34) yields

$$\frac{c}{\lambda} \tanh^{-1} \lambda = 1,$$

which is the characteristic Eq. (A.1) for the case of isotropic scattering.

If we exclude $\sigma_c(L)$ from the complex plane, Eq. (8.32) can also be considered the characteristic equation for the case of anisotropic scattering (dispersion relation).

Now, to verify statement (e), we let $\lambda = \pm i|\lambda|$ in (8.32). Then, considering the assumption $1 - c\beta_n > 0$, the continued fraction in (8.32) contains only positive members, and c must necessarily satisfy $c > 1$.

When (8.4) contains only a few terms, different approaches can be taken to determine $\sigma_p(L)$. If we insert (8.25) into (8.16), we obtain, as the characteristic equation,

$$\frac{c}{2} \sum_{n=0}^N (2n+1) \alpha_n \beta_n \int_{-1}^{+1} \frac{P_n(\mu)}{1 - \lambda\mu} d\mu = 1, \quad (\alpha_0 = \beta_0 = P_0(\mu) = 1) \quad (8.35)$$

where the α_n are given by (8.28) as functions of λ , say $\alpha_n = \alpha_n(\lambda)$. This equation corresponds to Eq. (2.11) of Ref. 8 and was known to Davison.⁽⁴⁾ The eigenfunctions for the roots $\pm\lambda_i$ of (8.35) are then given by (8.13), or explicitly

$$\phi_{\pm\lambda_i}(\mu) = \frac{c}{2} \frac{1}{1 \pm \lambda_i \mu} \left\{ 1 + \sum_{n=1}^N (2n+1) \alpha_n(\pm\lambda_i) \beta_n P_n(\mu) \right\}. \quad (8.36)$$

These functions are elements of S_0 . Hence we can apply the methods of the previous sections in this case.

In concluding this section, we add the following three remarks:

(a) Let

$$\phi = \frac{V(\mu)}{\sqrt{1 - \lambda\mu}}; \quad K_\lambda(\mu, \nu) = \frac{f(\mu, \nu)}{\sqrt{1 - \lambda\mu}\sqrt{1 - \lambda\nu}}; \quad (8.37)$$

and substitute in (8.3). We obtain

$$V(\mu) - c \int_{-1}^{+1} K_\lambda(\mu, \nu) V(\nu) d\nu = 0. \quad (8.38)$$

This equation is identical to (8.20). Hence $V \approx U$. Substitution of ϕ and J in (8.13), using (8.37) and (8.19), yields

$$V = cU. \quad (8.39)$$

(b) We note that, in (8.20) and (8.38),

$$K_\lambda(\mu, \nu) = K_\lambda(\nu, \mu).$$

This means, in the language of the theory of integral equations, that the kernel is complex symmetric, but not hermitian symmetric. Hence, (8.38) is not self-adjoint but is identical with its associated equation.

(c) One can give sufficient conditions in the case of $c > 1$ and anisotropic scattering for the existence of one pair of pure imaginary eigenvalues, and all other pairs real. If we start from the isotropic case we let the coefficients β_1 of (8.4) grow until one pair of additional points, which must be real, emerge from the continuous spectrum at the points ± 1 . Then we can vary β_1 and β_2 so that only one second pair, which must necessarily be real, appear at ± 1 , and so on. However, we were not able to prove the following conjecture:

If $f(\mu, \mu')$ is given by (8.4), then there are at most two pure imaginary points in $\sigma_p(L)$, while all other elements, if any, are real.

APPENDIX A

The Characteristic Equation (3.6)

It is well-known that the equation

$$\frac{c}{z} \tanh^{-1} z \equiv \frac{c}{2} \int_{-1}^{+1} \frac{d\mu}{1 - z\mu} = 1 \quad (\text{A1})$$

has exactly two solutions $z = \pm \lambda_0$ in the complex z -plane with cut from $-\infty$ to -1 and from 1 to ∞ . This statement is a special case of the following lemma.

Lemma A1.

If the operator $\oint \cdot d\mu$ means

$$\oint \psi(\mu) d\mu = \int_{-1/\epsilon}^{+\epsilon} \psi(\mu) d\mu + \int_{\epsilon}^1 \psi(\mu) d\mu,$$

then the equation

$$\frac{c}{2} \oint \frac{d\mu}{1 - z\mu} = 1 \quad (\text{A2})$$

has exactly two solutions, $\pm \lambda_0^\epsilon$, in the complex z -plane with the cuts from $-1/\epsilon$ to -1 and 1 to $1/\epsilon$ for $0 < \epsilon < 1$. For $c > 1/(1 - \epsilon)$, the solutions are pure imaginary; for $c = 1/(1 - \epsilon)$, the solutions are both zero; and for $c < 1/(1 - \epsilon)$, the solutions are real.

Our proof consists of three parts. First we conclude from the symmetry of (A2) that if $z = z_0$ is a solution of (A2), then $-z_0$, and $\pm \bar{z}_0$, the complex conjugates, are also solutions of (A2).

In the second part, we prove that if solutions to (A2) exist, they must be either pure imaginary or real, and that there are exactly two solutions.

In the third part, we deduce the conditions for which the two solutions are imaginary, zero, or real.

Proof.* Part one is evident.

*Ideas in this proof are due to Jerome Eisenfeld and Ibrahim Abu-Shumays.

For part two, we assume that z_1 and z_2 are solutions of (A2). Then

$$\oint \frac{d\mu}{1 - z_1\mu} - \oint \frac{d\mu}{1 - z_2\mu} = 0,$$

from which it follows that

$$(z_2 - z_1) \oint \frac{\mu d\mu}{(1 - z_1\mu)(1 - z_2\mu)} = 0; \quad (A3)$$

or explicitly

$$(z_2 - z_1) \left\{ \int_{-1}^{-\epsilon} \frac{\mu d\mu}{(1 - z_1\mu)(1 - z_2\mu)} + \int_{\epsilon}^1 \frac{\mu d\mu}{(1 - z_1\mu)(1 - z_2\mu)} \right\} = 0.$$

A change of variables ($\mu \rightarrow -\mu$) in the first integral yields

$$(z_2 - z_1) \left\{ - \int_{\epsilon}^1 \frac{\mu d\mu}{(1 + z_1\mu)(1 + z_2\mu)} + \int_{\epsilon}^1 \frac{\mu d\mu}{(1 - z_1\mu)(1 - z_2\mu)} \right\} = 0.$$

This is equivalent to

$$(z_2 - z_1)(z_1 + z_2) \left\{ \int_{\epsilon}^1 \frac{\mu^2 d\mu}{(1 - z_1^2\mu^2)(1 - z_2^2\mu^2)} \right\} = 0. \quad (A4)$$

Now let $z_2 = \bar{z}_1$, for the moment. In this case, the bracket of (A4) is always positive; viz.,

$$\left\{ \right\} = \int_{\epsilon}^1 \frac{\mu^2 d\mu}{|1 - z_1^2\mu^2|^2} > 0.$$

We conclude from (A4) that either $z_1 = \bar{z}_1$ or $z_1 = -\bar{z}_1$. This implies that any solution z_1 must be either zero, real, or pure imaginary. It follows that $(1 - z_1^2\mu^2)$ never changes sign in the cut z -plane.

For any two solutions z_1 and z_2 , it follows from the above that the integral in (A4) never vanishes. Thus

$$(z_2 - z_1)(z_1 + z_2) = 0,$$

from which we conclude that either $z_1 = z_2$, or $z_2 = -z_1$. Hence there can exist at most two distinct solutions. They differ only in sign and are denoted by $\pm \lambda_0^\epsilon$ in the text.

In part three, we prove that these two solutions of (A2) are zero if $c = 1/(1 - \epsilon)$, imaginary if $c > 1/(1 - \epsilon)$, and real if $c < 1/(1 - \epsilon)$.

We write Eq. (A2) in the following form:

$$\frac{1}{2} \oint \frac{d\mu}{1 - z\mu} = \int_{\epsilon}^1 \frac{d\mu}{1 - z^2\mu^2} = \frac{1}{c} \quad (\text{A5})$$

If $z = 0$, the $c = 1/(1 - \epsilon)$.

If z is imaginary, then the middle term of (A5),

$$r(y) = \int_{\epsilon}^1 \frac{d\mu}{1 + y^2\mu^2},$$

is a monotonically decreasing function of y with a range,

$$r(0) = 1 - \epsilon \geq r(y) \geq 0 = r(\infty).$$

Therefore, for $c > 1/(1 - \epsilon)$, the solutions of (A5), and hence of (A2), are imaginary and different for different values of c .

If z is real, then the middle term of (A5),

$$r(x) = \int_{\epsilon}^1 \frac{d\mu}{1 - x^2\mu^2},$$

is a monotonically increasing function of x with a range,

$$r(0) = (1 - \epsilon) \leq r(x) \leq r(1) = \infty \text{ for } |x| \leq 1.$$

If $|x| > 1/\epsilon$, $r(x)$ is monotonically decreasing and always negative. We conclude from (A5), and therefore for (A2), the unique existence of the real solutions for $c < 1/(1 - \epsilon)$. In practical application, the parameter c is always positive.

APPENDIX B

Convergence and Equivalence Transformation
of Continued Fractions

1. Convergence of Equation (8.32),

Given the continued fraction

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} \quad (B1)$$

Then,

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \dots + \frac{a_n}{b_n}}} = \frac{A_n}{B_n} \quad (B2)$$

is called the n^{th} approximant. We call A_n the n^{th} numerator, and B_n the n^{th} denominator.

Definition B1. The continued fraction (B1) is said to converge or to be convergent if at most a finite number of its denominators B_p vanish, and if the limit of its sequence of approximants,

$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n}, \quad (B3)$$

exists and is finite. Otherwise, the continued fraction is said to diverge or to be divergent. The value of a continued fraction is defined to be the limit (B3) of its sequence of approximants. No value is assigned to a divergent continued fraction.

Frequently, the elements a_p and b_p of the continued fraction depend upon one or more parameters, or may themselves be regarded as independent variables. In (8.32), the parameters are λ and β_i . In such cases, one is naturally concerned with the question of uniform convergence. We make the following definition:

Definition B2. If the elements a_p and b_p of a continued fraction are functions of one or more variables over a certain domain D , then the continued fraction is said to converge uniformly over D if it converges for all values of the variable or variables in D , and if its sequence of approximants converges uniformly over D .

The numerators and denominators satisfy the following recurrence formulas (see Wall, Ref. 12, p. 15):

$$A_{n+1} = b_{n+1} A_n + a_{n+1} A_{n-1}, \quad (B4)$$

$$B_{n+1} = b_{n+1} B_n + a_{n+1} B_{n-1},$$

where

$$A_{-1} = 1, A_0 = 0; \quad B_{-1} = 0, B_0 = 1. \quad (B5)$$

Hence A_n and B_n satisfy the same difference equation (called Poincaré difference equation) whose characteristic polynomial is

$$z^2 - bz + a = 0, \quad (B6)$$

where

$$\left. \begin{aligned} a &= \lim_{n \rightarrow \infty} a_n; \\ b &= \lim_{n \rightarrow \infty} b_n. \end{aligned} \right\} \quad (B7)$$

The continued fraction (B1) is convergent only if the continued fractions

$$\frac{a_{N+1}}{b_{N+1} + \frac{a_{N+2}}{b_{N+2} + \frac{a_{N+3}}{b_{N+3} + \dots}}} \quad (N = 0, 1, 2, \dots) \quad (B8)$$

are convergent also. Note that for $N = n$, (B8) is the truncation error in the n^{th} approximant (B2), which makes this statement self-evident.

We now investigate Eq. (8.32). Let

$$a_n = -\frac{n}{n+1} \lambda^2, \quad (n = 1, 2, \dots) \quad (B9)$$

$$b_n = \frac{2n+1}{n+1} (1 - c\beta_n).$$

Then (B1) represents $c = 1$ in (8.32). From (B7),

$$a = \lim_{n \rightarrow \infty} a_n = -\lambda^2, \quad (B10)$$

$$b = \lim_{n \rightarrow \infty} b_n = 2,$$

since $\lim_{n \rightarrow \infty} \beta_n = 0$. The polynomial Eq. (B6) is

$$z^2 - 2z + \lambda^2 = 0, \quad (B11)$$

with the roots

$$z_{1,2} = 1 \pm \sqrt{1 - \lambda^2}. \quad (B12)$$

For $N \rightarrow \infty$, (B8) becomes

$$\frac{\lambda^2}{2 - \frac{\lambda^2}{2 - \frac{\lambda^2}{2 - \dots}}} \quad (B13)$$

But by (B4) and (B10), the approximants of (B13), say $A_n^{(\infty)}$ and $B_n^{(\infty)}$, are given by the solutions of the difference equation

$$A_{n+1}^{(\infty)} = 2A_n^{(\infty)} - \lambda^2 A_{n-1}^{(\infty)}, \quad (B13)$$

$$B_{n+1}^{(\infty)} = 2B_n^{(\infty)} - \lambda^2 B_{n-1}^{(\infty)},$$

where

$$A_{-1}^{(\infty)} = 1, A_0^{(\infty)} = 0, B_{-1}^{(\infty)} = 0, \text{ and } B_0^{(\infty)} = 1. \quad (B14)$$

These solutions are linear combinations of the powers of the roots $z_{1,2}$ [Eqs. (B11) and (B12)], say

$$A_n^{(\infty)} = \alpha_1 z_1^n + \alpha_2 z_2^n,$$

$$B_n^\infty = \beta_1 z_1^n + \beta_2 z_2^n,$$

such that (B14) is satisfied. After some elementary algebraic manipulations, we obtain

$$\frac{A_n^{(\infty)}}{B_n^{(\infty)}} = -\lambda^2 \frac{z_1^n - z_2^n}{z_1^{n+1} - z_2^{n+1}}. \quad (\text{B15})$$

This equation satisfies the conditions of Definition B1 for the convergence of (B13), unless $|z_1| = \|z_2\|$. From (B12), we deduce that the moduli of z_1 and z_2 are equal if and only if λ lies in the interval $[-\infty, -1]$ or $[1, \infty]$. Since for $N \rightarrow \infty$, Eq. (B8) $\rightarrow$ Eq. (B15), we can conclude that (8.32) is uniformly convergent in λ on the complex plane with cuts from $-\infty$ to -1 and 1 to ∞ .

2. Equivalence Transformations

It is often convenient to throw the continued fraction (B1) into another form by means of a so-called equivalence transformation. This consists of multiplying numerators and denominators of successive fractions by numbers different from zero, as follows:

$$\frac{c_1 a_1}{c_1 b_1 + \frac{c_1 c_2 a_2}{c_2 b_2 + \frac{c_2 c_3 a_3}{c_3 b_3 + \dots}}} \quad (c_p \neq 0).$$

One may easily show by mathematical induction that this continued fraction has precisely the approximants of (B1) (see Wall, Ref. 12, p. 19).

To obtain (8.34) from (8.32), we use $c_n = n + 1$.

ACKNOWLEDGMENT

The author expresses his sincere thanks to P. M. Anselone and J. Eisenfeld for their enthusiastic discussions and questions during the stages of formulation of the problem. He also expresses his warm thanks to G. Leaf and especially to Ky Fan for their interest in the mathematical aspects of this paper and their most valuable comments. He is grateful to Garrett Birkhoff for his helpful suggestions, and to F. C. Shure for his comments on Section 8.

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