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Nonlinear Wave Conversion at the Lower Hybrid Resonance

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ABSTRACT

Excitation of ion cyclotron waves by nonlinear mixing of two waves with frequencies near the lower hybrid frequency is investigated. The conversion efficiency is estimated for moderately hot plasmas and the result indicates that the process can be of practical interest.

Excitation of low-frequency ion waves in warm plasmas is of great interest for the possibility of further heating of ions in a fusion device. In this note we investigate the excitation of ion cyclotron waves by the nonlinear mixing of two large amplitude waves with frequencies \( \omega_1, \omega_2 \) close to \( \omega_R \), the lower hybrid frequency. In the usual slab geometry with the direction of the density gradient normal to the slab faces and the external

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magnetic field \( \mathbf{B}_0 = \mathbf{B}_0 \hat{e}_z \), the lower hybrid layer is defined by
\[
\omega^2 = \omega_{\text{pi}}^2 (x_R), \quad \text{where} \quad \omega = \omega_1, \, \omega_2.
\]
Except very close to the slab boundaries (\( N_0 \to 0 \)) we assume \( \omega_{\text{hi}}^2 << \omega_{\text{pi}}^2 \leq \omega^2 << \omega_{\text{pe}}^2 \), where \( \omega_0 = q_0 B_0 / m_0 c \) and \( \omega_{\text{pa}}^2 = 4 \pi q_0^2 N_0 / m_0 \). The electric field of the interacting waves is written as
\[
\mathbf{E}(x,t) = \frac{1}{2} \mathbf{E}_j(x) \exp i(k_3 z - \omega_j t) + \text{c.c.},
\]
where we assume \( n_{jz}^2 = (ck_{jz}/\omega_j)^2 >> 1 \). For \( j = 3 \) we have
\[
\frac{d}{dx} \varepsilon_{zz} (\omega_3) E_{3x} + ik_3 z \varepsilon_{zz} (\omega_3) E_{3z} = \sum_{\alpha} 4 \pi q_\alpha N_{\alpha,3} \exp i \psi,
\]
(1)
\[
ik_3 z E_{3x} = \frac{d}{dx} E_{3z},
\]
(2)
where \( \varepsilon_{xx} = 1 - (\omega_{\text{pi}}^2 / \omega_3^2 - \omega_1^2) \), \( \varepsilon_{zz} = 1 - (\omega_{\text{pi}}^2 / \omega_3^2) + (m_i^\omega_{\text{pi}}^2 / k_{3z}^2 T_e) \), \( k_{3z} \psi_i << \omega_3 - \omega_i << k_{3z} \psi_e \), \( \psi_\alpha = 2 T_\alpha / m_\alpha \), \( \psi = k_{1z} - k_{2z} - k_{3z} \) and we assume \( \omega_3 = \omega_1 - \omega_2 << \omega \). The nonlinear density \( N_{\alpha,3}^{(2)} \) is obtained from the equations
\[
\frac{\partial N_{\alpha,3}^{(2)}}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \int d\mathbf{v} \mathbf{v} f_\alpha (2) = 0, \quad (3)
\]
\[
\frac{\partial f^{(2)}}{\partial t} + \mathbf{v} \cdot \frac{\partial f^{(2)}}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha c} + \mathbf{v} \times \mathbf{B}_0 \cdot \frac{\partial f^{(2)}}{\partial \mathbf{v}} = - \frac{q_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial f^{(1)}}{\partial \mathbf{v}}. \quad (4)
\]
In Eq. (1) the dominant contribution is given by \( N_{1,3}^{(2)} \).
Moreover, for \( \omega_3 \approx \omega_1 \) and \( \omega_3 \gg k_{3z} \psi_i \) in Eq. (4) we neglect the \( \mathbf{v} \cdot [\partial f_i^{(2)} / \partial \mathbf{r}] \) term, thus.
and we have used the fact that $E_{jx} \gg E_{jz}$, $j = 1, 2$. Now, $\omega_j \ll \omega_1, \omega_2 \approx \omega_i$ and $\omega_1, \omega_2 \gg k_{1z} v_\alpha, k_{2z} v_\alpha$, hence

$$N_{i,1}^{(1)} = \frac{q_i N_0}{m_i \omega_j} \frac{d}{dx} E_{jx}, \quad j = 1, 2,$$

and Eq. (1) becomes

$$\frac{d^2 E_{3z}}{dx^2} + \frac{\omega_j^2}{c^2} p_3 E_{3z} = -ik_{3z} \exp (i\psi) \frac{q_i}{2m_i \omega_j} \frac{d^2}{dx^2} (E_{1x} E_{2x})^*, \quad (5)$$

where $\omega^2 = \omega_1^2 \approx \omega_2^2$ and $p_3^2 = -n_j^2 \varepsilon_{zz}(\omega_3)/\varepsilon_{xx}(\omega_2) = (m_i c^2/T_e)^2 (\omega_3^2 - \omega_1^2/\omega_3^2)$.

Equation (5) must be solved simultaneously with the corresponding equations for $E_{1x}, E_{2x}$. To get an idea of the efficiency of the mixing process we solve Eq. (5) by an iteration method. For $x > x_j$ the field of the primary waves is given by

$$E_{jz} = A_j \varepsilon_{xx}^{-1/2}(\omega_j) p_j^{-1/2} \exp \left[ -i \frac{1}{c} \int_{x_j}^{x} p_j(x) dx \right], \quad (6)$$

$$E_{jx} = -\frac{p_j}{n_j} E_{jz},$$

where $p_j^2 = -n_j^2 \varepsilon_{zz}(\omega_j)/\varepsilon_{xx}(\omega_j) = n_j^2 (m_i/m_e)(\omega_i^2/\omega_j^2 - \omega_i^2)$, $\varepsilon_{xx}(\omega_j) = 1 - (\omega_i^2/\omega_j^2)$, $\varepsilon_{zz}(\omega_j) = -(\omega_i^2/\omega_j^2)$ and $x_j$ is determined by the relation $\omega_j^2 = \omega_i^2(x_j)$. The amplitude $A_j$ is determined by $E_{jz}(x = 0)$ and the density profile in the region $0 \leq x \leq x_j$. Equations (6) are valid as long as $x$ is not too close to $x_jR$ where
$$\omega_j^2 = \omega_{p_i}(x_j R) \text{ and } \varepsilon_{xx}(x_j R) \approx 0. \text{ For } x \text{ close to } x_j R \text{ linear wave conversion must be taken into account}^2 \text{ and we have}^1,^3$$

$$E_{jx} = -B_j \left(3\rho_j^2\right)^{1/3} F_j(\sigma_j, \eta_j), \quad i k_{jz} E_{jx} = \frac{dE_{jz}}{dx} \quad (7)$$

where

$$B_j = \frac{w_j \lambda_j}{\pi \eta_{jz}} \left( \int_{x_j}^{x_j R} \rho_j dx + \frac{\pi}{4} \right),$$

$$\eta_j = -(3\rho_j^2)^{1/3} \frac{w_j}{c} (x-x_j R), \quad \lambda_j^{-1} = \frac{1}{\omega_j^2} \left( \frac{d\omega_{p_i}^2}{dx} \right)_{x_j R} > 0,$$

$$\rho_j^2 = \frac{c}{w_j \lambda_j (3T_i/m_i c^2)}, \text{ and } \sigma_j = -\frac{w_j \varepsilon_{zz}(x_j R) \lambda_j n_{jz}}{(3\rho_j^2)^{1/3} c} > 0.$$}

For $\eta_j \gg 1$

$$F(\sigma, \eta) = i \pi \frac{\sigma}{\eta} \frac{1}{2} H_1(1) (2\sigma^{1/2} \eta^{1/2})$$

$$+ \exp \left(-\frac{3\pi i}{4}\right) \frac{\eta^{1/2}}{(3\eta)^{1/4}} \exp \left[i2(\eta^{3/2}) \right], \quad (8)$$

and it can be checked that for $2\sigma^{1/2} \eta^{1/2} >> 1$ the first term of Eq. (8) yields Eq. (6). The second term of Eq. (8) represents the converted plasma wave. For arbitrary values of $\eta$, $F(\sigma, \eta)$ can be written as a double series$^3$ in $\sigma$ and $\eta$. A simple analytical form of $F$ is obtained for $\sigma << 1$;

$$F(\sigma, \eta) = -i \int_0^\infty dt \exp \left[-i(t^3 - \eta t)\right]. \quad (9)$$
The right side of Eq. (6) has a localized behavior. For 
\[ |\eta| > 1 \] and \[ \eta < 0 \] \( x > x_R \) it is exponentially decreasing and for \( \eta > 0 \) we have
\[
\frac{d^2}{d\eta^2} (F_1^* P_2) \approx \frac{d^2}{d\eta^2} |F_1|^2 \approx \eta^{-5/2}.
\]

Since the source of \( E_3 \) is localized in a thin region around \( x_R \), the solutions of Eq. (5) are
\[
E_{3x}^{(\pm)} = -k_{3x} \frac{q_i}{4m_1 \omega^2} \left( \frac{c}{\omega_3 p_3} \right) \exp (\pm i \frac{\omega_3}{c} p_3 x)
+ i \psi \int_{-\infty}^{\infty} dx \exp (\mp i \frac{\omega_3}{c} p_3 x) \frac{d^2}{dx^2} (E_1 x E_2^*)
\]
where in \( E_3^x \) \( x \) is taken above the source layer and in \( E_3^- \) it is taken below. The energy flux is defined by
\[
\mathcal{S}_{3x} = -\frac{\omega_3}{16\pi} \frac{\partial}{\partial \omega_3} \left| \frac{\partial}{\partial \omega_3} E_3 \right|^2 , \quad \mathcal{S}_{3x} = \frac{k_{3x}^2 \varepsilon_{xx}(\omega_3) + k_{3x}^2 \varepsilon_{zz}(\omega_3)}{k_{3x}^2}
\]

hence \( S_{3x}/S_{3x} = -k_{3x}/k_{3x} \) and
\[
\frac{S_{3x}^+}{S_{1x}} = \pm \left( \frac{q_i |A_2|}{4\pi T_i k_{2z}} \right)^2 \frac{\omega_2}{p_3 (\omega_3^2 - \omega_1^2)} \left| \int_{-\infty}^{\infty} d\eta \exp (\pm i \eta \omega_3) \frac{d^2}{d\eta^2} F(\eta) F^*(\eta - \delta) \right|^2 ,
\]
(10)
where \( S_{1x} = (c/8\pi)(|A_1|^2/n_{1z}^2) \), \( \beta = (\omega_3/\omega)(p_3/(3p^2)^{1/3}) \) and 
\( \epsilon = 2(3p^2)^{1/3}(\omega_3/c)l \).

We calculate the integral in Eq. (10) for \( \sigma \ll 1 \), i.e. with \( F \) given by Eq. (9). Moreover, from Eq. (6) it is clear that \( |A_2| \) is approximately \( E_{2z}(x=0) \), the value of \( E_{2z} \) at the boundary, then \( q_i|A_2|/k_{2z} = q_i\phi_2 \) is the potential energy of the wave far away from \( x_R \). We get \( S_{3x}^{(+)} \approx 0 \) and

\[
\frac{S_{3}^{(-)}}{S_{1x}} = \frac{\pi}{12} \left( \frac{q_i\phi_2}{T_i} \right)^2 \frac{\omega_p^2 \beta^3}{|n_{1z}|(\omega_j^2 - \omega_i^2)}. \tag{11}
\]

Equation (11) has a limited range of application since it is for \( \sigma \ll 1 \) which can be fulfilled for low temperature and small \( \lambda \). For a plasma of 10 cm diameter we take \( \lambda = 3 \), \( T_e = 100 \text{ eV} = 10 \text{ } T_i \), 
\( N_0(x_R) = 10^{12} \text{ cm}^{-3} \), \( \omega = \omega_p(x_R) = 1.3 \cdot 10^9 \text{ sec}^{-1} \), \( \omega_i/\omega = 10^{-1} \),
\( (\omega_3 - \omega_i)/\omega_i = 1/5 \), which gives \( \beta = 1/3 \) and 

\[
\frac{S_{3}^{(-)}}{S_{1x}} \approx \frac{2}{|n_{1z}|} \left( \frac{q_i\phi_2}{T_i} \right)^2. \tag{12}
\]

It must be noted that in this case \( \sigma \approx (n_z^2/4) \) and the result must be considered an order of magnitude estimation. The interesting feature of Eq. (12) is that the numerical coefficient is of the order of one and therefore as \( q_i\phi_2 \) approaches \( T_i \) we have a good conversion. On the other hand for the validity of the perturbation theory we have assumed \( (q_i\phi_2)/T_i \ll 1 \).
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REFERENCES


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