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A Continuous Sampling Plan for the
Taking of Physical Inventories

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Abstract

One method of performing a periodic physical inventory involves the stopping of flows into and out of the material balance area being inventoried, determining that all items appearing in the records are actually in inventory, and weighing and sampling a subset of these items on a random basis for chemical and/or isotopic analyses. This procedure may interfere with production, will intermittently load the analytical laboratories, and results in delays in establishing the amount of special nuclear material that is in inventory. These undesirable effects become more onerous as the period between physical inventories is shortened.

In this paper a continuous sampling plan is described that eliminates these effects and in addition makes possible a more continual knowledge of the inventory. Further, with this plan and under proper conditions of flow and inventory size, a potential for reducing the total number of samples required to achieve equivalence with the usual method of establishing an inventory exists.

The continuous sampling plan is based on a stochastic process of shipping and sampling items that yields a probabilistic model for the number of items in inventory that have been sampled at any time. The expected value of the number of sampled items can be made equal to the number of sampled items in the usual inventory method by a suitable choice of parameters. Further, any function that depends on the number of sampled items in inventory also becomes a random variable and can be made to have the same expected value as it has in the periodic physical inventory method.

A Continuous Sampling Plan for the Taking of Physical Inventories

Introduction

Whenever it is necessary to take a physical inventory for the purpose of establishing the amount of special nuclear material contained in a particular material balance area two kinds of measurements are required; a bulk measurement (weight or volume) of the items containing the material and analyses to determine chemical and/or isotopic composition. One method used to establish the amount of SNM on hand at inventory time involves determining that all of the items appearing in the records are physically present, weighing (or measuring the volume) a subset containing a given composition of material and sampling this subset on a random basis for analysis. Although the weighing is accomplished quickly and inexpensively the analysis is relatively slow and expensive and so the number of samples drawn is generally small compared to the total number of items. Even so, the time required to do chemical and isotopic analyses results in a considerable delay before a conclusion can be reached regarding the total amount of SNM in inventory. In addition, if all inventory strata are established simultaneously, a heavy load is placed on the analytical laboratory at the time of inventory taking, thus further increasing the delay. In this paper a method is presented that would eliminate this time lag by doing the sampling on a continuous basis, with minimal effect on production, while the items in inventory are being shipped out and replenished. Although this weakens the confidence one has in the knowledge of the amount of SNM at the end of the accounting period, it strengthens the confidence in the intervening interval during which such samples are ordinarily not taken. The smoothing out of the intermittent load on the analytical laboratory that results from the continuous sampling procedure is in itself a desirable bonus.

It will be shown that with a few simple rules a model can be formulated with which the "continuous" sampling plan can be made equivalent to the "periodic" sampling plan from two alternative points of view. The first viewpoint requires that the expected number of sampled items in inventory

with the continuous plan be equal to the number sampled in the periodic plan.* The second viewpoint requires that the minimum number of sampled items in inventory with the continuous plan be equal to the number sampled in the periodic plan, with some degree of confidence (95% for example). The basic difference between the periodic and continuous sampling plans is that in the former the number of items sampled is known exactly whereas in the latter this number is a random variable having a known discrete probability distribution.

Description of the Sampling Plans

A brief description of what will be referred to as the "periodic" and "continuous" sampling plans is given in this section in order to point out the salient features of both plans. These are idealized versions of actual practice but serve to illustrate the prominent characteristics of each method.

Under the periodic sampling plan each of N items comprising a stratum of an inventory is verified to be present periodically. This is followed by a weighing and sampling (for analysis) of a subset of this stratum.

In the continuous sampling plan an analagous inventory of N items exists.[†] At an arbitrary time this inventory contains, in general, A items which have been sampled for inventory purposes previous to that time and $N-A$ items which have not been sampled. One can characterize this condition of the inventory as the "state" A . A cycle for this sampling plan is now defined as the following four steps:

- 1) n items out of N are selected randomly to be shipped from this inventory. In general this shipment is composed of " a " items that have been sampled during their tenure in inventory and $n-a$ items that have not been sampled.

*A related basis for equivalence would be to make the expected value of some function of the number of sampled items in inventory equal in both sampling plans.

[†]A steady state inventory of N items is assumed but this can be closely approximated by an inventory whose average level is N items and which has small fluctuations about this mean.

- 2) n replacement items are received in inventory, all of which, have as yet, not been sampled for inventory-taking purposes
- 3) A random selection of v items from inventory is made and each of these is sampled. In general this selection is composed of α items that have previously been sampled and $v-\alpha$ items that have not been sampled up to that time.
- 4) After all v items are sampled they are returned to inventory.

This sequence can be repeated indefinitely (beginning with some initial state) and will ultimately result in an inventory in which the expected number of sampled items (and indeed the probability that any given number of items have been sampled) stabilizes.

As stated in the introduction, two alternative viewpoints can be adopted as a basis for equivalence of the two sampling plans. The first involves assuring that the expected number of sampled items in inventory in the continuous sampling plan be equal to the number of items sampled in the periodic sampling plan.* The second alternative would be to require that with a stipulated degree of confidence, the minimum number of sampled items in inventory in the continuous plan be equal to the number of items sampled in the periodic plan.

Determination of the Probability Distribution for the Number of Sampled Items in Inventory

At any given time an inventory contains some number of items that have been sampled. This number is referred to as the state of the inventory. If the inventory is in say, state A, there exists a probability that the inventory state may change to another state, say A', during the subsequent cycle. This probability is called the transi-

*Or that the expected value of some function of the number of sampled items be equal to the value of the function in the periodic plan.

tion probability. This section shows how the transition probabilities are generated and how they are used to determine the probability distribution for the number of sampled items in inventory.

In the first step of the cycle, when n items are selected at random for shipment, the probability of choosing "a" sampled items out of the A that are in inventory (i.e., of going from state A to state $A-a$) is given by the hypergeometric probability distribution

$$P(A \text{ to } A-a) = \frac{\binom{A}{a} \binom{N-A}{n-a}}{\binom{N}{n}} \quad (1)*$$

The second step (replacement of the n items that were shipped out of inventory by n new, unsampled items) does nothing to change the state of the inventory and so Equation (1) is valid up to the beginning of the third step. During the third step, when v items are selected randomly for sampling, the probability that of the $A-a$ sampled items remaining in inventory, α will be included among the v items chosen (i.e., of temporarily decreasing the state from $A-a$ to $A-a-\alpha$) is

$$P(A-a \text{ to } A-a-\alpha) = \frac{\binom{A-a}{\alpha} \binom{N-A+a}{v-\alpha}}{\binom{N}{v}} \quad (2)$$

During the fourth step the inventory state will proceed from $A-a-\alpha$ to $A-a-\alpha+v$; i.e., the v sampled items are returned to inventory during the fourth and final step of the cycle and therefore the state of the inventory increases by v . Thus the probability of going from state A to state $A-a-\alpha+v$ is the product of the probabilities associated with each of the steps in the cycle, namely

$$P(A \text{ to } A-a-\alpha+v) = \frac{\binom{A}{a} \binom{N-A}{n-a} \binom{A-a}{\alpha} \binom{N-A+a}{v-\alpha}}{\binom{N}{n} \binom{N}{v}} \quad (3)$$

*The notation used is the one used in combinatorial analysis. For example,

$$\binom{A}{a} = \frac{A!}{a! (A-a)!}$$

Note that the difference between the initial state and the final state in the cycle is the quantity $v - \alpha - a$. Now, for any selected continuous sampling plan v will be a determined constant. Thus Equation (3) can be used with all physically possible combinations of α and a , for which $v - \alpha - a$ is a particular constant, in order to obtain the separate probabilities of all the independent routes from A to $A - a - \alpha + v$. The sum of all of these probabilities then, constitutes the total probability that there is a transition from state A to state $A - a - \alpha - v$.

The above transition probabilities can be computed for all possible combinations of starting states (0 through N) and ending states (also 0 through N). They can then be arranged in an $(N+1)$ by $(N+1)$ matrix that has for its elements the transition probability for going from the state defined by the row number of any element to the state defined by the column number of that element. This matrix is known as the transition matrix and is completely independent of the state of the inventory.

The probabilities of being in any particular state 0, 1, 2 ..., N can be represented by the corresponding elements of a row vector. Such a vector is known as a state vector and has, in general, $N+1$ elements. The rules of matrix multiplication dictate that the multiplication of the state vector by the transition matrix yields a new $N+1$ element row vector. Consideration of the physical meaning of the elements of the state vector and the transition matrix results in an appreciation of the fact that the new row vector represents the state vector after the completion of an additional cycle. The process of multiplying each new state vector by the transition matrix can be repeated indefinitely until the state vector converges to its steady state value.

Implementation

The foregoing theory is helpful in determining what the sample size (v) should be in order to satisfy the equivalence requirements from either point of view. To satisfy the first viewpoint it is simply necessary to find

the steady state vector corresponding to a given sample size and obtain the expected value of this frequency function. Actually, this is repeated for a range of sample sizes until the expected value is equal to the sample size in the periodic sampling plan.

Having carried this out for a range of sample sizes the data necessary to satisfy the second viewpoint is already at hand. It is merely necessary to choose that sample size that yields a state vector for which the cumulative probability of obtaining more than "x" sampled items in inventory is 95% (for example). The number "x" refers to the number of items sampled in the periodic sampling plan.

The implementation of the theory presents an arduous numerical task in practice, especially when the inventory contains a large number of items. Therefore a computer code was written to accomplish all of the calculations described in the previous section. The user specifies the number of items in inventory, the number of items shipped (and received) during each cycle, the initial state of the inventory, the convergence criterion on the steady state vector, and the range of values of sample sizes to be tried. For each sample size within the range specified, the program computes the transition matrix and finds the converged state vector. The code then calculates the expected value of the converged state.

Numerical Illustration

In order to clarify the concepts presented in this paper an illustrative example was run using the program outlined in the previous section. This case concerns an inventory of 60 items with a stipulated throughput (removals and replacements) of 8 items per cycle. The range of sample sizes considered was from 1 per cycle to 16 per cycle. Figure 1 shows how the expected value of the converged state increases with increase in sample size. It is evident that with a sample size of 2 per cycle, for example, the expected number of sampled items in inventory is about 12. If one adopts the first viewpoint concerning equivalent sampling plans one concludes that 2 samples per

cycle in the continuous plan is equivalent to 12 samples in the periodic plan. Therefore, if the stipulated throughput of eight items represents a weekly throughput, for example, the same confidence concerning the inventory could be obtained by sampling eight items (four weeks times two samples per week) during the month with the continuous sampling plan as could be obtained sampling twelve items at the end of the month with the periodic sampling plan. Furthermore, there would be additional confidence during the month that the inventory had not been tampered with than if there were no sampling at all during that month.

With the second point of view regarding equivalence, one would argue that the sample size with the continuous sampling plan should be about three items per cycle. Reference to the state vector for a sample size of three in the computer print-out (not shown here) indicates that with 95% probability, the number of items in inventory that will have been sampled is at least twelve. It is not surprising that this second viewpoint leads to a larger sample size than before since it requires that the continuous sampling plan yield an expected value of sampled items in inventory greater than the number used in the periodic sampling plan.

Another feature of the continuous sampling plan concerns the number of cycles necessary to reach the steady state. Besides depending somewhat on the other parameters of the system, the answer to this question is sensitive to the initial state of the inventory. This is briefly explored below.

The illustrative example was rerun three times using a value for the sample size of 2. In these runs all of the intermediate inventory states leading up to the asymptotic state were noted. Since the example showed that state 12 (i.e., having twelve sampled items in inventory) represents the expected value of the state of the inventory, the probability of having exactly twelve sampled items in inventory was chosen to indicate how quickly the state vector approaches the asymptotic state. This probability is shown as a function of cycle number in

Figure 2. The three cases shown correspond to three initial inventory conditions as follows:

- Case 1) It is equally probable that any number of items have been sampled
- Case 2) It is equally probable that 29, 30, 31, or 32 items have been sampled and there is no likelihood that any other number of items have been sampled
- Case 3) It is certain that twelve items have been sampled.

Figure 2 indicates that for the particular parameters of the illustrative example the steady state will be closely approached after about two dozen cycles in cases 1 and 2. If the continuous sampling plan is implemented with a beginning inventory that has the number of sampled items equal to the expected value of the steady state (case 3) the number of cycles necessary to approach the steady state can be appreciably reduced. In this particular illustration only about a half dozen cycles were necessary for the inventory to approximate the asymptotic value.

The last characteristic of the continuous sampling plan that is of primary interest here concerns the relationship of the independent variables (inventory size, throughput and sample size) to the dependent variables (the steady state vector) when the inventory problem is scaled up or down. For example, instead of an inventory situation in which there are 30 items, with a throughput of 4 items per cycle and a sample size of 2 per cycle, a study was made of inventory situations in which these three parameters were scaled up and down. The computed expected value of the steady state as a function of these variables (kept in constant proportion to one another) is shown in Figure 3. It is clear from this figure that a linear relationship between the input variables and the final state vector exists. This is a useful fact since it allows the user to scale problems to a convenient size. The transition matrix becomes large rather quickly as the number

of items in inventory increases, for example, and this effects the computer storage requirements and computing time needed. The possibility of scaling a problem down to manageable dimensions greatly enhances the utility of this analytic technique.

Conclusion

A continuous sampling plan has been developed that can be substituted for the usual periodic sampling plan. This plan has the following advantages:

- 1) More continual assurance that the inventory has not been compromised between inventory periods is provided
- 2) A smaller number of samples is required to attain equivalence with the periodic sampling plan in inventory situations that are not flow dominated
- 3) More timely results are available following the taking of the physical inventory
- 4) A smoothing of the work load on the analytical laboratory is accomplished
- 5) A decrease in the amount of interference with the normal production routine of the facility is attained.

PROBABILITY OF HAVING
EXACTLY 12 SAMPLED
ITEMS IN INVENTORY