Covariant Parton Dynamics

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Abstract

We propose a dynamical theory of hadrons in terms of partons. Partons are defined as free particle-like states in a reference frame moving with the velocity of light. Lorentz covariance properties of the theory are investigated; we give a method of constructing a class of covariant interactions directly in terms of parton operators. A transverse harmonic oscillator potential between parton pairs is suggested as an effective interaction giving rise to the observed meson spectrum. The parton-system exhibits collective excitations; in a Hartree-Fock approximation the latter correspond to free mesons. The elastic scattering amplitude of spinless mesons is calculated in an external field approximation. At moderate energies the amplitude exhibits local duality.

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1. Introduction and Summary

The objective of this paper is to explore some consequences of the following hypothesis:

Hadrons are composite objects; their "constituents" can be observed as "particles" in a reference frame moving with the velocity of light. (Feynman's "parton" hypothesis¹⁾)

This hypothesis receives support from several facts. As a reminder to the reader, we briefly list some of them.

a.) Low-lying (therefore, well-identifiable) levels of the hadron spectrum exhibit an "oscillator-like" structure, resembling low-energy excitations of a complicated many-body system (e.g. nuclei).

b.) Various successful phenomenological models^{1,2)} of hadronic reactions suggest an underlying many-body structure of hadrons.

c.) Duality both in hadron and lepton induced reactions can be qualitatively understood by invoking a compositeness hypothesis. (This was particularly emphasized by Lovelace for hadronic reactions³⁾ and by the present authors for lepton induced ones.⁴⁾)

d.) Gürsey and Orfanidis have shown that a large number of the observed features of hadronic reactions can be understood and unified by adapting the kinematics to a "lightlike" reference frame (LF).⁵⁾ Their results generalize earlier calculations performed in the framework of various specific models⁶⁾ and strongly suggest that a "lightlike" description of hadronic processes may considerably simplify matters or - possibly - it is their only correct description.

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The basic question to which we address ourselves in this paper is the following:

Can the parton hypothesis lead to a physically meaningful dynamical theory of hadrons?

Several remarks are in order at this point.

1.) Since the LF is not an inertial frame of reference, its explicit use in the description of dynamics should be avoided in order to preserve the Lorentz invariance of the theory. The correct procedure is to specify single "particle" states by means of measurements performed in a LF.^{5,7)} In a space-time picture this corresponds to the specification of initial conditions on a null plane, rather than on a space-like one.^{7,8)}

This procedure in turn leads to a front form of the dynamics, as proposed by Dirac.⁹⁾

2.) The concept of a "constituent" of a relativistic system is not a well-defined one, except perhaps in the limit of weak binding. In this paper we adopt a - rather vague - operational definition of the "constituents" of hadrons (the "partons" in Feynman's terminology). We assume that the Hamiltonian describing the time development in the front form of dynamics can be meaningfully split into two terms, one of which describes the free motion of partons, the other their interaction. The partons are tentatively identified with quarks (fermions of spin one-half); we pay, however, no attention to their internal symmetry properties in this paper.

3.) Before a meaningful investigation of a dynamical theory of partons can be undertaken, their kinematical properties have to be clarified. In more abstract terms, one has to construct an ir-

reducible realization of the inhomogeneous Lorentz group spanned by the single parton states. Further, reduction properties of product states have to be investigated; this leads to the construction of covariant interaction terms and/or S-matrix elements.

An outline of the plan of the paper and of the basic results follows.

Section 2 deals with the problem of the kinematics of free partons. While this problem has been treated before (cf. in particular Gürsey and Orfanidis for the most complete description¹⁰⁾), the essential steps are briefly reproduced here for further reference. We show in particular that single parton states span a basis of an irreducible unitary realization of the homogeneous Lorentz group,¹¹⁾ $SL(2,C)$. The realization of $SL(2,C)$ in question turns out to be identical to the Gelfand-Naimark one,¹²⁾ (As far as we know, this result is new, at least in its present form.) This result is a most comforting one, since it is relatively easy to investigate the transition between a LF and an inertial frame in this basis. As a consequence, one is able to choose a standard inertial frame connected with a LF by means of an infinite boost along a fixed direction. It follows, in particular, that no divergent (or vanishing) factors are picked up by the parton states when acted upon by such a boost.

After this, we construct the Fock-space of parton states. According to the parton hypothesis, vectors of this space form a basis for physical states. The action of appropriately defined discrete symmetry operations is also investigated in this section.

The problem of constructing covariant interactions is taken

up in Section 3. While in the conventional, instant form of dynamics⁹⁾ this problem is usually solved by introducing fields; that procedure is not practical here. As emphasized elsewhere⁸⁾ field operators are subject to constraints and, therefore, are inconvenient to use. Instead, we use a generalization of Wigner's classical procedure¹³⁾ of defining the spin of a particle in a covariant fashion. The interaction is first written down in terms of "particle" operators in a standard frame. Afterwards it is boosted up to an arbitrary frame of reference. This trick reduces the invariance requirements to invariance with respect to a subgroup of the inhomogeneous Lorentz group, which is relatively easy to satisfy. It is found that a large class of non-local, phenomenological interactions can be constructed.¹⁴⁾ Rather than searching for a "fundamental" interaction, in this paper we adopt an attitude similar to the one prevalent in the early days of nuclear physics, that at present too little may be known about the dynamics of hadrons. Thus, motivated by the properties of the spectrum (cf. point a) at the beginning of this section), an effective oscillator interaction between quark-antiquark pairs is written down. (We also come to the remarkable conclusion that no covariant pair interaction can be constructed which describes a quark-quark interaction.) In Section 4 it is shown that such an interaction leads, indeed, to "collective excitations" with a realistic-looking spectrum. We conjecture that - at least in some approximation - these excitations can be identified with free mesons.

An application of the formalism developed here is considered in Section 5. We add yet another term to our phenomenological

Hamiltonian which describes the excitation of the collective modes of the quark-parton system by means of scalar particles. The elastic scattering amplitude of the scalar mesons is then calculated in an "external field approximation" (i.e. only the quarks are assumed to be described by dynamical equations).

In this approximation the problem can be solved exactly and we find that at not too high energies, the amplitude of the elastic scattering of the scalar mesons exhibits local duality: the collective modes in the intermediate states add up to give rise to the appearance of Regge poles in the crossed channel. (This result is to be compared with the construction of optical potentials in nuclear physics by summing over a large number of nuclear levels.)

Finally, Section 6 is devoted to some concluding remarks.

2. Kinematics

2.1 The inhomogeneous Lorentz group in a lightlike basis

We use a lightlike basis (ℓ -basis), in Minkowski space; covariant components of a vector, v , are denoted by v_i ($i = 1, 2$), v_z, v_t , where v_z, v_t are the components in the two conjugate lightlike directions, cf.⁸⁾. The components of the metric tensor are:

$g_{ik} = \delta_{ik}$, $g_{zt} = g_{tz} = -1$, the others vanish. Correspondingly, generators of homogeneous Lorentz transformations, $M_{\mu\nu}$, as written in an ℓ -basis are:

$$M_{12} \equiv M, \quad M_{zt} \equiv N, \quad M_{zi} \equiv E_i, \quad M_{ti} \equiv F_i, \quad (2.1)$$

while we use the following notation for the translation generators:

$$P_i, \quad P_z \equiv P, \quad P_t \equiv H. \quad (2.2)$$

Eigenvalues of operators are denoted by the corresponding lower case letters, unless otherwise specified.

We denote the subgroup generated by E_i, M, N, P_i by G . It is easily seen from the standard commutators that it is isomorphic to the two dimensional Galilei group with dilatations. (Galilei boosts are generated by E_i , "dilatations" by N .) This subgroup plays a central role in what follows. In particular, it gives a precise formulation of the statement about the "slowing down" of the transverse motion¹⁾ in a LF. Notice that under G , the eigenvalue of P plays the role of a "Galilean mass," while H is the Hamiltonian. Independent components of a vector of given square

in the ℓ -basis are best characterized by a spinor, ξ_A ($A = 1, 2$). Choosing the eigenvector, p_μ , of (2.2), with $p_\mu p^\mu = -\mu^2$ (μ being the invariant mass), we have:

$$\vec{p} = \xi^\dagger \vec{\sigma} \xi, \quad p = \sqrt{2} \xi_1 \xi_2, \quad h = \frac{\vec{p}^2 + \mu^2}{2p}. \quad (2.3)$$

Here and in what follows, we denote: $\vec{p} = (p_1, p_2)$; $\vec{\sigma} = (\sigma_1, \sigma_2)$ are the standard Pauli matrices. Choose a standard spinor, η , by $\eta_1 = 0$, $\eta_2 = 2^{-1/4} \kappa$, so that the corresponding vector, $\tilde{p}_\mu$, has the components

$$\tilde{p} = 0, \quad \tilde{p} = \kappa, \quad \tilde{h} = \frac{\mu^2}{2\kappa}. \quad (2.4)$$

On choosing a three parameter subgroup, B , of G , with the general element:

$$B(\beta, \vec{u}) = e^{i\beta N} e^{i\vec{u} \cdot \vec{E}}, \quad (2.5)$$

one verifies that the orbit $B \tilde{p}_\mu$ is the hyperboloid $p_\mu p^\mu = \mu^2$, $p > 0$. The transformations (2.5) are called infinite momentum frame boosts by Gürsey and Orfanidis.⁵⁾ Evidently, the value of κ is immaterial (one can go from one "standard" vector, $\tilde{p}_\mu$ to another by means of a "dilatation," $\exp(i\gamma N)$). In what follows, we may put $\kappa = 1$ without loss of generality.

It is convenient to parametrize B by means of the components of the Galilean velocity, $\vec{v} = \vec{p}/p$ and p of the vector reached from

$\tilde{p}_\mu$ by the action of B. In this way we obtain the following form of B in the 2 x 2 matrix representation:

$$B(z, p) \rightarrow \begin{pmatrix} \frac{1}{\sqrt{p}} & z\sqrt{p} \\ 0 & \sqrt{p} \end{pmatrix}, \quad (2.6)$$

where

$$z = \frac{p_1 - i p_2}{\sqrt{2} p} = \frac{v_1 - i v_2}{\sqrt{2}}. \quad (2.7)$$

Given a function $f(p_\mu)$ of the momentum, we shall regard it either as a function of the momentum components $\vec{p}$, p and of the invariant $-p_\mu p^\mu = \mu^2$ or of z , p and μ^2 , which ever is convenient, without changing the symbol denoting the function. The argument μ^2 will be in general, suppressed.

Notice that by (2.3), the variable z is just a Gelfand-Naimark parameter,

$$z = \frac{\xi_1}{\xi_2}. \quad (2.8)$$

Therefore, under a proper, homogeneous Lorentz transformation,

$$\left. \begin{aligned} g &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, & \alpha\delta - \gamma\beta &= 1; \\ z' &\equiv gz = \frac{\alpha z + \beta}{\gamma z + \delta}, \\ p' &\equiv gp = p(\gamma z + \delta)(\bar{\gamma}\bar{z} + \bar{\delta}). \end{aligned} \right\} \quad (2.9)$$

The invariant volume element in terms of these variables becomes:

$$dB(z,p) = \frac{d^3p dp}{2p} = \frac{dz d\bar{z} p dp}{i\sqrt{2}}. \quad (2.10)$$

One checks easily that $dB(z,p) = dB(gz, gp)$ for $g \in SL(2, \mathbb{C})$. In what follows, we replace p by $|p|$ in (2.10), thus obtaining the invariant volume element for the entire hyperboloid $p_\mu p^\mu + \mu^2 = 0$.

2.2 Single particle states. Fock space. Induction of Lorentz transformations.

The description of single-"particle" states given in this subsection follows Wigner's classic treatment¹³⁾ adapted to the present physical situation, as described in Section 1. The calculations are explicitly carried out for the case of particles of spin one-half (quark states); generalization to arbitrary spin is straightforward.

A single-quark state is specified by measurements performed in a standard LF. The momentum there is given by (2.4), with $\kappa \rightarrow \infty$. The little group is generated by F_i, M , as the reader can immediately verify by using (2.3) and the 2×2 matrix representations:

$$e^{i\vec{w}\vec{F}} \rightarrow \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \quad w = \frac{w_1 + iw_2}{\sqrt{2}}, \quad (2.10)$$

$$e^{i\varphi M} \rightarrow \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}.$$

The generator M is the spin operator in the standard LF, while $\vec{F}$ acts trivially.¹³⁾ Thus, the standard quark state is characterized by the spin, m , (and by the sign of κ):

$$\begin{aligned} U(e^{iM\varphi})|m\rangle &= e^{im\varphi}|m\rangle, \\ U(e^{i\vec{z}\vec{F}})|m\rangle &= |m\rangle, \quad (m = \pm 1/2). \end{aligned} \quad (2.11)$$

Our standard frame ($\kappa = 1$) can be reached by an "infinite boost," viz.

$$e^{i\gamma N}, \quad \log \gamma = \kappa, \quad \kappa \rightarrow \infty.$$

A general state, $|z, p, m\rangle$, is obtained by letting the unitary operator $U(B(z, p))$ act on $|m\rangle$:

$$|z, p, m\rangle = U(B(z, p))|m\rangle. \quad (2.12)$$

The action of a general, proper Lorentz transformation is obtained by Wigner's induction procedure:

$$\begin{aligned} U(q)|z, p, m\rangle &= U(e^{iM\psi})|qz, qp, m\rangle \\ &= e^{im\psi}|qz, qp, m\rangle, \end{aligned} \quad (2.13)$$

in view of (2.11). The angle ψ can be determined by computing the element $W(g; z, p)$ of the little group given by the standard formula:

$$W(g; z, p) = B^{-1}(qz, qp) g B(z, p) \quad (2.14)$$

On using (2.6), (2.9), we find after a short calculation:

$$U(g) |z, p, m\rangle = (\gamma z + \delta)^{-m} (\bar{\gamma} \bar{z} + \bar{\delta})^m |qz, qp, m\rangle. \quad (2.15)$$

Equation (2.15) shows that the vectors $|z, p, m\rangle$ transform according to the irreducible, unitary realization, $j_1 = -m/2$, $j_2 = m/2$, of the homogeneous Lorentz group. The transformation law (2.15) is identical to the transformation of the elements of a Gelfand-Naimark basis.¹²⁾ This is not surprising. Indeed, one of the many ways of generating the Gelfand-Naimark realization of $SL(2, C)$ is by inducing the full group on one of its Euclidean subgroups; this coincides with the Wigner procedure in the present context. It should be observed, however, that in order to obtain a realization of the full inhomogeneous Lorentz group, we need the label p in addition to z .

There are several among the vectors (2.12) which are equivalent under $SL(2, C)$.

One observes that

$$\vec{v} = \frac{\vec{p}}{p} = \frac{-\vec{p}}{-p},$$

and using (2.15) one readily establishes the following equivalence classes:

$$|z, p, m\rangle \sim |z, -p, m\rangle \sim |\bar{z}, p, -m\rangle \quad (2.16)$$

from which all the others can be generated. These equivalence classes of vectors are mapped onto each other by discrete symmetry operations as it will become clear shortly.

It follows from this construction that under a boost along the 3^d axis ($\beta = \gamma = 0$, $\alpha\delta = 1$) the states (2.12) do not pick up any numerical factor. This proves the consistency of the procedure used throughout this paper: even though measurements specifying the parton (quark) states are supposed to be performed in a LF, one may safely use a standard frame with $\chi = 1$ even for a finite rest mass.

The Fock space of free quark states is generated by introducing the vacuum, $|0\rangle$, and creation and annihilation operators. We require that the latter satisfy the canonical anticommutation rules, viz.:

$$\{a_m(z, p), a_{m'}^\dagger(z', p')\} = \frac{\delta_{m, m'}}{|p|} \delta(p - p') \delta^{(2)}(z - z'), \quad (2.17)$$

$$\{a_m(z, p), a_{m'}(z', p')\} = 0,$$

where

$$\delta^{(2)}(z-z') = i\sqrt{2} \delta(z-z') \delta(\bar{z}-\bar{z}') = 2p^2 \delta^{(2)}(\vec{p}-\vec{p}').$$

In the usual way, for $p < 0$, $a_m(z, p)$ is interpreted as a creation operator of an antiquark. In an obvious matrix notation we write:

$$\Theta(-p) a(z, p) = \Theta(-p) \epsilon b^\dagger(z, -p), \quad (2.18)$$

where ϵ is the antisymmetric matrix:

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^T = \epsilon^\dagger = \epsilon^{-1}.$$

Using (2.16), one readily establishes that both sides of (2.18) behave in the same way under homogeneous Lorentz transformations.

Finally, we combine operators of both signs of p into the quark operator, φ :

$$\varphi(z, p) = \Theta(p) a(z, p) + \Theta(-p) \epsilon b^\dagger(z, -p). \quad (2.19)$$

(Crossing.)

The transformation law of the operator $\varphi(z, p)$ can be read off from (2.15). We find:

$$\begin{aligned}
U(q) \varphi_a(z, p) U(q)^\dagger &= (\gamma z + \delta)^a (\bar{\gamma} \bar{z} + \bar{\delta})^{-a} \varphi_a(z', p') = \\
&= (-\gamma z' + \alpha)^{-a} (-\bar{\gamma} \bar{z}' + \bar{\alpha})^a \varphi_a(z', p'), \\
&\quad (a = +1/2, -1/2).
\end{aligned}
\tag{2.20}$$

It follows, in particular, that $(\varphi^\dagger(z, p) \varphi(z, p))$ and $(\varphi^\dagger(z, p) \varphi(z, -p))$ are invariants. The generators of translations, the spin operator, Σ , and the operator of the Fermion charge, F , can be written in a particularly compact form in terms of φ . One finds:

$$\begin{aligned}
\vec{P} &= \frac{1}{i} \int dB : \varphi^\dagger(z, p) \varphi(z, p) : \vec{p}, \\
P &= \frac{1}{i} \int dB : \varphi^\dagger(z, p) \varphi(z, p) : p, \\
H_0 &= \frac{1}{i} \int dB : \varphi^\dagger(z, p) \varphi(z, p) : \left(p z \bar{z} + \frac{\mu^2}{2p} \right), \\
\Sigma &= \frac{1}{i} \int dB : \varphi^\dagger(z, p) \sigma_3 \varphi(z, p) : , \\
F &= \frac{1}{i} \int dB : \varphi^\dagger(z, p) \varphi(z, p) : ,
\end{aligned}
\tag{2.21}$$

whereas, for example, the Z -operator, which plays a central role in the treatment of the Lorentz group given by Bars and Gürsey,¹⁵⁾ can be written as:

$$Z = \frac{1}{i} \int dB : \varphi^+(z, p) \varphi(z, p) : \epsilon(p) z .$$

(In these formulae: $-\infty < p < \infty$.)

2.3. Discrete symmetry operations.

Charge conjugation is defined in the standard way:

$$U(C) a_m(\vec{\beta}, |p|) U^+(C) = \eta_c b_m(\vec{\beta}, |p|)$$

where η_c is a phase. However, space inversion (Π) and Wigner time reversal (T) are not convenient operations, since they change the direction of the velocity of a LF. Suitable operations are obtained by combining Π and T with a rotation through 180° around - say - the 2nd axis:

$$S = e^{-i\pi M_{13}} \Pi, \quad (\text{"reflection"}),$$

$$Q = e^{-i\pi M_{13}} T \quad (\text{"reversal"}).$$

The geometrical properties of these operations and, hence, their effect on creation and annihilation operators can be readily established. Without going into details, in Table 1, we summarize the action of these operations on the quark operators, φ .

In the following calculations, we set the quark mass equal to zero. This simplifies the formulae somewhat; a finite quark mass

can be taken into account in a trivial way.

3. Covariant interactions

In order to describe the dynamics of quarks, one has to construct covariant quantities describing the interaction. As a starting point, let us work in the interaction picture. This means that a quark operator $\varphi(\vec{p}, p)$ acquires an exponential time dependence, viz:

$$\begin{aligned}\varphi(\vec{p}_i, p_i, t) &= \varphi(\vec{p}_i, p_i) \exp i \frac{\vec{p}_i^2}{2p_i} t \\ &= \varphi(\vec{p}_i, p_i) \exp i h_i t.\end{aligned}\tag{3.1}$$

The interaction part, W_1 , of the action must be then of the form:¹⁶⁾

$$W_1 = -\frac{i}{2\pi} \int dt \int \prod_{i=1}^n \frac{d^2 \vec{p}_i dp_i}{2|p_i|} \exp i \sum_i h_i t \mathcal{H}_1(\vec{p}_i, p_i)\tag{3.2a}$$

or

$$W_1 = - \int \prod_{i=1}^n \frac{d^2 \vec{p}_i dp_i}{2|p_i|} \delta(\sum h_i) \mathcal{H}_1(\vec{p}_i, p_i)\tag{3.2b}$$

The operator $\mathcal{H}_1(\vec{p}_i, p_i)$ is the interaction density in momentum space. It describes the interaction of n quarks (antiquarks) of momentum components $(\vec{p}_i, p_i, h_i)$ $i = 1, 2, \dots, n$. Fermion number conservation requires that $\mathcal{H}_1$ contain $n/2$ operators φ and an equal number of the conjugate operators, φ^+ .

It will be convenient to define a light-like Schrödinger picture coinciding with the interaction picture on the null plane $t = 0$. Assuming that we found $\mathcal{H}_1$, and further that it does not contain expressions of the form $\vec{p}_i \varphi(\vec{p}_i, p_i)$, etc. (no "derivative interactions"), one immediately gets the interaction Hamiltonian in the Schrödinger picture as:

$$H_1 = \int \prod_{i=1}^n \frac{d\vec{p}_i dp_i}{2|p_i|} \mathcal{H}_1(\vec{p}_i, p_i) \quad (3.3)$$

(Formally, the expression of H_1 is obtained from (3.2) by omitting the factor, $-\delta(\sum h_i)$, from the integrand.) Evidently, H_1 must be translation invariant:

$$[\vec{P}, H_1] = [P, H_1] = 0, \quad (3.4)$$

where the operators $\vec{P}$, P are given by (2.21). Further, W_1 must be invariant with respect to proper Lorentz transformations; dealing with strong interactions, we also assume that it is invariant under reflections:

$$\begin{aligned} U(q) W_1 U^\dagger(q) &= U(C) W_1 U^\dagger(C) = U(S) W_1 U^\dagger(S) = \\ &= [U(Q) W_1 U^\dagger(Q)] = W_1, \\ q &\in SL(2, C). \end{aligned} \quad (3.5)$$

(The square bracket in the last expression means that the order of the operators $\varphi, \varphi^\dagger$ is to be reversed in the expression of W_1 .)

The requirement of invariance under proper Lorentz transformations (which is most difficult to satisfy, since $\varphi, \varphi^\dagger$ transform according to nonlinear realizations of $SL(2, C)$) can be reduced to invariance under a subgroup. Indeed by using (3.4), we can pick $n-1$ independent relative momenta, $\vec{q}_i$, and longitudinal momentum fractions, y_i , such that the total transverse momentum carried by the quarks adds up to zero and the total longitudinal momentum to one. (To demonstrate this, choose e.g.

$$\begin{aligned} \vec{P}_i &= \vec{q}_i, & P_i &= y_i \quad (i = 1, \dots, n-1) \\ \vec{P}_n &= -\sum_{i=1}^{n-1} \vec{q}_i, & P_n &= 1 - \sum_{i=1}^{n-1} y_i. \end{aligned}$$

With this choice, W_1 can be written as:

$$W_1 = \int dB(z, p) U(B(z, p)) \tilde{W}_1 U^\dagger(B(z, p)) \quad (3.6)$$

where $B(z, p) \in G$ is given by (2.6). Evidently, $\tilde{W}_1$ is the interaction in a standard frame of reference.

We now state the central result of this section.

The interaction W_1 is invariant under proper Lorentz transformations if $\tilde{W}_1$ is invariant under transformations generated by F_i, M (the little group of the standard light-like momentum, $\vec{p} = 0, p = 1$).

Proof. Consider $g \in \text{SL}(2, \mathbb{C})$.

$$\begin{aligned} U(g) W_1 U^\dagger(g) &= \int dB(zp) U(gB(zp)) \tilde{W}_1 U^\dagger(gB(zp)) \\ &= \int dB(qz, qp) U(B(qz, qp)) \times \\ &\quad \times U(W) \tilde{W}_1 U^\dagger(W) U^\dagger(B(qz, qp)). \end{aligned}$$

In the last line we used the invariance of the volume element, (2.10). The transformation W is defined by (2.14); it is evidently an element of the little group. Hence, $U(g) W_1 U^\dagger(g) = W_1$, provided $U(W) \tilde{W}_1 U^\dagger(W) = \tilde{W}_1$, q.e.d.

This result can be further simplified. Observe the matrix identity:

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.7)$$

which connects transformations of the form $\exp i \vec{w} \cdot \vec{P}$ with elements of the Galilei group, G . The transformation

$$R \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.8)$$

is itself a Lorentz transformation (a rotation through 180° around the 2nd axis). Hence, the previous result can be restated

in the more convenient form:

The interaction W_1 is invariant under proper Lorentz transformations, if $\tilde{W}_1$ is invariant under the two dimensional Galilei group (without dilatations) and under the single transformation, R .

Remark. The invariance requirement under the two dimensional Galilei group can be easily satisfied; in particular, one may use one's intuition gained in nonrelativistic quantum mechanics. Thus, the non-trivial requirement is the invariance under the transformation R . The transformation properties of the various kinematic quantities under R can be read off from eqs. (2.9), (2.20).

On restricting ourselves to two-body interactions without spin-orbit coupling, it is easy to verify that the only invariant combination of quark operators in the standard frame is the following:

$$\begin{aligned} I(\vec{q}'y'; \vec{q}y) = & :(\varphi^\dagger(\vec{q}', y') \varphi(\vec{q}, y)) (\bar{\psi}(\vec{q}, y-1) \varphi(\vec{q}', y'-1)) : \\ & + :(\bar{\psi}(\vec{q}', y') \sigma_3 \varphi(\vec{q}, y)) (\varphi^\dagger(\vec{q}, y-1) \sigma_3 \varphi(\vec{q}', y'-1)) : \end{aligned} \quad (3.9)$$

In the last equation we introduced the longitudinal momentum fractions y, y' , ($0 \leq y, y' \leq 1$) of the quarks. The transformation properties of the momentum components under R being:

$$\begin{aligned} R p_1 &= -p_1, & R p_2 &= -p_2 \\ R p &= \hbar, & R \hbar &= p, \end{aligned}$$

one easily verifies that an interaction of the form:

$$\tilde{W}_1 = \int d\vec{q} d\vec{q}' \int_0^1 dy I(\vec{q}', y, \vec{q}, y) v((\vec{q} - \vec{q}')^2, y(1-y)) \times \delta\left(\frac{\vec{q}^2}{2y} + \frac{\vec{q}'^2}{2(1-y)} - \frac{\vec{q}^2/2}{2y} - \frac{\vec{q}'^2/2}{2(1-y)}\right) \quad (3.10)$$

satisfies all the invariance requirements. The function $v(\tau, \vartheta)$, with $\tau = (\vec{q} - \vec{q}')^2$, $\vartheta = y(1-y)$, is in principle arbitrary: it is a "potential" describing the interaction between quark pairs. It is worth noticing at this point, that the operator expression (3.9), when written out in terms of the operators a , b , etc. contains only products of the form:

$$\begin{aligned} & : (a^\dagger(\vec{q}', y) a(\vec{q}, y)) (b(-\vec{q}', 1-y) b^\dagger(-\vec{q}, 1-y)) : \\ & = : (1 \rightarrow \sigma_3) : \end{aligned}$$

but no terms describing a quark-quark interaction. As a consequence, no invariant two-body interaction can be written down which would generate e.g. bound states of two quarks (rather than a quark-antiquark pair).

Motivated by the properties of the meson spectrum and by considerations of simplicity, we conjecture that a reasonable approximation to $v(\tau, \vartheta)$ may be a transverse oscillator potential:

$$v(\tau, \vartheta) = \left(\frac{q_0}{2}\right)^4 \nabla_{\vec{q}}^2 \delta^{(2)}(\vec{q} - \vec{q}'), \quad (3.11)$$

where the coupling constant g_0 is to be determined by fitting the meson spectrum.¹⁷⁾ In what follows, we explore consequences of the pair interaction (3.10) with the choice (3.11) for the potential.

The interaction Hamiltonian derived from (3.10) becomes:

$$\begin{aligned}
 H_1 &= \int d\mathbf{B} \frac{1}{P} \int d^2\vec{q} \int d^2\vec{q}' \int dy \, v((\vec{q}-\vec{q}')^2, y(1-y)) \\
 &\quad \times U(\mathbf{B}) I(\vec{q}', y; \vec{q}, y) U^\dagger(\mathbf{B}) \\
 &= \int d\mathbf{B} \frac{1}{P} U(\mathbf{B}) \tilde{H}_1 U^\dagger(\mathbf{B}).
 \end{aligned} \tag{3.12}$$

The explicit expression of H_1 can be easily worked out by observing that the Galilean boost, B , affects only the arguments of the quark operators, cf. eq. (2.20).

There is a further requirement which the interaction Hamiltonian has to satisfy. Working in the interaction picture, one must be able to write H_1 as the surface integral of an interaction density, $\mathcal{H}_1(\vec{x}, x; t)$ such that at equal times the density commutes on the surface $t = \text{const.}$:

$$[\mathcal{H}_1(\vec{x}, x; t), \mathcal{H}_1(\vec{x}', x'; t)] = 0 \tag{3.13}$$

(Condition (3.13) guarantees that the time ordered product of factors of $H_1(t)$, and hence, the scattering operator in the interaction picture is Lorentz invariant.)

In order to show the covariance of the time ordered product, it is evidently sufficient that (3.13) be satisfied at $t = 0$.

Introduce the Fourier transform of the quark operators:

$$\Phi(\vec{x}, x) = \frac{1}{(2\pi)^{3/2}} \int d^2\vec{p} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2|p|}} e^{i(\vec{p}\vec{x} + px)} \varphi(\vec{p}, p). \quad (3.14)$$

One verifies that $\Phi(\vec{x}, x)$ satisfies the following anticommutation relations:

$$\begin{aligned} \{\Phi(\vec{x}, x), \Phi(\vec{x}', x')\} &= 0, \\ \{\Phi(\vec{x}, x), \Phi^+(\vec{x}', x')\} &= \delta^{(2)}(\vec{x} - \vec{x}') \delta(x - x'). \end{aligned} \quad (3.15)$$

Using (3.14) and a standard change of variables, (3.12) can be rewritten as follows:

$$H_1 = \int d^2\vec{R} dR \mathcal{H}_1(\vec{R}, R), \quad (3.16)$$

where:

$$\begin{aligned} \mathcal{H}_1(\vec{R}, R) = \int d^2\vec{r} d\tau W(\vec{r}, \tau) & \left[(\Phi^+(\vec{R} + \frac{\vec{r}}{2}, R + \frac{\tau}{2}) \Phi(\vec{R} + \frac{\vec{r}}{2}, R + \frac{\tau}{2})) \right. \\ & \left. \times (\Phi^+(\vec{R} - \frac{\vec{r}}{2}, R - \frac{\tau}{2}) \Phi(\vec{R} - \frac{\vec{r}}{2}, R - \frac{\tau}{2})) + (1 \rightarrow \sigma_3) \right]. \end{aligned} \quad (3.17)$$

In the last equation, $W(\vec{r}, r)$ is the Fourier transform of the potential v . Using (3.15), one verifies that the expression (3.17) indeed satisfies (3.13) with any W . At present, the general class of invariant interactions which also satisfy (3.13) is not known to the present authors. However, by constructing some examples, one can see that the limitation imposed by (3.13) is a non-trivial one. (For example, a pair interaction of the form (3.17) but with an octet coupling between quarks at the same point would violate (3.13).)

4. Pair operators; collective excitations.

We conjecture that meson states are described - at least in some approximation - by bound quark-antiquark pairs. Therefore, let us consider the following superposition of quark pair operators:

$$A_{\mu N}^+(\vec{p}, p) = U(B(\vec{p}, p)) \tilde{A}_{\mu N}^+ U^\dagger(B(\vec{p}, p)) \quad (4.1)$$

and their Hermitean conjugates. The operators $\tilde{A}_{\mu N}^+$ (the quark pair operators in the standard frame) are given by the expression:

$$\tilde{A}_{\mu N}^+ = Z \int_0^1 \frac{dy}{y(1-y)} \int d\vec{q} f_N(\vec{q}, y) a^\dagger(\vec{q}, y) \sigma_\mu b^\dagger(-\vec{q}, 1-y), \quad (4.2)$$

($\mu = 0, 1, 2, 3$).

The functions f_N are determined from the requirement that $\tilde{A}_{\mu N}^+ |0\rangle$ be an eigenstate of mass M_N of the full Hamiltonian. The factor

$y(1-y)$ has been extracted for convenience, whereas Z is a normalization factor to be determined later.

The eigenvalue condition can be written in the following form:

$$[(H_0 + H_1), A_{\mu N}^+(\vec{p})] = \frac{\vec{p}^2 + M_N^2}{2p} A_{\mu N}^+(\vec{p}) \quad (4.3)$$

The expression of the Hamiltonian is given by eqs. (2.21), (3.9), (3.11) and (3.12). First one easily finds

$$[H_1, \tilde{A}_{iN}^+] = 0, \quad i = 1, 2.$$

By using the transformation properties of the interaction Hamiltonian, H_1 , under boosts one gets that

$$U(B(z, p)) H_1 U^\dagger(B(z, p)) = p H_1$$

and consequently

$$[H_1, A_{iN}^+] = 0, \quad i = 1, 2. \quad (4.4)$$

In other words, the "transverse" spin combinations of quark pairs are not bound by the interaction written down in the previous section.

Next, the equations for the "longitudinal" spin combinations, $\mu = 3, 0$, have to be worked out. As an example, we work out the equations for $\mu = 0$; the case $\mu = 3$ is similar. For the sake of brevity, the subscript "0" will be omitted from now on.

After some straightforward algebra we find:

$$\begin{aligned} [(H_b + H_h), A_N^+(\vec{p}, p)] = & -\frac{\vec{p}^2}{2p} A_N^+(\vec{p}, p) \\ & + \frac{1}{2p} Z \int_0^1 \frac{dz}{z(1-z)} d\vec{q} \left(\frac{\vec{q}^2}{z(1-z)} - g_o^4 z(1-z) \nabla_{\vec{q}}^2 \right) f_N(\vec{q}, z) \\ & \times a^+(z\vec{p} + \vec{q}, zp) b^+((1-z)\vec{p} - \vec{q}, (1-z)p). \end{aligned} \quad (4.5)$$

In order to obtain an equation for the pair operator. (4.1), we had to make a Hartree-Fock approximation to the right-hand-side of eq. (4.5). (Otherwise, we would have obtained an infinite set of equations, connecting n -quark operators to $(n+2)$ -quark operators.) In this approximation operator products e.g. of the form:

$$a_a^+(\vec{q}_1, y_1) b_b^+(\vec{q}_2, y_2) b_c^+(\vec{q}_3, y_3) b_d(\vec{q}_4, y_4)$$

are replaced by

$$a_a^+(\vec{q}_1, y_1) b_b^+(\vec{q}_2, y_2) \langle 0 | b_c^+ b_d | 0 \rangle - \bar{a}_a b_c^+ \langle 0 | b_b^+ b_d | 0 \rangle, \quad (4.6)$$

thus truncating the infinite set of equations for the n -quark operators. We assume for the moment that the vacuum expectation value of operators of the form $b_{(q,y)}^+ b_{(q,y)}$ and $a_{(q,y)}^+ a_{(q,y)}$ vanishes, i.e. that the Fock vacuum is the true ground state.¹⁸⁾

Thus, with the potential (3.11) we find that the eigenvalue condition (4.3) is satisfied if f_N obeys the Schrödinger equation:

$$\left(\frac{\vec{q}^2}{y(1-y)} - g_0^4 y(1-y) \nabla_{\vec{q}}^2 \right) f_N(\vec{q}, y) = M_N^2 f_N(\vec{q}, y). \quad (4.7)$$

This is the Schrödinger equation of a harmonic oscillator in two dimensions. Its normalised solution can be written in terms of associated Laguerre polynomials:

$$f_N(\vec{q}, y) \equiv u_{nm}(\vec{Q}) = \sqrt{\frac{n!}{(n+|m|)! \pi}} e^{-\frac{\vec{Q}^2}{2}} \times |\vec{Q}|^{|m|} e^{im\theta} L_n^{|m|}(\vec{Q}^2),$$

$$M_{n,m}^2 = g_0^2 (2n + |m| + 1), \quad (4.8)$$

$$(n = 0, 1, 2, \dots; m = 0, \pm 1, \pm 2, \dots),$$

where

$$\vec{Q} = \frac{\vec{q}}{q_0 \sqrt{y(1-y)}},$$

$$\tan \theta = \frac{q_1}{q_2} = \frac{Q_1}{Q_2}.$$

(4.9)

The normalization factor, Z , is determined from the requirement that in the same approximation as used in deriving eq. (4.7), the pair operators $A_{nm}^+(\vec{p}, p)$ create free - composite - particles. To this end we calculate the commutator of $A_{nm}(\vec{p}, p)$ with its Hermitean conjugate.

After a brief calculation we find:

$$[A_{nm}(\vec{p}, p), A_{n'm'}^+(\vec{p}', p')] =$$

$$= 4q_0^2 Z^2 \, 2|p| \delta(p-p') \delta^{(2)}(\vec{p}-\vec{p}') \times \\ \times \int d^3\vec{Q} \, \bar{U}_{nm}(\vec{Q}) U_{n'm'}(\vec{Q})$$

+ (terms containing $\bar{d}a$, b^+b).

Thus, on taking the Hartree-Fock approximation to the right-hand-side, we find that

$$Z = (2q_0)^{-1} \quad (4.10)$$

$$[A_{nm}(\vec{\beta}, p), A_{n'm'}^+(\vec{\beta}', p')] \approx \delta_{nn'} \delta_{mm'} 2|p| \delta(p-p') \delta^{(2)}(\vec{\beta}-\vec{\beta}'). \quad (4.11)$$

This tells us that the operators A , A^+ describe free "collective excitations" of the system, with a spectrum given by (4.8). In view of (4.11), one can write down an effective Hamiltonian, H' , for the collective excitations, viz:

$$H' = \sum_{nm} \int d^2\vec{\beta} \int_0^\infty \frac{dp}{2p} \frac{\vec{p}^2 + M_{nm}^2}{2p} A_{nm}^+(\vec{\beta}, p) A_{nm}(\vec{\beta}, p), \quad (4.12)$$

$$M_{nm}^2 = q_0^2 (2n + |m| + 1).$$

The reader will recognize that our treatment of the pair excitations bears a close resemblance to Bogoliubov's method of handling collective states in a superconductor.¹⁹⁾

The oscillator wave functions (4.8) tell us that the probability of finding a quark pair with very different longitudinal momenta ($y \rightarrow 0$ or $y \rightarrow 1$) vanishes exponentially, unless the relative transverse momentum is exactly zero. (Practically, the quark pairs share the longitudinal momentum in equal amounts, $y \approx \frac{1}{2}$.) For $y \approx \frac{1}{2}$, the characteristic spread of the wave packet in transverse coordinate space is of the order q_0^{-1} , which is the

fundamental length in the theory.

It can be observed that the wave functions (4.8) are irreducible tensors under rotations around the 3^d axis. For example, the wave functions for $m = \pm 1$ can be written in Cartesian form as:

$$\vec{u}_n(\vec{Q}) = \frac{\vec{Q}}{\sqrt{\pi(n+1)}} e^{-\frac{\vec{Q}^2}{2}} L_n^1(\vec{Q}^2). \quad (4.13)$$

Finally, we record the low-energy behavior ($|\vec{Q}| \rightarrow 0$) of the oscillator functions. Since

$$L_n^{1m}(0) = \binom{n+1m}{n}$$

we have:

$$u_{nm}(\vec{Q}) \sim \frac{|\vec{Q}|^m e^{im\theta}}{m!} \sqrt{\frac{(n+1m)!}{\pi n!}}, \quad (|\vec{Q}| \rightarrow 0). \quad (4.14)$$

5. Elastic scattering of spinless mesons. Duality.

We now add a phenomenological term to the Hamiltonian in order to obtain scattering. We assume that the dominant process in the scattering of mesons is that the two incoming mesons annihilate and thereby excite the collective modes of the system. The collective excitations then propagate according to the dynamics

described in the previous sections and finally generate the two outgoing mesons. In this approximation the incoming and outgoing mesons are treated as external sources for the collective excitations. The advantage of this simplification is that the scattering operator can be determined exactly. One hopes that - at least for certain problems - this gives a reasonable approximation to the physical scattering amplitude. The external mesons are taken to be spinless and we guess a simple form of the interaction. In order to simulate the selection rules in - say - the $\pi\pi\phi$ coupling, we allow only odd ($|m| = 1, 3, \dots$) two dimensional partial waves to contribute.

It is evident that one may rescale the energy units in such a way that g_0 can be chosen to be unity. Let $C(\vec{q}, q)$, $C^\dagger(\vec{q}, q)$ describe the absorption and emission of the external mesons, respectively. We try the following form for the effective interaction in the standard frame:

$$\begin{aligned} \tilde{H}'' = \lambda \int \frac{d\vec{q} d\vec{q}' dy dy'}{y(1-y) y'(1-y')} & (a^\dagger(\vec{q}, y) b^\dagger(-\vec{q}', 1-y) + h.c.) \\ & \times \mathcal{U}(\vec{q}, y; \vec{q}', y') (C(\vec{q}', y') C(-\vec{q}, y) + h.c.). \end{aligned} \quad (5.1)$$

We simply choose for the "potential," $\mathcal{U}$, an antisymmetrized δ -function:

$$\mathcal{U}(\vec{q}, y; \vec{q}', y') = \frac{1}{2} [\delta(\vec{Q} - \vec{Q}') - \delta(\vec{Q} + \vec{Q}')], \quad (5.2)$$

while the constant λ will be fixed later. Evidently, the potential (5.2) can be expanded in terms of oscillator functions (4.8) as follows:

$$U(\vec{q}, y, \vec{q}', y') = \sum_{n, (m=\text{odd})} U_{nm}(\vec{Q}) \bar{U}_{nm}(\vec{Q}')$$

so that

$$\begin{aligned} \tilde{H}'' = \frac{\lambda}{Z} \int \frac{d^2 \vec{q}' dy'}{y'(1-y')} \sum_{n, (m=\text{odd})} & \left(\tilde{A}_{nm}^+ \bar{U}_{nm}(\vec{Q}') + \text{h.c.} \right) \\ & \times \left(C(\vec{q}', y') C(-\vec{q}', 1-y') + \text{h.c.} \right). \end{aligned} \quad (5.3)$$

A general state of two incoming mesons can be written in terms of a standard two-meson state, boosted up to the desired center-of-mass momentum, viz.:

$$|1,2\rangle = U(B(\vec{p}, p)) C^+(\vec{q}, y) C^+(-\vec{q}, (1-y)) |0\rangle. \quad (5.4)$$

We calculate the scattering amplitude in the standard frame, i.e. $\vec{p}=0$, $p=1$ in eq. (5.4). Before the calculation is performed two remarks are in order.

First, the use of the phenomenological interaction (5.3) makes sense physically if the factor containing the external particles is replaced by its expectation value; thus one effectively deals with a classical source. (External field approximation.)

Second, using eqs. (4.11), (4.12), one observes: $[H', \tilde{A}_{nm}^+] = \tilde{A}_{nm}^+ \frac{M_{nm}^2}{2}$.

Therefore, as long as one stays in the standard frame, the calculation can be simplified by considering (4.11) and (4.12) to be built up from "effective oscillator operators", $B, B^\dagger$:

$$H' = \frac{1}{2} \sum_{nm} M_{nm}^2 B_{nm}^\dagger B_{nm}$$

$$[B_{nm}, B_{n'm'}^\dagger] = \delta_{nn'} \delta_{mm'} \quad (5.5)$$

We write for the matrix element of the transition operator:

$$T_{fi} = -(2\pi)^4 i \delta^{(4)}(\sum P_f^{(f)} - \sum P_i^{(i)}) \mathcal{T}, \quad (5.6)$$

where $\mathcal{T}$ is the scattering amplitude.

Notice that in the standard frame $2\sum h_i$ for the incoming particles is just the total invariant mass squared:

$$\sum h_i = \frac{s}{2},$$

while the longitudinal momentum fractions are fixed at $y=y'=\frac{1}{2}$ (equal mass scattering). In the external field approximation H'' is to be replaced by its expectation value between a two-meson state and the vacuum with respect to the external mesons, i.e.

$$\tilde{H}'' \rightarrow \frac{4\lambda}{Z} \sum_{n, (m=\text{odd})} (B_{nm}^\dagger \bar{u}_{nm}(\vec{q}) + \text{h.c.}). \quad (5.7)$$

Here $\vec{q}$ is the relative transverse momentum of the incident (outgoing) meson pair, and the condition $y=\frac{1}{2}$ was taken into account. (In what follows, we continue to denote the quantity (5.7) by H''). Next, take a low energy approximation to H'' , i.e. take

its limit as $|\vec{q}| \rightarrow 0$. (This is reasonable, since at high energies we cannot expect the scattering amplitude to be well described by the approximation used here, anyhow.)

Using (4.14) and going over to a Cartesian basis, we get:

$$H'' \approx \frac{8\lambda}{2\sqrt{\pi}} \vec{q} \sum_n \sqrt{n+1} (\vec{B}_n^\dagger + \vec{B}_n) \quad (5.8)$$

(Since from now on we will work exclusively in the standard frame, we omit the tilde from the notation.) Since H'' is approximated by the contribution of the dominant (lowest) partial wave, we may retain that wave only in the expression of the effective Hamiltonian, (4.12), of the collective excitations. In the standard frame we thus have:

$$H' \approx \sum_n (n+1) \vec{B}_n^\dagger \vec{B}_n \quad (5.9)$$

On choosing $\lambda = 2\pi^{1/2}/8$, the total effective Hamiltonian becomes

$$H' + H'' \approx \sum_{n=0}^{\infty} (n+1) \vec{B}_n^\dagger \vec{B}_n + \sqrt{n+1} \vec{q} (\vec{B}_n^\dagger + \vec{B}_n). \quad (5.10)$$

It is a straightforward matter to calculate the scattering amplitude with such a Hamiltonian. One has to remember that eigenstates of (5.10) are coherent states. Introduce the operator $U(q)$ by the equation:

$$U(\vec{q}) (H' + H'') U^\dagger(\vec{q}) = \sum_n (n+1) \vec{b}_n^\dagger \vec{b}_n$$

where

$$\vec{b}_n = \vec{B}_n - \vec{\lambda}_n$$

This gives

$$\vec{\lambda}_n = \frac{\vec{q}}{\sqrt{n+1}}$$

$$U(\vec{q}) = \exp \vec{q} \cdot \sum_n \frac{\vec{b}_n - \vec{b}_n^\dagger}{\sqrt{n+1}}$$

(The vacuum expectation values of the various operators are always subtracted.)

Thus, the scattering amplitude, $\mathcal{T}$, in the standard frame becomes:

$$\begin{aligned} \mathcal{T} \approx & \langle 0 | \sum_n \sqrt{n+1} \vec{q}' \cdot (\vec{b}_n^\dagger + \vec{b}_n) U(\vec{q}') R(s) U^\dagger(\vec{q}) \\ & \times \sum_n \sqrt{n+1} \vec{q} \cdot (\vec{b}_n^\dagger + \vec{b}_n) | 0 \rangle \end{aligned} \quad (5.11)$$

where $R(s)$ is the resolvent of the free Hamiltonian:

$$\begin{aligned} R(s) &= \left(\frac{s}{2} - \sum_n (n+1) \vec{b}_n^\dagger \cdot \vec{b}_n \right)^{-1} \equiv \left(\frac{s}{2} - H' \right)^{-1} \\ &= \int_0^1 \frac{dx}{x} \times \frac{s}{2} - \sum_n (n+1) \vec{b}_n^\dagger \cdot \vec{b}_n \end{aligned} \quad (5.12)$$

Remember that $U(\vec{q})$ satisfies the equation:

$$[H', U(\vec{q})] = H'' U(\vec{q}), \quad (5.13)$$

we derive:

$$\left(\xi \frac{d}{d\xi} e^{H' \log \xi} U e^{-H' \log \xi} \right)_{\xi=1} = H'' U \quad (5.14)$$

This, together with (5.12), gives the following representation for $\mathcal{T}$:

$$\mathcal{T} = \left[\xi \eta \frac{\partial^2}{\partial \xi \partial \eta} G(\xi, \eta) \right]_{\xi=\eta=1}, \quad (5.15)$$

where

$$G(\xi, \eta) = - \int_0^1 \frac{dx}{x} x^{-\frac{s}{2}} \langle 0 | U(\vec{q}') (\xi \eta x)^{H'} U^\dagger(\vec{q}) | 0 \rangle \quad (5.16)$$

The evaluation of (5.16) is straightforward. (In particle physics, expressions of this type became standard, after the harmonic oscillator formalism of dual resonance models was discovered by Fubini, Gordon, Nambu and Veneziano;^{20,21} the mathematical techniques have been, however, known for some time.) In evaluating (5.15), (5.16), we have to remember that the phenomenological interaction, (5.1) has been approximated by its low-energy limit. Therefore, every term of $O(\vec{q}^2)$ arising from the interaction has to be neglected in order to maintain the consistency of the approximation. Observing this rule, one derives the following expression for the amplitude, $\mathcal{T}$:

$$\mathcal{T} \approx -(\vec{q}\vec{q}') \lambda^2 \int_0^1 dx x^{-\frac{s}{2}+1} (1-x)^{-\vec{q}\vec{q}'-2} (x(\vec{q}\vec{q}') - 1).$$

On remembering that

$$q\bar{q}' = \frac{t}{2} + O(q^2)$$

this can be replaced by

$$\mathfrak{S} = \lambda^2 \left(\frac{t}{2}\right) \left[\left(\frac{t}{2}\right) \frac{\Gamma(-\frac{s}{2}+2) \Gamma(\frac{t}{2}+1)}{\Gamma(-\frac{s}{2}-\frac{t}{2}+3)} - \frac{\Gamma(-\frac{s}{2}+1) \Gamma(-\frac{t}{2}+1)}{\Gamma(-\frac{s}{2}-\frac{t}{2})} \right].$$

Thus, a Veneziano representation is obtained for the scattering amplitude of scalar mesons. Evidently, $\mathfrak{S}$ is not fully crossing symmetric. However, the purpose of this exercise has been merely to demonstrate that the contributions of the collective excitations can add up to give a Regge behavior of the amplitude. Given the fact that the scattering amplitude has been calculated in a very crude approximation, we believe that the result is quite satisfactory.

6. Concluding remarks.

In this paper we tried to make the first steps toward a dynamical theory of hadrons based on the parton hypothesis. We believe that the structure of dynamics emerging from these considerations is sufficiently interesting in order to warrant further investigation. In particular, the formulation of the theory directly in front form (without recourse to the use of fields and constraint equations) reveals a considerable freedom in setting up the dynamical equations. This may be particularly useful

in a search for a consistent theory of hadrons. In this respect the examples worked out in the last two sections are quite instructive. Even though the model chosen is a very primitive one (and also contains a certain amount of arbitrariness), it suggests a physical picture which may well be at least qualitatively correct.²²⁾

The results of the calculations suggest namely that our present concepts and methods used in high energy physics correspond essentially to a "low-energy approximation" to a future, more complete theory. The front form of dynamics is particularly appropriate from this point of view. Due to the Galilean structure of the theory in transverse space, quite a number of intuitively useful concepts (excitation energies, etc.) can be carried over from nonrelativistic theories in a Lorentz invariant way.

A final remark is in order concerning the nature of "constituents" of hadrons. Whereas in the present paper the theory was formulated as if quarks were ordinary "particles," it seems that such objects can never be seen as real particles by any observer in an inertial frame of reference. While this question is still under investigation at present, we wish to offer the following qualitative argument. Quarks play a role similar to the "bare quanta" in a field theory. It is known that the number of bare quanta is not a Lorentz invariant quantity in an interacting theory. Since quark states were defined in a LF, any inertial observer is connected with that frame via an infinite boost. This in turn changes a single-quark state into an infinite superposition of quarks. It appears then that parton (quark)

operators appear as convenient mathematical objects, in terms of which the theory can be formulated simply; however, they probably have no direct particle interpretation in an inertial frame of reference.

Acknowledgement.

We wish to thank Professor F. Gürsey for communicating to us several results prior to publication and for numerous discussions. We also wish to acknowledge fruitful conversations with Drs. L. N. Chang and P. Ramond.

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Research supported in part by the United States Atomic Energy Commission under contract No. AT(11-1)-3285.

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- 15.) I. Bars and F. Gürsey, Phys. Rev. D4, 1769 (1971). (Appendix and Yale preprint No. 2726-608 (1971) (unpublished).

- 16.) Eq. (3.2) does not give the most general form of the interaction. It seems possible to construct interactions which are non-local in the time-variable, t . In this paper, however, no such interactions are considered.

- 17.) **The quantity g_0 is the characteristic energy of the model. We shall see (Sec.5) that it also governs the slope of Regge-trajectories.**

- 18.) If this is not the case, the stable ground state can be found by performing a Bogoliubov transformation on the quark operators. All the following considerations remain essentially valid in the case of a degenerate ground state, provided the quark operators are reinterpreted as "quasiparticle" operators. We do not enter the question of a degenerate ground state in this paper.

- 19.) Cf. N. N. Bogoliubov, V. V. Tolmachov and D. V. Shirkov Fortsch. der Phys. 6, 605 (1958).

- 20.) S. Fubini, D. Gordon and G. Veneziano, Physics Letters 29B, 679 (1969).
- 21.) Y. Nambu, in Symmetries and Quark Models (Proc. International Conference, Wayne State University, 1969; R. Chand, Ed.) p. 269. (Gordon and Breach, New York, 1970)
- 22.) The reader will obviously notice the formal similarity between the model discussed in Sec.5. and the various "quasi-realistic" interpretations of dual resonance amplitudes in terms of elastic continua in Minkowski space, "fishnet"-diagrams, etc. Typical examples are: B.Sakita and M.Virasoro, Phys. Rev. Letters, 24, 1146 (1970); H.Nielsen and P. Olesen, Phys. Letters 32B, 203 (1970); L.Susskind, Nuovo Cim. 69A, 457 (1970); Y.Nambu, Univ. of Chicago preprint EFI 70-70 (1970) (unpublished); L.N.Chang and F.Mansouri, Univ. of Chicago preprint EFI 71-72 (1971) (unpublished). As far as the use of a null-plane formalism is concerned, our work is perhaps closest in its spirit to Bjorken's (J.D.Bjorken, SLAC-PUB-905 (1970); unpublished). As of this writing, however, we have been unable to discover a direct relationship between the present work and the ones just quoted. A possible exception may be : P.Ramond, Nuovo Cim. 4A, 544 (1971). This author does not specify detailed properties of the "internal motion" of a composite system; hence his scheme may probably accomodate a large class of concrete models.

Table 1

Action of discrete symmetry operations on the quark operators

Symmetry Operation	Remarks
$U(C) \varphi(z, p) U(C)^\dagger = \eta_C \epsilon \varphi^{\dagger T}(z, -p)$	Superscript T stands for "transposed."
$U(S) \varphi(z, p) U(S)^\dagger = \eta_S \epsilon \varphi(\bar{z}, p)$	
$U(Q) \varphi(z, p) U(Q)^\dagger = \eta_Q \varphi^{\dagger T}(z, p)$	Q is antiunitary: reverse order of operator products (Schwinger)
η_C, η_S, η_Q are arbitrary phase factors.	