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PION-NUCLEON INTERACTIONS IN THE INTERMEDIATE ENERGY RANGE

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Abstract

A detailed dynamical calculation is carried out to try to understand the formation of the resonances in elastic pion-nucleon scattering. The method used is a subtracted form of a two-channel N/D method, using Born terms for the dynamical cuts arising from rho meson exchange, 3-3 crossing, and crossed nucleon terms. The pion production at intermediate energies (below 1500 MeV) is represented by the formation of a rho meson with the one-pion exchange process, and the two-channel processes are required to satisfy unitarity approximately. It is found that the pattern of higher resonances is essentially determined by the elastic amplitude dynamical cuts, in particular the

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3-3 crossing term, but the inelastic process has a strong reactive damping effect which may determine the resonance widths. A vector meson model of strong interactions is used to estimate the $\pi\pi\rho$, e^{NN} and eee vertex strengths.

INTRODUCTION

A great deal of experimental data has been accumulated concerning the scattering of pions on nucleons, both elastic and inelastic, over a range of energies up to the multi-BeV region. This data presents many striking features, notably several peaks in the total cross sections, of which at least one can be identified with an elastic-scattering resonance (in the state $J = 3/2$, $T = 3/2$), while the interpretation of the other peaks is less clear at the moment. Many attempts have been made to calculate the pion-nucleon scattering and reaction amplitudes, using various models and attempting to reproduce some of the observed resonances in a qualitative fashion. The formation of the 3-3 resonance is now moderately well understood, although no quantitative calculation to date has been able to give its position correctly and unambiguously.

The latest publications concerning the pion-nucleon system^{1,2,3} treat the system as a multichannel reaction process, considering multiparticle production processes to be approximated by resonant states, thus reducing all states to two-particle states, which are amenable to simpler methods of analysis than three or more particle states. On the basis of this common formulation, it has become apparent that there exist a wealth of mechanisms which may be invoked to generate the higher resonances, and it is now imperative that one set out to do a quantitative calculation if one wishes to test a given set of assumptions

concerning the strong interactions involved in the pion-nucleon system.

The work of Bowcock, Cottingham and Lurie⁴, Frautschi and Walecka⁵, Hendry⁶, and Carruthers⁷ has made it abundantly clear that many of the important features of pion-nucleon reactions are intimately connected with the pion-pion interactions. Thus it is necessary to adopt a model for both of these strong interactions before an unambiguous test can be made through detailed calculations. Such a model has been proposed by Sakurai⁸; its central assumption involves the mediation of strong interactions by vector mesons, which have lately been identified with some of the observed resonant states in the two- and three-pion systems⁹. This is the model which we shall adopt in this calculation; it will be referred to as VTSI.

Although we do not know how to go about using VTSI as a model in the field-theoretic sense, since vector meson perturbation theory is nonrenormalizable, we can consider certain vertex parts in the dispersion-theory approach to the system and reject those which would be forbidden by VTSI. Furthermore, the universality hypothesis of VTSI allows us to determine (at least approximately) the values of the effective coupling for $e_{\pi\pi}$, e_{NN} , and $e_{\pi\pi\pi}$ which are required. The results of the calculation are not sensitive to the model used.

The weakest link in a full dynamical calculation at the present state of the art is the method used, given the above

hypothesis, to proceed to calculate the amplitudes. We will adopt a method which seems likely to reproduce reliably the gross features of the amplitudes, although the positions and widths of any resonances obtained through this calculation will not be reliable.

The inelastic processes of pion production will be taken into account only insofar as the final state pions are produced in a $T = 1$, $J = 1$ state (in the two-pion center of mass system), as was done by Cook and Lee¹. It is hoped that inclusion of the other processes, such as those states in which one of the final-state pions interacts with the nucleon in a resonant 3-3 configuration, would not alter the qualitative aspects of the amplitudes. This is indicated to some extent by the work of Hendry⁶.

The $J = 1$ resonant two-pion system will be identified with the ρ meson which couples to the isospin current of VTSl. This has been shown to be a reasonable assumption when carried out in the analysis of nucleon electromagnetic form factors (e.g. by Gell-Mann and Zachariasen¹⁰). This same resonant system will be used as part of the effective potential in the region of elastic pion-nucleon scattering, as well as the single-nucleon pole terms and the effect of the 3-3 resonance in the crossed channel, as in the work of Frautschi and Walecka⁵.

A formalism is used which guarantees exact unitarity, within the above approximation for inelastic processes, and the correct location for singularities in the complex variable W ; where for

real positive W above the elastic threshold, W represents the total energy of the system. A partial-wave dispersion formalism is set up for the two-channel reaction

$$\pi + N \rightarrow \pi + N, \quad \pi + N \rightarrow \rho + N$$

following the procedure of Cook and Lee¹, and analytically continued in the ρ mass to the physical value to give the expected locus of singularities, considering only peripheral (one-pion exchange) ρ production. Then we consider the mechanisms diagrammed in Fig. 1 to generate the left-hand singularities for the elastic process, those in Fig. 2 for the production process and for the process

$$\rho + N \rightarrow \rho + N$$

Note that we do not make the grossly simplified assumptions of Cook and Lee concerning the elastic amplitudes, but the price of including reasonable left-hand discontinuities is that all calculations will have to be done numerically (with the aid of a digital computer) if we intend to get quantitative results.

Calculations are carried out using a subtracted form of a determinantal method, in the one-channel elastic approximation first; then with the two-channel formalism described above an approximation in the same spirit is made, starting with the coupled integral equations for N and D matrices including anomalous-cut terms from the one-pion exchange ρ production process. A detailed description of the approximation is given below. The

inclusion of the explicit inelastic process, as we change from a one-channel to a two-channel process, has large effects on the characteristics of the elastic scattering amplitude.

ONE-CHANNEL ELASTIC APPROXIMATION

The first method used was quite similar in spirit to Hendry's⁶ approach, except that instead of replacing left-hand cuts with pole terms, the dynamical cuts were assigned the discontinuities given by the Born approximations corresponding to the diagrams in Fig. 1. Then instead of solving the integral equations for N and D in each angular momentum and parity state, we set D equal to unity on the left hand cuts in the q^2 plane as a first approximation and make one subtraction in the expression for D in terms of N on the right-hand cut. The resulting matrix element for each scattering eigenstate can then be expressed as follows:¹⁶

$$F_{\ell\pi\tau}(q) = N_{\ell\pi\tau}(q) / D_{\ell\pi\tau}(q^2) = \frac{\exp[i\delta_{\ell\pi\tau}(q)] \sin \delta_{\ell\pi\tau}(q)}{\rho(q)} \quad (2.1)$$

$$\text{Re } D_{\ell\pi\tau}(q^2) = 1 - \frac{(q^2 - q_0^2)}{\pi} P \int_0^\infty \frac{q'^2 N_{\ell\pi\tau}(q') \rho(q')}{(q'^2 - q^2)(q'^2 - q_0^2)} \quad (2.2)$$

$$\text{Im } D_{\ell\pi\tau}(q^2) = -\rho(q) N_{\ell\pi\tau}(q) \quad \text{for } q^2 > 0 \quad (2.3)$$

$$\rho(q) = 2(q^2/s)^{1/2}$$

In these expressions, q is the center-of-mass momentum of the pion, s is the square of the total center-of-mass energy, and the N functions are the sums of the partial wave projections of the Feynman diagrams of Fig. 1. The principal-value integrals must be done numerically. The subtraction point q_0 was chosen at the point where $s = M_n^2$; this is rather arbitrary, but some a fortiori justification is given by the reasonableness of the results.

Numerical study of these relations for states with $\ell \leq 4$ was carried out with an IBM 1620 computer. The coupling constants were varied for each of the contributions of Fig. 1, and a set *of phase shifts as a function of energy was obtained for each set* of coupling constants, over the range 150-1500 MeV pion lab energy. The crossed 3-3 resonance diagram was the greatest influence in this approximation; it was possible to generate a pattern of higher resonances in the pion-nucleon system roughly consistent with the experimental data by using only the diagrams b and c of Fig. 1. The addition of the ρ meson exchange (1d) strongly modified the resonances, but the addition of the ABC exchange was negligible in the energy range studied with coupling constants comparable to the pion-ABC vertex used for the nucleon-ABC vertex.

This dominance of the crossed 3-3 diagram, however, may be largely a consequence of the approximation method used. The left-hand cut contribution is large because of the high spin of the quasi-particle; thus, the contribution to the principal value integral in D from momenta higher than Q in the N function drops

off like $1/Q^2$ for the vector meson exchange, but like $1/Q$ for the 3-3 exchange. This difficulty persists in the two-channel treatment of the problem given below. If one desires to improve this kind of approximation, it would clearly be desirable to make use of some kind of Regge pole representation for the 3-3 resonance¹¹ as has been done by Wong in the pion-pion problem¹².

Explicit expressions for the Born terms used in N will be given in the discussion of the two-channel problem.

Numerical results for the principal-value integrations are presented in Fig. 4, including only the states which may be resonant below 2 BeV. The zeros of $\text{Re}(D)$ give the locations of resonances. No subtraction constants are introduced for the curves of Fig. 4. The widths of the resonances may be calculated in terms of the slope of these curves divided by the value of N at the resonance. Most of the widths of the higher resonances are rather wide, but the values of the N functions are not expected to be reliable at high energies with the rough approximation used.

In an effort to produce a set of resonances whose positions correspond roughly to the observed pattern of peaks in the elastic cross section, we may introduce arbitrary subtraction constants in the integrals representing the contribution of the 3-3 crossing term. These must have the same isospin dependence as the Born terms; thus, if we specify the position of the $T = 1/2$ resonances, the $T = 3/2$ resonance positions follow. For exploratory

purposes, subtraction constants were introduced for the $3-3$ crossing integrals to change the resonance positions in the states $P_{3/2}$, $D_{3/2}$ and $G_{7/2}$. The attempt was made to obtain the experimental $T = 1/2$ resonances at 600 and 900 MeV from $D_{3/2}$ and $G_{7/2}$, and force the $T = 3/2$, $P_{3/2}$ resonance to its experimental value. The $T = 1/2$, $P_{3/2}$ state and the $T = 3/2$, $D_{3/2}$ and $G_{7/2}$ states were thereby determined as to subtractions, and the resulting denominator behavior is given by the curves of Fig. 5. The dotted curves of Fig. 5 show the effect of removing the rho-meson exchange contributions.

Determining the $T = 3/2$, $P_{3/2}$ state in this way, one creates a bound state with $T = 1/2$, $P_{3/2}$ which is manifested by the vanishing of the denominator function in that state for some value of $q^2 < 0$. Since no such particle has been discovered, we may say that this is probably not a good way to choose the subtractions. In the two-channel case we use a reduction factor instead to adjust the $3-3$ resonance position.

It is evident that in our method the $T = 1/2$, $P_{1/2}$ state will be bound with total energy of the bound state less than M_n , as can be seen from Fig. 6. This is consistent with the interpretation of Chew¹³ of the nucleon itself as a bound state; our approximation method overestimates the binding by about 100 MeV, although the mechanism is the same as that proposed by Chew.

Other than the above difficulties, a pattern of resonances can be generated with some resemblance to the experimental data.

Table I summarizes the experimental situation. Note, however, that the assignment $G7/2$ is definitely preferred over $F5/2$ for the $T = 1/2$ resonance at 900 MeV; this is probably in conflict with the data, since a $G7/2$ resonance would be much sharper than the observed one. It is interesting to note that the N function for the $G7/2$ state has a zero at an energy slightly below the resonance, simulating an "elementary particle" character for such a resonance. There is no a priori reason why this might not happen in other, more realistic calculations. It seems probable, however, that the predominance of the G state over the F state is only a consequence of our representation of the 3-3 crossing term and its overestimation. In the two-channel formulation, where the W plane is used instead of the Q^2 plane, the F state is preferred.

The general character of the curves is largely determined by the 3-3 crossing term, although the rho contributes quite significantly. The ABC exchange effect is negligible above 150 MeV.

TWO-CHANNEL REACTION FORMULATION

Following Cook and Lee¹, we approximate the inclusion of single pion production processes by considering the two-channel reaction processes involving the pion, nucleon, and rho meson. The rho meson will be assumed to have a narrow width, thus eliminating the mass-distribution integrals in Ref. 1. Since

we do not ignore the left-hand cuts, our integral equations are rather more complicated than those of Ref. 1, but the additional terms have a structure simply dictated by the analytic properties of the N and D functions, as continued to physical mass values for the rho meson. Using the notation of Ref. 1 we write for the two-channel transition matrix

$$\underline{\underline{F}} = \underline{\underline{N}} \cdot \underline{\underline{D}}^{-1} \quad (3.1)$$

Here the matrix notation refers both to channels and to helicity states of the rho; in the equations below for N and D, only helicity indices are meant. As in Ref. 1, we use the notation (22) for the pion-nucleon channel (two-particle initial and final states), while (33) will index the rho-nucleon channel (three-particle states). Then (23) will index the production amplitude, and (32) the reverse reaction. The expressions for the elements of the N matrix become:

$$\underline{\underline{N}}_{22}(w) = \int_{D.C.} \frac{dw'}{w'-w} \left\{ \underline{\underline{A}}_{22}(w') \underline{\underline{D}}_{22}(w') + \underline{\underline{A}}_{23}(w') \cdot \underline{\underline{D}}_{32}(w') \right\} \quad (3.2)$$

$$\underline{\underline{N}}_{23}(w) = \int_{D.C.} \frac{dw'}{w'-w} \left\{ \underline{\underline{A}}_{22}(w') \underline{\underline{D}}_{23}(w') + \underline{\underline{A}}_{23}(w') \cdot \underline{\underline{D}}_{33}(w') \right\} \quad (3.3)$$

$$\underline{\underline{N}}_{32}(w) = \int_{D.C.} \frac{dw'}{w'-w} \left\{ \underline{\underline{A}}_{32}(w') \underline{\underline{D}}_{22}(w') + \underline{\underline{A}}_{33}(w') \cdot \underline{\underline{D}}_{32}(w') \right\} \quad (3.4)$$

$$\underline{\underline{N}}_{33}(w) = \int_{D.C.} \frac{dw'}{w'-w} \left\{ \underline{\underline{A}}_{32}(w') \cdot \underline{\underline{D}}_{23}(w') + \underline{\underline{A}}_{33}(w') \cdot \underline{\underline{D}}_{33}(w') \right\} \quad (3.5)$$

The notation D.C. is meant here as "dynamical cuts", including the left hand cuts and the anomalous cut arising from the one-pion-exchange process in the rho production matrix elements.

The anomalous branch cut extends from $W = M + \mu$ back along the real axis to $W = (M^2 + 2\mu^2)^{1/2}$, then down into the lower half of the complex W plane. Its endpoint is given by

$$W_0 = (1/\sqrt{2}) \left\{ 2M^2 + M_\rho^2 - i \left[M_\rho^2 (M_\rho^2 - 4\mu^2) (4M^2 - \mu^2) / \mu^2 \right]^{1/2} \right\}^{1/2}$$

$$-\pi/2 < \text{Arg}(W_0) < 0$$

This anomalous cut is the extension of a dynamical branch line through the physical cut; the intrusion occurs when we analytically continue the singularity given by the one-pion exchange diagram in the external mass M_ρ from small M_ρ until $M_\rho^2 \gg 2\mu^2(1 + \mu/2M)$. This process is described in Ref. 1. The branch cuts are pictured in Fig. 3.

The notation A.C. will be used to indicate the anomalous cut alone, while P.C. will denote the physical cut along the positive energy axis above the elastic threshold for pion-nucleon scattering.

The above equations follow from the equations (39) of Ref. 1, preserving their form after the analytic continuation. The A_{ij} are the discontinuities of the F matrix across the cuts, multiplied by $2\pi i$.

Referring to equations (37) and (53) of Ref. 1, we have in the same notation as above the equations for the D elements:

$$D_{22}(w) = 1 - \int_{P.C.} \frac{dw'}{w'-w} \rho_2(w') N_{22}(w') + 2\pi i \int_{A.C.} \frac{dw'}{w'-w} \rho_2(w') \underbrace{A_{23}(w')}_{\sim} \underbrace{D_{32}^*(w')}_{\sim} \quad (3.6)$$

$$\underbrace{D_{23}(w)}_{\sim} = - \int_{P.C.} \frac{dw'}{w'-w} \rho_2(w') \underbrace{N_{23}(w')}_{\sim} + 2\pi i \int_{A.C.} \frac{dw'}{w'-w} \rho_2(w') \underbrace{A_{23}(w')}_{\sim} \underbrace{D_{33}^*(w')}_{\sim} \quad (3.7)$$

$$\underbrace{D_{32}(w)}_{\sim} = - \int_{P.C.} \frac{dw'}{w'-w} \rho_3(w') \underbrace{N_{32}(w')}_{\sim} \quad (3.8)$$

$$\underbrace{D_{33}(w)}_{\sim} = \mathbf{I} - \int_{P.C.} \frac{dw'}{w'-w} \rho_3(w') \underbrace{N_{33}(w')}_{\sim} \quad (3.9)$$

The spirit of the approximation we desire would be achieved if we could set the D functions equal to constants under the integrals of equations (3.6) and equations (3.2)-(3.5), converting the coupled integral equations into a set of formulae for calculating the N and D functions explicitly in terms of the discontinuities A. This, we hope, would reproduce roughly the energy dependence of the exact solution, although we should be very careful about drawing detailed conclusions from the result.

It is possible to do this in all integrals except the anomalous cut terms in (3.6)-(3.7). There, the discontinuity functions A_{23} are singular at $W = M + \mu$, exhibiting poles in P of order $(\ell + 1)$, where ℓ is the angular momentum index of the πN system ($J = \ell \pm 1/2$). These singularities arise from the one-pion exchange diagram of Fig. 2, and were discussed by Cook and Lee¹. Kinematic factors of $(P/E + M)$ in A_{23} reduce the order of the singularity by unity. We assume that the formulation is

correct and nondivergent if the exact integral equation solutions are used for D_{32} and D_{33} ; this means that D_{32} and D_{33} must have zeros of order ℓ at $P = 0$. Such behavior is not inconsistent with their analytic properties.

Under this assumption, we see that the simplest approximation which will give an energy dependence for the anomalous integrals similar to the exact solution is the introduction of threshold factors with zeros of order ℓ which make the integrals just convergent. Thus, our approximation includes replacing (3.6) and (3.7) by:

$$D_{22}(W) = 1 - \int_{P.C.} \frac{dw'}{w' - W} \rho_2(w') N_{22}(w') + \left[2\pi i \int_{A.C.} \frac{dw'}{w' - W} \rho_2(w') \underline{A}_{23}(w') \left(\frac{P'}{E' + M} \right)^\ell \right] \cdot \hat{D}_{32}^*(W_0)$$

$$\underline{D}_{23}(W) = - \int_{P.C.} \frac{dw'}{w' - W} \rho_2(w') \underline{N}_{23}(w') + \left[2\pi i \int_{A.C.} \frac{dw'}{w' - W} \rho_2(w') \underline{A}_{23}(w') \left(\frac{P'}{E' + M} \right)^\ell \right] \cdot \hat{D}_{33}^*(W_0)$$

These define the constants $\hat{D}_{ij}(W_0)$.

The next step in the approximation is to make a subtraction of all denominator integrals at $W = W_0$; we will later choose W_0 to be M_N , for the sake of definiteness. Then the relations (3.6)-(3.9) are transformed into:

$$D_{22}(W) = [D_{22}(M) - 1] + 1 - (W - M_N) \int_{P.C.} \frac{dw' \rho_2(w') N_{22}(w')}{(w' - M)(w' - W)} + 2\pi i \left\{ (W - M_N) \int_{A.C.} \frac{dw' \rho_2(w') \left[P'/(E' + M) \right]^\ell}{(w' - M)(w' - W)} \underline{A}_{23}(w') \right\} \cdot \hat{D}_{32}^*(M) \quad (3.10)$$

$$\begin{aligned} \underline{D}_{23}(w) = \underline{D}_{23}(M) - (w-M) \int_{P.C.} \frac{dw' \rho_2(w')}{(w'-M)(w'-w)} \underline{N}_{23}(w') \\ + 2\pi i \left\{ (w-M) \int_{A.C.} \frac{dw' \rho_2(w') [P'/(E'+M)]^{\ell}}{(w'-M)(w'-w)} \underline{A}_{23}(w') \right\} \cdot \underline{\hat{D}}_{33}^*(M) \end{aligned} \quad (3.11)$$

$$\underline{D}_{32}(w) = \underline{D}_{32}(M) - (w-M) \int_{P.C.} \frac{dw' \rho_3(w')}{(w'-M)(w'-w)} \underline{N}_{32}(w') \quad (3.12)$$

$$\underline{D}_{33}(w) = \left[\underline{D}_{33}(M) - \underline{I} \right] + \underline{I} - (w-M) \int_{P.C.} \frac{dw' \rho_3(w')}{(w'-M)(w'-w)} \underline{N}_{33}(w') \quad (3.13)$$

For practical computations we will ignore the first terms in each of the equations (3.10)-(3.13) on the right hand side. As for the constants $\underline{\hat{D}}(M)$ introduced for the anomalous cut integrals, we take the off-diagonal elements here also to be zero, and use the identity matrix for $\underline{\hat{D}}_{33}(M)$.

The final approximation consists in keeping only the Born approximations, represented by the diagrams in Figs. 1 and 2, for the discontinuity functions A, and replacing the D functions in the integrals (3.2)-(3.5) by the subtraction-point values. This leaves the N functions as linear combinations of the Born approximation T-matrix elements, as follows:

$$\underline{N_{22}}(w) = \underline{T_{23}^{(B)}}(w) \cdot \underline{D_{32}}(M) + \underline{T_{22}^{(B)}}(w) \cdot \underline{D_{22}}(M) \quad (3.14)$$

$$\underline{N_{23}}(w) = \underline{T_{22}^{(B)}}(w) \cdot \underline{D_{23}}(M) + \underline{T_{23}^{(B)}}(w) \cdot \underline{D_{33}}(M) \quad (3.15)$$

$$\underline{N_{32}}(w) = \underline{T_{32}^{(B)}}(w) \cdot \underline{D_{22}}(M) + \underline{T_{33}^{(B)}}(w) \cdot \underline{D_{32}}(M) \quad (3.16)$$

$$\underline{N_{33}}(w) = \underline{T_{32}^{(B)}}(w) \cdot \underline{D_{23}}(M) + \underline{T_{33}^{(B)}}(w) \cdot \underline{D_{33}}(M) \quad (3.17)$$

If we adopt the approximation described in the preceding paragraph, the N functions reduce to exactly the Born terms without mixing. It is of some interest, however, to investigate the behavior of the amplitudes with some arbitrary subtractions used for the D functions to try to reproduce experimentally observed resonance positions. In this case, we should retain the form given above for the N functions if we wish to keep the spirit of the approximation consistently throughout the calculation.

The Born terms now are known in terms of linear combinations of Legendre functions $Q_n(z)$, and the discontinuity functions A resulting from the diagrams of Figs. 2 and 3 are linear combinations of the Legendre polynomials $P_n(z)$, where the dependence of z on w is algebraic and depends on the kinematics of the particular diagram considered. The integrals (3.10)-(3.13) may now be carried out numerically, for values of w in the physical energy range of interest, and the T matrix elements computed after

inverting the D matrices for each energy value and using equations (3.1) and (3.14)-(3.17) for each scattering eigenstate of isospin, parity, and angular momentum.

As the numerical computation involved in the two-channel formulation is much more complicated than in the one-channel elastic approximation, involving complex arithmetic and integration over nontrivial contours in the complex w plane, it was necessary to use a larger and faster computer for this problem. The Maniac III of the University of Chicago Institute for Computer Research was selected for reasons of convenience.

The values of the numerical integrals performed were used as input data, together with the chosen coupling constants, to a sequence of programs for the IBM 1620, which enabled exploration of the nature of the solutions. Output data included the N and D matrices, the T matrices, and phase shifts. The behavior of the matrix elements was then studied by changing the coupling constants to various selected values. The one-channel elastic approximation may be reattained by choosing extremely small coupling constants for the diagram of Fig. 2. Four energies were used, corresponding to $W/M_N = 1.25, 1.35, 1.55$ and 1.90 .

It is apparent, however, that the treatment of distant singularities is rather different using the w -plane formalism from that obtained by using the q^2 -plane formalism of equations (2.1)-(2.3). In the w plane, the cuts along the negative real

axis are not included in the D functions, but instead lumped into the subtraction constants in the approximation used here. To make the w plane treatment coincide with the q^2 plane formalism it would be necessary to make use of the reflection symmetry of the partial wave amplitudes⁵

$$q^{-1}(-w) \exp[i\delta_{\ell+}^T(-w)] \sin \delta_{\ell+}^T(-w) \equiv f_{\ell+}^T(-w) = -f_{(\ell+1)-}^T(+w)$$

and express the contributions from the negative real axis explicitly. This implies that the D functions would have a cut for negative real w as well as positive real w. Although such a formalism would probably be an improvement on the one used, it was not used for reasons of simplicity. Thus the one-channel limit of the two-channel formalism will not give exactly the same results as the formalism of (2.1)-(2.3), as far as the effect of distant singularities is concerned.

BORN TERMS

The expressions for the partial-wave projections of the contribution of Fig. 1 in terms of the helicity states of the rho meson have been given in Ref. 1, implicitly, in terms of integrals over the explicitly given discontinuity functions. We list here these amplitudes for completeness: after factoring out kinematic factors common to all angular momentum, parity, and helicity states we get these functions:

$$\hat{F}^{J1} = 2P \left\{ \left[Q/(Q_0+M) \right] \varphi_{l+1} - \left[P/(E+M) \right] \varphi_l \right\} \quad (4.1)$$

$$\hat{F}^{J3} = \left\{ \left[Q/(Q_0+M) \right] (\alpha - \psi_{l+1} + \beta \chi_{l+1}) - \left[P/(E+M) \right] (\alpha - \psi_l + \beta \chi_l) \right\}$$

$$\hat{F}^{J5} = 2P \left\{ \left[l/(l+2) \right]^{1/2} \left[Q/(Q_0+M) \right] \varphi_{l+1} - \left[(l+2)/l \right]^{1/2} \left[P/(E+M) \right] \varphi_l \right\}$$

$$\hat{F}^{J2} = 2P \left\{ \left[Q/(Q_0+M) \right] \varphi_l - \left[P/(E+M) \right] \varphi_{l+1} \right\}$$

$$\hat{F}^{J4} = \left\{ \left[Q/(Q_0+M) \right] (\alpha + \psi_l + \beta \chi_l) - \left[P/(E+M) \right] (\alpha + \psi_{l+1} + \beta \chi_{l+1}) \right\}$$

$$\hat{F}^{J6} = 2P \left\{ \left[(l+2)/l \right]^{1/2} \left[Q/(Q_0+M) \right] \varphi_l - \left[l/(l+2) \right] \left[P/(E+M) \right] \varphi_{l+1} \right\}$$

where Q is the center-of-mass momentum of the ρ meson, and P that of the pion; $Q_0 = (Q^2 + M^2)^{1/2}$, and $J = l - 1/2$.

The first 3 have the same parity, and are associated with the parity $J = l - 1/2$ in the pion-nucleon system, while the last 3 have the opposite parity;

$$\alpha_{\pm} = \sqrt{2} Q \left(\frac{W^2 \pm M^2 + \mu^2}{W M_P} \right) \quad \text{and} \quad \beta = -\sqrt{2} P \left(\frac{W^2 - M^2 + M_P^2}{W M_P} \right);$$

the superscripts 1 through 6 represent the superscript ξ of Ref. 1;

$$\varphi_l = \left[l/(2l+1) \right] \left[Q_{l-1}(x) - Q_{l+1}(x) \right], \quad \psi_l = Q_l(x);$$

$$\chi_\ell = \left[\frac{\ell+1}{2\ell+1} \right] Q_{\ell+1}(x) + \left[\frac{\ell}{2\ell+1} \right] Q_{\ell-1}(x) ;$$

$$x = (2EQ_0 - 2M^2 + \mu^2) / 2PQ .$$

The Born amplitudes used in the one-channel formulation are the same as those used in the (22) matrix element of the two-channel formulation. The contributions of Figs. 1a, 1b and 1c have been written down in Ref. 5. For the ABC exchange (1e) an arbitrary scalar-coupling Born term is used with a mass of 300 MeV assigned to the exchanged particle. The form of the rho exchange contribution is dictated by the universality hypothesis of Ref. 8 as explicated in Ref. 10; this will be explicitly constructed now, and the other terms will be listed for reference.

In a notation corresponding to equations (2.10) and (2.11) of Ref. 5, we represent the spinor structure of the Lorentz-invariant T matrix as:

$$\underline{T}^T = -A^T + \frac{1}{2}(\not{q}_1 + \not{q}_2)_\mu \gamma_\mu B^T \quad (4.2)$$

The isospin dependence of the rho exchange is given by the factor

$$\lambda_\rho^T = \begin{pmatrix} -1 \\ +1/2 \end{pmatrix} \quad \text{for} \quad T = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} ;$$

this follows from the unit isospin of the rho. Extracting a vector coupling constant f at each vertex in the diagram of Fig. 1d, we can write the rho exchange term as:

$$\underline{T}_\rho^T = \lambda_\rho^T f_\rho^2 \underline{T}_\rho$$

with

$$\underline{T}_\rho = -A_\rho + \frac{1}{2} (q_1 + q_2)_\mu \gamma_\mu B_\rho \quad (4.3)$$

Now under the assumption of universality concerning the isospin current coupling, the pole contribution of rho exchange must have exactly the same structure as the electromagnetic interactions of the nucleon and pion except for the occurrence of M_ρ^2 in the denominator:

$$\underline{T}_\rho = \frac{F_{\pi\rho}(-M_\rho^2)}{M_\rho^2} \left\{ (q_1 + q_2)_\mu \gamma_\mu F_{1N\rho}(-M_\rho^2) + \frac{1}{2M} (q_1 + q_2)_\mu \sum_{\mu\nu} (q_1 - q_2)_\nu F_{2N\rho}(-M_\rho^2) \right\} \quad (4.4)$$

The functions $F_{\pi\rho}$, $F_{1N\rho}$, $F_{2N\rho}$ are the form factors for ρ interactions.

We will ignore the isovector structure of the pion, and take

$$F_{\pi\rho}(-M_\rho^2) = 1. \quad (4.5)$$

As concerns the nucleon form factors, however, we can apply the universality criterion as in Ref. 10. There it is shown that under the assumption of strict universal coupling of the rho,

$$e F_{1N\rho}(-M_\rho^2) = \lim_{\tau \rightarrow M_\rho^2} \left[F_1^V(-\tau) \left(\frac{M_\rho^2 - \tau}{M_\rho^2} \right) \right] \quad (4.6)$$

and

$$e F_{2N\rho}(-M_\rho^2) = \lim_{\tau \rightarrow M_\rho^2} \left[F_2^V(-\tau) \left(\frac{M_\rho^2 - \tau}{M_\rho^2} \right) \right] \quad (4.7)$$

These relations allow us to determine the residues of the

pole terms T_e from the experimental data on nucleon structure. Analysis of the electron scattering data in terms of isoscalar and isovector terms leads to the representation of the electromagnetic form factors as¹⁴:

$$F_1^V(-\tau) = \frac{1}{2} \left\{ (1-a_v) + \frac{a_v M_p^2}{M_p^2 - \tau} \right\} \quad (4.8)$$

$$F_2^V(-\tau) = g_v \left\{ (1-b_v) + \frac{b_v M_p^2}{M_p^2 - \tau} \right\}$$

Identification of the pole residues and application of the relations (4.6)-(4.7) leads to:

$$F_{1p}(-M_p^2) = a_v/2 \quad ; \quad F_{2p}(-M_p^2) = b_v g_v \quad (4.9)$$

Thus we can write the contribution of the diagram in Fig. 1d as:

$$\tilde{T}_p = \frac{a_v}{M_p^2 - \tau} \left[\frac{1}{2} (q_1 + q_2)_\mu \gamma_\mu \right] + \frac{b_v g_v}{2M} \left(\frac{1}{M_p^2 - \tau} \right) \left[(q_1 + q_2)_\mu \Sigma_{\mu\nu} (q_1 - q_2)_\nu \right] \quad (4.10)$$

Separation into A and B terms is accomplished by decomposing the last term with application of the Dirac equation for the free incoming and outgoing nucleons. The result may be written as

$$\tilde{T}_p = \frac{a_v + 2g_v b_v}{M_p^2 - \tau} \left[\frac{1}{2} (q_1 + q_2)_\mu \gamma_\mu \right] + \frac{-b_v g_v}{M_p^2 - \tau} \left(\frac{\tau - 4M^2}{2M} \right) \quad (4.11)$$

where the first term is to be identified as B_e and the second as A_e .

The experimental data¹⁴ is fairly well approximated by the assumptions $a_v = b_v = 1$. Since we cannot trust the calculation

to a high degree of accuracy, this will be assumed hereafter.

The value of the vector coupling constant f is taken from data on the width of the pion-pion resonance observed in pion production at medium energies^{15,16}. A value consistent with available data is such that $f^2/4\pi = 2.0$; although this may be in error by 20 or 30%, again the accuracy of the calculation is not great enough so that we will be concerned about it. The relation between observed resonance width and f^2 may be found in references 10 and 16.

The form adopted for the ABC exchange is given, in the same notation as for rho exchange, by:

$$A_{ABC} = \frac{1}{M_A^2 - T}, \quad B_{ABC} = 0 \quad (4.12)$$

This is multiplied by unity for either isospin state, since the ABC resonance is isoscalar, and by $\frac{a^2}{4\pi}$, the square of a ^{rationalized} scalar coupling constant of order unity.

For the crossed nucleon term (Fig. 1b) we have, from Ref. 5,

$$A_{CRN} = 0, \quad B_{CRN} = \frac{1}{u(S, T) - M^2} \quad (4.3)$$

these have the isospin dependence $\lambda_{CRN}^T = \begin{pmatrix} -1 \\ +2 \end{pmatrix}$ for $T = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$, and are multiplied by the square of the pseudoscalar pion-nucleon coupling constant, G^2 . Note that if only this contribution is retained, our work on the one-channel approximation duplicates that of Bali et al¹⁷, and we indeed reproduce their results.

The nucleon pole diagram (Fig. 1a) is important in the p-1/2 pion-nucleon states only. Its contribution is given by

$$A_N = 0, \quad B_N = 1/(s-M^2) \quad (4.14)$$

with isospin factor $\lambda_N^T = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and multiplied by the square of the pseudoscalar coupling constant. This term is not included in the data of Fig. 6, although its influence is not great.

The expression for the crossed 3-3 quasiparticle state, taken from Ref. 5, is:

$$A_{33} = (+4/9) g_R^2 \left[\frac{(M_R + M) \cdot 3C^X}{(M_R + M)^2 - \mu^2} + \frac{(M_R - M)}{(M_R - M)^2 - \mu^2} \right] \cdot \frac{1}{M_R^2 - u(s, \tau)} \quad (4.15)$$

and

$$B_{33} = (-4/9) g_R^2 \left[\frac{3C^X}{(M_R + M)^2 - \mu^2} - \frac{1}{(M_R - M)^2 - \mu^2} \right] \cdot \frac{1}{M_R^2 - u(s, \tau)}$$

These terms are multiplied by the factor G^2 , and have the isospin factors

$$\lambda_{33}^T = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{for} \quad T = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} ;$$

q_R is the c. of m. momentum of the pion evaluated at the 3-3 resonance, and M_R the total energy of the 3-3 resonance:

$$C^X = 1 + \frac{\tau_R}{2g_R^2} ; \quad \tau_R = 2(M^2 + \mu^2) - s - M_R^2$$

The A_{33} term used in the two-channel case was reduced arbitrarily by a factor of 8; this will be discussed later.

The kinematic factors connecting these expressions with the F amplitudes of equation (3.1) in the (22) matrix elements, which we choose to be the same as the amplitudes of equation (2.1), are exhibited by the following relations:

$$4\pi F_{\ell\pm}^T(w) = \frac{1}{8w} \left\{ [(w+m)^2 - \mu^2] \cdot [A_{\ell}^T + (w-m)B_{\ell}^T] + [(w+m)^2 - \mu^2] \cdot [-A_{\ell\pm 1}^T + (w-m)B_{\ell\pm 1}^T] \right\} \quad (4.16)$$

Here the A_{ℓ} and B_{ℓ} are the projections of the functions of equation (4.2):

$$A_{\ell}^T = \frac{1}{2} \int_{-1}^{+1} dx P_{\ell}(x) A^T(w, x) \quad , \quad x = \cos \theta_{c.m.}$$

and similarly for B_{ℓ}^T . Note that this notation differs by a factor of two from that of Ref. 5. The x variable is related to the t variable by

$$t = -2q^2(1-x) \quad .$$

Now the Born approximations for the F_{22} functions are given by (4.16) with

$$A_{BORN}^T = \lambda_p^T f^2 A_p + \lambda_{33}^T G^2 A_{33} + a^2 A_{ABC}$$

and

$$B_{BORN}^T = \lambda_p^T f^2 B_p + \lambda_{33}^T G^2 B_{33} + \lambda_N^T G^2 B_N + \lambda_{cRN}^T G^2 B_{cRN}$$

The Born approximations to the off-diagonal F matrix elements are obtained from the expressions (4.1) by multiplication

with appropriate kinematic factors, as follows:

$$4\pi \left[F_{23}^{JS}(w) \right]_{\text{BORN}} = \left(\frac{E+M}{8} \right) \left(\frac{1}{2PQ} \right) (fG) \hat{F}^{JS}(w)$$

Invariance of the interactions under time reversal allows the conclusion that

$$\left[F_{32}^{JS}(w) \right]_{\text{BORN}} = \left[F_{23}^{JS} \right]_{\text{BORN}}^* \quad (\text{which is clear from direct computation also}).$$

In terms of w ,

$$E+M = [(w+M)^2 - \mu^2] / 2w \quad ;$$

P is the c.m. momentum of the pion, and Q that of the rho.

The elements $\left[F_{33}^{JSS'} \right]_{\text{BORN}}$ may be computed explicitly, but since in this calculation we are principally interested in the (22) channel resonances, only the zero-momentum limit was used; the important factors here are the isospin dependence of the diagram of Fig. 3,

$$\lambda_{\rho N}^T = \begin{pmatrix} -1 \\ +1/2 \end{pmatrix} \quad \text{for} \quad T = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

and the magnitude of the coupling constant, which is the same f^2 as in the rho interaction in the (22) channel. In fact the integrals over the (33) N functions are not carried out, but assumed zero. This leads to the scattering-length form for F_{33} .

We use definitions for the matrix elements which do not include angular momentum threshold factors, since use of the

Born approximations for the N matrix automatically gives the desired power-law dependence on momentum for each angular momentum state. The phase space function for the rho-nucleon states is taken simply as

$$\rho_3 = (4Q^2/s)^{1/2} \theta(W-M-m_\rho)$$

where Q is the ρ meson C.M. momentum, thus making the treatment of the (33) channel resemble the (22) channel as much as possible kinematically.

RESULTS OF THE TWO-CHANNEL CALCULATION

One serious defect of this calculation method is the violation of time reversal invariance in the resulting amplitude. Even if the N_{ij} functions form a Hermitian matrix, the resulting S matrix will in general not be unitary. (The D matrix is obtained by integration over the N_{ij} functions; the connection between N and T is not algebraic.) In the present calculation, this defect is reflected in the imaginary parts of the T_{22} elements; they are not positive definite, as required in the true physical situation. This means that unless the off-diagonal D matrix elements are quite small, the representation of the resulting elastic amplitudes T_{22} in terms of phase shifts will be meaningless.

Faced with this problem, we can look at other characteristics of T_{22} , such as $|T_{22}|^2$ and $\text{Re}(T_{22})$. The former will give

some indication of the elastic cross section, while the latter will give indication of resonance positions: $\text{Re}(T_{22}) = 0$ is necessary for a resonance.

Many of the off-diagonal D integrals turned out to have very large values. Some of these are shown in Figs. 7, 8 and 9. The notation D^{13} refers to the representation of each $D_{\ell\pi}^T$ matrix as a 4×4 complex matrix, in which the (1,1) element gives the D_{22} of equation (3.6); the (1,2), (1,3), (1,4) elements are $(D_{23})_{\ell\pi}^T$ with helicity state index $(\xi) = 1, 3, 5$ for $J = \ell - 1/2$ and $(\xi) = 2, 4, 6$ for $J = \ell + 1/2$. (This is the smallest irreducible set of quantum numbers.)

The elements D^{k2} and D^{kj} represent similarly the D_{23}^{ξ} and $D_{33}^{\xi\xi'}$ of equations (3.8) and (3.9) with the ξ, ξ' replaced by k and j as in D^{1k} . An identical notation was used for the N and T matrices.

In Figs. 8 and 9 are illustrated the types of contribution given by the anomalous cut integrals. Evidently this is quite large for some amplitudes, not so for others, and very strongly energy-dependent in all cases. It is apparent that no numerical results involving such large values of the integrals can be relied upon. However, it is instructive to continue the analysis further, as some general features emerge of the relation between elastic and strong inelastic processes.

Some kind of modification in the values of the 3-3 crossing

term was necessary before any reasonable results could be obtained, as this term contributes much too strongly. If we integrate the term as written in equations (4.15), all the interesting resonances will be found to be either bound states or at a very small pion energy, and almost every amplitude becomes resonant at some energy. The situation is worse in the W plane formalism than in the q^2 -plane formalism treated previously. To obtain a 3-3 resonance which was somewhere near the experimental value, it was decided to reduce the term A_{33} (equation 4.15) by an arbitrary factor of 8. All of the following discussion is based on such a modified amplitude.

To find resonances in the elastic cross section we look first for intervals of energy W within which $\text{Re}(T^{11})$ changes sign, indicating a zero lies in that interval, which possibly signifies a resonance. Such a search readily yields two resonances, with quantum numbers $D_{5/2}^{1/2}$ and $D_{5/2}^{3/2}$, both located near 400 MeV. There are, however, other resonances which are located at energies lower than 110 MeV in this calculation. These actually will be those which correspond to the resonances in the experimental amplitude around 800 MeV, since the attractive 3-3 crossing term has still been grossly overestimated. We locate these by looking first for positive N^{11} functions at 110 MeV, representing attractive potentials; then by plotting $\text{Re}(T^{11})$ for these amplitudes at the computed energies, and applying our knowledge of the ℓ^{th} wave threshold behavior of T^{11} near $p^2 = 0$, we can sketch the behavior of $\text{Re}(T^{11})$ between 0 and 110 MeV. We

have a resonance-type behavior whenever $\text{Re}(T^{11}) < 0$ at 110 MeV under the above conditions. These resonances are most easily located by setting the off-diagonal matrix elements to zero, then tracing the behavior as we restore the inelastic couplings.

In this way we find additional resonances below 100 MeV in the $D_{3/2}^{1/2}$, $P_{3/2}^{3/2}$, and $F_{5/2}^{1/2}$ states, a set suggested by experimental data. Furthermore, the amplitudes $P_{3/2}^{1/2}$, $P_{1/2}^{3/2}$ are both resonant below 100 MeV. We interpret these last two resonances as additional evidence of overestimation of the 3-3 crossing. The higher angular momentum states $F_{7/2}^{1/2}$, $G_{7/2}^{3/2}$, $G_{7/2}^{1/2}$, $G_{9/2}^{1/2}$ are not resonant in the absence of inelastic coupling, and the accuracy of the calculation is not sufficient to allow making any statements about the real parts of these amplitudes when the inelastic contributions are included.

The unambiguously resonant states are displayed in Figs. 10, 11, 14 and 15. The damping effect of adding the inelastic matrix elements is quite striking in all cases.

An attempt was made to construct a set of phase shifts including inelastic matrix elements which represented the essential features of the elastic amplitudes between 100 and 1000 MeV. This was done by running the computer program through which would have produced the correct phase shifts if the T matrix had come out properly symmetric. This produced a set C of phase shifts, given in Table III. Phase shifts which corresponded to those T^{11} elements with $\text{Im}(T^{11}) < 0$ then were reversed in sign. The S-wave

phase shifts then were arbitrarily set to zero to minimize non-resonant behavior, and the resulting set of phase shifts, called set C', was utilized to compute elastic cross sections and angular distributions in the region 100-10000 MeV, using a 4-parameter interpolation form for each state:

$$[P_2(w)]^{2\ell+1} \cot \delta_{\ell\pi}^T(w) = (A_1 + A_2 P^2 + A_3 P^4) / (1 + A_4 P^2)$$

This was fitted at the four energies computed, and the result was in general an excellent interpolation. Cross sections are plotted in Fig. 12 for pure isospin states, and the angular distributions of Fig. 13 are examples of the type of results to be expected with such a phase shift set.

A set of phase shifts with zero off-diagonal elements was also run through; this is denoted by set A, and given in Table II.

The (ρ N) channel interaction was found to have rather small influence on the (π N) amplitudes. A set of rather large scattering lengths were used, corresponding to the right order of magnitude for the ρ -exchange diagram of Fig. 2, but at one point in the calculation a run was made setting these to zero; the resulting (π N) phase shifts were altered by an insignificant amount. Furthermore, the final amplitudes for T_{33} were essentially determined by the large off-diagonal terms, and not by the scattering lengths used for the N_{33} elements.

DISCUSSION

As the numerical results of these calculations in general have little to do with the experimental situation on πN scattering, we must exercise some care in picking out those features of the results which will tell something about what one would expect from a "true" dynamical calculation. The difficulty with the imaginary parts of the amplitudes resulting from strong inelastic coupling has been covered above. Now we must discuss the resonances found, their locations, and the effects on them of the production channel. Firstly, the 3-3 crossing mechanism which is the principal generating agent for resonances has been overestimated strongly. This means that all resonance positions will be considerably higher than that given by this calculation. Furthermore, we may expect the $P_{3/2}^{1/2}$ and $P_{1/2}^{3/2}$ resonances to disappear, hopefully before the others, if we reduce the 3-3 crossing term.

Secondly, the reactive damping of the closed (eN) channel considerably below the inelastic threshold has a very important effect on the widths of the resonances, as can be inferred from Figs. 10, 11, 14 and 15, although the qualitative character of the resonance pattern is not significantly changed; all the resonances move up in energy and increase in width together as the off-diagonal terms are included.

Thirdly, if the resonance positions fall near the threshold for the inelastic process, threshold effects will completely dominate the resonance formations; this has been anticipated

by Cook and Lee¹.

Fourthly, the 3-3 crossing term determines the pattern of resonances in the elastic scattering amplitude, in a way which is consistent with the current interpretation of scattering data. We would further predict that the hump at 1300 MeV in the $T = 3/2$ amplitude is a $D_{5/2}^{3/2}$ broad resonance, and that a $D_{5/2}^{1/2}$ broad resonance is concealed in the $T = 1/2$ amplitude somewhere around 1000 MeV.

It is possible to see how it comes about that our calculation provides that the inelastic channel would have little influence on the qualitative resonance pattern below the inelastic threshold by considering a simple model of three scalar particles in a two-channel reaction. There we have only 2×2 matrices and we can explicitly construct T_{11} ; (here (11) refers to the πN channel, and (22) to the pN channel)

$$T_{11} = (N_{11} D_{22} - N_{12} D_{12}) / \Delta$$

$$\text{with } \Delta = \det(D) = D_{11} D_{22} - D_{12} D_{21}.$$

Now take the masses, etc., to correspond to (π, p, N) , and use the same assumptions as the full calculation. Consider, below the inelastic threshold ($W < W_2$),

$$\begin{aligned} |\Delta|^2 \operatorname{Re} T_{11} &= N_{11} \operatorname{Re} (D_{22} \Delta^*) - \operatorname{Re} (N_{12} D_{12} \Delta^*) \\ &= N_{11} D_{22} \operatorname{Re} (D_{11} D_{22} - D_{12} D_{21})^* - \operatorname{Re} (N_{12} D_{12} \Delta^*) \end{aligned}$$

Now assume

$$\begin{aligned} D_{12} &\cong (W - W_0) \lambda_{12} - i \rho_1 N_{12} \\ D_{21} &\cong (W - W_0) \lambda_{21} - i \rho_2 \theta(W - W_2) N_{21} \end{aligned}$$

where λ_{21} is real.

For $W \ll W_2$, we have $|N_{12}| \ll |N_{11}|$ usually (if we do not consider S waves) and we can assume that $\operatorname{Re}(T_{11}) = 0$ when $\operatorname{Re}(\Delta^*) = 0$; then $D_{22} \operatorname{Re} D_{11} - (W - W_0)^2 \operatorname{Re}(\lambda_{12}^* \lambda_{21}) = 0$; for $(W - W_0)^2$ small, then, the zeros of $\operatorname{Re}(T_{11}(W))$ approximately coincide with the zeros of $\operatorname{Re}(D_{11}(W))$. But this is exactly the condition for a resonance in the one-channel approximation. Note that $\operatorname{Im}(T_{11}(W))$ may be very different, however. This means that the positions of resonances will be affected little, but their widths will certainly depend on the inelastic couplings. This depends on the subtraction being done at W_0 , or some similar point close to $(M + \mu)$.

CONCLUSIONS

We have carried out an exploratory calculation of the scattering amplitudes in the $\pi N \ell$ system, assuming a model of strong interactions to estimate pion effects and using a very approximate

dispersion-theoretic method. The method has some serious defects: violation of time-reversal invariance and the necessity of introducing ad hoc reduction factors and/or subtraction constants for integrals involving the most important term, the 3-3 crossing effect. In spite of these, we have been able to show plausibly how the observed resonances in the elastic pion-nucleon scattering come about, and qualitatively how they are affected by inelastic processes.

This limited success points the way to the possible development of a more satisfactory calculational procedure in strong interaction physics. If the exchanged rho meson and the 3-3 crossing term were represented by Regge poles, and if the full set of integral equations (3.2)-(3.9) were solved (numerically) using the Regge representations for the discontinuity functions, one could clear up the main difficulties of this present calculation, provided one had some a priori idea of the Regge trajectories.

One must be careful, however, if numerically reliable results are to be obtained, that the approximation method does not violate the assumed analytic properties of the T matrix. For instance, in the one-channel case one may find complex zeros of the denominator function; if these lie close to the real axis, one does not have a consistent solution for the T matrix.

The use of digital computers for investigating the full set of phase shifts simultaneously is very advantageous in cases such

as the present calculation, where interpretation of the results was relatively difficult.

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FIGURE CAPTIONS

1. Pole contributions to the process $\pi + N \rightarrow \pi + N$
2. Pole contribution to the processes $\pi + N \rightarrow \rho + N$,
 $\rho + N \rightarrow \rho + N$
3. Singularities in the complex plane for the amplitudes involving one-pion-exchange diagrams
4. Real parts of denominator functions in the one-channel approximation with no arbitrary subtraction constants
5. Real parts of denominator functions in the one-channel approximation with one choice of subtraction constants, with and without the ρ exchange contributions
6. Real part of $P_{1/2}^{1/2}$ denominator function of the one-channel approximation
7. Energy dependence of off-diagonal D matrix elements D^{13} , in $T = 1/2$, $J = L-1/2$ states
8. Contribution of anomalous cut integral to D^{13} for the $P_{3/2}^{1/2}$ state
9. Contribution of anomalous cut integral to D^{13} for the $D_{5/2}^{1/2}$ state
10. Damping effect of coupling to inelastic process of $P_{3/2}^{3/2}$ resonant state; no off-diagonal elements (A), one-fourth full value for off-diagonal elements (B), and full value of off-diagonal elements (C)
11. Damping effect on $D_{5/2}^{1/2}$ state, same labels as Fig. 10

12. Elastic scattering cross sections using interpolation of modified phase shift set C'
13. Angular distributions for elastic scattering based on phase shift set C'
14. Real parts of $D_{3/2}^{1/2}$, $F_{5/2}^{1/2}$ amplitudes with and without inelastic coupling terms; notation as in Fig. 10. The $F_{5/2}^{1/2}$ (B) curve is infinitesimal on this scale.
15. Real parts of $P_{3/2}^{1/2}$, $P_{1/2}^{3/2}$ amplitudes with and without inelastic coupling terms; notation as in Fig. 10

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Pion Lab Kinetic Energy

Fig. 7

Fig. 8

Fig. 9

Fig.10

Fig. II

Fig. 12

Fig.13

Fig. 14

Fig.15

Table 1

Experimental Resonances in the πN System

[from Maglic et al., Phys. Rev. 123, 1444 (1961)]

<u>Pion Lab</u> <u>Kinetic Energy</u>	<u>Isospin</u>	<u>Experimental</u> <u>Angular Momentum-Parity</u>
200 MeV	$3/2$	$P_{3/2}$
600	$1/2$	$D_{3/2}$ or $P_{3/2}$
900	$1/2$	$D_{5/2}$ or $F_{5/2}$
1350	$3/2$	? (broad hump)

Table 2

Phase Shifts A from Two-channel Formalism, no Inelastic Coupling

[Negative phase shifts may be supplemented by 180°]

l	=	0	1	2	3	4	Pion Lab Energy
$\delta_{l+}^{3/2} =$		86°	-56	1.63	.00	.00	112 MeV
		53	-70	4.3	.01	-2.78	234
		40	+74	38.6	.27	.13	507
		37	+68	63.5	1.28	.63	1072
$\delta_{l+}^{1/2} =$		82°	-18	6.2	.02	.00	112
		47	-53	3.9	.12	-1.37	234
		39	-82	-34.0	1.28	.07	507
		35	+79	-48.0	3.86	.40	1072
$\delta_{l-}^{3/2} =$	-	-78°	.52	.02	.00	112	
	-	+75	2.78	.21	.00	234	
	-	1.8	15.5	1.95	.05	507	
	-	63	2.4	-46.5	-.91	1072	
$\delta_{l-}^{1/2} =$	-	-23°	-8.1	+.11	.00	112	
	-	-40	-8.0	-3.38	.02	234	
	-	+.66	-12.2	-2.6	.37	507	
	-	+21.2	-12.4	-3.32	3.17	1072	

Table 3

Phase Shifts C from Two-channel Formulation, Full Inelastic Coupling

Parentheses () denote $\text{Im}(T_{22}) < 0$

[Negative phase shifts may be supplemented by 180°]

$l =$	0	1	2	3	4	Pion Lab Energy
$\delta_{\ell+}^{3/2} =$	77	42	(20.7)	-88	-46.9	112 MeV
	41	(-59)	24.5	(+22.8)	-5.91	234
	17	(-60)	(20.5)	44.7	-10.42	507
	(-47)	39	(-6.0)	54.0	(+25.86)	1072
$\delta_{\ell+}^{1/2} =$	58	+36.8	(13.3)	-87.6	-46.9	112
	10.5	(-65.2)	42.7	(+22.9)	(-5.9)	234
	(-3.7)	(-57.6)	(11.2)	44.2	10.5	507
	(-17.2)	+17.1	(-4.85)	54.0	(25.9)	1072
$\delta_{\ell-}^{3/2} =$	-	(-16.6)	(6.66)	82.6	-87.7	112
	-	(-39.9)	(4.52)	28.8	-46.9	234
	-	(-18.8)	(3.68)	54.3	-22.2	507
	-	(-18.4)	(83.5)	10.3	+46.3	1072
$\delta_{\ell-}^{1/2} =$	-	(-27.6)	(3.7)	82.3	-87.7	112
	-	(-71.2)	(3.5)	28.7	-46.6	234
	-	(-44.6)	(4.7)	53.8	-22.3	507
	-	(-40.0)	(82.4)	10.3	+47.4	1072