THE UNIVERSITY OF WISCONSIN
MATHEMATICS RESEARCH CENTER

Contract No. : DA-31-124-ARO-D-462

SCATTERING THEORY FOR ELLIPTIC
DIFFERENTIAL OPERATORS IN
UNBOUNDED DOMAINS

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MRC Technical Summary Report # 1218

Received February 10, 1972

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ABSTRACT

The purpose of this paper is to develop a new method for meromorphically continuing the S -matrix and distorted waves associated with a general class of elliptic partial differential operators and for relating the poles to "resonant states". The method is applied to non-selfadjoint second order perturbations of the Laplacian acting in wedge-like regions contained in R^N . The main analytic tools employed are the limiting absorption principle and the Schwarz reflection principle.

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Section 1. Introduction

It is the purpose of this report to demonstrate a method of meromorphically continuing the $\mathcal{S}$ -matrix and distorted plane waves associated with elliptic differential operators acting in unbounded domains and to relate the poles to "resonant states". The particular problem we shall deal with involves scattering in a wedge shaped region contained in $\mathbb{R}^N$, N -dimensional Euclidean space. We constructed the $\mathcal{S}$ -matrix and distorted waves in [1] for self-adjoint second order perturbations and in [2] for non-selfadjoint second order perturbations of the Laplacian. In the remainder of this section we shall review these results and briefly describe the remainder of this paper. For the sake of simplicity, the details will be given for $\mathbb{R}^2$. In Section 4, we shall indicate how they are generalized to $\mathbb{R}^N$ (using spherical harmonics), as well as to more general problems.

First, we briefly describe how the work of this paper differs from previous work on the $\mathcal{S}$ -matrix. Our construction of the $\mathcal{S}$ -matrix and distorted plane waves is based on the "limiting absorption principle". This differs from the other two methods of treating this problem. The Lax-Phillips method, [3], relies heavily on the use of Huyghens principle, whereas

Sponsored in part by the United States Army under Contract No.: DA-31-124-ARO-D-462, and Brookhaven National Laboratory, Upton, Long Island, New York.

the integral equation method (see, e.g., [4]) requires a detailed knowledge of Green's function for the unperturbed operator. In [3] and [4], the Laplacian is perturbed by a real-valued potential and both operators act in a domain in R^N with at most a finite boundary. The limiting absorption principle, however, is based on elliptic estimates and enables us to treat domains with infinite boundaries, as well as higher order perturbations. Finally we remark that by applying the abstract results of [2], we need not require A to be self-adjoint as in [3] and [4].

Let S denote the wedge shaped region in R^2 , consisting of the points $x = (x_1, x_2) = (r, \theta)$ for which $r > 0$, $0 < \theta < 2\pi\alpha$, $0 < \alpha \leq 1$, where (x_1, x_2) denotes Cartesian coordinates and (r, θ) denotes polar coordinates for the point x . Let the domain Ω be obtained from S by perturbing a finite part of $\dot{S}$, the boundary of S , and by deleting at most a finite number (say m) of bounded subsets, $D_j \subseteq S$, $j = 1, \dots, m$. We assume that $\dot{\Omega}$ is a C^2 curve with the possible exception of the point $r = 0$. If $r = 0$ is a singular point, then we assume this point to be bounded away from $\dot{\Omega} - \dot{\Omega} \cap \dot{S}$.

We next define the following operators. Let $\tilde{A}_0(\tilde{\Lambda})$ denote the formal uniformly elliptic partial differential operator given by $-\Delta(-\Delta + \sum_{|\beta| \leq 2} a_\beta(x) D^\beta)$. Here $\beta = (\beta_1, \beta_2)$, β_1 and β_2 are non-negative integers,

$$|\beta| = (\beta_1^2 + \beta_2^2)^{\frac{1}{2}}, \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

consists of those complex valued functions with continuous second order derivatives. We assume that $a_\beta(x) \in C_0^2(\Omega)$ for each multi-index β (i.e., $a_\beta(x) \in C^2(\Omega)$ and has bounded support). We may prove using elliptic estimates exactly as in [2]* that $\tilde{A}_0(\tilde{A})$ has a self-adjoint (closed) extension, $A_0(A)$, acting in $H_0 = L_2(S)$ ($H = L_2(\Omega)$) and that $D(A) = D(A^*)$. Note that A^* may also be defined as the closure of A' , the formal adjoint of $\tilde{A}$, given by

$$\tilde{A} = (-\Delta + \sum_{|\beta| \leq 2} a'_\beta D^\beta).$$

Before proceeding further, we shall define certain outgoing and incoming radiation conditions at infinity. Let $J_{n/\alpha}(r)$ and $Y_{n/\alpha}(r)$ denote the Bessel and Hankel functions of order n/α , respectively, and set

$$H_{n/\alpha}^{(1)}(r) = J_{n/\alpha}(r) + iY_{n/\alpha}(r) \quad \text{and}$$

$$H_{n/\alpha}^{(2)}(r) = J_{n/\alpha}(r) - iY_{n/\alpha}(r).$$

We shall say that $u(x)$ is an outgoing solution of the boundary value problem :

* In [1] and [2] we did not delete the domains D_j from S . We do this now so as to generalize the "exterior problem" ($\alpha = 1$, $|\beta| = 0$). The proofs remain unchanged.

$$(\tilde{\Lambda} - \lambda)u(x) = (-\Delta + \sum_{|\beta| \leq 2} a_\beta(x) D^\beta - \lambda)u(x) = 0 \quad \text{in } \Omega, \quad (1.1)$$

$$u(x) = 0 \quad \text{on } \dot{\Omega} \quad \text{for } \lambda \in (0, \infty),$$

provided $u(x)$ satisfies the following outgoing radiation condition for each $r \geq r_0^*$.

$$u(x) = \sum_{n=1}^{\infty} c_n(\lambda) H_{n/\alpha}^{(1)}(\lambda^{\frac{1}{2}} r) \sin \frac{n}{\alpha} \theta \quad \text{and} \quad (1.2)$$

$$\frac{\partial u(x)}{\partial r} = \sum_{n=1}^{\infty} c_n(\lambda) \frac{d}{dr} H_{n/\alpha}^{(1)}(\lambda^{\frac{1}{2}} r) \sin \frac{n}{\alpha} \theta.$$

We define the incoming radiation condition similarly with $H_{n/\alpha}^{(1)}$ replaced by $H_{n/\alpha}^{(2)}$.

Now suppose that $\mathcal{J}$ is an arbitrary compact interval contained in $(0, \infty)$ such that the following condition holds.

(C $_{\mathcal{J}}$) (i) Suppose that for no point λ in $\mathcal{J}$ does there exist a non-trivial outgoing or incoming solution of (1.1) and similarly with $\tilde{\Lambda}$ replaced by $\tilde{\Lambda}'$.

(ii) Suppose that there exists a number $\epsilon_0 > 0$ such that

$\mathcal{J}_{\epsilon_0}^+ \equiv_{\text{df.}} \{z = \lambda \pm i\epsilon \mid \lambda \in \mathcal{J}, \epsilon \in (0, \epsilon_0]\} \subseteq \rho(A)$, where $\rho(A)$ denotes the resolvent set of A .

*We assume that the perturbation is contained in the sphere $C_{r_0} = \{x \mid |x| < r_0\}$. Thus each $a_\beta(x) = 0$ for $r = |x| \geq r_0$ and $\Omega = \Omega \cap C_{r_0} = S - S \cap C_{r_0}$.

We may combine the methods of [1] and [2] to prove that there exists a set of projection operators, $E_{\mathscr{J}'}$, associated with arbitrary measurable subsets, $\mathscr{J}' \subseteq \mathscr{J}$, such that $\{E_{\lambda}\} \equiv_{\text{df.}} \{E_{(-\infty, \lambda) \cap \mathscr{J}}\}$, $\lambda \in \mathscr{J}$, yields a spectral resolution for the spectral operator $A_{\mathscr{J}} = \Lambda E_{\mathscr{J}}$. The operator Λ , however, need not have a spectral resolution. Let $\{E_{0\lambda}\}$ denote the spectral resolution for the self-adjoint operator A_0 . As in [2], we may construct spectral mappings for the operators $A_{0\mathscr{J}} = A_0 E_{0\mathscr{J}}$ and $A_{\mathscr{J}}$ in the following way.

Set $\tilde{w}_n^0(x; \lambda) = (2\pi\alpha)^{-\frac{1}{2}} J_{n/\alpha}(\lambda^{\frac{1}{2}} r) \sin \frac{n\theta}{\alpha}$, $\lambda \in (0, \infty)$, and set $\hat{u}_n^0(\lambda) = \int_S u(x) \overline{\tilde{w}_n^0(x; \lambda)} dx$ for each $u(x) \in C_0^\infty(S)$. Each $\tilde{w}_n^0(x; \lambda)$ is a "generalized eigenfunction" for the operator A_0 associated with the spectral point λ . We define Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_{\mathscr{J}}$ by

$$\mathscr{H} = \bigoplus_{n=1}^{\infty} L_2((0, \infty); d\lambda) \quad \text{and} \quad \mathscr{H}_{\mathscr{J}} = \bigoplus_{n=1}^{\infty} L_2(\mathscr{J}; d\lambda).$$

Thus $h(\lambda) = (h_1(\lambda), h_2(\lambda), \dots) = \{h_n(\lambda)\} \in \mathscr{H}$ if and only if $\|h\|_{\mathscr{H}}^2 \equiv_{\text{df.}} \sum_{n=1}^{\infty} \int_0^{\infty} |h_n(\lambda)|^2 d\lambda < \infty$. If $h = \{h_n(\lambda)\}$ and $k = \{k_n(\lambda)\} \in \mathscr{H}$, then

$(h, k)_{\mathscr{H}}$ denotes $\sum_{n=1}^{\infty} \int_0^{\infty} h_n(\lambda) \overline{k_n(\lambda)} d\lambda$. $\|\cdot\|_{\mathscr{H}_{\mathscr{J}}}$ and $(\cdot, \cdot)_{\mathscr{H}_{\mathscr{J}}}$ are defined similarly with $\int_0^{\infty}$ replaced by $\int_{\mathscr{J}}$. Finally, we define

$$(1.3) \quad T^0 u(\lambda) = \{\hat{u}_n^0(\lambda)\} = \{(T^0 u)_n(\lambda)\}$$

for each $u(x) \in C_0^\infty(S)$ and $\lambda \in (0, \infty)$.

Lemma 1.1: The mapping T^0 , defined on $C_0^\infty(S)$ by (1.3), has a unitary extension mapping H_0 onto $\mathcal{H}$. Furthermore $(\Lambda_0 u)_n^{\hat{0}}(\lambda) = \lambda \hat{u}_n^0(\lambda)$ for each $u \in D(\Lambda_0)$, $n = 1, 2, \dots$, and $\lambda \in \mathcal{D}$. Thus T^0 "diagonalizes" Λ_0 .

Lemma 1.1 is easily proved and says that T^0 is a spectral mapping for Λ_0 . In order to construct "spectral mappings" for Λ , we shall obtain two sets of "generalized eigenfunctions", $\tilde{w}_n^\pm(x; \lambda)$, for the operator Λ , where $\lambda \in \mathcal{D}$. To do this, we first need the following result.

Lemma 1.2: Suppose that condition (C_2) holds and $F(x) \in C_0(\Omega)$.

Set

$$u_\epsilon^\pm(x, \lambda) = (A - \lambda \mp i\epsilon)^{-1} F(x).$$

Then

$$\lim_{\epsilon \downarrow 0} u_\epsilon^\pm(x; \lambda) = u^\pm(x; \lambda)$$

exists in $L_2^{\text{loc}}(\Omega)^*$, the convergence being uniform with respect to λ in $\mathcal{D}$.

Finally, $u^+(x; \lambda)(u^-(x; \lambda))$ is the unique outgoing (incoming) solution of the boundary value problem

$$(1.4) \quad (\tilde{A} - \lambda)u^\pm(x; \lambda) = F(x) \text{ in } \Omega, \quad u^\pm(x; \lambda) = 0 \text{ on } \dot{\Omega}.$$

Analogous results hold with Λ replaced by Λ^* .

*The definition of $L_2^{\text{loc}}(\Omega)$ is given at the end of this section.

The proof of Lemma 1.2 follows exactly as in [2] using the "limiting absorption principle", which relies heavily on the theory of elliptic equations. Using Lemma 1.2, we may now construct our "eigenfunctions", $\tilde{w}_n^\pm(x; \lambda)$ (also referred to as "distorted plane waves"). If we set $\tilde{w}_n^\pm(x; \lambda) = \tilde{w}_n^0(x; \lambda) + \tilde{v}_n^\pm(x; \lambda)$, we see that it suffices to construct functions $\tilde{v}_n^\pm(x; \lambda)$ satisfying the boundary value problem

$$(1.5) \quad (\tilde{A} - \lambda)\tilde{v}_n^\pm(x; \lambda) = - \sum_{|\beta| \leq 2} a_\beta(x) D^\beta \tilde{w}_n^0(x; \lambda) \text{ in } \Omega,$$

$$\tilde{v}_n^\pm(x; \lambda) = -\tilde{w}_n^0(x; \lambda) \text{ on } \dot{\Omega}.$$

We shall determine $\tilde{v}_n^\pm(x; \lambda)$ uniquely by specifying that $\tilde{v}_n^+(x; \lambda)(\tilde{v}_n^-(x; \lambda))$ be outgoing (incoming).

We reduce this problem to that of Lemma 1.2 as follows. Let $\chi(x)$ be a cutoff function defined in Ω satisfying the following conditions.

$$(i) \quad \chi(x) \in C^\infty(\Omega),$$

(ii) $\chi(x) \equiv 0$ outside of some neighborhood of $\dot{\Omega} - \dot{S} \cap \dot{\Omega}$ and in some neighborhood of any singular point of $\dot{\Omega}$, and

$$(iii) \quad \chi(x) \equiv 1 \text{ in some neighborhood of } \dot{\Omega} - \dot{S} \cap \dot{\Omega} \text{ and in } \Omega - S \cap \Omega.$$

Now suppose $z \in \mathcal{D}_0^\pm \subseteq \rho(\Lambda)$ and set

$$\tilde{G}_n(x; z) = f(x) \tilde{w}_n^0(x; z),$$

$$\tilde{\Gamma}_n(x; z) = (\tilde{\Lambda} - z) \tilde{G}_n(x; z),$$

$$(1.6) \quad \tilde{u}_n(x; z) = (A - z)^{-1} (\tilde{\Gamma}_n(\cdot; z) - \sum_{|\beta| \leq 2} a_\beta(\cdot) D^\beta \tilde{w}_n^0(\cdot; z)) (x),$$

$$\tilde{v}_n(x; z) = \tilde{u}_n(x; z) - \tilde{G}_n(x; z), \text{ and}$$

$$\tilde{w}_n(x; z) = \tilde{w}_n^0(x; z) + \tilde{v}_n(x; z).$$

We define

$$\tilde{v}_n^\pm(x; \lambda) = \lim_{\epsilon \downarrow 0} \tilde{v}_n(x; \lambda \pm i\epsilon) \text{ and}$$

(1.7)

$$\tilde{w}_n^\pm(x; \lambda) = \tilde{w}_n^0(x; \lambda) + \tilde{v}_n^\pm(x; \lambda).$$

We may readily employ Lemma 1.2 to obtain the limit in (1.7). It is clear, using Lemma 1.2, that $\tilde{v}_n^\pm(x; \lambda)$ satisfies the required conditions. We may similarly construct the functions $\tilde{w}_n^{\pm,1}(x; \lambda) = \tilde{w}_n^0(x; \lambda) + \tilde{v}_n^{\pm,1}(x; \lambda)$ with A replaced by Λ^* . We now set

$$(1.8) \quad T^\pm u(\lambda) = \{\hat{u}_n^\pm(\lambda)\}$$

for each $u(x) \in C_0^\infty(\Omega)$ and $\lambda \in \mathscr{L}$, where

$$\hat{u}_n^\pm(\lambda) = \int_{\Omega} u(x) \overline{\tilde{w}_n^\pm(x; \lambda)} dx.$$

We define $T^{\pm,1}$ and $\hat{u}_n^{\pm,1}(\lambda)$ analogously with $\tilde{w}_n^\pm(x; \lambda)$ replaced by $\tilde{w}_n^{\pm,1}(x; \lambda)$.

Theorem 1.1. Suppose that condition $(C_{\mathcal{J}})$ holds. Then the mappings $T^{\pm}$ and $T'^{\pm}$ defined on $C_0^{\infty}(\Omega)$ by (1.8) and its analogue may be extended to bounded mappings from H onto $\mathcal{H}_{\mathcal{J}}$ such that $T'^{\pm} E_{\mathcal{J}} u = T'^{\pm} u$ and $T^{\pm} E_{\mathcal{J}}^* u = T^{\pm} u$ for each $u \in H$. Furthermore $T'^{\pm*} T'^{\pm} = I$ on $H_{\mathcal{J}} = E_{\mathcal{J}} H$ and $T'^{\pm} T'^{\pm*} = I$ on $\mathcal{H}_{\mathcal{J}}$. Finally we have $(\Lambda_{\mathcal{J}} u)_n^{\pm}(\lambda) = \hat{\lambda} \hat{u}_n^{\pm}(\lambda)$ and $(\Lambda_{\mathcal{J}}^* u)_n^{\pm}(\lambda) = \hat{\lambda} \hat{u}_n^{\pm}(\lambda)$ for each $u \in D(A) = D(A^*)$, $n = 1, 2, \dots$, and $\lambda \in \mathcal{J}$.

Theorem 1.1 follows exactly as in [2]. We now define "wave operators", $W^{\pm}$ and $W'^{\pm}$, by

$$W^{\pm} = T'^{\pm*} T^0 \quad \text{and} \quad W'^{\pm} = T'^{\pm*} T^0.$$

We see from Theorem 1.1 and Lemma 1.1 that $W^{\pm}(W'^{\pm})$ are bounded mappings from $H_{0_{\mathcal{J}}} = F_{0_{\mathcal{J}}} H$ onto $H_{\mathcal{J}} (H'_{\mathcal{J}} = E_{\mathcal{J}}^* H)$ and they establish the similarity of $A_{0_{\mathcal{J}}}$ and $A_{\mathcal{J}} (A_{\mathcal{J}}^*)$. These wave operators may also be expressed in terms of a time-dependent formulation as in [5]. The scattering operator $S(S')$ is defined by $S = W^{+-1} W^{-}$ ($S' = W'^{+-1} W'^{-}$). $S(S')$ clearly maps $H_{0_{\mathcal{J}}}$ onto itself.

We next construct the S -matrices, $\tilde{S}(\lambda)$ and $\tilde{S}'(\lambda)$. We obtained this result in an abstract setting in [5] (see Remark 1.1 below). Set

$$\mathfrak{D} = \{u \in H_{0_{\mathcal{J}}} \mid T'^{+-1} T^0 u = E_{\mathcal{J}} \tilde{u}, \tilde{u}(x) \in C_0^{\infty}(\Omega)\}.$$

Lemma 1.3: $\mathfrak{D}$ is dense in $H_{0_{\mathcal{J}}}$.

Proof: Suppose $u \in H_{0_{\mathcal{J}}}$ and set $\tilde{u} = T'^{+-1} T^0 u \in H$. There clearly exists a sequence of functions $\tilde{u}_n(x) \in C_0^{\infty}(\Omega)$ such that $\int_{\Omega} |\tilde{u}_n(x) - \tilde{u}(x)|^2 dx \rightarrow 0$

as $n \rightarrow \infty$. Set $u_n = T^{0-1} T^{'+1} \tilde{u}_n$. Since T^0 and $T^{'+1}$ are bounded mappings, we see that $\int |u_n(x) - u(x)|^2 dx \rightarrow 0$ as $n \rightarrow \infty$. Furthermore $u_n \in \mathfrak{D}$ since $T^{'+1-1} T^0 u_n = T^{'+1-1} T^0 T^{0-1} T^{'+1} \tilde{u}_n = T^{'+1-1} T^{'+1} \tilde{u}_n = \tilde{u}_n$ using Theorem 1.1.

Q.E.D.

Remark 1.1: If we were to apply the method of [5] (Lemma 2.5) to the concrete operators considered here, we would define $\mathfrak{D}$ as $\{u \in H_0 \mid T^{'+1-1} T^0 u \in C_0^\infty(\Omega)\}$. This set is not dense in H_0 and hence the proof in [5] breaks down. However, that proof may be fixed up by redefining $\mathfrak{D}$ in the way we have done here and then applying the proof of Theorem 1.2 below (which is very similar to that of [5]).

Theorem 1.2: Suppose that condition $(C_{\mathcal{L}})$ holds. Then for each

$u \in \mathfrak{D}$, $n = 1, 2, \dots$, and $\lambda \in \mathcal{L}$, we have

$$(1.9) \quad (\mathfrak{S}u)_n^0(\lambda) = \hat{u}_n^0(\lambda) + \sum_{n'=1}^{\infty} \tilde{t}_{n,n'}(\lambda) \hat{u}_{n'}^0(\lambda),$$

where

$$(1.10) \quad \tilde{t}_{n,n'}(\lambda) = (8\pi\alpha)^{\frac{1}{2}} \tilde{c}_n^{n'+1}(\lambda), \quad n, n' = 1, 2, \dots$$

and

$$(1.11) \quad \tilde{v}_{n'}^+(x; \lambda) = \sum_{n=1}^{\infty} \tilde{c}_n^{n'+1}(\lambda) H_{n/\alpha}^{(1)}(\lambda^{\frac{1}{2}} r) \sin \frac{n}{\alpha} \theta \quad \text{for } r \geq r_0.$$

An analogous result holds for $T^0 \mathfrak{S}$ with $\tilde{t}_{n,n'}(\lambda)$, $\tilde{c}_n^{n'+1}(\lambda)$ and $\tilde{v}_{n'}^+(x; \lambda)$ replaced by $\tilde{t}_{n,n'}^1(\lambda)$, $\tilde{c}_n^{n'+1}(\lambda)$ and $\tilde{v}_{n'}^{1+}(x; \lambda)$, respectively.

Proof: Suppose $u \in \mathcal{D}$. It follows from the definitions of $\mathcal{G}$ and

$W^\pm$ and Theorem 1.1 that

$$(\mathcal{T}^0 \mathcal{G} u)_n(\lambda) = (\mathcal{T}^+ \mathcal{T}^{+,-1} \mathcal{T}^0 u)_n(\lambda) = (\mathcal{T}^0 u)_n(\lambda) + ((\mathcal{T}^+ - \mathcal{T}^{+,-1}) \mathcal{T}^{+,-1} \mathcal{T}^0 u)_n(\lambda).$$

Since $\mathcal{T}^{+,-1} \mathcal{T}^0 u = E_{\mathcal{D}} \tilde{u}$ for some $\tilde{u}(x) \in C_0^\infty(\Omega)$ by the definition of $\mathcal{D}$, and $(\mathcal{T}^{+,-1} E_{\mathcal{D}} \tilde{u}^2)_n(\lambda) = (\mathcal{T}^{+,-1} \tilde{u})_n(\lambda)$ by Theorem 1.1, we see that

$$(\mathcal{G} u)_n^{\hat{0}}(\lambda) = \hat{u}_n^0(\lambda) + \int_{\Omega} \overline{(\tilde{v}_n^{+,-}(x; \lambda) - \tilde{v}_n^{+,+}(x; \lambda)) \tilde{u}(x)} dx.$$

We see from the construction of $\tilde{v}_n^{+,+}(x; \lambda)$ that

$$\tilde{v}_n^{+,+}(x; \lambda) = \lim_{\epsilon \downarrow 0} R(\lambda \mp i\epsilon)^* \tilde{H}_n^+(x; \lambda) - \tilde{G}_n^+(x; \lambda),$$

where

$$\tilde{H}_n^+(x; \lambda) = \tilde{F}_n^+(x; \lambda) - \sum_{|\beta| \leq 2} a_\beta^+(x) D^\beta \tilde{w}_n^0(x; \lambda)$$

and $\tilde{G}_n^+(x; \lambda)$, $\tilde{F}_n^+(x; \lambda)$ are defined by (1.6) with z replaced by λ and $\tilde{A}$ by $\tilde{A}'$. We now obtain from Theorem 1.1, the equation

$$\begin{aligned} (\mathcal{G} u)_n^{\hat{0}}(\lambda) &= \hat{u}_n^0(\lambda) + \int_{\Omega} \lim_{\epsilon \downarrow 0} 2i\epsilon \sum_{n'=1}^{\infty} \int_0^{\infty} \frac{\overline{(\tilde{H}_n^+(\cdot; \lambda))^{\hat{+}}_{n',(\lambda)} w_{n',(\lambda)}^{+,+}(x; \lambda)} d\lambda \tilde{u}(x) dx}{(\lambda' - \lambda)^2 + \epsilon^2} \\ &= \hat{u}_n^0(\lambda) + \lim_{\epsilon \downarrow 0} 2i\epsilon \sum_{n'=1}^{\infty} \int_0^{\infty} \frac{\overline{(\tilde{H}_n^+(\cdot; \lambda))^{\hat{+}}_{n',(\lambda')}}}{(\lambda' - \lambda)^2 + \epsilon^2} \int_{\Omega} \tilde{u}(x) w_{n',(\lambda')}^{+,+}(x; \lambda') dx d\lambda' \end{aligned}$$

The last step is justified by the limiting absorption principle and the fact that $\tilde{u}(x) \in C_0^\infty(\Omega)$. Since $\int_{\Omega} \tilde{u}(x) w_{n',(\lambda')}^{+,+}(x; \lambda') dx = (\mathcal{T}^{+,-1} \tilde{u})_{n',(\lambda')} =$

$$= (T^{+1} L_{\mathcal{L}} \tilde{u})_{n'}(\lambda) = (T^{+1} T^{+1-1} T^0 u)_{n'}(\lambda) = \hat{u}_{n'}^0(\lambda), \text{ we see that}$$

$$(\mathcal{S} u)_{n'}^{\hat{0}}(\lambda) = \hat{u}_{n'}^0(\lambda) + \lim_{\epsilon \downarrow 0} 2i\epsilon \sum_{n'=1}^{\infty} \int_0^{\infty} \frac{(\tilde{H}_{n'}^1(\cdot; \lambda))_{n'}^{\hat{+}}(\lambda') \hat{u}_{n'}^0(\lambda') d\lambda'}{(\lambda' - \lambda)^2 + \epsilon^2}$$

Using a well-known result on singular integrals, [6] (pg. 30), we conclude from the last equation, the fact that $\tilde{H}_{n'}^1(x; \lambda) \in C_0(\Omega)$ and the properties of $w_{n'}^0(x; \lambda)$ that

$$(\mathcal{S} u)_{n'}^{\hat{0}}(\lambda) = \hat{u}_{n'}^0(\lambda) + 2\pi i \sum_{n'=1}^{\infty} \overline{(\tilde{H}_{n'}^1(\cdot; \lambda))_{n'}^{\hat{+}}(\lambda)} \hat{u}_{n'}^0(\lambda).$$

We are thus left with evaluating

$$(\tilde{H}_{n'}^1(\cdot; \lambda))_{n'}^{\hat{+}}(\lambda) = \int_{\Omega} ((\tilde{A}' - \lambda)(\mathcal{L}(x) \tilde{w}_{n'}^0(x; \lambda)) - \tilde{V}' w_{n'}^0(x; \lambda)) w_{n'}^+(x; \lambda) dx,$$

where $\tilde{V}' = \tilde{A}' - \tilde{A}_0$. If we observe that $(\tilde{A}' - \lambda) \tilde{w}_{n'}^0(x; \lambda) = ((\tilde{A}_0 - \lambda) + \tilde{V}') \tilde{w}_{n'}^0(x; \lambda) = \tilde{V}' \tilde{w}_{n'}^0(x; \lambda)$, we obtain the following expression for the right hand side.

$$\int_{\Omega} (\tilde{A}' - \lambda)((\mathcal{L}(x) - 1) \tilde{w}_{n'}^0(x; \lambda)) w_{n'}^+(x; \lambda) dx.$$

Since $\mathcal{L}(x) - 1 \equiv 0$ in a neighborhood of $\dot{\Omega} - \dot{\Omega} \cap \dot{S}$, we see that

$(\mathcal{L}(x) - 1) \tilde{w}_{n'}^0(x; \lambda) = 0$ on $\dot{\Omega}$. We thus obtain from the divergence theorem:

$$\begin{aligned} (\tilde{H}_{n'}^1(\cdot; \lambda))_{n'}^{\hat{+}}(\lambda) &= \left(\frac{1}{2\pi\alpha}\right)^{\frac{1}{2}} \int_0^{2\pi\alpha} \left(\frac{\partial}{\partial r} J_{n/\alpha}(\lambda^{\frac{1}{2}} r)\right) \sin \frac{n}{\alpha} \theta \overline{w_{n'}^+(x; \lambda)} \\ &\quad - J_{n/\alpha}(\lambda^{\frac{1}{2}} r) \sin \frac{n}{\alpha} \theta \frac{\partial}{\partial r} \overline{w_{n'}^+(x; \lambda)} r d\theta \end{aligned}$$

for $r \geq r_0$, where we have used the definition of $w_n^0(x; \lambda)$ and the properties of $g(x)$, $\tilde{V}$ and $w_{n'}^+(x; \lambda)$.

Finally, we employ the outgoing radiation condition, the definition of $H_{n/\alpha}^{(1)}(\lambda^{\frac{1}{2}}r)$ and the equations

$$\int_0^{2\pi\alpha} \sin \frac{n}{\alpha} \theta \sin \frac{n'}{\alpha} \theta d\theta = \pi\alpha \delta_n^{n'}$$

($\delta_n^{n'}$ denoting the Kronecker delta) and

$$(1.12) \quad J_{n/\alpha}(r) \frac{d}{dr} Y_{n/\alpha}(r) - Y_{n/\alpha}(r) \frac{d}{dr} J_{n/\alpha}(r) = \frac{2}{\pi r}$$

(see [7]) to obtain

$$(\tilde{H}_n(\cdot; \lambda))_{n'}^{\hat{+}}(\lambda) = \frac{2\pi\alpha i}{\frac{1}{2}\pi r} \overline{\tilde{c}_n^{n'+}(\lambda)} = \frac{(2\pi\alpha)^{\frac{1}{2}} i}{\pi} \overline{\tilde{c}_n^{n'+}(\lambda)}.$$

We thus conclude that $\tilde{t}_{n,n'}(\lambda) = 2\pi i H_n(\cdot; \lambda)_{n'}^{\hat{+}}(\lambda) = (8\pi\alpha)^{\frac{1}{2}} \tilde{c}_n^{n'+}(\lambda)$.

This proves (1.9). The proof is similar for $T^0 g'$.

Q.E.D.

We now set $\tilde{S}(\lambda) = I + \tilde{T}(\lambda)(\tilde{S}'(\lambda) = I + \tilde{T}'(\lambda))$, where the matrix $\tilde{T}(\lambda)(\tilde{T}'(\lambda))$ is given by $\tilde{T}(\lambda) = (\tilde{t}_{n,n'}(\lambda))(\tilde{T}'(\lambda) = (\tilde{t}'_{n,n'}(\lambda)))$ for each $\lambda \in \mathcal{S}$.

The operators $\tilde{S}(\lambda)$ and $\tilde{S}'(\lambda)$ act in the Hilbert space ℓ_2^∞ , consisting of infinite sequences of complex numbers and topologized by the Euclidean norm. $\tilde{S}(\lambda)(\tilde{S}'(\lambda))$ will be referred to as the $\tilde{S}$ -matrix associated with Λ_0 and $\Lambda(\Lambda^*)$.

Remark 1.2: When Λ is not self-adjoint, $\tilde{S}(\lambda)$ is not unitary (nor even normal). Instead, the following equation follows readily from Theorems 1.1 and 1.2:

$$(1.13) \quad \tilde{S}(\lambda)^{-1} = \tilde{S}'(\lambda)^* .$$

Let us set $\kappa = \sqrt{\lambda}$ for each $\lambda \in \mathcal{J} = [a, b]$ ($0 < a < b$) and define $S(\kappa) = \tilde{S}(\lambda)$, $w_n^\pm(x; \kappa) = \tilde{w}_n^\pm(x; \lambda)$, $c_n^{n'+}(\kappa) = \tilde{c}_n^{n'+}(\lambda)$, etc. for $\kappa \in I_{\mathcal{J}}^\pm$, where

$$I_{\mathcal{J}}^+ = [\sqrt{a}, \sqrt{b}] \text{ and } I_{\mathcal{J}}^- = [-\sqrt{b}, -\sqrt{a}] .$$

It is the purpose of this paper to meromorphically continue the functions $w_n^\pm(x; \kappa)$, $S(\kappa)$, etc. off the intervals $I_{\mathcal{J}}^\pm$ and to study these continuations. In Section 2 we shall establish the existence of these continuations. In Section 3, we characterize the poles of $S(\kappa)$ in terms of "resonant states". In Section 4, we compare our results with those already known for the special case of the "exterior problem" (see, e.g., [3] and [4]). In the Appendix, we construct Green's function for the operator $A_0 - \kappa^2$.

We close this section by setting down some notation that will be needed in the next section. If $\mathcal{S}$ denotes an arbitrary domain contained in R^N , then by $H_m(\mathcal{S})$ ($H_m^{\text{loc}}(\mathcal{S})$), we mean those functions $u(x)$ with $L_2(\mathcal{S})(L_2(\mathcal{S}'))$ derivatives of order up to and including m for each bounded subdomain $\mathcal{S}' \subseteq \mathcal{S}$. We set

$$\|u\|_{m(\mathcal{S})} = \left(\sum_{|\beta| \leq m} \int_{\mathcal{S}} |D^\beta u(x)|^2 dx \right)^{\frac{1}{2}} \text{ for } u \in H_m(\mathcal{S}) .$$

By convergence in $H_m^{\text{loc}}(\mathcal{D})$, we mean convergence in $H_m(\mathcal{D}')$ for each bounded domain $\mathcal{D}' \subseteq \mathcal{D}$. Note that $H_0(\mathcal{D}) = L_2(\mathcal{D})$ and $H_0^{\text{loc}}(\mathcal{D}) = \text{df. } L_2^{\text{loc}}(\mathcal{D})$. We denote by Ω_{r_0} , the set of points $x \in \Omega$ for which $|x| < r_0$. By ℓ_r , we mean the intersection of Ω with the circle $|x| = r$. Finally, we set $\|u\|_{m(r)} = \|u\|_{m(\Omega_r)}$ for each $u \in H_m(\Omega_r)$.

Section 2. Meromorphic Continuation of the S-matrix and Distorted Waves

In this section we shall meromorphically continue the S-matrix, $S(\kappa)(S'(\kappa))$, and the distorted plane waves, $w_n^+(x; \kappa)(w_n'^+(x; \kappa))$, from I_2^+ first onto the upper half plane and then onto the lower half plane. We begin by defining these functions for $\text{Im } \kappa > 0$. Let $\rho(A)$ denote the resolvent set of A and suppose $\kappa^2 \in \rho(A)$. We set $w_n^+(x; \kappa) = w_n^0(x; \kappa) + v_n^+(x; \kappa) = \tilde{w}_n^+(x; z) = \tilde{w}_n^0(x; z) + \tilde{v}_n^+(x; z)$, where $z = \kappa^2$ and the right hand side is defined by (1.6). Similarly, we define $G_n(x; \kappa) = \tilde{G}_n(x; z)$, etc.

It is clearly seen that $v_n^+(x; \kappa)$ satisfies:

$$(\tilde{A} - \kappa^2)v_n^+(x; \kappa) = - \sum_{|\beta| \leq 2} a_\beta(x) D^\beta w_n^0(x; \kappa) \text{ in } \Omega \quad (2.1)$$

and

$$v_n^+(x; \kappa) = -w_n^0(x; \kappa) \text{ on } \dot{\Omega}.$$

Furthermore we have

$$(2.2) \quad v_n^+(x; \kappa) = \sum_{n'=1}^{\infty} c_{n', (n)}^+ \Pi_{n'/\alpha}^{(1)}(\kappa r) \sin \frac{n'}{\alpha} \theta \text{ for } r \geq r_0.$$

This follows from the facts that $v_n^+(\cdot; \kappa) \in H = L_2(\Omega)$ and the perturbation is contained in S_{r_0} . Again we have analogous results for Λ^* with $v_n^+(x; \kappa)$, etc., replaced by $v_n^{*+}(x; \kappa)$, etc.

Let $\sigma(\Lambda)$ denote the spectrum of Λ and $\sigma'(\Lambda)$ consist of that part of $\sigma(\Lambda)$ outside of $[0, \infty)$. Since the constructions of (1.6) are only possible when $z = \kappa^2 \notin \sigma(\Lambda)$, we add the following condition.

(C) Suppose that $\sigma'(\Lambda)$ consists of a discrete set of eigenvalues of Λ of finite index.

Remark 2.1: While we do not know that (C) holds a priori for the general operators under consideration, (C) does hold for perturbations of order less than 2 (i.e. $a_\beta(x) \equiv 0$ for $|\beta| = 2$). This follows from a result of Gohberg and Krein ([8], Theorem 5.1) and the fact that $(\Lambda - \Lambda'_0)(\Lambda'_0 - z)^{-1}$ is a compact operator in this case, where Λ'_0 is the self-adjoint operator given by $-\Delta$ associated with the zero boundary condition acting in $L_2(\Omega)$ and $\text{Im } z \neq 0$. For "small" second order perturbations, see Corollary 2.2.

Remark 2.2: When V is a general second order non-selfadjoint perturbation with bounded support, an interesting open question is what can be said about the non-real spectrum of Λ . This is unanswered even when Λ_0 and Λ act in $L_2(\mathbb{R}^N)$.

It follows from condition (C) that $(\Lambda - \kappa^2)^{-1}((\Lambda^* - \kappa^2)^{-1})$ is a meromorphic function of κ for $\text{Im } \kappa > 0$ with poles only occurring at the points κ_0 for which $\kappa_0^2 \in \sigma'(\Lambda)(\kappa_0^2 \in \sigma'(\Lambda^*))$. We thus see from (1.6) that $v_n^+(\cdot; \kappa)$ is an H -valued meromorphic function of κ in $\text{Im } \kappa > 0$. If

$\kappa^2 \in \rho(A)$ and $\text{Im } \kappa > 0$, we define the operator $S(\kappa)$ acting on l_2^∞ by

$$(2.3) \quad S(\kappa) = I + T(\kappa),$$

where the infinite matrix $T(\kappa)$ is given by

$$T(\kappa) = (t_{n,n'}(\kappa)), \quad t_{n,n'}(\kappa) = (8\pi\alpha)^{\frac{1}{2}} c_n^{n'+}(\kappa), \quad n, n' = 1, 2, \dots$$

The coefficients $c_n^{n'+}(\kappa)$ are defined by (2.2). We construct $S'(\kappa)$ similarly for $\kappa^2 \in \rho(A^*)$ and $\text{Im } \kappa > 0$.

Lemma 2.1: Suppose that condition (C) holds. Then the operator $S(\kappa)$ and $S'(\kappa)$ are meromorphic functions of κ in the operator topology on l_2^∞ for $\text{Im } \kappa > 0$. κ_0 is a pole of $S(\kappa)S'(\kappa)$ only if κ_0^2 is an eigenvalue of $A(A^*)$. Furthermore $T(\kappa) = S(\kappa)(T'(\kappa) = (S'(\kappa) - I))$ is a Hilbert Schmidt operator if $\kappa^2 \in \rho(A)(\kappa^2 \in \rho(A^*))$.

Proof: It clearly suffices to consider $S(\kappa)$. Suppose $\kappa_0^2 \in \rho(A)$ and $\text{Im } \kappa_0 > 0$ (i.e. κ_0^2 is not an eigenvalue of A). Then for some $\delta > 0$ sufficiently small, we have $\kappa^2 \in \rho(A)$ and $\text{Im } \kappa > 0$ for each

$$\kappa \in N_\delta \equiv \text{df. } \{\kappa \mid |\kappa - \kappa_0| \leq \delta\}.$$

We wish first to estimate the components, $t_{n,n'}(\kappa)$, of $S(\kappa)$ for n, n' large and $\kappa \in N_\delta$. By virtue of the definition of $t_{n,n'}(\kappa)$, it suffices to consider $c_n^{n'+}(\kappa)$. It is clear from (2.2) that

$$(2.4) \quad \begin{aligned} \pi\alpha \sum_{n=1}^{\infty} |c_n^{n'+}(\kappa)|^2 |h_{n/\alpha}^{(1)}(\kappa r)|^2 &= \int_0^{2\pi\alpha} |v_n^+(x; \kappa)|^2 d\theta = \\ &= \int_{l_r} |v_n^+(x; \kappa)|^2 d\theta \quad \text{for each } r = |x| \geq r_0. \end{aligned}$$

We shall employ the following asymptotic formula for $H_{n/\alpha}^{(1)}(z)$ (see, [7], page 197):

$$(2.5) \quad H_{n/\alpha}^{(1)}(z) = \left(\frac{z}{\pi}\right)^{\frac{1}{2}} e^{\frac{1}{2}i(z - \frac{n}{\alpha}\pi - \frac{1}{4}\pi)} \cdot \left[1 + \frac{c_n}{z}\right], \quad n = 1, 2, \dots,$$

where the constant c_n depends only on n . Suppose that $r_1 > r_0$ and $\frac{1}{2} > \eta > 0$ are fixed. Set

$$P_1 = \left\{ \text{positive integers } n \mid \left| 1 + \frac{c_n}{\kappa r_1} \right| > \eta \text{ for each } \kappa \in N_\delta \right\} \text{ and}$$

$$P_2 = \{ \text{positive integers } n \mid n \notin P_1 \}.$$

It is clear that we may choose an $r_2 > r_1$ such that

$$\left| 1 + \frac{c_n}{\kappa r_2} \right| > \eta \text{ for each } n \in P_2 \text{ and } \kappa \in N_\delta.$$

We thus conclude from (2.5) that

$$\max_{\substack{\kappa \in N_\delta \\ n \in P_1}} |H_{n/\alpha}^{(1)}(\kappa r_1)|^{-1} < \infty \quad \text{and}$$

(2.6)

$$\max_{\substack{\kappa \in N_\delta \\ n \in P_2}} |H_{n/\alpha}^{(1)}(\kappa r_2)|^{-1} < \infty.$$

Combining (2.4) and (2.6), we deduce

$$\begin{aligned}
 (2.7) \quad \sum_{n=1}^{\infty} |c_n^{n'+}(x)|^2 &= \sum_{n \in P_1} |c_n^{n'+}(x)|^2 + \sum_{n \in P_2} |c_n^{n'+}(x)|^2 \\
 &\leq C \left(\int_{\ell_{r_1}} |v_n^+(x; \kappa)|^2 d\theta + \int_{\ell_{r_2}} |v_n^+(x; \kappa)|^2 d\theta \right), \quad \kappa \in N_\delta,
 \end{aligned}$$

where C denotes a constant. (We shall denote different constants by the same letter C when there is no danger of confusion).

It thus suffices to consider

$$\int_{\ell_r} |v_n^+(x; \kappa)|^2 d\theta$$

for $\kappa \in N_\delta$ and fixed $r > r_0$. We first employ the following inequality:

$$(2.8) \quad \int_{\ell_r} |v_n^+(x; \kappa)|^2 d\theta \leq C \|v_n^+(\cdot; \kappa)\|_{1(r_1)}^2$$

provided $r_1 > r_0$. Now (1.6) implies

$$\begin{aligned}
 \|v_n^+(\cdot; \kappa)\|_{1(r_1)} &\leq \|(A - \kappa^2)^{-1}(F_n(\cdot; \kappa) - \sum_{|\beta| \leq 2} a_\beta(x) D^\beta w_n^0(\cdot; \kappa))\|_{1(r_1)} \\
 &\quad + \|G_n(\cdot; \kappa)\|_{1(r_1)}.
 \end{aligned}$$

We may choose r_0 such that the "cutoff function", $\chi(x)$, has its support in S_{r_0} . We shall need the following result from the theory of elliptic equations, [9]:

$$(2.9) \quad \|u\|_{j(r)} \leq C[\|u\|_{0(r')} + \|\Delta u\|_{j-2(r')}] , \quad r' > r, \quad j = 2, 3, \dots,$$

for each $u \in H_0^{\text{loc}}(\Omega)$ such that $u(x) = 0$ on Ω and $\Lambda u \in H_{j-2}^{\text{loc}}(\Omega)$, where C is independent of $u(x)$ but depends on r' .

Using the properties of $v_{n'}^+(x; \kappa)$, $F_{n'}(x; \kappa)$, $G_{n'}(x; \kappa)$ and $a_\beta(x)$, as well as estimate (2.9), we obtain

$$(2.10) \quad \|v_{n'}^+(\cdot; \kappa)\|_{1(r_1)} \leq C \|w_{n'}^0(\cdot; \kappa)\|_{2(r_0)}, \quad n' = 1, 2, \dots, r_1 > r_0.$$

Since $\frac{\partial^2 w_{n'}^0(x; \kappa)}{\partial \theta^2} = \frac{n'^2}{\alpha^2} w_{n'}^0(x; \kappa)$, we obtain

$$(2.11) \quad \frac{n'^4}{\alpha^4} \|w_{n'}^0(\cdot; \kappa)\|_{2(r_0)}^2 = \left\| \frac{\partial^2 w_{n'}^0(\cdot; \kappa)}{\partial \theta^2} \right\|_{2(r_0)}^2 \leq C \|w_{n'}^0(\cdot; \kappa)\|_{4(S_{r_0})}^2,$$

$$n' = 1, 2, \dots$$

Combining (2.9), (2.11) and the properties of $w_{n'}^0(x; \kappa)$, we deduce

$$(2.12) \quad \max_{\kappa \in N_\delta} \|w_{n'}^0(\cdot; \kappa)\|_{2(r_0)}^2 \leq \frac{C}{n'^4}, \quad n' = 1, 2, \dots$$

Finally, we may combine (2.7), (2.8), (2.10) and (2.12) to obtain

$$(2.13) \quad \max_{\kappa \in N_\delta} \sum_{n=1}^{\infty} |c_n^{n'+}(\kappa)|^2 \leq \frac{C}{n'^4}, \quad n' = 1, 2, \dots$$

It now follows easily from (2.13) and the analyticity of each

$$c_n^{n'+}(\kappa) = \frac{1}{\pi \alpha H_{n/\alpha}^{(1)}(\kappa r)} \int_{\ell_r} v_{n'}^+(x; \kappa) \sin \frac{n}{\alpha} \theta d\theta$$

that $T(\kappa)$ is Hilbert-Schmidt and analytic at κ_0 in the operator topology

on ℓ_2^∞ . Finally, if κ_1^2 is an eigenvalue of Λ , it follows from the discussion

preceding Lemma 2.1 that $(A - \kappa^2)^{-1}$ has a pole at $\kappa = \kappa_1$. Hence each of the elements $t_{n,n'}(\kappa)$ has at worst a pole at $\kappa = \kappa_1$. It follows from this that $\mathfrak{S}(\kappa)$ has at worst a pole at $\kappa = \kappa_1$. To see this, we simply combine equations (3.8) and (3.9) of Section 3 with the preceding arguments. Q.E.D.

Before defining $\mathfrak{S}(\kappa)\mathfrak{S}'(\kappa)$ for $\text{Im } \kappa < 0$, we state the following result, due to Ribaric and Vidav, [10], and Steinberg, [11].

Lemma 2.2: Suppose that $L(\kappa)$ is a compact operator-valued function of κ acting in a Banach space, B , for each κ in a domain, D , of the complex plane. Suppose also that $L(\kappa)$ is an analytic function of κ in the operator topology on B . Then either $M(\kappa) \equiv_{\text{df.}} (I + L(\kappa))^{-1}$ exists nowhere in D or else $M(\kappa)$ is analytic everywhere in D except for a discrete set of points, where it has poles.

We now define $\mathfrak{S}(\kappa)\mathfrak{S}'(\kappa)$ for $\text{Im } \kappa < 0$ and $\overline{\kappa}^2 \in \rho(A^*) (\overline{\kappa}^2 \in \rho(A))$ by means of the equation

$$\begin{aligned} \mathfrak{S}(\kappa) &= \mathfrak{S}'(\overline{\kappa})^{*-1} \\ (2.14) \quad \mathfrak{S}'(\kappa) &= \mathfrak{S}(\overline{\kappa})^{*-1} \end{aligned}$$

We may easily see using the "limiting absorption principle" as in [2] that $w_n^+(x; \kappa)(w_n'^+(x; \kappa))$ and hence each $t_{n,n'}(\kappa)(t_{n,n'}'(\kappa))$ is a continuous

function of κ at $\kappa_0 \in D_{\mathcal{J}}^{\pm} \equiv_{\text{df.}} \text{Im } \kappa > 0 \cup I_{\mathcal{J}}^{\pm}$, provided $\kappa_0^2 \in \rho(\Lambda)(\kappa_0^2 \in \rho(\Lambda^*))$.

Furthermore it follows as before with the aid of condition $(C_{\mathcal{J}})$ and the limiting absorption principle, that (2.13) holds uniformly with respect to κ in a neighborhood, $N_{\mathcal{J}}^{\pm}$, of $I_{\mathcal{J}}^{\pm}$, contained in $D_{\mathcal{J}}^{\pm}$. Hence we conclude with the aid of (2.13) that $\mathcal{S}(\kappa)$ is a continuous operator-valued function of κ in $N_{\mathcal{J}}^{\pm}$ and similarly for $\mathcal{S}'(\kappa)$. It follows from (1.13) that (2.14) holds if $\kappa \in I_{\mathcal{J}}^{\pm}$. We may therefore employ lemmas 2.1 and 2.2, $(C_{\mathcal{J}})$, (2.14) and the Cauchy integral formula as in the proof of the Schwarz reflection principle) to obtain the following result.

Theorem 2.1: Suppose that $(C_{\mathcal{J}})$ and (C) hold. Then the operator $\mathcal{S}(\kappa)$ defined by (2.3) and (2.14) yields a meromorphic continuation of the $\mathcal{S}$ -matrix associated with A_0 and A from $I_{\mathcal{J}}^{\pm}$ onto the domain $\Gamma_{\mathcal{J}}^{\pm} \equiv_{\text{df.}} \text{Im } \kappa > 0 \cup I_{\mathcal{J}}^{\pm} \cup \text{Im } \kappa < 0 - \{\kappa \mid \text{Im } \kappa < 0 \text{ and } \overline{\kappa}^2 \in \sigma(A^*)\}$. Furthermore $\mathcal{S}(\kappa)$ is analytic in a neighborhood of $I_{\mathcal{J}}^{\pm}$. Analogous results hold for $\mathcal{S}'(\kappa)$.

Note: The continuation, $\mathcal{S}(\kappa)$, defined by (2.3) and (2.14) is clearly independent of the interval $\mathcal{J}$.

In general, we do not have any information about the number of "singular points," $\lambda > 0$, at which (1.1) has a non-trivial outgoing or incoming solution. However, if A is self-adjoint the situation is clearer.

Corollary 2.1: If A is self-adjoint and $\lambda > 0$, then λ is not a singular point.

Proof: We shall prove that Λ has no non-trivial outgoing solution.

The proof is similar for the case of incoming solutions. Hence, suppose that $u(x)$ satisfies (1.1) and (1.2). We employ the divergence theorem and the properties of $u(x)$ and $\tilde{\Lambda}$ to obtain for each $r \geq r'_0$.

$$\begin{aligned} 0 &= \int_{\Omega_r} (u(x)(\tilde{\Lambda} - \lambda)u(x) - (\tilde{\Lambda} - \lambda)u(x)\overline{u(x)})dx = \\ &= - \int_0^{2\pi\alpha} (u(r, 0) \frac{\partial \overline{u(r, 0)}}{\partial r} - \frac{\partial u(r, 0)}{\partial r} \overline{u(r, 0)})r d\theta = \\ &= -\pi\alpha \sum_{n=1}^{\infty} r |c_n(\lambda)|^2 H_{n/\alpha}^{(1)}(\sqrt{\lambda} r) \frac{d}{dr} \overline{H_{n/\alpha}^{(1)}(\sqrt{\lambda} r)} + \\ &+ \pi\alpha \sum_{n=1}^{\infty} r |c_n(\lambda)|^2 \overline{H_{n/\alpha}^{(1)}(\sqrt{\lambda} r)} \frac{d}{dr} H_{n/\alpha}^{(1)}(\sqrt{\lambda} r). \end{aligned}$$

Using (1.12), we conclude that $0 = \sum_{n=1}^{\infty} |c_n(\lambda)|^2$ and so each $c_n(\lambda) = 0$, $n = 1, 2, \dots$. Thus $u(x) \equiv 0$ in $\Omega - \Omega \cap S_{r'_0}$. Employing the unique continuation theorem for elliptic equations, [9], we conclude that $u(x) \equiv 0$ in Ω .

Q.E.D.

Corollary 2.2: (a) Suppose that condition $(C_{\mathcal{J}})(i)$ holds for the compact interval $\mathcal{J} \subset (0, \infty)$ and $\rho(A) \neq 0$. Then $(C_{\mathcal{J}})(ii)$ holds for some $\epsilon_0 > 0$.

(b) Given an arbitrary compact interval $\mathcal{J} \subset (0, \infty)$, there exists a $\delta_0 > 0$ such that $(C_{\mathcal{J}})$ holds with ϵ_0 replaced by any $\epsilon_1 > 0$ and Λ replaced by $\Lambda^\delta = A_0 + \delta V$, where δ is any complex number such that $|\delta| < \delta_0$.

Hence

Hence Λ_δ has no eigenvalues in each rectangle $\mathcal{R}_1^\pm$.

Proof: (a) Suppose that $z_0 \in \rho(\Lambda)$. It follows readily from (2.9) and the Rellich selection theorem that $R(z) - R'_0(z) = -R'_0(z)VR(z)$ is a compact operator, where $R'_0(z) = (\Lambda'_0 - z)^{-1}$ and Λ'_0 was defined in Remark 2.1. Hence Λ'_0 and A have the same essential spectrum. From this

we conclude with the aid of [12] (Lemma 2) that if $z \in \sigma(A)$ ($\text{Im } z \neq 0$), then either z is an eigenvalue of A or $\bar{z}$ is an eigenvalue of A^* . Thus if $(C_2)(ii)$ is not true, there exists an infinite sequence of numbers $\lambda_n \in \mathcal{J}$ and $\epsilon_n > 0$ such that $\epsilon_n \downarrow 0$ as $n \rightarrow \infty$ and either $\lambda_n + i\epsilon_n$ or $\lambda_n - i\epsilon_n$ are eigenvalues of either A or of A^* , $n = 1, 2, \dots$. In any case we obtain a contradiction to $(C_2)(i)$ using the arguments employed in the proof of the limiting absorption principle in [2].

(b) Suppose $\text{Im } z_0 \neq 0$. We may clearly choose a $\delta_1 > 0$ sufficiently small that $\|\delta_1 VR_0(z_0)\| < 1$ ($\|\cdot\|$ denoting the operator norm in H). It is now clear that $R_\delta(z_0) = (\Lambda_\delta - z_0)^{-1}$ exists and is given by

$$(2.15) \quad R_\delta(z_0) = R_0(z_0)(I + \delta VR_0(z_0))^{-1} \quad \text{for } |\delta| \leq \delta_1.$$

Hence $\rho(\Lambda_\delta) \neq \mathcal{J}$ for each δ such that $|\delta| < \delta_1$. We may now apply limiting absorption principle as in part (a) to show that given an arbitrary compact interval $\mathcal{J}$, there exists a $\delta_s > 0$ and an $\epsilon_0 > 0$ such that (C_2) holds for the operator Λ_δ for each complex number δ such that $|\delta| < \delta_2$. (This argument uses the

obvious fact that $(C_{\mathcal{D}})$ holds for Λ_0). Since $\|VR_0(z_0)\|$ is uniformly bounded for each z_0 in $\mathcal{D}_{\epsilon_0}^{\pm} \equiv \text{df. } \{z = \lambda \pm i\epsilon \mid \lambda \in \mathcal{D}, \epsilon \geq \epsilon_0\}$, the result now follows readily using (2.15). Q.E.D.

Remark 2.3: Corollary 2.2 and the proof of Theorem 2.1 implies that the S -matrix may have an analytic continuation off of the real axis even if (C) fails to hold. Let us also observe that part (b) is a small perturbation result.

We next meromorphically continue the functions $w_n^+(x; \kappa)$ and $w_n^+(x; \kappa)$ onto $\Gamma_{\mathcal{D}}^{\pm}$. To simplify the details we shall assume in the next theorem that $\tilde{A} = -\Delta$ acting in $L_2(\Omega)$. We shall discuss the more general case in Remark 3.4.

Theorem 2.2: Suppose that the hypotheses of Theorem 2.1 hold and $\tilde{A} = -\Delta$. Then the functions $w_n^+(x; \kappa)$, $n = 1, 2, \dots$, defined earlier for $\text{Im } \kappa > 0$, have a meromorphic continuation onto $\Gamma_{\mathcal{D}}^{\pm}$ in $L_2^{\text{loc}}(\Omega)$. Furthermore, $\kappa_0 \in \Gamma_{\mathcal{D}}^{\pm}$ is a pole of $\mathfrak{S}(\kappa)$ if and only if κ_0 is a pole of some $w_n^+(x; \kappa)$. Similar results hold for the functions $w_n^-(x; \kappa)$, where $w_n^-(x; \kappa)$ may be constructed analogously to $w_n^+(x; \kappa)$.

Proof: To begin with, we observe that it follows readily from (2.2) and Theorem 2.1 that each $w_n^+(\cdot; \kappa)$ is meromorphic $L_2^{\text{loc}}(\Omega - \Omega_{r_0})$ and hence in $H_2^{\text{loc}}(\Omega - \Omega_{r_1})$ for $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$ and $r_1 > r_0$ (using estimate (2.9)). We thus wish to express $w_n^+(x; \kappa)$, $x \in \Omega_{r_1}$, $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$, in terms of its values on Γ_{r_1} . To this end we proceed as follows. Let $G_0^+(x, y, \kappa)$ denote the outgoing Green's function for the operator $-\Delta - \kappa^2$ acting in $L_2(\mathbb{R}^2)$, where $\text{Im } \kappa > 0$ and $x, y \in \mathbb{R}^2$.

$G_0^+(\cdot, y, \kappa)$ clearly has an analytic continuation onto $\Gamma_{\mathcal{D}}^{\pm}$ in $L_2^{\text{loc}}(\mathbb{R}^2)$

for each fixed $y \in \mathbb{R}^2$.

We shall construct Green's function, $G^+(x, y, \kappa)$, for the operator $\Lambda_{r_1} - \kappa^2$, given by $\tilde{\Lambda} - \kappa^2$ associated with the zero boundary condition in Ω_{r_1} , for each $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$, $x, y \in \Omega_{r_1}$ and $r_1 > r_0$ fixed. Having made this construction, we shall define functions $w_n^{+*}(x; \kappa)$ by the equation

$$(2.16) \quad w_n^{+*}(x; \kappa) = - \int_{\ell_{r_1}} w_n^+(x', \kappa) \frac{\partial G^+(x, x', \kappa)}{\partial r'} r_1 d\theta', \quad n = 1, 2, \dots,$$

where $x' = (r', \theta') \in \ell_{r_1}$, $x \in \Omega_{r_1}$, and $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$. We shall then show that

$w_n^{+*}(x; \kappa) = w_n^+(x; \kappa)$, $x \in \Omega_{r_1}$, $\kappa \in D_{\mathcal{D}}^{\pm}$, and that $w_n^{+*}(\cdot, \kappa)$ is a meromorphic function of $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$ in $L_2^{\text{loc}}(\Omega_{r_1})$ for each $r_1' < r_1$. The theorem will then follow easily.

Suppose that $y \in \Omega_{r_1}$, where $r_1 > r_0$ is fixed. We wish to construct a solution, $G^+(x, y, \kappa) = G_0^+(x, y, \kappa) + H^+(x, y, \kappa)$, of the boundary value problem

$$(2.17) \quad (\tilde{\Lambda}_x - \kappa^2)G^+(x, y, \kappa) = \delta(x - y) \text{ for } x \in \Omega_{r_1}, G^+(x, y, \kappa) = 0$$

$$\text{for } x \in \dot{\Omega}_{r_1},$$

where $\delta(x - y)$ is the Dirac delta function, $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$ and $\tilde{\Lambda}_x$ denotes differentiation with respect to the variables $x = (x_1, x_2)$. To construct $H^+(x, y, \kappa)$, we proceed as follows.

Set $T_{r_1} = A_{r_1}^{-1}$. It is clear that T_{r_1} exists and is a compact integral operator acting in $L_2(\Omega_{r_1})$. Hence there exists a discrete set of numbers, $\mathfrak{D}$, such that $(I - \kappa^2 T_{r_1})^{-1}$ exists for each $\kappa \notin \mathfrak{D}$. Let $\chi(x, y, \kappa)$ denote the unique solution in $L_2(\Omega_{r_1})$ of the Dirichlet problem

$$(2.18) \quad \tilde{A}_x \chi(x, y, \kappa) = 0 \text{ for } x \in \Omega_{r_1}, \quad \chi(x, y, \kappa) = -G_0^+(x, y, \kappa) \text{ for } x \in \dot{\Omega}_{r_1},$$

where $\kappa \in \Gamma_{\mathfrak{D}}^{\pm}$. Finally we set

$$(2.19) \quad H^+(x, y, \kappa) = (I - \kappa^2 T_{r_1})^{-1} \chi(x, y, \kappa), \text{ where } x, y \in \Omega_{r_1},$$

and $\kappa \in \Gamma_{\mathfrak{D}}^{\pm} - \mathfrak{D}$. It follows easily from (2.18) and (2.19) that $G^+(x, y, \kappa)$ satisfies (2.17) for each $\kappa \in \Gamma_{\mathfrak{D}}^{\pm} - \mathfrak{D}$. Since $G_0^+(x, y, \kappa)$ is an analytic function of $\kappa \in \Gamma_{\mathfrak{D}}^{\pm}$ for $x \neq y$, we see from (2.18) that so is $\chi(x, y, \kappa)$ for each $x \in \Omega_{r_1}$. Hence $H^+(x, y, \kappa)$ is analytic for $\kappa \in \Gamma_{\mathfrak{D}}^{\pm} - \mathfrak{D}$ by (2.19).

Using definition (2.16), the properties of $G^+(x, y, \kappa)$ and $w_n^+(x; \kappa)$ and the divergence theorem, we deduce that $w_n^+(x; \kappa) = w_n^{+*}(x; \kappa)$ for each $x \in \Omega_{r_1}$, $\kappa \in \Gamma_{\mathfrak{D}}^{\pm}$. We have already observed that $w_n^+(\cdot; \kappa)$ is a meromorphic function of $\kappa \in \Gamma_{\mathfrak{D}}^{\pm}$ in the topology of $H_2^{\text{loc}}(\Omega - S_{r_1'})$ for each $r_1' > r_0$. Hence we may readily conclude using the properties of $G^+(x, y, \kappa)$ that (2.16) defines a meromorphic $L_2(\Omega_{r_1'})$ -valued function of $\kappa \in \Gamma_{\mathfrak{D}}^{\pm} - \mathfrak{D}$ for each $r_1' < r_1$. Thus $v_n^+(\cdot; \kappa)$ is meromorphic in $L_2^{\text{loc}}(\Omega)$ and hence in $H_2^{\text{loc}}(\Omega)$.

for $\kappa \in \Gamma_{\mathcal{D}}^{\pm} = \emptyset$, where

$$v_n^+(x; \kappa) \equiv_{\text{df.}} w_n^{+*}(x; \kappa) - w_n^0(x; \kappa) \text{ for each } x \in \Omega_{r_1} \text{ and } v_n^+(x; \kappa)$$

is defined by (2.2) otherwise.

It is easily seen that we may select a suitable r_1 such that $\frac{1}{\kappa^2} \in \rho(T_{r_1})$ for each $\kappa \in \emptyset$. Thus $\Gamma_{\mathcal{D}}^{\pm} = \emptyset$ may be replaced by $\Gamma_{\mathcal{D}}^{\pm}$. The functions $v_n^+(x; \kappa)$ satisfy (2.1) for each $\kappa \in \Gamma_{\mathcal{D}}^{\pm}$ by the unique continuation theorem for analytic functions since (2.1) holds for $\kappa \in D_{\mathcal{D}}^{\pm}$. Finally, we see from (2.2), (2.3), (2.16) and the unique continuation theorem that each $w_n^+(x; \kappa) = w_n^0(x; \kappa) + v_n^+(x; \kappa)$ is analytic at $\kappa = \kappa_0$ if and only if $g(\kappa)$ is analytic at $\kappa = \kappa_0$. We have thus proved the theorem for $w_n^+(x; \kappa)$.

In order to construct $w_n^-(x; \kappa)$, we may proceed as before to obtain a function $w_n^-(x; \kappa) = w_n^0(x; \kappa) + v_n^-(x; \kappa)$ satisfying (2.1), $n = 1, 2, \dots$, and such that

$$v_n^-(x; \kappa) = \sum_{n'=1}^{\infty} c_{n'}^{n-}(\kappa) H_{n'/\alpha}^{(2)}(\kappa r) \sin \frac{n'}{\alpha} \theta \text{ for } r \geq r_0,$$

where $\text{Im } \kappa < 0$ and $\kappa^2 \in \rho(A)$. It again follows from the limiting absorption principle that $w_n^-(x; \kappa)$ and $c_{n'}^{n-}(\kappa)$ are continuous functions of κ in $\tilde{D}_{\mathcal{D}}^{\pm} \equiv_{\text{df.}} I_{\mathcal{D}}^{\pm} \cup \text{Im } \kappa < 0$, except when $\kappa^2 \in \sigma'(A)$. For $\kappa \in I_{\mathcal{D}}^{\pm}$, $w_n^-(x; \kappa)$ denotes an incoming distorted plane wave and for $\text{Im } \kappa < 0$ ($\kappa^2 \in \rho(A)$), $w_n^-(x; \kappa)$ is analytic in κ .

Since A is self-adjoint, we have the equation $g = g^{*-1}$. If we now evaluate both sides of the equation

$$T^0 S T^{0^{-1}} = T^0 S^{-1*} T^{0^{-1}},$$

using the method of Theorem 1.2, we readily obtain the relation

$$(2.20) \quad \overline{c_{n'}^{n-}(\kappa)} = c_n^{n'+}(\bar{\kappa}), \quad n, n' = 1, 2, \dots, \quad \kappa \in I_{\mathcal{D}}^{\pm}.$$

We now define the operator $\hat{S}(\kappa)$ acting in l_2^{∞} by

$$(2.21) \quad \hat{S}(\kappa) = I + \hat{T}(\kappa),$$

where

$$\hat{T}(\kappa) = (\hat{t}_{n,n'}(\kappa)), \quad \hat{t}_{n,n'}(\kappa) = (8\pi\alpha)^{\frac{1}{2}} c_{n'}^{n-}(\kappa) \quad \text{for } \kappa \in \tilde{D}_{\mathcal{D}}^{\pm}.$$

We readily conclude using (1.13), (2.20), (2.21) and Theorem 1.2 that

$$(2.22) \quad \hat{S}(\kappa) = \hat{S}(\bar{\kappa})^{*-1}, \quad \kappa \in I_{\mathcal{D}}^{\pm}.$$

Employing (2.22) and the proof of Theorem 2.1, we conclude that

$\hat{S}(\kappa)$ has a meromorphic continuation from $\tilde{D}_{\mathcal{D}}^{\pm}$ onto all of $\tilde{\Gamma}_{\mathcal{D}}^{\pm} \equiv_{\text{df}} \tilde{D}_{\mathcal{D}}^{\pm} \cup \{\kappa \mid \text{Im } \kappa > 0 \text{ and } \bar{\kappa}^2 \in \sigma(A)\}$. Defining $c_{n'}^{n-}(\kappa)$ in terms of $\hat{S}(\kappa)$ as in (2.21) for $\text{Im } \kappa > 0$, we thus have a meromorphic continuation of $c_{n'}^{n-}(\kappa)$ onto all of $\tilde{\Gamma}_{\mathcal{D}}^{\pm}$, $n, n' = 1, 2, \dots$. Furthermore, it follows from (2.20) and the unique continuation theorem for analytic functions that this equation holds for $\kappa \in \tilde{\Gamma}_{\mathcal{D}}^{\pm}$. Finally, we apply the same arguments as in the first part of this theorem to meromorphically continue $w_n^-(x; \kappa)$ onto $\tilde{\Gamma}_{\mathcal{D}}^{\pm}$. The proof is thus complete. Q.E.D.

Remark 2.4: Theorem 2.2 holds in the more general case,

$$\tilde{A} = -\Delta + \sum_{|\beta| \leq 2} a_\beta D^\beta, \text{ using the same method. In this case, however,}$$

we must replace (2.18) by the boundary value problem

$$(2.23) \quad \tilde{A}_x \chi(x, y; \kappa) = - \sum_{|\beta| \leq 2} a'_\beta(x) D_x^\beta G_0^+(x, y; \kappa) \text{ for } x \in \Omega_{r_1},$$

$$\chi(x, y; \kappa) = - G_0^+(x, y; \kappa) \text{ for } x \in \dot{\Omega}_{r_1},$$

where $y \in \Omega_{r_1}$. Since the right side of (2.23) has a singularity at $x = y$,

the theory of distributions must be invoked to obtain $\chi(x, y; \kappa)$. The

remainder of the proof of Theorem 2.2 remains unchanged provided we

take into account the fact that A need not equal A^* , so that the formulas

must be suitably modified, as in the proof of Theorem 2.1.

Section 3. The Resonant States

In this section, we characterize the non-real poles of the S -matrix in terms of "resonant states".

Definition 3.1: Suppose that $\text{Im } \kappa_0 < 0$ and there exists a non-trivial solution, $u(x)$, of the boundary value problem

$$(3.1) \quad (\tilde{A} - \kappa_0^2)u(x) = 0 \text{ in } \Omega, \quad u(x) = 0 \text{ on } \dot{\Omega},$$

such that

$$(3.2) \quad u(x) = \sum_{n=1}^{\infty} c_n H_{n/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta \text{ for } r \geq r_0.$$

Then we shall say that $u(x)$ is a resonant state for Λ at κ_0 . We define resonant states for Λ^* similarly.

Theorem 3.1: Suppose that $\text{Im } \kappa_0 < 0$ and $\bar{\kappa}_0^2 \in \rho(\Lambda^*)$. Then there exists a resonant state for Λ at κ_0 if and only if κ_0 is a pole of $\mathcal{S}(\kappa)$ (defined by (2.3) and (2.14)). An analogous result holds with Λ replaced by Λ^* .

We first prove a lemma.

Lemma 3.1: Suppose that $\text{Im } \kappa_0 < 0$ and $\bar{\kappa}_0^2 \in \rho(\Lambda^*)$. Then κ_0 is a pole of $\mathcal{S}(\kappa)$ if and only if $\mathcal{N}(\mathcal{S}'(\bar{\kappa}_0)^*) \neq \{0\}$, where $\mathcal{N}(\mathcal{S}'(\bar{\kappa}_0)^*)$ denotes the null space of $\mathcal{S}'(\bar{\kappa}_0)^*$.

Proof: Since $\bar{\kappa}_0^2 \in \rho(\Lambda^*)$, we see that $\mathcal{S}'(\bar{\kappa})^*$ is clearly analytic at $\kappa = \kappa_0$. If $\mathcal{N}(\mathcal{S}'(\bar{\kappa}_0)^*) = \{0\}$, then by the compactness of $I - \mathcal{S}'(\bar{\kappa}_0)^*$ (see Lemma 2.1), it follows that $\mathcal{S}(\kappa) = \mathcal{S}'(\bar{\kappa})^{*-1}$ is analytic at $\kappa = \kappa_0$.

Conversely suppose $\mathcal{S}'(\bar{\kappa}_0)^* f = 0$ for some $0 \neq f \in l_2^\infty$. We clearly have $f = \mathcal{S}(\kappa) \mathcal{S}'(\bar{\kappa})^* f$ for each κ in a (possibly deleted) neighborhood of κ_0 .

We see that $\mathcal{S}'(\bar{\kappa})^* f \rightarrow 0$ as $\kappa \rightarrow \kappa_0$, since $\mathcal{S}'(\bar{\kappa})^*$ is analytic at $\kappa = \kappa_0$.

Hence $\mathcal{S}(\kappa)$ must have a pole at $\kappa = \kappa_0$.

Q.E.D.

Proof of Theorem 3.1: Suppose that κ_0 is a pole of $\mathcal{S}(\kappa)$, so that by Lemma 3.1 we have $\mathcal{S}'(\bar{\kappa}_0)^* h = 0$ for some $0 \neq h = (h_1, h_2, \dots) \in l_2^\infty$.

It follows from (2.3) and an analogue of equation (2.20) (with Λ not necessarily self-adjoint) that

$$(3.3) \quad \sum_{n=1}^{\infty} (8\pi\alpha)^{\frac{1}{2}} c_n^{n'}(\kappa_0) h_{n'} = -h_n, \quad n = 1, 2, \dots$$

Set

$$(3.4) \quad w(x; \kappa_0) = \sum_{n=1}^{\infty} h_n \bar{w}_n(x; \kappa_0) \text{ in } \Omega.$$

Hence

$$w(x; \kappa_0) = \sum_{n=1}^{\infty} h_n w_n^0(x; \kappa_0) + \sum_{n=1}^{\infty} h_n \sum_{n'=1}^{\infty} c_{n'}^{n-}(\kappa_0) H_{n'/\alpha}^{(2)}(\kappa_0 r) + \sin \frac{n'}{\alpha} \theta$$

in $\Omega - \Omega_{r_0}$. Note that each $\bar{w}_n(x; \kappa_0)$ is defined since $\kappa_0^2 < \rho(\Lambda)$. The function $w(x; \kappa_0)$ clearly satisfies (3.1) since each $\bar{w}_n(x; \kappa_0)$ does. (The right hand side of (3.4) may be shown to converge in $H_2^{\text{loc}}(\Omega)$ by the argument used to derive (2.13)).

Since we have by definition

$$w_n^0(x; \kappa_0) = \left(\frac{1}{2\pi\alpha}\right)^{\frac{1}{2}} J_{n/\alpha}(\kappa_0 r) \sin \frac{n}{\alpha} \theta = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} (H_{n/\alpha}^{(1)}(\kappa_0 r) + H_{n/\alpha}^{(2)}(\kappa_0 r)) \sin \frac{n}{\alpha} \theta,$$

we deduce from (3.3) that

$$(3.5) \quad \begin{aligned} w(x; \kappa_0) &= \sum_{n=1}^{\infty} \frac{h_n}{(8\pi\alpha)^{\frac{1}{2}}} H_{n/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta + \\ &+ \sum_{n=1}^{\infty} \frac{h_n}{(8\pi\alpha)^{\frac{1}{2}}} H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta + \sum_{n=1}^{\infty} \left(\sum_{n'=1}^{\infty} c_{n'}^{n-}(\kappa_0) h_{n'} \right) H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta \\ &= \sum_{n=1}^{\infty} \frac{h_n}{(8\pi\alpha)^{\frac{1}{2}}} H_{n/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta \text{ for } r \geq r_0. \end{aligned}$$

Finally we conclude from (3.5) that $w(x; \kappa_0) \neq 0$ (since $h \neq 0$). Hence $w(x; \kappa_0)$ is a resonant state.

Conversely, suppose that $w(x; \kappa_0)$ is a resonant state for A at κ_0 and $w(x; \kappa_0)$ is given by the right hand side of (3.5) for $r \geq r_0$. Set

$$(3.6) \quad \begin{aligned} \tilde{w}(x; \kappa_0) &= \sum_{n=1}^{\infty} h_n w_n^-(x; \kappa_0) \quad \text{in } \Omega = \sum_{n=1}^{\infty} h_n w_n^0(x; \kappa_0) \\ &+ \sum_{n=1}^{\infty} h_n \sum_{n'=1}^{\infty} c_{n'}^-(\kappa_0) H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta \quad \text{in } \Omega - \Omega_{r_0}. \end{aligned}$$

and set

$$\hat{w}(x; \kappa_0) = w(x; \kappa_0) - \tilde{w}(x; \kappa_0).$$

It is clear that $\hat{w}(\cdot; \kappa_0) \in L_2(\Omega)$ and

$$(3.7) \quad (A - \kappa_0^2) \hat{w}(\cdot; \kappa_0) = 0$$

using (3.1), (3.5) and (3.6). Since $\kappa_0^2 \in \rho(A)$, we see from (3.7) that $w(x; \kappa_0) \equiv \tilde{w}(x; \kappa_0)$ in Ω . Comparing the coefficients of $H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta$, $n = 1, 2, \dots$, on the right hand sides of (3.5) and (3.6), we obtain (3.3).

If $h = 0$, then $w(x; \kappa_0) \equiv 0$ in Ω by (3.5) and the unique continuation theorem for elliptic equations. Hence $h \neq 0$ and κ_0 is a pole of $S(\kappa)$ by (3.3) and Lemma 3.1. The same arguments clearly apply with A replaced by A^* . Q.E.D.

We next wish to drop the condition $-\kappa_0^2 \in \rho(A^*)$. Thus, suppose that $-\kappa_0^2 \in \sigma(A^*)$, or equivalently $z_0 = \kappa_0^2 \in \sigma(A)$. We shall assume in the remainder of this section that the following condition holds.

(C_{κ₀}) Suppose that $z_0 = \kappa_0^2$ is an isolated eigenvalue of Λ with finite index and finite dimensional root subspaces.

The following results are well-known consequences of (C_{κ₀}) (see, e.g., [13], pp. 228-230). In a neighborhood of $z = z_0$, we have

$$(3.8) \quad (z - A)^{-1} = \sum_{j=J}^{\infty} R_j (z - z_0)^j = \frac{R_{-J}}{(z - z_0)^J} + \dots + \frac{R_{-1}}{z - z_0} + R_0 + R_1 (z - z_0)$$

for some positive integer J , where the R_j 's are mutually commutative bounded linear operators mapping H into $D(A)$ and $R_1(z)$ is analytic at $z = z_0$. Furthermore, the following equations hold:

$$(3.9) \quad R_j = \frac{1}{2\pi i} \int_C \frac{(z' - A)^{-1} dz'}{(z' - z_0)^{j+1}},$$

where C is a sufficiently small curve surrounding z_0 ,

$$(3.10) \quad R_0 R_{-j} = 0 \quad (j \geq 1)$$

and

$$R_{-j-k+1} = R_{-j} R_{-k} \quad (j, k \geq 1),$$

$$(3.11) \quad R_{-(j+1)} = (A - z_0 I) R_{-j} = (A - z_0 I)^j R_{-1} \quad (j \geq 1),$$

$$(3.12) \quad (A - z_0 I) R_0 = R_{-1} - I$$

and

$$(3.13) \quad \text{Range}(R_{-1}) = N((A - z_0 I)^J) \text{ and } \text{Range}(I - R_{-1}) = \text{Range}((A - z_0 I)^J) \quad (J \geq 1).$$

We shall assume that $J = 1$ and treat this case in detail. We shall briefly describe in remarks 3.1 and 3.2 how to generalize these results to arbitrary J using the same method. Since $J = 1$ whenever A is self-adjoint, our next theorem extends the Shenk results [4]. We begin by again setting $\kappa^2 = z$ and writing (3.8) as

$$(3.14) \quad (\kappa^2 - A)^{-1} = \frac{R_{-1}}{\kappa^2 - \kappa_0^2} + R_0 + R_1(\kappa)(\kappa^2 - \kappa_0^2)$$

for κ near κ_0 , where $R_1(\kappa)$ is analytic at $\kappa = \kappa_0$. It follows readily that

$$(3.15) \quad (\bar{\kappa}^2 - A^*)^{-1} = \frac{R_{-1}^*}{\bar{\kappa}^2 - \bar{\kappa}_0^2} + R_0^* + R_1(\kappa)^*(\bar{\kappa}^2 - \bar{\kappa}_0^2)$$

for κ near κ_0 .

It follows from condition $(C_{\kappa_0}^*)$ that $\dim \text{Range}(R_{-1}) = \dim \text{Range}(R_{-1}^*) = m < \infty$. Hence we may choose m linearly independent eigenfunctions, $w_1(x), \dots, w_m(x)$, of A corresponding to κ_0^2 and m linearly independent eigenfunctions, $w_1^*(x), \dots, w_m^*(x)$, of A^* corresponding to $\bar{\kappa}_0^2$ such that the $\{w_j\}$ span R_{-1} , the $\{w_j^*\}$ span R_{-1}^* and $(w_j, w_k^*) = \int_{\Omega} w_j(x) \overline{w_k^*(x)} dx = \delta_j^k$, $j, k = 1, \dots, m$. We thus have

$$(3.16) \quad R_{-1}u = \sum_{j=1}^m (u, w_j^*) w_j(x) \quad \text{and} \quad R_{-1}^*u = \sum_{j=1}^m (w_j, u) \cdot w_j^*(x)$$

for each $u \in H$. Since each function $w_j(x)(w_j^*(x))$ satisfies the equation

$$(3.17) \quad Aw_j = \kappa_0^2 w_j \quad (A^* w_j^* = \bar{\kappa}_0^2 w_j^*),$$

we easily deduce

$$(3.18) \quad \begin{aligned} w_j(r, 0) &= \sum_{n=1}^{\infty} c_{jn} H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} 0 \\ (w_j^*(r, 0) &= \sum_{n=1}^{\infty} c_{jn}^* H_{n/\alpha}^{(1)}(\bar{\kappa}_0 r) \sin \frac{n}{\alpha} 0 \end{aligned}$$

for some constants c_{jn} and c_{jn}^* , where $j = 1, \dots, m$ and $r \geq r_0$.

We next wish to express $S'(\bar{\kappa})^*$ in the form

$$(3.19) \quad S'(\bar{\kappa})^* = \frac{S_{-1}}{\kappa - \kappa_0} + S_0 + S_1(\kappa)(\kappa - \kappa_0),$$

where S_{-1} , S_0 and $S_1(\kappa)$ act in ℓ_2^∞ and $S_1(\kappa)$ is analytic at $\kappa = \kappa_0$.

In order to find a convenient expression for S_{-1} and S_0 , we proceed as follows. By (2.3) and (2.20), we have for each $f = (f_n) = (f_1, f_2, \dots) \in \ell_2^\infty$ and $\kappa^2 \in \rho(A)$:

$$(3.20) \quad (S'(\bar{\kappa})^* f)_n = f_n + \sum_{n'=1}^{\infty} (8\pi\alpha)^{\frac{1}{2}} c_n^{n'}(\kappa) f_{n'}, \quad n = 1, 2, \dots$$

Set $H_n'(x; \kappa) = F_n'(x; \kappa) - \tilde{V}' w_n^0(x; \kappa) = (\tilde{A}' - \kappa^2)(f(x) \cdot w_n^0(x; \kappa) - \tilde{V}' w_n^0(x; \kappa))$ and define $H_n(x; \kappa)$ similarly with $\tilde{A}'$ and $\tilde{V}'$ replaced by $\tilde{A}$ and $\tilde{V}$.

Using the same arguments as in the proof of Theorem 1.2, we obtain:

$$(3.21) \quad c_n^{n'}(\kappa) = -i\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} \int_{\Omega} \overline{H_n'(x; \bar{\kappa})} w_{n'}^-(x; \kappa) dx$$

We recall that

$$(3.22) \quad w_n^-(x; \kappa) = (1 - \zeta(x)) w_n^0(x; \kappa) + (\Lambda - \kappa^2)^{-1} (H_n(\cdot; \kappa))(x).$$

Substituting (3.14) and (3.22) into (3.21), we obtain

$$(3.23) \quad c_n^{n, \sim}(\kappa) = -i\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} \int_{\Omega} \overline{H'_n(x; \bar{\kappa})} (\phi_n(x; \kappa) - \frac{1}{\kappa - \kappa_0} R_{-1}\left(\frac{H_{n,1}(\cdot; \kappa)}{\kappa + \kappa_0}\right)(x)) dx + r(\kappa)(\kappa - \kappa_0),$$

where $r(\kappa)$ is analytic at $\kappa = \kappa_0$ and

$$(3.24) \quad \phi_n(x; \kappa) = (1 - \zeta(x)) w_n^0(x; \kappa) - R_0(H_{n,1}(\cdot; \kappa))(x).$$

Note that in a neighborhood of $\kappa = \kappa_0$, we have

$$(3.25) \quad \begin{aligned} \frac{1}{\kappa - \kappa_0} \cdot \overline{H'_n(x; \bar{\kappa})} \cdot R_{-1}\left(\frac{H_{n,1}(\cdot; \kappa)}{\kappa + \kappa_0}\right)(x) = \\ = \frac{1}{\kappa - \kappa_0} \cdot \overline{H'_n(x; \bar{\kappa}_0)} R_{-1}\left(\frac{H_{n,1}(\cdot; \kappa_0)}{2\kappa_0}\right)(x) \\ + \left[\frac{d}{d\kappa} \left(\overline{H'_n(x; \bar{\kappa})} R_{-1}\left(\frac{H_{n,1}(\cdot; \kappa)}{\kappa + \kappa_0}\right)(x) \right) \right]_{\kappa=\kappa_0} + \tilde{r}(x; \kappa)(\kappa - \kappa_0). \end{aligned}$$

where $\tilde{r}(x; \kappa)$ is analytic at $\kappa = \kappa_0$.

It follows readily from (3.17) and (3.18) that

$$(3.26) \quad \int_{\Omega} \overline{H'_n(x; \bar{\kappa}_0)} w_j(x) dx = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i c_{jn}$$

and

$$(3.27) \quad \int_{\Omega} H_{n,1}(x; \kappa_0) \overline{w_j^*(x)} dx = -\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i c_{jn}^*,$$

using the same arguments as in the derivation of (3.21). Combining (3.16) (3.23) and (3.25)-(3.27), we deduce (using the analyticity at κ_0 of all functions involved):

$$(3.28) \quad c_n^{n'}(\kappa) = \frac{s_{-1,n,n'}}{\kappa - \kappa_0} + s_{0,n,n'} + s_{1,n,n'}(\kappa)(\kappa - \kappa_0)$$

for κ near κ_0 , where $s_{1,n,n'}(\kappa)$ is analytic at κ_0 ,

$$(3.29) \quad s_{-1,n,n'} = -\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{i}{2\kappa_0} \sum_{j=1}^m c_{jn} \overline{c_{jn'}^*},$$

$$(3.30) \quad s_{0,n,n'} = -i\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} \left\{ \int_{\Omega} \overline{H_n'(x; \kappa_0)} \phi_{n'}(x; \kappa_0) dx - \right. \\ \left. - \frac{1}{4\kappa_0^2} \sum_{j=1}^m \left(\frac{2\alpha}{\pi}\right) c_{jn} \overline{c_{jn'}^*} + \frac{i}{2\kappa_0} \sum_{j=1}^m \left(\frac{2\alpha}{\pi}\right) (c_{jn}' \overline{c_{jn'}^*} + c_{jn} \overline{c_{jn'}^{**}}) \right\}$$

$$c_{jn}' = \left[-\frac{d}{d\kappa} \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} i \int_{\Omega} \overline{H_n'(x; \kappa)} w_j(x) dx \right]_{\kappa=\kappa_0}$$

and

$$\overline{c_{jn'}^{**}} = \left[-\frac{d}{d\kappa} \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} i \int_{\Omega} H_{n'}(x; \kappa) \overline{w_j^*(x)} dx \right]_{\kappa=\kappa_0}.$$

We finally conclude from (3.19), (3.20) and (3.28) that the matrices S_{-1} and S_0 are given by

$$(3.31) \quad S_{-1} = ((8\pi\alpha)^{\frac{1}{2}} s_{-1,n,n'})$$

and

$$(3.32) \quad S_0 = (\delta_n^{n'} + (8\pi\alpha)^{\frac{1}{2}} s_{0,n,n'}), \quad n, n' = 1, 2, \dots$$

We are now ready to state and prove the main result.

Theorem 3.2: Suppose that condition (C_{κ_0}) holds, $\text{Im } \kappa_0 < 0$ and κ_0^2 is a pole of $(A - \kappa^2)^{-1}$ of order one. Then we have the following:

(a) Suppose that there exists an element $h = (h_n) \in l_2^\infty$ such that $h \neq 0$ and

$$(P) \quad S_{-1}h = 0 \text{ and } S_0h \in \text{Range } S_{-1}.$$

Then κ_0 is a singular point of $\mathcal{S}(\kappa)$, the function $u(x)$ defined by (3.34) below is a resonant state for A at κ_0 and we have

$$(3.33) \quad u(x) = u(r, 0) = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} h_n H_{n/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta$$

for $r \geq r_0$, where

$$(3.34) \quad u(x) \equiv_{\text{df.}} U(x) = \sum_{j=1}^m (k_j^{(1)} + k_j^{(2)}) w_j(x), \quad x \in \Omega,$$

$$(3.35) \quad U(x) = \sum_{n=1}^{\infty} h_n \phi_n(x; \kappa_0), \quad x \in \Omega,$$

and $k_j^{(1)}$ and $k_j^{(2)}$ are defined by

$$(3.36) \quad k_j^{(1)} = i\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{2\kappa_0} \sum_{n=1}^{\infty} h_n \overline{c_{jn}^{*'}}, \quad j = 1, \dots, m,$$

and

$$(3.37) \quad \sum_{j=1}^m k_j^{(2)} c_{jn} = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} (S_0 h)_n, \quad n = 1, 2, \dots$$

(b) Suppose that $u(x)$ is a resonant state for A at κ_0 and

$h = (h_n)$ is defined by equation (3.33).

Then h satisfies condition (P), $h \neq 0$ and $u(x)$ is given by (3.34).

We first prove the following lemmas.

Lemma 3.2: Suppose that (C_{κ_0}) holds, $\text{Im } \kappa_0 < 0$, κ_0^2 is a simple pole of $(A - \kappa_0^2)^{-1}$ and $f = (f_n) \in \ell_2^\infty$.

Then there exists unique complex numbers $k_1, \dots, k_m$ such that

$$(3.38) \quad (S_{-1}f)_n = \sum_{j=1}^m k_j c_{jn}, \quad n = 1, 2, \dots$$

Proof: It follows from (3.29) and (3.31) that

$$(3.39) \quad S_{-1}f = -\frac{2\alpha i}{\kappa_0} \sum_{j=1}^m \sum_{n'=1}^{\infty} c_{jn'} \overline{c_{jn'}^*} f_{n'}, \quad n = 1, 2, \dots$$

Hence (3.38) holds with $k_j = -\sum_{n'=1}^{\infty} \frac{2i\alpha}{\kappa_0} \overline{c_{jn'}^*} f_{n'}$, $j = 1, \dots, m$.

To see that the k_j are uniquely defined, suppose there exists constants

γ_j , $j = 1, \dots, m$, such that

$$(3.40) \quad 0 = \sum_{j=1}^m \gamma_j c_{jn}, \quad n = 1, 2, \dots$$

We shall show that each $\gamma_j = 0$. Set $w(x) = \sum_{j=1}^m \gamma_j w_j(x)$. Then $(A - \kappa_0^2)w = 0$

and we have by (3.18) and (3.40):

$$w(x) = \sum_{n=1}^{\infty} \left(\sum_{j=1}^m c_{jn} \gamma_j \right) H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta = 0$$

for $r = |x| \geq r_0$. Hence by the unique continuation theorem, we conclude that

$w(x) \equiv 0$ in Ω . Since the $w_j(x)$ are linearly independent, we have

$\gamma_j = 0$, $j = 1, \dots, m$.

Q.E.D.

Note: It follows easily from (3.18), (3.29), (3.31), Lemma 3.2

and the unique continuation theorem applied to $w_j^*(x)$ that the matrix $S_{-1} \neq 0$. Hence $S'(\bar{\kappa})^*$ has a pole at $\kappa = \kappa_0$.

We next give a sufficient condition for κ_0^2 to be a singular point of $S(\kappa)$.

Lemma 3.3: Suppose that $\text{Im } \kappa_0 < 0$, κ_0 is a simple pole of $(A - \kappa^2)^{-1}$ and condition (P) holds for some $h \neq 0$. Then κ_0 is a singular point of $S(\kappa)$.

Proof: Suppose that $S(\kappa)$ is analytic at $\kappa = \kappa_0$. Then in a neighborhood of κ_0 , we have

$$S(\kappa) = G_0 + (\kappa - \kappa_0)G_1 + (\kappa - \kappa_0)^2 G_2(\kappa),$$

where G_0, G_1 and $G_2(\kappa)$ are bounded operators acting in ℓ_2^∞ and $G_2(\kappa)$ is analytic at $\kappa = \kappa_0$. It follows from (2.14) and (3.19) that

$$(3.41) \quad I = S(\kappa)S'(\bar{\kappa})^* = \frac{G_0 S_{-1}}{\kappa - \kappa_0} + G_0 S_0 + G_1 S_{-1} + G_3(\kappa)(\kappa - \kappa_0)$$

for κ near κ_0 , where $G_3(\kappa)$ is analytic at $\kappa = \kappa_0$. If we apply both sides of (3.41) to h , let $\kappa \rightarrow \kappa_0$ and employ condition (P), we obtain

$h = G_0 S_0 h = G_0 S_{-1} f$ for some $f \in \ell_2^\infty$. But (3.41) implies the equation

$G_0 S_{-1} = 0$. Thus $h = 0$. This contradiction proves the lemma. Q.E.D.

Remark 3.1: If $\kappa_0^2 (\text{Im } \kappa_0 < 0)$ is a pole of order J of $(A - \kappa^2)^{-1}$, then it follows from (3.8) and (3.20)-(3.22) that

$$(3.42) \quad S'(\kappa)^{-1} = \frac{S_{-J}}{(\kappa - \kappa_0)^J} + \dots + \frac{S_{-1}}{(\kappa - \kappa_0)} + S_0 + S_1(\kappa)(\kappa - \kappa_0)$$

for κ near κ_0 , where each S_j is a bounded operator acting in ℓ_2^∞ and $S_1(\kappa)$ is analytic at $\kappa = \kappa_0$. We now have the following generalization of Lemma 3.3. κ_0 is a singular point of $S(\kappa)$ if there exists an element $h \in \ell_2^\infty$ such that $h \neq 0$ and

$$(P') \quad S_{-J}h = 0 \text{ and } S_{-k}h = \sum_{j=k+1}^J S_{-j}f, \quad k = 0, \dots, J-1,$$

for some $f \in \ell_2^\infty$. The proof of this assertion is essentially the same as that of Lemma 3.3.

Proof of Theorem 3.2: (a) It follows from condition (P) that $S_0h = S_{-1}f$ for some $f \in \ell_2^\infty$. Hence we conclude from Lemma 3.2 that there exist unique constants $k_1^{(2)}, \dots, k_m^{(2)}$ for which (3.37) holds. We next observe that the right hand side of (3.35) converges in $H_2^{\text{loc}}(\Omega)$ (and hence uniformly on bounded subsets of Ω by Sobolev's inequality). This follows readily from (3.12), (3.24) and (2.9) (as in the derivation of (2.13)). By virtue of Lemma 3.3, it remains to prove that $u(x)$ is a resonant state and (3.33) holds.

Each $\phi_n(x; \kappa_0) = 0$ on $\dot{\Omega}$ using the properties of $\mathcal{L}(x)$ and $w_n^0(x; \kappa)$ and the fact that $R_0(H_n(\cdot; \kappa_0))(x) \in D(\Lambda)$. Since each $w_j(x) = 0$ on $\dot{\Omega}$, we conclude that $u(x) = 0$ on $\dot{\Omega}$. We next wish to prove:

$$(3.43) \quad (\tilde{\Lambda} - \kappa_0^2)u(x) \equiv_{\text{def}} (-\Delta - \kappa_0^2 + \sum_{|\beta| \leq 2} a_\beta(x) D^\beta)u(x) = 0 \text{ in } \Omega.$$

It clearly suffices to prove (3.43) for $U(x)$. Using (3.12), (3.16) and the definitions of $\phi_n(x; \kappa_0)$ and $H_n(x; \kappa_0)$, we obtain

$$\begin{aligned} (\tilde{A} - \kappa_0^2)\phi_n(x; \kappa_0) &= -H_n(x; \kappa_0) - (R_{-1} - I)(H_n(\cdot; \kappa_0))(x) = \\ &= -\sum_{j=1}^m w_j(x) \int_{\Omega} H_n(y; \kappa_0) \overline{w_j^*(y)} dy. \end{aligned}$$

Combining (3.18) and (3.27), we have

$$(3.44) \quad (\tilde{A} - \kappa_0^2)\phi_n(x; \kappa_0) = -\sum_{j=1}^m \sum_{n'=1}^{\infty} \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i c_{jn'} \overline{c_{jn'}^*} H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta$$

Finally, we employ (3.29), (3.31), (3.35), (3.44) and (P) to deduce

$$\begin{aligned} (\tilde{A} - \kappa_0^2)U(x) &= -\sum_{n=1}^{\infty} h_n \sum_{j=1}^m \sum_{n'=1}^{\infty} \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i c_{jn'} \overline{c_{jn'}^*} H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta = \\ (3.45) \quad &= -\sum_{n'=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i \sum_{j=1}^m c_{jn'} \overline{c_{jn'}^*} h_n H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta = \\ &= 2\kappa_0 \sum_{n'=1}^{\infty} \left(\sum_{n=1}^{\infty} S_{-1, n'} h_n\right) H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta = \\ &= \sum_{n'=1}^{\infty} \frac{2\kappa_0}{(8\pi\alpha)^{\frac{1}{2}}} = (S_{-1} h)_{n'} H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta = 0. \end{aligned}$$

(Since the constants, $\pi\alpha c_{jn'} = \frac{1}{H_{n'/\alpha}^{(2)}(\kappa_0 r)} \int_{\Gamma_r} w_j(r, \theta) \sin \frac{n'}{\alpha} \theta d\theta$ and $c_{jn'}^*$,

may be easily estimated for large n using (2.5) and integration by parts, there is no difficulty in justifying the interchanging of summation signs in (3.45)). We have thus proved (3.43).

We next prove equation (3.33). It follows readily from (3.24),

(3.35) and (3.43) that for $r \geq r_0$, we have

$$(3.46) \quad U(x) = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} \sum_{n'=1}^{\infty} h_{n'} H_{n'/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta + \sum_{n'=1}^{\infty} d_{n'} H_{n'/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n'}{\alpha} \theta$$

for some constants $d_1, d_2, \dots$. Once again using the method of Theorem 1.2 and the properties of $U(x)$, we see that

$$(3.47) \quad \int_{\Omega} U(x) \overline{H_{n'}^{(1)}(x; \overline{\kappa_0})} dx = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} i \left(d_{n'} - \frac{h_{n'}}{(8\pi\alpha)^{\frac{1}{2}}} \right), \quad n' = 1, 2, \dots$$

Equations (3.30), (3.35) and (3.47) now yield

$$(3.48) \quad d_{n'} = \frac{h_{n'}}{(8\pi\alpha)^{\frac{1}{2}}} + \frac{1}{i} \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} \int_{\Omega} U(x) \overline{H_{n'}^{(1)}(x; \overline{\kappa_0})} dx$$

$$= \frac{h_{n'}}{(8\pi\alpha)^{\frac{1}{2}}} + \sum_{n=1}^{\infty} h_n s_{0, n', n} - i \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \sum_{j=1}^m \left\{ \frac{1}{4\kappa_0} \sum_{n=1}^{\infty} h_n c_{jn'} \overline{c_{jn}^*} \right.$$

$$\left. - \frac{1}{2\kappa_0} \sum_{n=1}^{\infty} h_n (c_{jn'} \overline{c_{jn}^*} + c_{jn'} \overline{c_{jn}^{*'}}) \right\}, \quad n' = 1, 2, \dots$$

The equations

$$0 = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} (S_{-1}h)_{n'} = -\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \frac{i}{2\kappa_0} \sum_{n=1}^{\infty} \sum_{j=1}^m c_{jn'} \overline{c_{jn}^*} h_n, \quad n' = 1, 2, \dots,$$

combined with Lemma 3.2 (with $k_j = \sum_{n=1}^{\infty} -\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \frac{i}{2\kappa_0} \overline{c_{jn}^*} h_n$, $j = 1, \dots, m$)

implies

$$(3.49) \quad \sum_{n=1}^{\infty} \overline{c_{jn}^*} h_n = 0, \quad j = 1, \dots, m.$$

Equations (3.32), (3.48) and (3.49) yield

$$(3.50) \quad d_{n'} = \left(\frac{1}{8\pi\alpha}\right)^{\frac{1}{2}} (S_0 h)_{n'} + i\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} \sum_{j=1}^m \sum_{n=1}^{\infty} \frac{1}{2\kappa_0} h_n c_{jn} \overline{c_{jn}^{*}}, \quad n' = 1, 2, \dots$$

Finally, we combine (3.18), (3.34), (3.36), (3.37), (3.46), and (3.50) to obtain

(3.33). To see that $u \neq 0$, we simply observe that if $u(x) \equiv 0$ in Ω , then by (3.33) we have

$$0 = \int_{\Gamma_r} u(r, 0) \sin \frac{n}{\alpha} \theta d\theta = \frac{\pi \alpha h_n H_{n/\alpha}^{(1)}(\kappa_0 r)}{(8\pi\alpha)^{\frac{1}{2}}} \quad \text{for } r \geq r_0, \quad n = 1, 2, \dots$$

Hence $h = 0$. This contradiction proves that $u \neq 0$ and hence $u(x)$ is a resonant state.

(b) Suppose that $u(x)$ is a resonant state for A at κ_0 and that $u(x)$ satisfies (3.33). We first show that

$$(3.51) \quad S_{-1} h = 0.$$

It follows from (3.18), the divergence theorem, the arguments of Theorem 1.2 and the properties of $u(x)$ and each $w_j^*(x)$ that for $r \geq r_0$, we have

$$(3.52) \quad \begin{aligned} 0 &= \int_{\Omega} (\tilde{A} - \kappa_0^2) u(x) \overline{w_j^*(x)} dx = \int_{\Omega} u(x) \overline{(\tilde{A} - \kappa_0^2) w_j^*(x)} \\ &+ \int_{\Gamma_r} \left(-\frac{\partial u}{\partial r} \overline{w_j^*(x)} + u(x) \overline{\frac{\partial w_j^*(x)}{\partial r}} \right) r d\theta = \text{Const.} \sum_{n=1}^{\infty} h_n \overline{c_{jn}^{*}}, \quad j = 1, \dots, m. \end{aligned}$$

Equation (3.51) now follows from (2.29), (3.31) and (3.52).

We now define $U(x)$ by (3.35) and set $v(x) = u(x) - U(x)$. It follows from (3.33) and (3.35) that $v \in H$. Using (3.51) and the arguments employed to derive (3.43), we obtain $(A - \kappa_0^2)v = 0$. Therefore we have

$$(3.53) \quad u(x) = U(x) - \sum_{j=1}^m \gamma_j w_j(x)$$

for some constants γ_j , $j = 1, \dots, m$. We next combine (3.46), (3.48) and (3.52) as in part (a) to obtain

$$(3.54) \quad \begin{aligned} U(x) = & \sum_{n=1}^{\infty} \left(\frac{1}{8\pi\alpha} \right)^{\frac{1}{2}} h_n H_{n/\alpha}^{(1)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta + \\ & + \sum_{n=1}^{\infty} \left\{ \left(\frac{1}{8\pi\alpha} \right)^{\frac{1}{2}} (S_0 h)_n + i \left(\frac{2\alpha}{\pi} \right)^{\frac{1}{2}} \sum_{j=1}^m \sum_{n'=1}^{\infty} \frac{h_{n'}}{2\kappa_0} c_{jn} \overline{c_{jn'}^{*1}} \right\} H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta \end{aligned}$$

for $r \geq r_0$.

Let us define $\beta_j = \gamma_j - k_j^{(1)}$. We may compare the coefficients of $H_{n/\alpha}^{(2)}(\kappa_0 r) \sin \frac{n}{\alpha} \theta$ for $r \geq r_0$ in (3.33) and (3.53) (with $U(x)$ given by (3.54)) and recall (3.18) to see that

$$\sum_{j=1}^m \beta_j c_{jn} = \left(\frac{1}{8\pi\alpha} \right)^{\frac{1}{2}} (S_0 h)_n, \quad n = 1, 2, \dots$$

It thus follows from (3.37) and Lemma 3.2 that $\beta_j = k_j^{(2)}$ is defined uniquely and (3.34) holds. It is immediate from (3.33) and the unique continuation theorem that $h \neq 0$ (since the resonant state $u(x) \neq 0$). To complete the proof of (P), we must show that $S_0 h \in \text{range } S_{-1}$. Using (3.29), (3.31) and (3.37), it suffices to construct a vector $f = (f_n) \in l_2^\infty$ such that

$$-(\frac{2\alpha}{\pi})^2 \frac{1}{2\kappa_0} \sum_{n=1}^{\infty} \overline{c_{jn}^* f_n} = k_j^{(2)}, \quad j = 1, \dots, m.$$

Since this involves m equations and an infinite number of unknowns, $f_1, f_2, \dots$, we may certainly construct such a vector f and the proof is complete. Q.E.D

Remark 3.2: As we mentioned earlier, κ_0^2 need not be a simple pole of $(A - \kappa^2)^{-1}$ when A is not self-adjoint, but may have arbitrary order J . The results and proof of Theorem 3.2 may be generalized to this case. There are no serious theoretical difficulties involved, however, the calculations obviously become very complicated. Hence we shall merely make the following observations. Since (3.8) holds in a neighborhood of $\kappa = \kappa_0$ (with $z = \kappa^2$ and $z_0 = \kappa_0^2$), so does (3.42). It follows from (3.11), (3.13), and condition $(C_{\kappa_0}^*)$ that the Range of $R_{-(J-j)}$ ($0 \leq j \leq J-1$) is spanned by a finite number (say m_j) of linearly independent "generalized eigenfunctions", $w_{jk}(x)$, $k = 1, \dots, m_j$, such that $(A - \kappa_0^2)^{j+1} w_{jk} = 0$, and similarly for $R_{-(J-j)}^*$ with corresponding "generalized eigenfunctions", $w_{jk}^*(x)$, of $(A - \kappa_0^2)^*$.

The functions $w_{jk}(x)$ and $w_{jk}^*(x)$ satisfy "radiation conditions" analogous to (although more general than) (3.18), which give rise to constants $\{c_{jkl n}\}$ and $\{c_{jkl n}^*\}$, $j = 0, \dots, J-1$, $k = 1, \dots, m_j$, $\ell = 1, \dots, j+1$ and $n = 1, 2, \dots$. The matrices $S_{-j'}$, $j' = 0, \dots, J$, of (3.42) may then be expressed in terms of $\{c_{jkl n}\}$ and $\{c_{jkl n}^*\}$ by formulas analogous to (3.29) - (3.32). An analogue of Theorem 3.2 may then be formulated and proved using the same method as before. In particular, a resonant state,

$u(x)$, will in this case have the form:

$$u(x) = U(x) - \sum_{j=0}^{J-1} \sum_{k=1}^m \alpha_{jk} w_{jk}(x),$$

where $U(x)$ is defined by (3.35) and the α_{jk} are constants.

Section 4. The Exterior Problem.

The exterior problem refers to the case in which $S = R^N$, $N \geq 2$, Ω is the exterior of a finite number of bounded domains and $A = -\Delta + q(x)$. The results of sections 2 and 3 have been established for the exterior problem in [4] when $q(x)$ is real-valued. (In [14], the results were shown to hold when $q(x)$ is also complex-valued). However, the setting employed in [4] (and [14]) differs from that of this paper. The aim of this section is to compare our formulation with that of [4]. For the sake of simplicity, we shall assume that $\Omega = R^N$.

In [4], the resonant states were defined in terms of the "Sommerfeld radiation condition". We shall say that $u(x)$ satisfies the outgoing Sommerfeld radiation condition (associated with the complex number κ) if for each $x \in R^N$, we have:

$$(4.1) \quad \lim_{N \rightarrow \infty} \int_{|y|=N} \left[u(y) \frac{\partial}{\partial |y|} F^+(x, y; \kappa) - F^+(x, y; \kappa) \frac{\partial}{\partial |y|} u(y) \right] dy = 0,$$

where $F^+(x, y; \kappa) = F^+(x - y; \kappa)$ denotes the outgoing fundamental solution for $\Delta_0 - \kappa^2$. We define the incoming radiation condition similarly with

$F^+(x-y; \kappa)$ replaced by $F^-(x-y; \kappa)$, the incoming fundamental solution.*

In [4], a non-trivial solution, $u(x)$, of the equation

$$u(x) = - \int_{R^N} F^+(x-y; \kappa) q(y) u(y) dy$$

was referred to as a "resonant state" for A at the point κ ($\text{Im } \kappa < 0$).

It may be shown, [4], that $u(x)$ is outgoing in the sense of (4.1) and that $u(x)$ satisfies the equation

$$(4.2) \quad (-\Delta + q(x) - \kappa^2)u(x) = 0.$$

As we mentioned before, the results of sections 2 and 3 may be proved in R^N , $N > 2$, with the aid of spherical harmonics. We now briefly give the formulation of these results for the special case $S = R^N$,

$A = A_0 + V = -\Delta + q(x)$; where $q(x)$ is real-valued and $q(x) \in C_0(R^N)$;***

First of all, we define our "plane waves" as follows. We consider polar coordinates $x = (r, 0)$ in R^N , where $r = |x|$ and 0 represents the angular variables. Let $\{Y_{m\ell}(0)\}$ denote a system of $N-1$ dimensional spherical harmonics which forms a complete, orthonormal system in $L_2(S^{N-1})$, where S^{N-1} is the unit sphere in R^N , $\ell = 0, 1, \dots$, and m ranges over a finite set, $F_{\ell, N}$, that depends on N and ℓ . For each ℓ , the set $\{Y_{m\ell}(0)\}$ forms a maximal family of linearly independent $N-1$ dimensional spherical harmonics of order ℓ . Each $Y_{m\ell}(0)$ is an

*The function $\bar{F}^+(x-y; \kappa)(F^-(x-y; \kappa))$ is given by

$\frac{1}{4} \left(\frac{\kappa}{2\pi |x-y|} \right)^p H_p^{(1)}(\kappa |x-y|) \left(-\frac{1}{4} \left(\frac{\kappa}{2\pi |x-y|} \right)^p H_p^{(2)}(\kappa |x-y|) \right)$, where $p = \frac{N-2}{2}$.

***By $C_0^j(D)$, we mean those functions, $u(x) \in C^j(D)$, such that $u(x)$ has compact support, where $\infty, j = 0, 1, \dots, \infty$, and D is an arbitrary domain.

eigenfunction of the angular part of the Laplace operator corresponding to the eigenvalue $-\ell(\ell + N - 2)$. We now set

$$(4.3) \quad w_{m,\ell}^0(x; \kappa) = \text{Const. } J_{\ell + \frac{N-2}{2}}(\kappa r) Y_{m\ell}(0).$$

Our radiation conditions are defined similarly to before with $\sin \frac{n}{\alpha} 0$ replaced by $Y_{m\ell}(0)$ and $H_{n/\alpha}^{(j)}$ replaced by $H_{\ell + \frac{N-2}{2}}^{(j)}$, $j = 1, 2$.

All of our previous arguments and results now go through with very little change. We may thus obtain two complete sets of generalized eigenfunctions for A ,

$$w_{m,\ell}^{\pm}(x; \kappa) = w_{m,\ell}^0(x; \kappa) + v_{m,\ell}^{\pm}(x; \kappa),$$

where

$$v_{m\ell}^+(x; \kappa) = \sum_{m' \in F_{\ell,N}} \sum_{\ell'=0}^{\infty} c_{m'\ell'}^{m\ell+}(\kappa) H_{\ell' + \frac{N-2}{2}}^{(1)}(\kappa r) Y_{m'\ell'}(0)$$

for r sufficiently large, and similarly for $v_{m\ell}^-(x; \kappa)$. Again the S -matrix, $g(\kappa)$, may be easily expressed in terms of the coefficients, $c_{m\ell}^{m'\ell'+}(\kappa)$, as before.

Lemma 4.1: Suppose $\kappa \neq 0$ is a fixed complex number and $u(x) \not\equiv 0$ satisfies (4.2). Then $u(x)$ satisfies (4.1) if and only if $u(x)$ satisfies an analogue of (1.2) (with λ , $\sin \frac{n}{\alpha} 0$ and $H_{n/\alpha}^{(1)}(\sqrt{\lambda} r)$ replaced by κ^2 , $Y_{m\ell}(0)$ and $H_{\ell + \frac{N-2}{2}}(\kappa r)$, respectively).

Proof: We first observe that

$$(4.4) \quad F^+(x-y; \kappa) = \begin{cases} \text{Const.} \sum_{m \in F_{\ell, N}} \sum_{\ell=0}^{\infty} J_{\ell + \frac{N-2}{2}}(\kappa|x|) H_{\ell + \frac{N-2}{2}}^{(1)}(\kappa|y|) \cdot \\ \quad \cdot Y_{m\ell}(\theta_x) \cdot Y_{m\ell}(\theta_y) \quad \text{if } |x| < |y| \\ \text{Const.} \sum_{m \in F_{\ell, N}} \sum_{\ell=0}^{\infty} H_{\ell + \frac{N-2}{2}}^{(1)}(\kappa|x|) J_{\ell + \frac{N-2}{2}}(\kappa|y|) \cdot \\ \quad \cdot Y_{m\ell}(\theta_x) \cdot Y_{m\ell}(\theta_y) \quad \text{if } |y| < |x|, \end{cases}$$

where $x = (|x|, \theta_x)$, $y = (|y|, \theta_y)$ and $\kappa \neq 0$. This follows from the argument as in the proof of Theorem A.1 in the Appendix.

Now if $u(x)$ satisfies the analogue of (1.2), then we

immediately obtain (4.1), using (4.4). Conversely, suppose that $u(x)$ satisfies (4.1) and (4.2). Then we readily obtain

$$(4.5) \quad u(x) = \sum_{m \in F_{\ell, N}} \sum_{\ell=0}^{\infty} (c_{m\ell}^{(1)} H_{\ell + \frac{N-2}{2}}^{(1)}(\kappa r) + c_{m\ell}^{(2)} H_{\ell + \frac{N-2}{2}}^{(2)}(\kappa r)) Y_{m\ell}(\theta)$$

for r sufficiently large. Combining (4.1), (4.4), (4.5) and (1.12), we see that

$$c_{m\ell}^{(2)} = 0, \quad m \in F_{\ell, N}, \quad \ell = 0, 1, \dots \quad \text{Q.E.D.}$$

Note: Lemma 4.1 clearly holds for the more general operators considered in Section 2 by the same arguments.

In the setting of [4], the $\mathcal{S}$ -matrix, $\hat{\mathcal{S}}(\kappa)$, was constructed as a unitary operator acting in $L_2(S^{N-1})$. We may now prove that the poles of $\mathcal{S}(\kappa)$ and $\hat{\mathcal{S}}(\kappa)$ coincide.

Theorem 4.1: Suppose that κ_0 is a non-zero complex number.

Then κ_0 is a pole of $S(\kappa)$ if and only if κ_0 is a pole of $\hat{S}(\kappa)$. Furthermore each non-zero singular point of $S(\kappa)$ is a pole.

Proof: Let T_κ^+ denote the integral operator given by

$$T_\kappa^+ u(x) = \int_{R^N} F^+(x-y; \kappa) q(y) u(y) dy,$$

$$\text{for } x \in R^N, \kappa \neq 0 \text{ and } u \in H^\alpha \equiv_{\text{df.}} L_2(R^N; e^{-\alpha|x|} dx),$$

where $\alpha > 0$ is arbitrary. It was shown in [4] that T_κ^+ is a compact operator acting in H^α and is an analytic function of κ for $\kappa \neq 0$ and $\text{Im } \kappa > -\frac{\alpha}{2}$. Hence we see from Lemma 2.1 that $(I + T_\kappa^+)^{-1}$ is a meromorphic function of κ in $\text{Im } \kappa > -\frac{\alpha}{2}$ ($\kappa \neq 0$).

Now suppose that κ_0 is a singular point of $S(\kappa)$ and choose

$\alpha > 2|\text{Im } \kappa_0|$. We claim that $(I + T_\kappa^+)^{-1}$ (acting in H^α) has a pole at κ_0 .

For if $(I + T_\kappa^+)^{-1}$ is analytic at κ_0 , then we have

$$(4.6) \quad w_{m\ell}^+(x; \kappa) = (I + T_\kappa^+)^{-1} (w_{m\ell}^0(\cdot; \kappa))(x), \quad \ell = 0, 1, \dots, \text{ and } m \in F_{\ell, N}$$

for κ near κ_0 . (This equation follows readily from the properties of

$F^+(x-y; \kappa)$). Hence $S(\kappa)$ must be analytic at κ_0 using (4.6) and (2.9) as

in the proof of Lemma 2.1. This contradiction proves the assertion. We

now conclude from [4] that $\hat{S}(\kappa)$ has a pole at $\kappa = \kappa_0$. It also follows from

(4.6) and (2.9) that κ_0 is a pole of $S(\kappa)$.

Conversely, suppose that κ_0 is a singular point of $\hat{S}(\kappa)$. It follows from [4] that κ_0 is a pole of $\hat{S}(\kappa)$ and equations (4.1) and (4.2) hold for some $u(x)$. Hence by Lemma 4.1 and analogues of theorems 3.1 and 3.2, we conclude that κ_0 is a singular point of $S(\kappa)$. Q.E.D.

Q.E.D.

Remark 4.1: While the domain, S , discussed in this paper is wedge shaped, we may treat a great many other domains, such as infinite cylinders, in the same way. The main requirement is that we may separate variables to obtain "plane waves". In addition, our results may also be proved when A_0 and A are more general elliptic operators, again provided we may separate variables and $A - A_0$ has bounded support.

APPENDIX

We shall now construct the outgoing Green's function, $G_0^+(x, y; \kappa)$, for the operator $\Delta_0 - \kappa^2$, where $x, y \in S$ and $\text{Im } \kappa > 0$. Again without loss of generality we assume that $N = 2$, so that Δ_0 is defined as in Section 2.

Theorem A.1: For each pair of points $x = (|x|, \theta_x)$, $y = (|y|, \theta_y) \in S$ ($|x| = |y|$), we have

$$(A.1) \quad G_0^+(x, y; \kappa) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{2\alpha} J_{n/\alpha}(\kappa|x|) H_{n/\alpha}^{(1)}(\kappa|y|) \sin \frac{n}{\alpha} \theta_x \sin \frac{n}{\alpha} \theta_y \\ \quad \text{for } |x| < |y| \\ \text{and} \\ \sum_{n=1}^{\infty} \frac{1}{2\alpha} H_{n/\alpha}^{(1)}(\kappa|x|) J_{n/\alpha}(\kappa|y|) \sin \frac{n}{\alpha} \theta_x \sin \frac{n}{\alpha} \theta_y \\ \quad \text{for } |y| < |x|, \end{cases}$$

where $\text{Im } \kappa > 0$.

This result is typical for those unbounded domains for which our methods apply (i.e. for which we may separate variables) and is obtained by representing $H = L_2(S)$ in terms of a direct sum of Hilbert spaces, $H_n = L_2(0, \infty; r dr)$, as follows. It is well-known that each function $u \in H$ may be written as

$$u(x) = u(r, \theta) = \sum_{n=1}^{\infty} u_n(r) \sin \frac{n}{\alpha} \theta,$$

the convergence occurring in Π . Each $u_n \in H_n$ and the mapping $u(x) \rightarrow \{\sqrt{\pi\alpha} u_n(r)\}$ defines an isometry from Π onto $\bigoplus_{n=1}^{\infty} H_n$.

This mapping takes A_0 into the direct sum of the operators:

$$A_{0n} = -\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right) + \frac{n^2}{\alpha^2 r^2}$$

acting in H_n . It is also known that

$$(A.2) \quad f(r) = \int_0^{\infty} (A_{0n} - \kappa^2) f(r') G_{0n}^+(r', r; \kappa) r' dr', \quad n = 1, 2, \dots$$

for each $f \in C_0^{\infty}(0, \infty)$, where

$$G_{0n}^+(r', r; \kappa) = \begin{cases} \frac{i\pi}{2} J_{n/\alpha}(\kappa r) H_{n/\alpha}^{(1)}(\kappa r') & \text{for } r < r' \\ \text{and} \\ \frac{i\pi}{2} H_{n/\alpha}^{(1)}(\kappa r) J_{n/\alpha}(\kappa r') & \text{for } r' < r. \end{cases}$$

The above facts together imply the theorem, as we shall now show.

Proof of Theorem A.1: We first observe that the right side of (A.1) is well-defined and $G_0^+(\cdot, y; \kappa) \in L_2^{\text{loc}}(S)$. This follows since each term in the series in (A.1) may be estimated for large n in the same way that we estimated $w_n^0(x; \kappa)$ in the derivation of (2.12). Since we clearly have $G_0^+(x, y; \kappa) = 0$ for $x \in \dot{S}$, $y \in S$, it suffices to prove that

$$(A.3) \quad \int_S (A_0 - \kappa^2) u(x) G_0^+(x, y; \kappa) dx = u(y), \quad y \in S,$$

for each $u \in C_0^{\infty}(S)$.

First, suppose that $u(x) = u(|x|, 0_x) = u_n(|x|) \sin \frac{n}{\alpha} 0_x$, where n is fixed and $u_n \in C_0^\infty(0, \infty)$. Using (A.1) and (A.2), we have

$$\begin{aligned} \int_S (A_0 - \kappa^2) u(x) G_0^+(x, y; \kappa) dx &= \\ &= \frac{1}{\pi \alpha} \int_0^\infty \int_{|x|} (A_{0_n} - \kappa^2) u_n(|x|) G_{0_n}^+(|x|, |y|; \kappa) \sin^2 \frac{n}{\alpha} 0_x \sin \frac{n}{\alpha} 0_y |x| dx d0_x \\ &= \frac{\pi \alpha}{\pi \alpha} \sin \frac{n}{\alpha} 0_y \int_0^\infty (A_{0_n} - \kappa^2) u_n(|x|) G_{0_n}^+(|x|, |y|; \kappa) |x| dx \\ &= u_n(|y|) \sin \frac{n}{\alpha} 0_y = u(y). \end{aligned}$$

Hence (A.3) holds for all finite linear combination of these functions.

Next suppose that $u(x) = \sum_{n=1}^\infty u_n(r) \sin \frac{n}{\alpha} 0 \in C_0^\infty(S)$. Since

$$\tilde{u}(x) \equiv_{\text{df.}} (-\Delta - \kappa^2) u(x) = \sum_{n=1}^\infty \tilde{u}_n(r) \sin \frac{n}{\alpha} 0 \in C_0^\infty(S), \text{ we see from integration}$$

by parts that

$$(A.4) \quad \begin{cases} \max_{r \in (0, \infty)} |u_n(r)| \leq \frac{\text{Const}}{n^k} & \text{and} \\ \max_{r \in (0, \infty)} |\tilde{u}_n(r)| \leq \frac{\text{Const}}{n^k} & \text{for each } k = 1, 2, \dots, \end{cases}$$

and furthermore

$$(A.5) \quad \tilde{u}_n(r) = (A_{0_n} - \kappa^2) u_n(r).$$

By the previous result, (A.3) holds for each function $u_M(x) \equiv_{\text{df.}} \sum_{n=1}^M u_n(r) \sin \frac{n}{\alpha} 0$.

Since $u(x) \in C_0^\infty(S)$ and $G_0^+(\cdot, y; \kappa) \in L_2^{\text{loc}}(S)$, we easily deduce (A.3) from

(A.4) and (A.5).

Q.E.D.

Remark A.1: We may also prove Theorem A.1 by a method analogous to that of [15].

Remark A.2: We may construct the incoming Green's function, $G_0^-(x, y; \kappa)$, analogously to $G_0^+(x, y; \kappa)$ with $H_{n/\alpha}^{(1)}$ replaced by $H_{n/\alpha}^{(2)}$.

Remark A.3: In order to obtain single-valued continuations in Section 2, we were forced to assume that $0 \notin \mathcal{L}$. This restriction is a consequence of the form of our radiation condition, or equivalently, of our Green's function, $G_0^+(x, y; \kappa)$. To see this, we simply recall the fact that $J_{n/\alpha}(z)$ and $H_{n/\alpha}^{(1)}(z)$ are not definable as single-valued functions at $z = 0$ (see [7], pg. 75).

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AR 70-31 Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing notation must be entered when the overall report is classified.)

1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION	
Mathematics Research Center University of Wisconsin, Madison, Wis. 53706		Unclassified	
		2b. GROUP	
		None	
3. REPORT TITLE			
SCATTERING THEORY FOR ELLIPTIC DIFFERENTIAL OPERATORS IN UNBOUNDED DOMAINS			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)			
Summary Report: no specific reporting period.			
5. AUTHOR(S) (First name, middle initial, last name)			
Charles I. Goldstein			
6. REPORT DATE		7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
		59	15
8a. CONTRACT OR GRANT NO.		9a. ORIGINATOR'S REPORT NUMBER(S)	
Contract No. DA-31-124-ARO-D-462			
8b. PROJECT NO.			
None			
c.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.		None	
10. DISTRIBUTION STATEMENT			
Distribution of this document is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY	
None		Army Research Office-Durham, N.C.	
13. ABSTRACT			
<p>A method is developed for meromorphically continuing the S-matrix associated with elliptic partial differential operators acting in domains with infinite boundaries. The poles are related to "resonant states".</p>			

DD FORM 1473
1 NOV 66Unclassified

Security Classification

A00 448A

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