MASTER

Argonne National Laboratory

RSSR ROUTINE,
A ROOT-SQUARING AND
SUBRESULTANT PROCEDURE FOR
FINDING ZEROS OF REAL POLYNOMIALS

by

Erwin H. Bareiss and Ronald Hamelink

RELEASED FOR ANNOUNCEMENT

IN NUCLEAR SCIENCE ABSTRACTS

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

ANL-6987

Mathematics and Computers

(TID-4500)

AEC Research and

Development Report

ARGONNE NATIONAL LABORATORY

9700 South Cass Avenue

Argonne, Illinois 60440

CFSTI PRICES

H.C. \$ 3.00; MN. 25

RSSR ROUTINE, A ROOT-SQUARING AND SUBRESULTANT PROCEDURE FOR FINDING ZEROS OF REAL POLYNOMIALS

by

Erwin H. Bareiss

Applied Mathematics Division

Argonne National Laboratory

and

Ronald Hamelink

Michigan State University

East Lansing, Michigan

RELEASED FOR ANNOUNCEMENT
IN NUCLEAR SCIENCE ABSTRACTS

October 1965

Operated by The University of Chicago

under

Contract W-31-109-eng-38

with the

U. S. Atomic Energy Commission

THIS PAGE
WAS THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	5
I. INTRODUCTION	5
A. Motivation	5
B. The Basic Idea of the Resultant Procedures	5
C. Outline	7
D. Notation	7
II. DESCRIPTION OF THE METHOD	9
III. DESCRIPTION OF THE RSSR CODE	11
IV. COMPARISON WITH OTHER METHODS	17
APPENDIXES	
A: Numerical Examples	19
1. Some General Remarks	19
2. Polynomials Solved	19
B. FORTRAN Statements	28
1. FORTRAN II Statements of the RSSR Routine as Coded for the IBM-704 Computer	28
2. FORTRAN 63 Statements of the RSSR Routine as Coded for the CDC-3600 Computer	49
C: Operating Instructions	71
1. Operating Instructions for IBM-704	71
2. Operating Instructions for CDC-3600	73
ACKNOWLEDGMENTS	74
REFERENCES	74

LIST OF FIGURES

<u>No.</u>	<u>Title</u>	<u>Page</u>
1.	RSSR Subroutine.	11
2.	Finding of Moduli of Roots and Real Roots	12
3.	Check Routine	12
4.	Flow Chart for Finding the Reduced Subresultant Polynomial . .	13

RSSR ROUTINE, A ROOT-SQUARING AND SUBRESULTANT PROCEDURE FOR FINDING ZEROS OF REAL POLYNOMIALS

by

Erwin H. Bareiss and Ronald Hamelink

ABSTRACT

This report describes the theoretical background of the RSSR Routine, a program for finding simultaneously all roots of a real polynomial equation. Codes for the IBM-704 and CDC-3600 are described, including complete listings of FORTRAN statements. These codes are designed to solve polynomials of degree up to order 20 to 30, but dimensioning allows the possible solution of polynomial equations of degree up to 50. A set of sample problems is presented which may also serve to test other root-finding routines.

I. INTRODUCTION

A. Motivation

In 1957, one of the authors conceived the concept of the "Resultant Procedures."^{1,2} The corresponding code for the IBM-704 proved very successful and has not undergone any changes in the past 7 years. With the acquisition by the Applied Mathematics Division of a new computer, the CDC-3600, it was felt the code should be rewritten and possibly improved. Based on the performance of the code, it was felt that a less conservative approach could be tolerated, and therefore an even faster code designed. This was indeed accomplished. The idea was to make the code very fast for frequently encountered distributions of the zeros of polynomials as they occurred in the past, and not to worry too much over other types, such as multiple complex zeros, as long as the routine was able to obtain them all.

B. The Basic Idea of the Resultant Procedures

Assume that a real polynomial

$$P_n(x) = \sum_{i=0}^n a_i x^{n-i}$$

and a quadratic factor

$$Q(p, q) = x^2 + px + q$$

are given. Then

$$P_n = QP_{n-2} + R_1x + R,$$

where

$$P_{n-2} = \sum_{i=0}^{n-2} b_i x^{n-2-i}$$

is a polynomial $P_{n-2}(x)$ of order $n - 2$. The remainder terms R and R_1 are functions of p and q and can be shown to be related to the resultant and the subresultant [e.g., $R_1 = R'(p, q)$] of Reference 1. In order that Q be a factor of $P_n(x)$, both R and R_1 must vanish. This can be achieved in different ways. In applying the gradient method or Newton-Raphson method simultaneously to

$$R(p, q) = 0$$

and

$$R_1(p, q) = 0,$$

one obtains the Hitchcock-Bairstow iteration (Reference 5, page 472). Other procedures are due to Lin, Friedman, and others (Reference 1, page 381; Reference 4, page 468). In all methods, the purpose of finding a quadratic factor, say $Q(p, q)$, is to determine pairs of complex conjugate roots of a real polynomial, thus avoiding arithmetic with complex numbers. The idea of the resultant procedures as discussed in detail in Reference 1 is to first find the real q by a different method, and then to solve either

$$R(p, q) = 0,$$

or

$$\text{Resultant} = 0 \text{ (Resultant Procedure)}$$

or

$$R_1(p, q) = 0 \text{ (Subresultant Procedure)}$$

with q known. Referring again to Reference 1 or 2, it can be shown that $R(p, q)$ with known real q is a real polynomial of order n in p , and $R_1(p, q)$ with known real q is a real polynomial of order $n - 1$. If $Q(p, q)$ is a

quadratic factor of $P_n(x)$ such that its zeros are α and $\bar{\alpha}$, then $q = \alpha\bar{\alpha}$. If α has modulus ρ , then

$$q = \rho^2.$$

As explained in Reference 1, the modules to any zeros of a real polynomial can be found efficiently and accurately by a modification of the well-known Graeffe root-squaring method. It is further shown that the set of real roots p_i of $R(p, \rho^2) = 0$ and $R_1(p, \rho^2) = 0$ contain those p that make $Q(p, \rho^2)$ a factor of $P_n(x)$. It is sufficient either to solve $R(p, \rho^2) = 0$; or to find p such that the resultant vanishes, which leads to the resultant procedure as worked out in detail in Reference 1; or to solve $R_1(p, \rho^2) = 0$, which leads to a subresultant procedure as briefly discussed in Reference 1. Since in either case we have reduced the problem to finding only moduli of roots, we can use the modified root-squaring method mentioned above again as a subroutine. From the quadratic equation $Q(p_i, \rho^2) = 0$, the corresponding complex zeros α and $\bar{\alpha}$ are easily found.

By combining the $R(p, q)$ and $R_1(p, q)$, it is possible by elementary elimination to reduce the order in p of $R(p, q)$ from n to $n - 2$ (see Reference 6), and the order in p of the subresultant $R_1(p, q)$ from $n - 1$ to $n - 3$. The code described in this paper uses the reduced subresultant.

C. Outline

In the next sections, we discuss the general plan of the RSSR code, which has been in successful operation since summer 1963, and motivate some of the refinements. Then we demonstrate by a simple example that the Graeffe method as usually known exhibits a numerical stability behavior less favorable than the resultant procedures.

In Appendix A, a selection of problems is discussed that may also be used to test the efficiency or reliability of other root-finding routines.

In Appendix B, we present FORTRAN listings of the present RSSR code.

In Appendix C, we present the operating instructions for the IBM-704 and CDC-3600.

D. Notation

The following symbols will be used in what follows:

M = number of root-squaring iterations;

$P(x) = \sum_{i=0}^n a_i x^{n-i}$, polynomial with real coefficients a_i and zeros $(\alpha_j)_t$;

$P^{(m)}(x) = \sum_{i=0}^n a_i^{(m)} x^{n-i}$, polynomial with roots $(\alpha_j^{(2m)})_t$;

$$-p = \alpha + \bar{\alpha} = 2\operatorname{Re}(\alpha);$$

$$q = \alpha \bar{\alpha} = \rho^2;$$

$$Q = x^2 + px + q;$$

$$T(x) = \sum_{i=0}^n |a_i| x^{n-i};$$

α_k is an approximate root of $P(x) = 0$;

δ = the pivotal bound;

ϵ = the tolerated relative error of α_k with respect to $(\rho_k)_t$;

η = the separation coefficient;

ν = the multiplicity of roots of modulus $(\rho)_t$ of $P(x)$;

$$\rho = |\alpha|;$$

t as a subscript indicates the true value of a number instead of its approximation;

n as a subscript indicates the degree of a polynomial.

II. DESCRIPTION OF THE METHOD

The solution is obtained in four major steps.

Step 1

We first determine the moduli ρ_k and their multiplicity ν_k of the zeros of the given polynomial $P_n(x)$ of degree n in the same way as in the Resultant Procedure described in Reference 1.

Step 2

Starting with the smallest ρ_k , we test whether $\alpha_k = \pm\rho_k$ is a real root of $P_n(x)$. If it is, we divide the factor $x - \alpha_k$ into $P_n(x)$ and thereby reduce the degree of $P_n(x)$ by one:

$$P_n(x) = (x - \alpha_k) P_{n-1}(x).$$

If the multiplicity of ρ_k is larger than 1, we test whether $\alpha_k = \pm\rho_k$ is a real root of $P_{n-1}(x)$, etc. This is continued until the multiplicity of ρ_k is satisfied, or until $\pm\rho_k$ fails the test and hence is the modulus of a complex root. Then we take the next smallest ρ_k and so on, and repeat this process for each ρ_k until no real zeros are left. Therefore we are left with a polynomial $P_\ell(x)$ ($\ell \leq n$) with only complex zeros.

Remark: Dividing the linear factors out is a departure from the conservative approach used in Reference 1 where each linear factor was tested on the given polynomial $P(x)$. The reason for doing this is to make the final code faster. To minimize the effects of the round-off errors (see References 8 and 9) in the factorization process, we started with the smallest ρ_k . Also, all arithmetic operations are done in double precision.

Step 3

To determine the quadratic factors (i.e., the pairs of complex zeros), we construct the reduced subresultant for $P_\ell(x)$ and $Q(x) = x^2 + px + \rho_k^2$, where ρ_k is the smallest modulus whose multiplicity ν_k after Step 2 indicates the existence of such a quadratic factor. This reduced subresultant is a polynomial of order $\ell - 3$ in p . Since p must be real, we obtain its absolute value by the root-squaring routine used in Step 1. We test whether $x^2 \pm |p|x + \rho_k^2$ is a factor of $P_\ell(x)$. If it is, we divide this factor into $P_\ell(x)$ and reduce the polynomial to one of degree $\ell - 2$. If the multiplicity of ρ_k is not satisfied, we repeat Step 3 with ρ_k and $P_{\ell-2}(x)$. Then we proceed to the next smallest ρ_k that indicates the existence of a quadratic factor, and so on.

Remark: The reasons for this flow of operations are the same as those given for Step 2. In addition, we do not check immediately for quadratic factors of higher multiplicity since, in our experience, they do not occur frequently enough in practical problems but are still obtained in the following repetitions of Step 3. Thus, the most unlikely polynomials will take more time for their complete factorization than the more common ones. Should we, for some reason, be left with a nonconstant remainder, say a polynomial of degree S , the code solves the remaining polynomial equation $P_S(x) = 0$ as a new problem and adds the roots to the set of already-determined zeros of $P_n(x)$.

If there should still be some roots missing, an adjustment to the standard setting is made automatically by the computer.

Step 4

After the given polynomial $P_n(x)$ is completely factorized, all factors are multiplied together in double precision to generate a polynomial of order n , whose coefficients are adjusted by the factor a_0 of $P_n(x)$. By comparing the input coefficients a_i and the generated coefficients of the polynomials, one can estimate whether the factorization represents a satisfactory approximation to the true solution of $P_n(x)$. References 3, 4, 8, 9 contain detailed information concerning this.

Remark: Most polynomials encountered at Argonne contain approximate coefficients a_j . The users seem to be pleased to have a check for the solution in the form described in Step 4.

III. DESCRIPTION OF THE RSSR CODE

In this section, we list the actual mathematical formulas as they are used in the RSSR code. The logical flow of operations is indicated by the flow charts (Figs. 1 to 4).

Fig. 1. RSSR Subroutine

Fig. 2. Finding of Moduli of Roots and Real Roots

Fig. 3
Check Routine

Fig. 4. Flow Chart for Finding the Reduced Subresultant Polynomial

Step 1

Preliminary Considerations

M , δ , and ϵ have to be chosen. Reference 1 contains theoretical discussions and tables on these quantities. The standard setting in RSSR is $M = 10$, $\delta = 10^{-4}$, and $\epsilon = 10^{-6}$. When n is large and the roots are close together, i.e., a_j/a_{j+1} is near one, an ϵ considerably smaller may be needed with $M = 20$ or more. Examples are included in Appendix A.

Determination of the Moduli ρ of the Roots α

(1) The squaring operations are

$$a_j^{(m+1)} = (-1)^j \left\{ \left(a_j^{(m)} \right)^2 + 2 \sum_{\ell=1}^{\min(n-j, j)} (-1)^\ell a_{j-\ell}^{(m)} a_{j+\ell}^{(m)} \right\}$$

where $a_j^{(0)} = a_j$; $j = 0, 1, \dots, n$; $m = 0, 1, \dots, M-1$. The terms $\left(a_j^{(m)} \right)^2$ are added last to reduce the rounding errors (see Reference 1).

(2) Next, the pivotal coefficients are found. A coefficient $a_j^{(m)}$ is pivotal if

$$\left| 1 - (-1)^j \frac{[a_j^{(m-1)}]^2}{a_j^{(m)}} \right| < \delta.$$

Notice that $a_0^{(m)}$ and $a_n^{(m)}$ are always pivotal and need not be tested.

(3) The multiplicity ν of the moduli ρ is determined from successive pivotal coefficients. The moduli ρ of the roots α are

$$\rho = \left[\frac{(-1)^\nu a_{k+\nu}^{(M)}}{a_k^{(M)}} \right]^{\frac{1}{2M\nu}}$$

Step 2

$\alpha = \pm\rho$ is tested as a root. The acceptance test is shown under Step 3b below.

Step 3.

a. Determination of the Complex Roots α of Modulus ρ

(1) The subresultant $R_1(p)$ is computed as follows: Assume $Q = x^2 + px + q$ to be a factor of $P_n(x)$. Then

$$P_n(x) = (x^2 + px + q) P_{n-2}(x),$$

and

$$P_{n-2}(x) = \sum_{i=0}^{n-2} b_i x^{n-2-i},$$

where

$$b_{-1} = 0,$$

$$b_0 = a_0,$$

and

$$b_k = a_k - pb_{k-1} - qb_{k-2}, \quad k = 1, 2, \dots, n-2.$$

(2) The remainder in the division of $P(x)$ by $x^2 + px + q$ is zero;
i.e.,

$$R = a_n - b_{n-2}q = 0,$$

and

$$R_1 = a_{n-1} - b_{n-3}q - b_{n-2}p = 0 \text{ (subresultant).}$$

Hence,

$$b_{n-2} = a_n/q.$$

Substituting this in the expression for R_1 yields

$$R_1 = a_{n-1} - b_{n-3}q - \frac{a_n}{q} p = 0.$$

Since we know $q = \rho^2$ in the quadratic factor, the last equation is a polynomial equation in p ,

$$R_1(p) = 0.$$

We call this polynomial, $R_1(p)$, the reduced subresultant. Since b_k is a polynomial of degree k in p , $R_1(p)$ is of degree $n - 3$ for $n \geq 4$; otherwise it is linear in p because of the term $(a_n/q)p$. When $n = 4$, the reduced subresultant is

$$a_3 - b_1q - (a_4/q)p = 0.$$

Since p enters linearly in b_1 as well as in $(a_4/q)p$, the coefficient of p can vanish. To avoid this, we use

$$a_4 - b_2q = 0$$

instead of the reduced subresultant whenever the degree of $P(x)$ is $n = 4$.

(3) We solve $R(p) = 0$ for real roots p by the method of Step 1. The roots p such that $|p| < 2\rho$ then lead to complex roots of $P_n(x)$ as follows:

If $Q = x^2 \pm |p|x + \rho^2$ is a quadratic factor of $P_n(x)$, then

$$\alpha = \mp \frac{1}{2}(|p| \pm i\sqrt{4\rho^2 - p^2}).$$

The acceptance test is shown in paragraph b below.

Remark: In the actual code, the coefficients of $P_n(x)$ are modified so that

$$P_n \Rightarrow \sum_{i=0}^n \tilde{a}_i x^{n-i},$$

where

$$\tilde{a}_i = a_i \rho^{n-i} (i = 0, 1, \dots, n).$$

Then the corresponding root $\tilde{\alpha}$ has a modulus of unit length, $|\tilde{\alpha}| = \tilde{\rho} = 1$, and

$$Q = x^2 + \tilde{p}x + 1.$$

Thus the calculation of the coefficients for $R_1(p)$ can be reduced to shift operations and additions only, since $q = 1$.

b. Testing of Roots

(1) We compute for $k = 1, 2, \dots, n, n+1$,

$$b_k = a_k - w b_{k-1} - q b_{k-2} (b_{-1} = 0, b_0 = a_0, a_{n+1} = 0).$$

To test if $\begin{Bmatrix} x - \alpha \\ x^2 + px + \rho^2 \end{Bmatrix}$ is a factor of $P(x)$, set

$$\begin{cases} w = -\alpha, q = 0 \\ w = p, q = \rho^2 \end{cases}.$$

(2) The approximation is considered sufficient when

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) > \begin{cases} |b_n| & \text{for } x - \alpha \\ \sqrt{b_n^2 - b_{n-1}b_{n+1}} & \text{for } x^2 + px + \rho^2 \end{cases}.$$

Step 4.

The regeneration of the coefficients of $P_n(x)$ is done by nesting as follows:

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the computed roots obtained for $P_n(x) = 0$. They are arranged so that complex roots and their conjugates are in adjacent locations. Then, approximately,

$$P_n(x) = \left\{ \dots [(x^2 - 2x \operatorname{Re} \alpha_n + |\alpha_n|^2)(x^2 - 2x \operatorname{Re} \alpha_{n-2} + |\alpha_{n-2}|^2)] \dots (x - \alpha_2)(x - \alpha_1) \right\} a_0.$$

IV. COMPARISON WITH OTHER METHODS

The remarks in the introduction indicate that many of the well-known iterative methods, such as the Newton-Raphson, Bairstow-Hitchcock, Lin, and others, are also resultant procedures.

The special feature of the procedures discussed in Reference 1 and in this paper is that we intersect the surface of the algebraic functions

$$R(p, q) = 0$$

and

$$R_1(p, q) = 0$$

by a plane $q = \rho^2$ and thus are able to reduce the entire problem to finding real roots of real polynomial equations. One can see from geometrical consideration that such an approach yields a better overall numerical stability.

Unfortunately the title of Reference 1 led some readers to believe that the Resultant Procedure is a mechanization of the Graeffe process. We should like to demonstrate on a simple example that this is not so.

Take the accurate polynomial equation

$$P^{(m)}(x) = x^2 - 2ax + a^2 = 0.$$

Assume now that in the process of root squaring, a small error was introduced in the constant term, such that

$$P^{(m)}(x) = x^2 - 2ax + a^2(1 - \epsilon).$$

Then, with the methods described in the literature, we have, by the Graeffe method,

$$\alpha_{1,2}^{(2^m)} = a(1 \pm \sqrt{\epsilon}),$$

while by the Resultant Procedure and RSSR Routine,

$$\alpha_{1,2}^{(2^m)} = a\sqrt{1 - \epsilon} \approx a\left(1 - \frac{\epsilon}{2}\right).$$

Assume now, that in the process of root squaring, a small error is introduced in the middle term such that

$$P^{(m)}(x) = x^2 - 2(1 + \epsilon)ax + a^2.$$

Then, with the methods described in the literature we have, by the Graeffe method,

$$\alpha_{1,2}^{(2^m)} = a(1 + \epsilon \pm \sqrt{2\epsilon + \epsilon^2}),$$

while by the Resultant Procedure and RSSR Routine,

$$\alpha_{1,2}^{(2^m)} = a.$$

This exhibits quite clearly that the behavior with respect to numerical stability is quite different than in the Graeffe method and in favor of the resultant procedures of Reference 1.

In addition, we have already given in Reference 1 examples that cannot be solved by the "Graeffe" method and by many of the well-known "standard" iterative methods, but can be solved by the resultant procedures. Such problems are generally polynomial equations which contain cyclotomic factors and/or multiple real and/or complex roots.

We believe that the resultant procedures belong to the better "all-purpose methods" for practical use in research centers when all zeros of a polynomial are desired. We plan to improve our codes as time goes on, taking advantage of other research results such as presented in References 3, 4, 7, etc., should our experience show such changes to be desirable for practical applications.

APPENDIX A

Numerical Examples

1. Some General Remarks

We present here a number of test polynomials and some short remarks relating to their particular point of interest. While we endeavored to design a routine that accepts any polynomial of practical interest, it will always be possible to find polynomials that cannot be solved with any given code.

It is always desirable that all coefficients be of the same order of magnitude. This can often be achieved by a transformation of the roots such that

$$x \rightarrow 10^r x'$$

or

$$x \rightarrow 2^s x'.$$

If one suspects clustering of roots in the neighborhood of $z_0 = x_0 + iy_0$, the transformation

$$x \rightarrow x_0 - x$$

will increase the separation coefficient and thus reduce M .

Sometimes the transformation

$$x \rightarrow \frac{1}{x}$$

leads to somewhat better numerical behavior.

Such transformations must be considered as an exception, and for the solution of the following problems these transformations were not used.

2. Polynomials Solved

Example 1. The difficulty in the following polynomial is the high multiplicity of the modulus 2:

$$\begin{aligned} x^{12} - x^{11} + 6x^{10} - 4x^9 - 29x^8 + 45x^7 - 82x^6 + 64x^5 - 384x^4 + 256x^3 \\ + 1856x^2 - 2880x + 1152 = (x^6 - 2^6)(x+2)(x^2+3^2)(x-1)^3 = 0. \end{aligned}$$

The roots found with $M = 9$, $\delta = 10^{-5}$, and $\epsilon = 10^{-8}$ are as follows:

<u>Real Part</u>					<u>Imaginary Part</u>				
1.1993	9865	3312	0892	801×10^{-17}	-3.0000	0000	0000	0000	031
1.1993	9865	3312	0892	801×10^{-17}	3.0000	0000	0000	0000	031
9.9999	9999	9999	9999	399×10^{-1}	-1.7320	5080	7568	8773	020
9.9999	9999	9999	9999	399×10^{-1}	1.7320	5080	7568	8773	020
-1.0000	0000	0000	0000	082	-1.7320	5080	7568	8772	887
-1.0000	0000	0000	0000	082	1.7320	5080	7568	8772	887
-1.9999	9999	9999	9999	999	0.0				
-1.9999	9999	9999	9999	999	0.0				
2.0000	0000	0000	0000	044	0.0				
1.0000	0000	0000	0000	000	0.0				
1.0000	0000	0000	0000	000	0.0				
1.0000	0000	0000	0000	000	0.0				

When the polynomial was reconstructed from these roots, the new coefficients agreed with the original ones to at least 16 digits.

Example 2. The difficult part of this polynomial is that the separation coefficient for the two largest roots is 0.001:

$$x^4 - 2020x^3 + 1,039,109x^2 - 19,199,090x + 90,090,000 = 0.$$

The roots found with $M = 16$, $\delta = 10^{-6}$, and $\epsilon = 10^{-8}$, truncated, but not rounded, to 20 significant figures, are:

<u>Real Part</u>					<u>Imaginary Part</u>
1.0009	9999	9999	9999	999×10^3	0.0
1.0000	0000	0000	0000	000×10^3	0.0
9.9999	9999	9999	9999	999	0.0
8.9999	9999	9999	9999	999	0.0

When the coefficients were reconstructed from these roots, the new coefficients agreed with the original ones to at least 19 significant digits. The true roots are 9, 10, 1000, and 1001.

Example 3. This is the transformed Chebyshev polynomial $\frac{1}{2}T_{20}(x/2)$. The difficulty here is that the smallest separation coefficient, η , is $1 - \alpha_{k+1}/\alpha_k = 0.0246$:

$$x^{20} - 20x^{18} + 170x^{16} - 800x^{14} + 2275x^{12} - 4004x^{10} + 4290x^8 - 2640x^6 + 825x^4 - 100x^2 + 2 = 0.$$

The roots found (all real) with $M = 10$, $\delta = 10^{-6}$, and $\epsilon = 10^{-8}$ are as follows:

-1.9938	3466	7466	2556	788
+1.9938	3466	7466	2556	788
-1.9447	3984	0795	3535	493
+1.9447	3984	0795	3535	493
-1.8477	5906	5022	5734	865
+1.8477	5906	5022	5734	865
-1.7052	8032	8708	1843	987
+1.7052	8032	8708	1843	987
-1.5208	1193	1200	0618	698
+1.5208	1193	1200	0618	698
-1.2988	9609	6660	3673	133
+1.2988	9609	6660	3673	133
-1.0449	9712	9431	8977	302
+1.0449	9712	9431	8977	302
-7.6536	6864	7301	7954	346×10^{-1}
+7.6536	6864	7301	7954	346×10^{-1}
-4.6689	0727	7118	1082	353×10^{-1}
+4.6689	0727	7118	1082	353×10^{-1}
-1.5691	8191	4556	8989	006×10^{-1}
+1.5691	8191	4556	8989	006×10^{-1}

The coefficients found from the roots agree to at least 17 significant digits with the original ones.

Example 4. No particular difficulty is presented by this polynomial. It simply demonstrates that a quadratic can be handled by the resultant procedure without any trouble.

$$x^2 - 2x + 2 = 0.$$

Roots found, $M = 4$, $\delta = 10^{-4}$, $\epsilon = 10^{-6}$.

$$0.999\ 999 \pm 0.9999\ 999i$$

Example 5

$$x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 \equiv x^2 \frac{x^8 - 1}{x - 1} = 0.$$

This polynomial demonstrates several aspects of the RSSR Routine.

- a) It contains a cyclotomic factor whose zeros are the primitive 8th roots. Such a polynomial cannot be solved with the usual Graeffe procedure. It represents no difficulty with the resultant procedures (see Reference 1).
- b) It contains a high multiplicity in the modulus of the roots with absolute value 1.
- c) It contains pure imaginary zeros.
- d) It contains a double zero as a factor.

The accurate values for the zeros are

$$(1 \pm i) \sqrt{1/2}, \pm i, (-1 \pm i) \sqrt{1/2}, -1, 0, 0.$$

The results found by the CDC-3600 with $M = 5$, $\delta = 0.005$, and $\epsilon = 10^{-6}$ are as follows:

<u>Real Part</u>					<u>Imaginary Part</u>				
7.07106	78118	65475	24400	84453×10^{-1}	-7.07106	78118	65475	24400	84416×10^{-1}
7.07106	78118	65475	24400	84453×10^{-1}	7.07106	78118	65475	24400	84416×10^{-1}
-7.31131	27550	28601	08107	97448×10^{-26}	-1.00000	00000	00000	00000	00000
-7.31131	27550	28601	08107	97448×10^{-26}	1.00000	00000	00000	00000	00000
-7.07106	78118	65475	24400	84437×10^{-1}	-7.07106	78118	65475	24400	84428×10^{-1}
-7.07106	78118	65475	24400	84437×10^{-1}	7.07106	78118	65475	24400	84428×10^{-1}
-1.00000	00000	00000	00000	00000	0.00000	00000	00000	00000	00000
0.00000	00000	00000	00000	00000	0.00000	00000	00000	00000	00000
0.00000	00000	00000	00000	00000	0.00000	00000	00000	00000	00000

These roots are accurate to at least 23 decimal places (the limitations of the log x, e^x subroutines). Total calculation time on CDC-3600 is less than 2 sec.

The check routine yields the following results:

<u>Input Coefficients</u>		<u>Reconstructed Polynomial</u>						
$a_0 = 1.00000$	00000	$a_0 = 1.00000$	00000	00000	00000	00000	00000	0
$a_1 = 1.00000$	00000	$a_1 = 1.00000$	00000	00000	00000	00000	00000	0
$a_2 = 1.00000$	00000	$a_2 = 0.99999$	99999	99999	99999	99999	99988	4
$a_3 = 1.00000$	00000	$a_3 = 0.99999$	99999	99999	99999	99999	99983	4
$a_4 = 1.00000$	00000	$a_4 = 0.99999$	99999	99999	99999	99999	99983	4
$a_5 = 1.00000$	00000	$a_5 = 0.99999$	99999	99999	99999	99999	99986	7
$a_6 = 1.00000$	00000	$a_6 = 0.99999$	99999	99999	99999	99999	99996	6
$a_7 = 1.00000$	00000	$a_7 = 1.00000$	00000	00000	00000	00000	00000	0
$a_8 = 0.00000$	00000	$a_8 = -0.00000$	00000	00000	00000	00000	00000	0
$a_9 = 0.00000$	00000	$a_9 = 0.00000$	00000	00000	00000	00000	00000	0

The following three polynomial equations (Examples 6, 7, and 8) are given in Wilkinson^{8,9} and known to be extremely difficult to solve.

Example 6. This is a polynomial which Oliver and Wilkinson⁸ consider too difficult for the Graeffe process. It can be solved by the RSSR Routine.

$$\begin{aligned}
 &2.0325 \ 3121x^{16} + 3.4356 \ 048x^{15} + 25.1783 \ 048x^{14} + 37.6510 \ 96x^{13} \\
 &128.2187 \ 48x^{12} + 166.4476 \ 8x^{11} + 345.07256x^{10} + 378.908x^9 \\
 &+ 524.327x^8 + 468.88x^7 + 443.576x^6 + 304.08x^5 + 190.68x^4 + 89.6x^3 \\
 &+ 32.8x^2 + 8x + 1 = 0.
 \end{aligned}$$

The solution was found with $M = 30$, $\delta = 10^{-6}$, and $\epsilon = 10^{-7}$. We give a complete reproduction of the CDC-3600 output. The roots found are as follows:

REAL PART	IMAGINARY PART
-2.4892024459300534091983319-003	-1.6671203612177835792153141+000
-2.4892024459300534091983319-003	1.6671203612177835792153141+000
-1.0493550094786246139124231-002	-1.5962954955540975249059991+000
-1.0493550094786246139124231-002	1.5962954955540975249059991+000
-2.5668710501835624507950315-002	-1.4743771438266904040684607+000
-2.5668710501835624507950315-002	1.4743771438266904040684607+000
-5.0864435604355134377307702-002	-1.2969112790733189623891615+000
-5.0864435604355134377307702-002	1.2969112790733189623891615+000
-9.0039988741533156123763808-002	-1.0611920598431853135487569+000
-9.0039988741533156123763808-002	1.0611920598431853135487569+000
-1.4762378022667001866562007-001	-7.7175720106385244613338246-001
-1.4762378022667001866562007-001	7.7175720106385244613338246-001
-2.2447005787902159556593484-001	-4.5092795830149006994849992-001
-2.2447005787902159556593484-001	4.5092795830149006994849992-001
-2.9350452923438473849175751-001	-1.4349929692750051258853104-001
-2.9350452923438473849175751-001	1.4349929692750051258853104-001

INPUT COEFFICIENTS

A(0)= 2.03253120999999999999999999999999+000
 A(1)= 3.43560479999999999999999999999998+000
 A(2)= 2.51783048000000000000000000000000+001
 A(3)= 3.76510960000000000000000000000002+001
 A(4)= 1.28218748000000000000000000000001+002
 A(5)= 1.66447679999999999999999999999999+002
 A(6)= 3.45072559999999999999999999999999+002
 A(7)= 3.78907999999999999999999999999998+002
 A(8)= 5.24327000000000000000000000000000+002
 A(9)= 4.68879999999999999999999999999999+002
 A(10)= 4.43576000000000000000000000000005+002
 A(11)= 3.04080000000000000000000000000004+002
 A(12)= 1.90679999999999999999999999999999+002
 A(13)= 8.960000000000000000000000000000016+001
 A(14)= 3.27999999999999999999999999999999+001
 A(15)= 8.00000000000000000000000000000000+000
 A(16)= 1.00000000000000000000000000000000+000

RECONSTRUCTED POLYNOMIAL

A(0)= 2.03253120999999999999999999999999+000
 A(1)= 3.43560479999999999999999999999994+000
 A(2)= 2.51783047999999999999999999999999+001
 A(3)= 3.765109600000000000000000000000037341495+001
 A(4)= 1.28218748000000000000000000000006004599+002
 A(5)= 1.664476800000000000000000000000026143183+002
 A(6)= 3.450725600000000000000000000000034586439+002
 A(7)= 3.789080000000000000000000000000067958806+002
 A(8)= 5.243270000000000000000000000000070546032+002
 A(9)= 4.688800000000000000000000000000080750189+002
 A(10)= 4.435760000000000000000000000000061799823+002
 A(11)= 3.040800000000000000000000000000043061875+002
 A(12)= 1.906800000000000000000000000000021698943+002
 A(13)= 8.960000000000000000000000000000083556061+001
 A(14)= 3.280000000000000000000000000000020944971+001
 A(15)= 8.000000000000000000000000000000026291505+000
 A(16)= 1.0000000000000000000000000000000002+000

From the reconstructed coefficients, we can estimate an accuracy of 18 significant figures in the zeros.

Example 7. The difficulty in the following example is that all roots are complex with absolute values close together. Five pairs of roots are clustered around the imaginary axis.

$$\begin{aligned} &12\ 5016\ 2561x^{16} + 3\ 8545\ 5882x^{15} + 8\ 4594\ 7696x^{14} + 2\ 4077\ 5148x^{13} \\ &+ 2\ 4792\ 6664x^{12} + 6424\ 9356x^{11} + 4101\ 8752x^{10} + 949\ 0840x^9 \\ &+ 417\ 8260x^8 + 83\ 7860x^7 + 26\ 7232x^6 + 4\ 4184x^5 + 1\ 0416x^4 \\ &+ 1288x^3 + 224x^2 + 16x + 2 = 0. \end{aligned}$$

The roots found with $M = 24$, $\delta = 10^{-8}$, and $\epsilon = 10^{-11}$ on the CDC-3600 are as follows:

$$\begin{aligned} &-0.13244\ 72469\ 90246\ 20179 \pm 0.13600\ 55079\ 51377\ 63785\ i \\ &-0.01869\ 49953\ 44576\ 20320 \pm 0.25304\ 56818\ 77088\ 48212\ i \\ &-0.00232\ 1\ 00000\ 00000 \pm 0.29258\ 374\ 00000\ i \\ &-0.00049\ 1\ 00000\ 00000 \pm 0.30418\ 2\ 00000\ i \\ &-0.00000\ 00000\ 00000 \pm 0.30857\ 00000\ i \\ &-0.00006\ 00000\ 00000 \pm 0.3105\ 00000\ i \\ &-0.00003\ 00000\ 00000 \pm 0.3125\ 00000\ i \\ &-0.00007\ 00000\ 00000 \pm 0.3116\ 00000\ i \end{aligned}$$

Figures which are believed doubtful (by comparison with the results given in Reference 8) are replaced by dots. The reconstruction of the original polynomial from the calculated roots gave an agreement in the coefficients from 25 significant figures for a_1 down to 10 significant figures for a_{16} . We attribute the gradual loss of accuracy in the roots, to the deflation process and the use of the reduced subresultant. This is the price we may sometimes pay for high speed. If greater accuracy should be needed than to at least four decimal places, as given above, one makes the transformation $x = z - 0.3i$ and solves the complex polynomial $P_{16}(z - 0.3i) = 0$ (see Reference 1). (Of course, higher-precision arithmetic could be applied too.)

Example 8. This polynomial was constructed by Wilkinson⁸ to demonstrate the possible sensitivity of roots to small changes in the coefficients. It is presented here to demonstrate that the RSSR Routine can handle such problems.

$$P_{20}(x) = (x+1)(x+2)(x+3)\dots(x+19)(x+20) + 2^{-23}x^{19}.$$

The coefficients as computed by the CDC-3600 are listed below in the left column. The coefficients of the polynomial as reconstructed by the roots found with the RSSR Routine are listed in the right column for comparison. The agreement is to at least 18 significant decimal figures.

Input Coefficients					Reconstructed Polynomial				
a_0	= 1.00000	00000	00000	00000	a_0	= 1.00000	00000	00000	00000
a_1	= 2.10000	00011	92092	89550	a_1	= 2.10000	00011	92092	89550
a_2	= 2.06149	99999	99999	99999	a_2	= 2.06149	99999	99999	99999
a_3	= 1.25685	00000	00000	00000	a_3	= 1.25685	00000	00000	00001
a_4	= 5.33279	45999	99999	99999	a_4	= 5.33279	45999	99999	99992
a_5	= 1.67228	08199	99999	99999	a_5	= 1.67228	08199	99999	91167
a_6	= 4.01717	71630	00000	00000	a_6	= 4.01717	71629	99999	99998
a_7	= 7.56111	18449	99999	99999	a_7	= 7.56111	18449	99999	99993
a_8	= 1.13102	76995	38099	99999	a_8	= 1.13102	76995	38099	99997
a_9	= 1.35585	18289	95299	99999	a_9	= 1.35585	18289	95299	99994
a_{10}	= 1.30753	50105	40395	00000	a_{10}	= 1.30753	50105	40394	99990
a_{11}	= 1.01422	99865	51145	00000	a_{11}	= 1.01422	99865	51144	99988
a_{12}	= 6.30308	12099	29489	60000	a_{12}	= 6.30308	12099	29489	59897
a_{13}	= 3.11333	64316	13906	39999	a_{13}	= 3.11333	64316	13906	39929
a_{14}	= 1.20664	78037	80373	35999	a_{14}	= 1.20664	78037	80373	35964
a_{15}	= 3.59997	95179	47607	19999	a_{15}	= 3.59997	95179	47607	19864
a_{16}	= 8.03781	18226	45051	77599	a_{16}	= 8.03781	18226	45051	77229
a_{17}	= 1.28709	31245	15098	88000	a_{17}	= 1.28709	31245	15098	87929
a_{18}	= 1.38037	59753	64070	39999	a_{18}	= 1.38037	59753	64070	39912
a_{19}	= 8.75294	80367	61599	99999	a_{19}	= 8.75294	80367	61599	99380
a_{20}	= 2.43290	20081	76639	99999	a_{20}	= 2.43290	20081	76639	99811

The roots were found with $M = 12$, $\delta = 1.5 \times 10^{-4}$, and $\epsilon = 10^{-11}$ as follows:

Real Part					Imaginary Part				
-1.39923	58137	23567	28477	71031×10^1	-2.51883	00696	30280	51923	36702
-1.39923	58137	23567	28477	71031×10^1	2.51883	00696	30280	51923	36702
-1.17936	33881	07941	69417	55319×10^1	-1.65232	97281	60931	55985	89243
-1.17936	33881	07941	69417	55319×10^1	1.65232	97281	60931	55985	89243
-1.95024	39400	49368	25733	05975×10^1	-1.94033	03466	64480	50888	58385
-1.95024	39400	49368	25733	05975×10^1	1.94033	03466	64480	50888	58385
-1.67307	37466	09070	62233	93250×10^1	-2.81262	48942	70039	28684	73528
-1.67307	37466	09070	62233	93250×10^1	2.81262	48942	70039	28684	73528
-1.00952	66145	12995	74203	55494×10^1	-6.43500	90386	35443	15616	19311×10^{-1}
-1.00952	66145	12995	74203	55494×10^1	6.43500	90386	35443	15616	19311×10^{-1}
-2.08469	08101	48225	74613	16694×10^1					
-8.91725	02485	17118	77991	69465					
-8.00726	76034	50363	83908	58431					
-6.99969	72339	36015	54449	91110					
-6.00000	69439	57295	55757	68494					
-4.99999	99275	51537	92460	23621					
-4.00000	00002	61023	18783	43012					
-2.99999	99999	99805	23302	53595					
-2.00000	00000	00000	00976	16303					
-9.99999	99999	99999	99999	99693×10^{-1}					

The roots are believed accurate to at least 17 significant figures.

To conclude, we like to emphasize that some of these examples are presented merely to demonstrate the inherent power of the code. Most requests for the solution of polynomials specify accuracy less than six significant digits.

APPENDIX B
FORTRAN Statements

1. FORTRAN II Statements of the RSSR Routine as Coded for the IBM-704 Computer
 - a. Main Program
(Input and output procedures and call of subroutine RSSR)
 - b. Subroutine RSSR
(Master subroutine: Checks for zero roots. Calls root-squaring subroutine, RTSQ. Calls subroutine for finding moduli of roots and finding real roots, RERO. If complex roots exist, it calls subroutine COMPRT to find them. Calls subroutine RECON to reconstruct the polynomial.)
 - c. Subroutine RTSQ
(Root-squaring process)
 - d. Subroutine RERO
(Finds moduli and real roots, calls for test subroutine, TEST)
 - e. Subroutine TEST
(Tests roots for accuracy)
 - f. Subroutine COMPRT
(Finds complex roots and calls TEST)
 - g. Subroutine SUBRES
(Constructs the reduced subresultant polynomial)
 - h. Subroutine RECON
(Reconstructs the original polynomial from the computed zeros)
 - i. Subroutine QUADIV
(Divides out quadratic factors)
 - j. Subroutine DPOUTP
(Outputs double-precision numbers)
 - k. Subroutine SPCON
(Converts single-precision floating-point numbers to double precision)

- l. Subroutine DPEX
(Computes e^x in double precision for double-precision x)
- m. Subroutine DPLOG
(Computes double-precision natural logs for double-precision arguments)

a. Main Program

(Input and output procedures and call of subroutine RSSR)

```

5  DIMENSION A(3,51,3),ROMOD(3,50),ROOTS(3,2,50),T(3),U(3),B(3,51),
   1SKIP(12),D(3,51),CON104(3),SPACE(1401),CONTEN(3),
   2JA(5)
10  COMMON A,ROMOD,ROOTS,T,U,SKIP,D,CON104,SPACE,B,
   1CONTEN
15  CALL LLNPRC(CONTEN)
16  CONTEN=10.0
17  CALL SPCON(CONTEN)
18  CON104=10000.0
19  CALL SPCON(CON104)
20  READ INPUT TAPE 7,25
25  FORMAT (72H
   1
30  WRITE OUTPUT TAPE 6,25
35  READ INPUT TAPE 7,40,NDEGRE,M,MMAX,DELTA,EPSIL
36  IF (M) 37,37,41
37  EPSIL=0.000001
   DELTA=0.0001
38  M=10
39  MMAX=15
40  FORMAT (3I6,2E12.6)
41  WRITE OUTPUT TAPE 6,42,NDEGRE,M,DELTA,EPSIL
42  FORMAT (8HODEGREE=I2,17H  NO OF ROOT SQ=I2,9H  DELTA=E16.8,11H
   1 EPSILON=E16.8)
45  IF (NDEGRE) 50,50,55
50  CALL SYSTEM
55  N1=NDEGRE+1
56  READ INPUT TAPE 7,120,IPI,IPO,IFALCO
60  WRITE OUTPUT TAPE 6,65
65  FORMAT (19HINPUT COEFFICIENTS)
66  IF (IPI-1) 70,70,110
70  DO 100 I=1,N1
75  READ INPUT TAPE 7,80,A(1,I,1)
80  FORMAT (E12.6)
85  J=I-1
90  WRITE OUTPUT TAPE 6,95,J,A(1,I,1)
95  FORMAT (3HOA(I2,3H)= E16.8)
100 CALL SPCON(A(1,I,1))
105 GO TO 285
110 DO 260 I=1,N1
115 READ INPUT TAPE 7,120,(JA(II),II=1,5),J
120 FORMAT (6I6)
125 A(1,I,1)=JA(1)
130 CALL SPCON(A(1,I,1))
135 DO 160 II=2,5
140 T=JA(II)
145 CALL SPCON(T)
150 CALL NPREC
155 A(1,I,1)=A(1,I,1)*CON104+T
160 CALL NPOUT
165 IF (J) 170,260,185
170 J=-J
175 K=-1
180 GO TO 190
185 K=1
190 T=1.0
195 CALL SPCON(T)
200 DO 215 JJ=1,J
205 CALL NPREC

```

```

210 T=T*CONTEN
215 CALL NPOUT
220 IF (K) 225,225,245
225 CALL NPREC
230 A(1,I,1)=A(1,I,1)/T
235 CALL NPOUT
240 GO TO 260
245 CALL NPREC
250 A(1,I,1)=A(1,I,1)*T
255 CALL NPOUT
260 CONTINUE
265 DO 280 I=1,N1
269 J=I-1
270 WRITE OUTPUT TAPE 6,275,J
275 FORMAT (3H0A(I2,1H))
280 CALL DPOUTP(A(1,I,1))
285 IF (IFALCO) 477,477,290
290 DO 300 I=1,N1
295 DO 300 J=1,3
300 B(J,I)=A(J,I,1)
477 N2=NDEGRE
480 CALL RSSR(NDEGRE,M,MMAX,DELTA,EPSIL)
485 IF (IPO-1) 490,490,600
490 IF (N1-NDEGRE-1) 585,585,495
495 WRITE OUTPUT TAPE 6,500
500 FORMAT (24H0ROOTS OF THE POLYNOMIAL)
505 WRITE OUTPUT TAPE 6,510
510 FORMAT (33H0REAL PART          IMAGINARY PART)
515 K=NDEGRE+1
520 DO 535 I=K,N2
525 CALL NP2SP(ROOTS(1,1,I),ROOTS(1,1,I))
530 CALL NP2SP(ROOTS(1,2,I),ROOTS(1,2,I))
535 WRITE OUTPUT TAPE 6,540,ROOTS(1,1,I),ROOTS(1,2,I)
540 FORMAT (1H0E18.8,E18.8)
545 IF (NDEGRE) 550,550,585
550 WRITE OUTPUT TAPE 6,555
555 FORMAT (25H0RECONSTRUCTED POLYNOMIAL)
560 DO 575 I=1,N1
561 T(1)=D(1,I)
562 T(2)=D(2,I)
563 T(3)=D(3,I)
565 CALL NP2SP(T,T)
570 J=I-1
575 WRITE OUTPUT TAPE 6,95,J,T(1)
580 GO TO 645
585 WRITE OUTPUT TAPE 6,590
590 FORMAT (21H0SOME ROOTS NOT FOUND)
595 GO TO 20
600 IF (N1-NDEGRE-1) 585,585,605
605 WRITE OUTPUT TAPE 6,500
607 K=NDEGRE+1
610 DO 640 I=K,N2
615 WRITE OUTPUT TAPE 6,620
620 FORMAT (10H0REAL PART)
625 CALL DPOUTP(ROOTS(1,1,I))
630 WRITE OUTPUT TAPE 6,635
635 FORMAT (15H0IMAGINARY PART)
640 CALL DPOUTP(ROOTS(1,2,I))
641 IF (NDEGRE) 6420,6420,6410
6410 WRITE OUTPUT TAPE 6,590
6415 GO TO 20
6420 WRITE OUTPUT TAPE 6,555

```

```
6421 DO 6424 I=1,N1
6422 J=I-1
6423 WRITE OUTPUT TAPE 6,275,J
6424 CALL DPOUTP(D(1,I))
645 IF (IFALCO) 20,20,650
650 IFALCO=0
655 KK=0
660 DO 705 I=1,N1
665 CALL NPREC
670 T=B(1,I)-D(1,I)
680 A(1,I,1)=B(1,I)+T
685 CALL NPOUT
690 KK=1
705 CONTINUE
710 IF (KK) 20,20,715
715 WRITE OUTPUT TAPE 6,720
720 FORMAT (19HOFALSE COEFFICIENTS)
725 DO 735 I=1,N1
726 J=I-1
730 WRITE OUTPUT TAPE 6,275,J
735 CALL DPOUTP(A(1,I,1))
737 NDEGRE=N2
740 GO TO 480
```

b. Subroutine RSSR

(Master subroutine: Checks for zero roots. Calls root-squaring subroutine, RTSQ. Calls subroutine for finding moduli of roots and finding real roots, RERO. If complex roots exist, it calls subroutine COMPT to find them. Calls subroutine RECON to reconstruct the polynomial.)

```

      3 SUBROUTINE RSSR(NDEGRE,M,MMAX,DELTA,EPSIL)
      5 DIMENSION A(3,51,3),ROMOD(3,50),ROOTS(3,2,50),CON0(3),CON1(3),
      1 CON2(3),CON4(3),T(3),U(3),D(3,51),SPACE(1398),E(3),MROMOD(50),
      2 NONRT(50),MNONRT(50),C(51),CON104(3),JA(4)
      10 COMMON A,ROMOD,ROOTS,CON0,CON1,CON2,CON4,T,U,D,CON104,SPACE,E
S   AA OCT 000000000000
S   BB OCT 000000000001
S   CC OCT 200000000000
S   DD OCT 000000000002
S   EE OCT 000000000003
      16 CON0(1)=AA
      17 CON0(2)=AA
      18 CON0(3)=AA
      19 CON1(1)=BB
      20 CON1(2)=CC
      21 CON1(3)=AA
      22 CON2(1)=DD
      23 CON2(2)=CC
      24 CON2(3)=AA
      25 CON4(1)=EE
      26 CON4(2)=CC
      27 CON4(3)=AA
      28 N=NDEGRE
      29 IF (N) 37,37,45
      30 NDEGRE=NCUR
      31 RETURN
      32 N1=N+1
      33 N2=N1+1
      34 DO 62 I=1,N
      35 K=N2-I
      36 IF (A(2,K,1)) 65,58,65
      37 J=N1-I
      38 CALL NPREC
      39 ROOTS(1,1,J)=CON0
      40 ROOTS(1,2,J)=CON0
      41 CALL NPOUT
      42 NDEGRE =0
      43 GO TO 38
      44 N1=K
      45 N=K-1
      46 NCUR=N
      47 NL=N
      48 CALL RTSQ(A,NCUR,M)
      49 CALL RERO(A,M,NCUR,DELTA,EPSIL,ROMOD,MROMOD,NONRT,MNONRT,NCO,ROOTS
      50 1)
      51 IF (NCO) 195,195,115
      52 N1=NCUR+1
      53 CALL COMPT(M,MNONRT,NONRT,MROMOD,NCO,DELTA,EPSIL,NCUR)
      54 IF (NCUR) 195,195,130
      55 IF (NL-NCUR) 135,135,131
      56 NL=NCUR
      57 GO TO 105
      58 M=M+1
      59 IF (MMAX-M) 175,105,105
      60 N1=NCUR+1
      61 GO TO 37
      62 CALL RECON(ROOTS,A,D,NDEGRE)
      63 GO TO 37

```


c. Subroutine RTSQ
(Root-squaring process)

```

5 SUBROUTINE RTSQ(A,N,MM)
10 DIMENSION A(3,51,3),SPACE(450),CON0(3),CON1(3),CON2(3)
15 COMMON A,SPACE,CON0,CON1,CON2
20 N1=N+1
21 DO 24 J=1,N1
22 DO 24 I=1,3
23 A(I,J,2)=A(I,J,1)
24 A(I,J,3)=CON0(I)
25 CO 200 M=1,MM
30 DO 170 J=1,N1
40 K1=N1-J
45 K2=J-1
50 KM=XMINOF(K1,K2)
70 IF (KM) 130,130,75
75 DO 125 L=1,KM
80 LR=XMODF(L,2)
82 JL=J-L
83 JLP=J+L
85 IF (LR) 90,90,110
90 CALL NPREC
95 A(1,J,3)=A(1,J,3)+A(1,JL,2)*A(1,JLP,2)
100 CALL NPGUT
105 GO TO 125
110 CALL NPREC
115 A(1,J,3)=A(1,J,3)-A(1,JL,2)*A(1,JLP,2)
120 CALL NPGUT
125 CONTINUE
130 CALL NPREC
135 A(1,J,3)=CON2(1)*A(1,J,3)
136 A(1,J,3)=A(1,J,3)+A(1,J,2)*A(1,J,2)
140 CALL NPGUT
145 JR=XMODF(J,2)
150 IF (JR) 155,155,170
155 CALL NPREC
160 A(1,J,3)=CON0(1)-A(1,J,3)
165 CALL NPGUT
170 CONTINUE
175 IF (MM-M) 200,200,180
180 DO 195 J=1,N1
185 DO 195 I=1,3
190 A(I,J,2)=A(I,J,3)
195 A(I,J,3)=CON0(I)
200 CONTINUE
205 RETURN

```

d. Subroutine RERO

(Finds moduli and real roots, calls for test subroutine, TEST)

```

5 SUBROUTINE RERO(A,M,NCUR,DELTA,EPSIL,ROMOD,MROMOD,NONRT,
  1MNONRT,NCO,ROOTS)
10 DIMENSION A(3,51,3),RATIO(50),T(3),IPIV(50),ROMOD(3,50),MROMOD(50)
  1,Q(3),W(3),ARED(3,49),NONRT(50),MNONRT(50),SPACE1(909),XN(3),
  2ROOTS(3,2,50),CGNO(3),SPACE2(933)
12 COMMON SPACE1,CGNO,SPACE2,T,Q,W,ARED,XN
S MN OCT 0000000000001
15 RATIO(1)=1.0
16 IF (2-NCUR) 20,20,82
20 DO 65 I=2,NCUR
25 I1=XMODF(I,2)
30 CALL NPREC
31 IF (A(1,I,3)) 35,32,35
32 CALL NPOUT
33 RATIO(I)=0.0
34 GO TO 65
35 T=A(1,I,2)**2
36 T=T/A(1,I,3)
40 CALL NPOUT
45 CALL NP2SP(T(1),T(1))
50 RATIO(I)=T(1)
55 IF (I1) 65,60,65
60 RATIO(I)=-RATIO(I)
65 CONTINUE
82 RATIO(NCUR+1)=1.0
85 IPIV(1)=1
90 IPIV(NCUR+1)=1
91 IF (2-NCUR) 95,95,135
95 DO 130 I=2,NCUR
105 T(1)=ABSF(RATIO(I)-1.0)
110 IF (T(1)-DELTA) 115,125,125
115 IPIV(I)=1
120 GO TO 130
125 IPIV(I)=0
130 CONTINUE
135 I1=0
140 MULT=0
145 I=1
150 I4=1
155 I1=I1+1
160 I2=I1+I
165 MULT=MULT+1
170 IF (IPIV(I2)) 155,155,180
180 CALL NPREC
185 ROMOD(1,I4)=A(1,I2,3)/A(1,I,3)
190 IF(ROMOD(1,I4)) 195,200,200
195 ROMOD(1,I4)=-ROMOD(1,I4)
200 CALL NPOUT
210 CALL CPLOG(ROMOD(1,I4),XN)
215 MM=M
S 216 ARS 18
S 217 STO MM
S 218 CLA XN(1)
S 219 SUB MM
S 220 STO XN(1)
  221 IF (I1-2) 240,222,226
S 222 CLA XN(1)
S 223 SUB MN
S 224 STO XN(1)

```

```

225 GO TO 240
226 IF (I1-4) 232,227,232
S 227 CLA XN(1)
S 228 SUB MN
S 229 SUB MN
S 230 STO XN(1)
231 GO TO 240
232 T=I1
233 CALL SPCON(T)
234 CALL NPREC
235 XN=XN/T
236 CALL NPOUT
240 CALL DPEX(XN,ROMOD(1,I4))
245 MROMOD(I4)=MULT
250 IF (NCUR+1-I2) 290,290,255
255 I=I2
260 I4=I4+1
265 MULT=0
270 I1=0
275 GO TO 155
290 Q(1)=0.0
295 CALL SPCON(Q(1))
300 NCO=0
305 DO 455 I=1,I4
306 KL=I4+1-I
310 CALL NPREC
312 W=-ROMOD(1,KL)
315 CALL NPOUT
320 I5=MROMOD(KL)
330 DO 430 J=1,I5
332 J=J
335 CALL TEST(A,W,Q,NCUR,ROMOD(1,KL),EPSIL,K)
340 IF (K) 410,410,345
345 CALL NPREC
350 ROOTS(1,1,NCUR)=-W
355 ROOTS(1,2,NCUR)=CONO
360 ARED(1,1)=A(1,1,1)
365 CALL NPOUT
370 IF (NCUR-2) 430,375,375
375 DO 395 L=2,NCUR
380 CALL NPREC
385 ARED(1,L)=A(1,L,1)-ARED(1,L-1)*W
390 A(1,L,1)=ARED(1,L)
395 CALL NPOUT
405 GO TO 430
410 IF (W(2)) 415,440,440
415 W(2)=-W(2)
420 W(3)=-W(3)
425 GO TO 335
430 NCUR = NCUR-1
435 GO TO 455
440 NCO=NCO+1
445 NONRT(NCO)=KL
450 MNONRT(NCO)=I5+1-J
455 CONTINUE
460 RETURN

```

e. Subroutine TEST

```

5 SUBROUTINE TEST(A,W,Q,N,ROMOD,EPSIL,K)
8 DIMENSION A(3,51,3),B(3,3),W(3),Q(3),ROMOD(3),SPACE(1089),CEPSIL(3
1),T(3,2),C(3,51),CON2(3),E(3,2),DIF(3),X(3),Y(3),BLANK(915)
10 COMMON BLANK,CON2,SPACE,B,CEPSIL,T,E,DIF,X,Y,C
15 B(1,1)=0.0
20 CALL SPCON(B(1,1))
25 DO 30 I=1,3
30 B(I,2)=A(I,1,1)
35 DO 75 I=1,N
40 CALL NPREC
45 X=W*B(1,2)
46 Y=Q*B(1,1)
47 B(1,3)=A(1,I+1,1)-X-Y
50 CALL NPOUT
55 IF (N-I) 75,75,60
60 DO 70 J=1,3
65 B(J,1)=B(J,2)
70 B(J,2)=B(J,3)
75 CONTINUE
76 IF (SENSE LIGHT 2) 78,77
77 KOUNT=1
   GO TO 80
78 CALL NPREC
79 GO TO 210
80 CEPSIL(1)=EPSIL
85 CALL SPCON(CEPSIL(1))
90 T(1,1)=0.0
95 T(1,2)=0.0
100 CALL SPCON(T(1,1))
105 CALL SPCON(T(1,2))
110 N1=N+1
115 CALL NPREC
120 X=CON2*CEPSIL
121 Y=X*ROMOD
122 E(1,1)=ROMOD+Y
125 E(1,2)=ROMOD(1)+CEPSIL(1)*ROMOD(1)
130 CALL NPOUT
135 DO 175 I=1,N1
140 CALL NPREC
145 IF (A(1,I,1)) 150,160,160
150 C(1,I)=-A(1,I,1)
155 GO TO 165
160 C(1,I)=A(1,I,1)
165 T(1,1)=T(1,1)*E(1,1)+C(1,I)
170 T(1,2)=T(1,2)*E(1,2)+C(1,I)
175 CALL NPOUT
200 CALL NPREC
205 DIF(1)=T(1,1)-T(1,2)
210 IF (Q(1)) 290,215,290
215 IF (B(1,3)) 220,255,255
220 B(1,3)=-B(1,3)
255 IF (DIF(1)-B(1,3)) 260,260,275
260 CALL NPOUT
265 K=0
271 IF (KOUNT-2) 272,274,274
272 IF (Q(2)) 2721,2731,2721
2731 IF (W(2)) 2721,274,274
2721 SENSE LIGHT 2
273 KOUNT=KOUNT+1
274 RETURN
275 CALL NPOUT
280 K=1
285 GO TO 274
290 X=B(1,2)/CON2
291 Y=B(1,3)+W*X
295 IF (Y) 300,335,335
300 Y=-Y
335 IF (DIF-Y) 260,260,275

```

f. Subroutine COMPRT

(Finds complex roots and calls TEST)

```

5 SUBROUTINE COMPRT(MM,MNONRT,NCUR,MROMOD,NCUR,DELTA,EPIL,NCUR)
10 DIMENSION A(3,51,3),ROMOD(3,50),RCOTS(3,2,50),CON0(3),CON1(3),
  1CON2(3),CON4(3),SR(3,51,3),SROMOD(3,50),SROOTS(3,2,50),U(3),T(3),
  2W(3),Q(3),WE(3),AA(3),BB(3),R(3,2),D(3,2),B(3,49),MNONRT(50),
  3NONRT(50),MSROMD(50),NSONRT(49),MSNORT(49),MROMOD(50)
15 COMMON A,ROMOD,ROOTS,CON0,CON1,CON2,CON4,SR,SROMOD,SROOTS,U,T,W,Q,
  1WE,AA,BB,R,D,B
20 DO 295 I=1,NCUR
22 JA=NONRT(I)
25 I1=MNONRT(I)
26 I1=I1/2
27 IF (I1) 39,39,40
39 I1=1
40 IF (ROMOD(2,JA)) 45,295,45
45 Q(1)=ROMOD(1,JA)
48 Q(2)=ROMOD(2,JA)
50 Q(3)=ROMOD(3,JA)
55 DO 290 J=1,I1
60 CALL SUBRES(NCUR)
65 IF (NCUR-4) 67,67,72
67 NSCUR=1
68 J2=1
70 GO TO 80
72 NSCUR=NCUR-3
73 J2=NSCUR
80 LL=NSCUR+1
113 IF (NSCUR-1) 305,305,115
115 CALL RTSQ(SR,NSCUR,MM)
120 CALL RERO(SR,MM,NSCUR,DELTA,EPIL,SROMOD,MSROMD,
  1INSONRT,MSNORT,NSCO,SROOTS)
125 IF (J2-NSCUR) 126,126,140
126 CALL NPREC
127 SROOTS(1,1,J2)=CON0
128 CALL NPOUT
140 CALL NPREC
  SROOTS(1,1,J2)=SROOTS(1,1,J2)*ROMOD(1,JA)
145 T=CON2*ROMOD(1,JA)
146 IF (SROOTS(1,1,J2)-T) 165,150,150
150 CALL NPOUT
155 GO TO 275
165 W(1)=SROOTS(1,1,J2)
167 WE(1)=ROMOD(1,JA)**2
169 CALL NPOUT
170 CALL TEST(A,W,WE,NCUR,ROMOD(1,JA),EPIL,K)
175 IF (K) 255,255,180
180 CALL NPREC
181 ROOTS(1,1,NCUR)=-W/CON2
182 T=CON4*WE
183 U=W**2
184 T=T-U
185 U=DPSQT(T)
186 ROOTS(1,2,NCUR)=U/CON2
190 ROOTS(1,1,NCUR-1)=ROOTS(1,1,NCUR)
195 ROOTS(1,2,NCUR-1)=-ROOTS(1,2,NCUR)
196 CALL NPOUT
200 DO 210 IQ=1,3
205 D(IQ,1)=W(IQ)
210 D(IQ,2)=WE(IQ)
225 CALL QUADIV(NCUR)

```

```
228 JX=NCUR-1
230 DO 240 JY=1,JX
235 DO 240 JW=1,3
240 A(JW,JY,1)=B(JW,JY)
245 NCUR=NCUR-2
250 GO TO 290
255 W(2)=-W(2)
260 W(3)=-W(3)
265 CALL TEST(A,W,WE,NCUR,ROMOD(1,JA),EPSIL,K)
270 IF (K) 275,275,180
275 IF (J2-NSCUR) 295,276,280
276 IF (J2-1) 295,295,277
277 J2=J2-1
278 SROOTS(1,1,J2)=CON0(1)
      SROOTS(2,1,J2)=CON0(2)
      SROOTS(3,1,J2)=CON0(3)
      CALL NPREC
279 GO TO 165
280 CALL NPREC
281 IF (SROOTS(1,1,J2)-SROOTS(1,1,J2-1))282,286,282
282 CALL NPOUT
283 J2=J2-1
284 CALL NPREC
285 GO TO 140
286 CALL NPOUT
287 J2=J2-1
288 GO TO 275
290 CONTINUE
295 CONTINUE
300 RETURN
305 IF (SR(2,1,1)) 306,250,306
306 CALL NPREC
310 SROOTS(1,1,1)=-SR(1,2,1)/SR(1,1,1)
315 CALL NPOUT
320 NSCUR=0
325 GO TO 140
```

g. Subroutine SUBRES

(Constructs the reduced subresultant polynomial)

```

5 SUBROUTINE SUBRES(N)
10 DIMENSION A(3,51,3),SPACE(450),CON1(3),BLANK(6),
   1SR(3,51,3),OPEN(459),ROMOD(3),X(3),T(3),C(3,51),U(3),
   2B(3,50,3),CONO(3)
15 COMMON A,SPACE,CONO,CON1,BLANK,SR,OPEN,ROMOD,X,T,C,U,B
20 N1=N+1
25 T(1)=CON1(1)
30 T(2)=CON1(2)
35 T(3)=CON1(3)
40 DO 65 I=1,N
45 J=N1-I
50 CALL NPREC
55 T=T*ROMOD
60 C(1,J)=A(1,J,1)*T
65 CALL NPOUT
70 C(1,N1)=A(1,N1,1)
75 C(2,N1)=A(2,N1,1)
80 C(3,N1)=A(3,N1,1)
84 IF (N-2) 260,260,85
85 N2=N-2
90 DO 110 I=1,N2
95 CALL NPREC
100 B(1,1,1)=CONO
105 B(1,I,2)=CONO
110 CALL NPOUT
115 I=2
120 B(1,1,2)=C(1,1)
125 B(2,1,2)=C(2,1)
130 B(3,1,2)=C(3,1)
135 CALL NPREC
140 B(1,1,3)=C(1,I)-B(1,1,1)
145 CALL NPOUT
150 DO 165 J=2,N2
155 CALL NPREC
160 B(1,J,3)=-B(1,J-1,2)-B(1,J,1)
165 CALL NPOUT
170 IF (N-3-I) 205,175,175
175 I=I+1
180 DO 195 J=1,N2
185 DO 195 K=1,3
190 B(K,J,1)=B(K,J,2)
195 B(K,J,2)=B(K,J,3)
200 GO TO 135
205 CALL NPREC
210 SR(1,N2,1)=C(1,N)-B(1,1,3)
215 SR(1,N2-1,1)=-C(1,N1)-B(1,2,3)
220 CALL NPOUT
222 IF (N2-2) 245,245,225
225 DO 240 J=3,N2
228 K=N2+1-J
230 CALL NPREC
235 SR(1,K,1)=-B(1,J,3)
240 CALL NPOUT
245 RETURN
260 CALL NPREC
265 SR(1,1,1)=C(1,1)
270 SR(1,2,1)=-C(1,2)
275 CALL NPOUT
280 GO TO 245

```

h. Subroutine RECON

(Reconstructs the original polynomial from the computer zeros)

```

5 SUBROUTINE RECON(ROOTS,A,D,N)
10 DIMENSION SPACE(909),ROOTS(3,2,50),CON0(3),CON1(3),SKIP(3),
   IT(3),U(3),D(3,51),A(3,51,3),CON2(3)
15 COMMON SPACE,CON0,CON1,CON2,SKIP,T,U
195 DO 210 J=1,3
200 DO 205 I=1,N
205 D(J,I)=CON0(J)
210 D(J,N+1)=CON1(J)
225 I=1
230 NL=N-1
232 IF (ROOTS(2,2,I)) 235,330,235
235 CALL NPREC
240 T=ROOTS(1,1,I)**2
245 U=ROOTS(1,2,I)**2
250 T=T+U
255 U=CON2*ROOTS(1,1,I)
260 CALL NPOUT
265 DO 295 J=1,NL
270 IF (I+J-N) 295,275,275
275 CALL NPREC
280 D(1,J)=D(1,J+2)+T*D(1,J)
285 D(1,J)=D(1,J)-U*D(1,J+1)
290 CALL NPOUT
295 CONTINUE
300 CALL NPREC
305 D(1,N)=T*D(1,N)
307 D(1,N)=D(1,N)-U*D(1,N+1)
310 D(1,N+1)=T*D(1,N+1)
315 CALL NPOUT
320 I=I+2
325 IF (N-I) 385,232,232
330 DO 355 J=1,N
335 IF (J+I-N) 355,355,340
340 CALL NPREC
345 D(1,J)=D(1,J+1)-D(1,J)*ROOTS(1,1,I)
350 CALL NPOUT
355 CONTINUE
360 CALL NPREC
365 D(1,N+1)=-D(1,N+1)*ROOTS(1,1,I)
370 CALL NPOUT
375 I=I+1
380 GO TO 325
385 NS=N+1
390 DO 405 IA=1,NS
395 CALL NPREC
400 D(1,IA)=D(1,IA)*A(1,1,1)
405 CALL NPOUT
445 RETURN

```


i. Subroutine QUADIV

(Divides out quadratic factors)

```

 5 SUBROUTINE QUADIV(N)
10 DIMENSION A(3,51,3),SPACE(1392),R(3,2),D(3,2),B(3,49),XN(3),YN(3)
12 COMMON A,SPACE,R,D,B,XN,YN
15 CALL NPREC
20 B(1,1)=A(1,1,1)
21 CALL NPOUT
22 IF (N-2) 60,60,23
23 CALL NPREC
25 B(1,2)=A(1,2,1)-B(1,1)*D(1,1)
26 CALL NPOUT
30 IF (N-3) 60,60,35
35 NT=N-1
40 DO 55 I=3,NT
45 CALL NPREC
50 XN=B(1,I-1)*D(1,1)
51 YN=B(1,I-2)*D(1,2)
52 B(1,I)=A(1,I,1)-XN-YN
55 CALL NPOUT
60 CALL NPREC
65 XN=B(1,N-1)*D(1,1)
66 YN=B(1,N-2)*D(1,2)
67 R(1,1)=A(1,N,1)-XN-YN
70 R(1,2)=A(1,N+1,1)-B(1,N-1)*D(1,2)
72 CALL NPOUT
75 RETURN

```

j. Subroutine DPOUTP

(Outputs double-precision numbers)

```

5 SUBROUTINE DPOUTP(A)
10 DIMENSION A(3),T(3),X(3),ID(21),SPACE(1083)
15 COMMON SPACE,X,T
S C4 OCT 0000000000004
S A1 OCT 0000000000001
S A2 OCT 314631463146
S A3 OCT 146314631463
  T(1)=A1
  T(2)=A2
  T(3)=A3
19 KK=0
20 X(1)=A(1)
21 X(2)=A(2)
22 X(3)=A(3)
25 I=0
30 IF (X(2)) 35,495,35
35 IF (X(1)) 40,100,55
40 K=-1
S 41 CLA X(1)
S 42 ADD C4
S 43 STO X(1)
44 CALL NPREC
45 X=X/T
46 CALL NPOUT
47 I=I-1
48 IF (X(1)) 41,100,100
55 K=1
S 60 CLA X(1)
S 61 SUB C4
S 62 STO NDIF
63 IF (NDIF) 100,65,65
S 65 CLA NDIF
S 66 STO X(1)
70 I=I+1
75 CALL NPREC
80 X=X*T
82 CALL NPOUT
85 GO TO 60
100 IF (X(2)) 105,120,120
105 KK=1
110 X(2)=-X(2)
115 X(3)=-X(3)
S 120 CLA X(1)
S 125 ALS 18
S 130 STO N
135 N=N+1
140 GO TO (145,155,195,255),N
145 ID(1)=0
150 GO TO 310
155 ID(1)=1
S 160 CLA X(2)
S 165 LDQ X(3)
S 170 LLS 1
S 180 STO X(2)
S 185 STQ X(3)
190 GO TO 310
S 195 CLA X(2)
S 200 ARS 33
S 210 ALS 18

```

```

S 215 STO ID(1)
S 220 CLA X(2)
S 225 LDQ X(3)
S 230 LLS 2
S 240 STO X(2)
S 245 STQ X(3)
  250 GO TO 310
S 255 CLA X(2)
S 260 ARS 32
S 270 ALS 18
S 275 STO ID(1)
S 280 CLA X(2)
S 285 LDQ X(3)
S 290 LLS 3
S 300 STO X(2)
S 305 STQ X(3)
  310 IF ACCUMULATOR OVERFLOW 315,315
  315 DO 455 J=2,21
S 320 CLA X(2)
S 325 LDQ X(3)
S 330 LRS 2
S 335 STO T(2)
S 340 LLS 37
S 341 LRS 2
  342 IF ACCUMULATOR OVERFLOW 345,345
S 345 ADD X(3)
S 350 STO X(3)
  355 IF ACCUMULATOR OVERFLOW 375,360
S 360 CLA X(2)
S 365 ADD T(2)
  370 GO TO 390
S 375 CLA X(2)
S 380 ADD T(2)
S 385 ADD A1
S 390 STO X(2)
  391 IF ACCUMULATOR OVERFLOW 392,394
  392 ID(J)=0
  393 GO TO 395
  394 ID(J)=0
S 395 CLA X(2)
S 400 ARS 32
S 410 ALS 18
S 412 ADD ID(J)
S 415 STO ID(J)
S 430 CLA X(2)
S 435 LDQ X(3)
S 440 LLS 3
S 446 STO X(2)
S 447 STQ X(3)
  450 IF ACCUMULATOR OVERFLOW 455,455
  455 CONTINUE
  460 IF (ID(1)) 476,465,476
  465 I=I-1
  470 DO 475 J=1,20
  475 ID(J)=ID(J+1)
  476 IF (KK) 480,480,477
  477 ID(1)=-ID(1)
  480 WRITE OUTPUT TAPE 6,485,(ID(J),J=1,20),I
  485 FORMAT (1H0I2,1H.19I1,3H EI4)
  490 RETURN
  495 T(1)=0.0
  500 WRITE OUTPUT TAPE 6,505,T(1)
  505 FORMAT (1H0E10.2)
  510 GO TO 490

```

k. Subroutine SPCON

(Converts single-precision floating-point numbers to double precision)

```
5 SUBROUTINE SPCON(IA)
10 DIMENSION IA(3)
15 IA(2)=0
20 IA(3)=0
S 25 CLM
S 30 LDQ IA(1)
S 35 LLS 0
S 40 STO IA(3)
S 45 LLS 8
S 50 SSP
S 55 STO IA(1)
S 60 STQ IA(2)
65 I=1
S 70 ARS 11
S 75 STO I
80 IF (IA(1)) 85,90,85
85 IA(1)=IA(1)-I
90 RETURN
```

1. Subroutine DPEX(Computes e^x in double precision for double-precision x)

```

5 SUBROUTINE DPEX(X,Y)
10 DIMENSION SPACE(912),CON1(3),BLANK(1092),X(3),Y(3),TEST(3),TEMP(3)
1,EF(3),T(3),R(3),Z(3),U(3),FAC(3),E(3)
15 COMMON SPACE,CON1,BLANK,TEST,TEMP,T,R,Z,U,FAC,EF,E
S E1 OCT 000000000000
S E2 OCT 261344137676
S E3 OCT 216367465365
S F1 OCT 000000000013
S F2 OCT 200000000000
EF(1)=E1
EF(2)=E2
EF(3)=E3
20 IF (X(2)) 25,45,45
25 K=1
30 T(2)=-X(2)
35 T(3)=-X(3)
40 GO TO 60
45 T(2)=X(2)
50 T(3)=X(3)
55 K=0
60 T(1)=X(1)
65 CALL NPREC
66 TEMP=CON1
70 R=T/EF
75 CALL NPOUT
80 CALL NP2SP(R,R)
85 IQ=R
90 U=IQ
95 CALL SPCON(U)
100 CALL NPREC
105 TEST=U*EF
110 IF (T-TEST) 115,140,140
115 U=U-CON1
120 TEST=U*EF
125 CALL NPOUT
130 IQ=IQ-1
135 CALL NPREC
140 T=T-TEST
145 CALL NPOUT
150 IQQ=IQ/10
155 IQ=IQ-IQQ*10
160 J=2**IQ
161 EF(1)=CON1(1)
162 EF(2)=CON1(2)
163 EF(3)=CON1(3)
164 U(1)=F1
165 U(2)=F2
166 U(3)=F3
167 IF (IQQ) 180,180,168
168 DO 171 I=1,IQQ
169 CALL NPREC
170 EF=EF*U
171 CALL NPOUT
180 Z=J
185 CALL SPCON(Z)
190 CALL NPREC
195 E=Z*EF
205 Z=CON1
210 CALL NPOUT

```

```
215 DO 245 I=1,20
220 FAC=I
225 CALL SPCON(FAC)
230 CALL NPREC
435 TEMP=TEMP*T
437 TEMP=TEMP/FAC
240 Z(1)=Z(1)+TEMP(1)
245 CALL NPOUT
255 CALL NPREC
260 Y=E*Z
    CALL NPOUT
270 IF (K) 286,286,275
275 CALL NPREC
    Z=CON1/Y
    Y=Z
285 CALL NPOUT
286 RETURN
```

m. Subroutine DPLOG

(Computes double-precision natural logs for double-precision arguments)

```

5 SUBROUTINE DPLOG(A,AL)
10 DIMENSION A(3),CON1(3),CON2(3),X(3),Y(3),Z(3),U(3),AL(3),V(3),W(3)
1,DC(3),SPACE(909),BLANK(1089),CONC(3)
15 COMMON SPACE,CON0,CON1,CON2,BLANK,X,Y,Z,U,V,W,DC
S M OCT 000000000001
S C OCT 261344137676
S D OCT 216367465366
45 T=A(1)
50 A(1)=0.0
55 CALL NPREC
57 Y=A-CON1
60 Z=A+CON1
62 X=Y/Z
65 CALL NPOUT
66 A(1)=T
70 DO 80 I=1,3
75 Y(I)=X(I)
80 Z(I)=X(I)
81 CALL NPREC
82 V=X**2
83 CALL NPOUT
85 DO 115 I=2,14
90 U(I)=FLOATF(I)
95 CALL SPCON(U(I))
100 CALL NPREC
102 Y=Y*V
103 U=CON2*U
104 DC=U-CON1
105 W=Y/DC
106 Z=Z+W
115 CALL NPOUT
120 CALL NPREC
121 Y=CON0
122 Z=CON2*Z
123 CALL NPOUT
130 X(1)=CON0(1)
135 X(2)=C
140 X(3)=D
145 IF (A(1)) 150,225,150
150 N=M
155 DO 220 I=1,35
S 160 CLA A(1)
S 165 ANA N
S 170 STO R
175 IF (R) 180,195,180
180 CALL NPREC
185 Y(1)=Y(1)+X(1)
190 CALL NPOUT
195 CALL NPREC
200 X(1)=CON2(1)*X(1)
205 CALL NPOUT
S 210 CLA N
S 215 ALS 1
S 220 STO N
221 IF (A(1)) 245,225,225
225 CALL NPREC
230 AL(1)=Z(1)+Y(1)
235 CALL NPOUT
240 GO TO 260
245 CALL NPREC
250 AL=Z-Y
255 CALL NPOUT
260 RETURN

```

2. FORTRAN 63 Statements of the RSSR Routine as Coded for the CDC-3600 Computer

- a. Main Program
(Does input and output, and calls RSSR)
- b. Subroutine RSSR
(Calls subroutines for root squaring, finding moduli of roots and real roots, finding complex roots, and reconstructing the polynomial)
- c. Subroutine ROOTSQ
(Performs the root squarings)
- d. Subroutine REALROOT
(Finds moduli of roots and real roots)
- e. Subroutine COMPROOT
(Finds the complex roots)
- f. Subroutine TEST
(Tests roots for accuracy)
- g. Subroutine SUBRES
(Computes reduced subresultant polynomial or, if $n = 4$, the subresultant polynomial)
- h. Subroutine RECON
(Reconstructs the polynomial from the roots and the first coefficient)
- i. Subroutine QUADIV
(Divides out quadratic factors)
- j. Subroutine DOUBLOG
(Computes double-precision log for numbers of the form x times 64^{Ix})
- k. Subroutine DOUBLEXP
(Computes in double precision e^t , where $t = x(64^{Ix})$)

- l. Subroutine ADD
(Adds numbers of the form $x(64^{Ix})$ and $y(64^{Iy})$, where x and y are in double precision)
- m. Subroutine SUBTRACT
(Subtracts numbers of the form $x(64^{Ix})$ and $y(64^{Iy})$, where x and y are in double precision)
- n. Subroutine SCALE
(Scales double-precision numbers of the form x times 64^{Ix} so that $1/64 \leq |x| \leq 64$ ($x \neq 0$))
- o. Subroutine UNSCALE
(Unscales numbers of the form $x(64^{Ix})$ to the form $\bar{X}(64^{\bar{I}x})$ such that $\bar{I}x = 0$)

a. Main Program

(Does input and output, and calls RSSR)

c

```

PROGRAM RSSRMAIN
DIMENSION A(51,3),IA(51,3),ROOTS(2,50),TITLE(10),D(51)
TYPE DOUBLE A,ROOTS,D
TYPE INTEGER DEGREE
1 READ (60,2) TITLE
2 FORMAT (10A8)
  IF (EOF,60) 3,4
3 CALL EXIT
4 WRITE (61,5) TITLE
5 FORMAT (1H1,10A8)
  READ (60,6) DEGREE,M,MMAX,DELTA,EPSILON
6 FORMAT (3I6,2E12.4)
  IF (M) 7,7,8
7 M=10
  MMAX=15
  DELTA=0.0001
  EPSILON=0.000001
8 WRITE (61,9) DEGREE,M,MMAX,DELTA,EPSILON
9 FORMAT (8H0DEGREE=13,17H NO. OF ROOT SQ=13,7H MMAX=13, /
17H DELTA=E12.3,10H EPSILON=E12.3)
  N1=DEGREE+1
  READ (60,6) IPI,IPO
  WRITE (61,10)
10 FORMAT (19H0INPUT COEFFICIENTS)
  IF (IPI-1) 11,11,15
11 DO 14 I=1,N1
  READ (60,12) X
12 FORMAT (E18.10)
  A(I,1)=X
  IA(I,1)=0
  J=I-1
  WRITE (61,13) J,X
13 FORMAT (3H0A(I2,3H)= E18.10)
  CALL SCALE (A(I,1),IA(I,1))
14 CONTINUE
  GO TO 19
15 DO 18 I=1,N1
  J=I-1
  READ (60,16) A(I,1)
16 FORMAT (D36.25)
  WRITE (61,17) J,A(I,1)
17 FORMAT (3H0A(I2,3H)= D33.25)
  IA(I,1)=0
  CALL SCALE (A(I,1),IA(I,1))
18 CONTINUE
19 N2=DEGREE
  CALL RSSR (A,IA,ROOTS,DEGREE,M,MMAX,DELTA,EPSILON,D)
  IF (IPO-1) 20,20,31

```

```

20 IF (N1-(DEGREE+1)) 29,29,21
21 WRITE (61,22)
22 FORMAT (24H0ROOTS OF THE POLYNOMIAL)
   WRITE (61,23)
23 FORMAT (33H0REAL PART          IMAGINARY PART)
   K=DEGREE+1
   DO 25 I=K,N2
   X=ROOTS(1,I)
   Y=ROOTS(2,I)
   WRITE (61,24) X,Y
24 FORMAT (1H0E18.10,E20.10)
25 CONTINUE
   IF (DEGREE) 26,26,29
26 WRITE (61,27)
27 FORMAT (25H0RECONSTRUCTED POLYNOMIAL)
   DO 28 I=1,N1
   X=D(I)
   J=I-1
   WRITE (61,13) J,X
28 CONTINUE
   GO TO 1
29 WRITE (61,30)
30 FORMAT (21H0SOME ROOTS NOT FOUND)
   GO TO 1
31 IF (N1-(DEGREE+1)) 29,29,32
32 WRITE (61,33)
33 FORMAT (62H0          REAL PART          IMAGINA
   IRY PART)
   K=DEGREE+1
   DO 35 I=K,N2
   WRITE (61,34) ROOTS(1,I),ROOTS(2,I)
34 FORMAT (1H0D33.25,D35.25)
35 CONTINUE
   IF (DEGREE) 36,36,29
36 WRITE (61,27)
   DO 37 I=1,N1
   J=I-1
   WRITE (61,17) J,D(I)
37 CONTINUE
   GO TO 1
   END

```

b. Subroutine RSSR

(Calls subroutines for rootsquaring, finding moduli of roots and real roots, finding complex roots, and reconstructing the polynomial)

```

C      SUBROUTINE RSSR (A,IA,ROOTS,DEGREE,M,MMAX,DELTA,EPSILON,D)

      DIMENSION A(51,3),IA(51,3),ROOTS(2,50),D(51),ROMOD(50),MROMOD(50),
1      INGNRT(50),MNONRT(50)
      TYPE DOUBLE A,ROOTS,D,ROMOD
      TYPE INTEGER DEGREE
      N=DEGREE
      IF (N) 1,1,3
1     DEGREE=NCUR
2     RETURN
3     N1=N+1
      N2=N1+1
      DO 5 I=1,N
      K=N2-I
      IF (A(K,1)) 6,4,6
4     J=N1-I
      ROOTS(1,J)=0.0
      ROOTS(2,J)=0.0
5     CONTINUE
      DEGREE=0
      GO TO 2
6     N1=K
      N=K-1
      NCUR=N
      NL=N
7     CALL ROOTSQ (A,IA,NCUR,M)
      CALL REALROOT (A,IA,M,NCUR,DELTA,EPSILON,ROMOD,MROMOD,
1      INGNRT,MNONRT,NCO,ROOTS)
      IF (NCO) 12,12,8
8     N1=NCUR+1
      CALL COMPROOT (A,IA,ROMOD,ROOTS,M,MNONRT,NONRT,
1      MROMOD,NCO,DELTA,EPSILON,NCUR)
      IF (NCUR) 12,12,9
9     IF (NL-NCUR) 11,11,10
10    NL=NCUR
      GO TO 7
11    M=M+1
      IF (MMAX-M) 1,7,7
12    CALL RECON (ROOTS,A(1,1),IA(1,1),D,DEGREE)
      GO TO 1
      END

```

c. Subroutine ROOTSQ
(Performs the root squarings)

```

C      SUBROUTINE ROOTSQ (A,IA,NCUR,MM)

      DIMENSION A(51,3),IA(51,3)
      TYPE DOUBLE A,X
      N1=NCUR+1
      DO 1 J=1,N1
        A(J,2)=A(J,1)
        IA(J,2)=IA(J,1)
        A(J,3)=0.0
        IA(J,3)=0
1      CONTINUE
      DO 9 M=1,MM
        DO 6 J=1,N1
          K1=N1-J
          K2=J-1
          KM=XMINOF(K1,K2)
          DO 4 L=1,KM
            LR=XMODF(L,2)
            JL=J-L
            JLP=J+L
            IF (LR) 2,2,3
2          X=A(JL,2)*A(JLP,2)
            IX=IA(JL,2)+IA(JLP,2)
            CALL SCALE (X,IX)
            CALL ADD (A(J,3),IA(J,3),X,IX,A(J,3),IA(J,3))
            GO TO 4
3          X=A(JL,2)*A(JLP,2)
            IX=IA(JL,2)+IA(JLP,2)
            CALL SCALE (X,IX)
            CALL SUBTRACT (A(J,3),IA(J,3),X,IX,A(J,3),IA(J,3))
4          CONTINUE
            A(J,3)=2.0*A(J,3)
            CALL SCALE (A(J,3),IA(J,3))
            X=A(J,2)**2
            IX=IA(J,2)+IA(J,2)
            CALL SCALE (X,IX)
            CALL ADD (A(J,3),IA(J,3),X,IX,A(J,3),IA(J,3))
            JR=XMODF (J,2)
            IF (JR) 5,5,6
5          A(J,3)=-A(J,3)
6          CONTINUE
            IF (MM-M) 9,9,7
7          DO 8 J=1,N1
            A(J,2)=A(J,3)
            IA(J,2)=IA(J,3)
            A(J,3)=0.0
            IA(J,3)=0
8          CONTINUE
9          CONTINUE
      RETURN
      END

```

d. Subroutine REALROOT

(Finds moduli of roots and real roots)

```

SUBROUTINE REALROOT (A,IA,M,NCUR,DELTA,EPSILON,ROMOD,MROMOD,
1NONRT,MNONRT,NCO,ROOTS)
C
  DIMENSION A(51,3),IA(51,3),ROOTS(2,50),ROMOD(50),MROMOD(50),
1NONRT(50),MNONRT(50),RATIO(51),IPIV(51),ARED(50),IARED(50)
  TYPE DOUBLE A,ROOTS,ROMOD,T,XN,W,ARED,Y,Q
  RATIO(1)=1.0
  DO 4 I=2,NCUR
    I1=XMODF(I,2)
    IF (A(I,3)) 2,1,2
1  RATIO(I)=0.0
    GO TO 4
2  T=A(I,2)*A(I,2)
    IT=IA(I,2)+IA(I,2)
    CALL SCALE (T,IT)
    T=T/A(I,3)
    IT=IT-IA(I,3)
    IF (IT-2) 50,50,1
50 IF (IT+2) 1,51,51
51 CALL UNSCALE (T,IT)
    RATIO(I)=T
    IF (I1) 3,3,4
3  RATIO(I)=-RATIO(I)
4  CONTINUE
    RATIO(NCUR+1)=1.0
    IPIV(1)=1
    IPIV(NCUR+1)=1
    DO 7 I=2,NCUR
      X=ABSF(RATIO(I)-1.0)
      IF (X-DELTA) 5,6,6
5  IPIV(I)=1
      GO TO 7
6  IPIV(I)=0
7  CONTINUE
    NCUR1=NCUR+1
    I1=0
    MULT=0
    I=1
    I4=1
8  I1=I1+1
    I2=I1+I
    MULT=MULT+1
    IF (IPIV(I2)) 8,8,9
9  ROMOD(I4)=A(I2,3)/A(I,3)
    IROMOD=IA(I2,3)-IA(I,3)
    CALL SCALE (ROMOD(I4),IROMOD)
    IF (ROMOD(I4)) 10,11,11
10 ROMOD(I4)=-ROMOD(I4)

```

```

11 CALL DOUBLOG (ROMOD(I4),IROMOD,XN,IXN)
   T=2.0**M*I1
   XN=XN/T
   CALL SCALE (XN,IXN)
   CALL DOUBLEXP (XN,IXN,ROMOD(I4),IROMOD)
   IF (IROMOD-150) 100,100,101
100 IF (IROMOD+150) 101,101,102
101 ROMOD(I4)=0.0
   IROMOD=0
   GO TO 103
102 CALL UNSCALE (ROMOD(I4),IROMOD)
103 MROMOD(I4)=MULT
   IF (NCUR+1-I2) 13,13,12
12 I=I2
   I4=I4+1
   MULT=0
   I1=0
   GO TO 8
13 Q=0.0
   NCO=0
   DO 22 I=1,I4
   KL=I4+1-I
   W=-ROMOD(KL)
   I5=MROMOD(KL)
   DO 20 J=1,I5
   J=J
14 CALL TEST (A,IA,W,Q,NCUR,ROMOD(KL),EPSILON,K)
   IF (K) 17,17,15
15 ROOTS(1,NCUR)=-W
   ROOTS(2,NCUR)=0.0
   ARED(1)=A(1,1)
   IARED(1)=IA(1,1)
   DO 16 L=2,NCUR
   Y=ARED(L-1)*W
   IY=IARED(L-1)
   CALL SCALE (Y,IY)
   CALL SUBTRACT (A(L,1),IA(L,1),Y,IY,ARED(L),IARED(L))
   A(L,1)=ARED(L)
   IA(L,1)=IARED(L)
16 CONTINUE
   GO TO 19
17 IF (W) 18,21,21
18 W=-W
   GO TO 14
19 NCUR=NCUR-1
20 CONTINUE
   GO TO 22
21 NCO=NCO+1
   NONRT(NCO)=KL
   MNONRT(NCO)=I5+1-J
22 CONTINUE
   RETURN
   END

```

e. Subroutine COMPROOT

(Finds the complex roots)

```

SUBROUTINE COMPROOT (A,IA,ROMOD,ROOTS,M,MNONRT,NONRT,
1MROMOD,NCO,DELTA,EPSILON,NCUR)
C
  DIMENSION A(51,3),IA(51,3),ROMOD(50),ROOTS(2,50),SR(51,3),ISR(51,3
1),SROMOD(50),SROOTS(2,50),MNONRT(50),NONRT(50),MSROMOD(50),
2NSNONRT(49),MSNONRT(49),MROMOD(50),D(2),R(2),B(49)
  TYPE DOUBLE A,ROMOD,ROOTS,SR,SROMOD,SROOTS,X,Y,T,W,Q,
1WE,U,D,R,B,DCNEX
  DO 23 I=1,NCO
    JA=NONRT(I)
    I1=MNONRT(I)
    I1=I1/2
    IF (I1) 1,1,2
  1 I1=1
  2 IF (ROMOD(JA)) 3,23,3
  3 Q=ROMOD(JA)
  DO 22 J=1,I1
300 CALL SUBRES (A,IA,NCUR,SR,ISR,Q)
    IF (NCUR-4) 4,100,5
100 NSCUR=2
    GO TO 101
  4 NSCUR=1
    J2=1
    GO TO 6
  5 NSCUR=NCUR-3
101 J2=NSCUR
  6 LL=NSCUR+1
    IF (NSCUR-1) 7,7,9
  7 IF (SR(1,1)) 8,10,8
  8 X=SR(1,1) $ IX=ISR(1,1)
    Y=SR(2,1) $ IY=ISR(2,1)
    CALL UNSCALE (X,IX)
    CALL UNSCALE (Y,IY)
    SROOTS(1,1)=-Y/X
    NSCUR=0
    GO TO 11
  9 CALL ROOTSQ (SR,ISR,NSCUR,M)
    CALL REALROOT (SR,ISR,M,NSCUR,DELTA,EPSILON,SROMOD,MSROMOD,
1NSNONRT,MSNONRT,NSCO,SROOTS)
    IF (J2-NSCUR) 10,10,11
10 SROOTS(1,J2)=0.0
11 SROOTS(1,J2)=SROOTS(1,J2)*ROMOD(JA)
    T=ROMOD(JA)+ROMOD(JA)
    IF (SROOTS(1,J2)-T) 12,16,16
12 W=SROOTS(1,J2)
    WE=ROMOD(JA)*ROMOD(JA)
    CALL TEST (A,IA,W,WE,NCUR,ROMOD(JA),EPSILON,K)
    IF (K) 15,15,13

```



```

13  ROOTS(1,NCUR)=-W/2.0
    T=4.0*WE
    U=W*W
    T=T-U
    IF (T) 201,201,202
201  T=-T
    U=DSQRT (T)
    ROOTS(1,NCUR)=ROOTS(1,NCUR)-U/2.0
    ROOTS(1,NCUR-1)=- (W-U)/2.0
    ROOTS(2,NCUR)=0.0
    ROOTS(2,NCUR-1)=0.0
    GO TO 204
202  U=DSQRT(T)
203  ROOTS(2,NCUR)=U/2.0
    ROOTS(1,NCUR-1)=ROOTS(1,NCUR)
    ROOTS(2,NCUR-1)=-ROOTS(2,NCUR)
204  D(1)=W
    D(2)=WE
    CALL QUADIV (NCUR,A,IA,R,D,B)
    JX=NCUR-1
    DO 14 JY=1,JX
    A(JY,1)=B(JY)
    IA(JY,1)=0
    CALL SCALE (A(JY,1),IA(JY,1))
14  CONTINUE
    NCUR=NCUR-2
    GO TO 22
15  W=-W
    CALL TEST (A,IA,W,WE,NCUR,ROMOD(JA),EPSILON,K)
    IF (K) 16,16,13
16  IF (J2-(NSCUR+1)) 23,17,19
17  IF (J2-1) 23,23,18
18  J2=J2-1
    SROOTS(1,J2)=0.0
    GU TO 12
19  IF (SROOTS(1,J2)-SROOTS(1,J2-1)) 20,21,20
20  J2=J2-1
    GO TO 11
21  J2=J2-1
    GO TO 16
22  CONTINUE
23  CONTINUE
    RETURN
    END

```

f. Subroutine TEST

(Tests roots for accuracy)

C

```

SUBROUTINE TEST (A,IA,W,Q,N,ROMOD,EPSILON,K)
  DIMENSION A(51,3),IA(51,3),B(3),IB(3),T(2),E(2),C(51)
  TYPE DOUBLE A,B,W,X,Y,Z,CEPSIL,E,C,DIF,Q,ROMOD
  B(1)=0.0$IX=0$IW=0
  IB(1)=0
  B(2)=A(1,1)
  IB(2)=IA(1,1)
  DO 2 I=1,N
  X=W*B(2)
  IX=IB(2)
  CALL SCALE (X,IX)
  Y=Q*B(1)
  IY=IB(1)
  CALL SCALE (Y,IY)
  CALL ADD (X,IX,Y,IY,Z,IZ)
  CALL SUBTRACT (A(I+1,1),IA(I+1,1),Z,IZ,B(3),IB(3))
  IF (N-I) 2,2,1
1 B(1)=B(2)
  IB(1)=IB(2)
  B(2)=B(3)
  IB(2)=IB(3)
2 CONTINUE
3 KOUNT=1
  CEPSIL=EPSILON
  T(1)=0.0$T(2)=0.0
  N1=N+1
  X=2.0*EPSILON
  Y=X*ROMOD
  E(1)=ROMOD+Y
  E(2)=ROMOD+CEPSIL*ROMOD
  DO 7 I=1,N1
  IF (A(I,1)) 4,5,5
4 C(I)=-A(I,1)$IC=IA(I,1)
  GO TO 6
5 C(I)=A(I,1)$IC=IA(I,1)
6 CALL UNSCALE (C(I),IC)
  T(1)=T(1)*E(1)+C(I)
  T(2)=T(2)*E(2)+C(I)
7 CONTINUE
  DIF=T(1)-T(2)
8 IF (Q) 18,9,18
9 IF (B(3)) 10,11,11
10 B(3)=-B(3)
11 IF (IB(3)-150) 100,100,12
100 IF (IB(3)+150) 12,101,101
101 CALL UNSCALE (B(3),IB(3))
  IF (DIF-B(3)) 12,12,17

```

```
12 K=0
   IF (KOUNT-2) 13,16,16
13 IF (Q) 15,14,15
14 IF (W) 15,16,16
15 SENSE LIGHT 2
   KOUNT=KOUNT+1
16 RETURN
17 K=1
   GO TO 16
18 IF (IB(2)-150) 102,102,12
102 IF (IB(2)+150) 12,103,103
103 CALL UNSCALE (B(2),IB(2))
   IF (IB(3)-150) 104,104,12
104 IF (IB(3)+150) 12,105,105
105 CALL UNSCALE (B(3),IB(3))
   X=Q*B(2)*B(2)
   Y=W*B(2)*B(3)
   Z=B(3)*B(3)
   Y=X-Y+Z
   IF (Y) 19,17,20
19 Y=-Y
20 DIF=DIF*DIF
   IF (DIF-Y) 12,17,17
   END
```

2

g. Subroutine SUBRES

(Computes reduced subresultant polynomial or,
if $n = 4$, the subresultant polynomial)

```

C
SUBROUTINE SUBRES (A,IA,N,SR,ISR,ROMOD)
  DIMENSION A(51,3),IA(51,3),SR(51,3),ISR(51,3),C(51),B(50,3)
  TYPE DOUBLE A,SR,ROMOD,X,T,C,U,B
  N1=N+1
  T=1.0
  DO 1 I=1,N
    J=N1-I
    T=T*ROMOD
    C(J)=A(J,1)*T
    IC=IA(J,1)
    CALL UNSCALE (C(J),IC)
1  CONTINUE
    C(N1)=A(N1,1)
    IC=IA(N1,1)
    CALL UNSCALE (C(N1),IC)
    IF (N-2) 12,12,2
2  N2=N-2
    DO 3 I=1,N2
      B(I,1)=0.0
      B(I,2)=0.0
3  CONTINUE
      I=2
      B(1,2)=C(1)
4  B(1,3)=C(1)-B(1,1)
      DO 5 J=2,N2
        B(J,3)=-B(J-1,2)-B(J,1)
5  CONTINUE
      IF (N-(3+I)) 100,6,6
6  I=I+1
      DO 7 J=1,N2
        B(J,1)=B(J,2)
        B(J,2)=B(J,3)
7  CONTINUE
      GO TO 4
100 IF (N-4) 14,101,8
101 IF (N-(2+I)) 104,102,102
102 I=I+1
      DO 103 J=1,2
        B(J,1)=B(J,2)
        B(J,2)=B(J,3)
103 CONTINUE
      GO TO 4
104 B(3,3)=-B(2,2)
      SR(3,1)=-C(5)+B(1,3) $ ISR(3,1)=0
      SR(2,1)=B(2,3) $ ISR(2,1)=0
      SR(1,1)=B(3,3) $ ISR(1,1)=0
      CALL SCALE (SR(1,1),ISR(1,1))
      CALL SCALE (SR(2,1),ISR(2,1))
      CALL SCALE (SR(3,1),ISR(3,1))
      GO TO 11

```

```
8 SR(N2,1)=C(N)-B(1,3) $ ISR(N2,1)=0
  SR(N2-1,1)=-C(N1)-B(2,3) $ ISR(N2-1,1)=0
  CALL SCALE (SR(N2,1),ISR(N2,1))
  CALL SCALE (SR(N2-1,1),ISR(N2-1,1))
  IF (N2-2) 11,11,9
9 DO 10 J=3,N2
  K=N2+1-J
  SR(K,1)=-B(J,3) $ ISR(K,1)=0
  CALL SCALE (SR(K,1),ISR(K,1))
10 CONTINUE
11 RETURN
12 SR(1,1)=C(1) $ ISR(1,1)=0
  SR(2,1)=-C(2) $ ISR(2,1)=0
13 CALL SCALE (SR(1,1),ISR(1,1))
  CALL SCALE (SR(2,1),ISR(2,1))
  GO TO 11
14 SR(1,1)=-C(4)
  SR(2,1)=C(3)-C(1)
  ISR(1,1)=0
  ISR(2,1)=0
  GO TO 13
END
```

h. Subroutine RECON

(Reconstructs the polynomial from the roots and the first coefficient)

C

```

SUBROUTINE RECON (ROOTS,A,IA,D,N)
  DIMENSION ROOTS(2,50),D(51)
  TYPE DOUBLE ROOTS,A,D,X,T,U
  X=A$IX=IA
  CALL UNSCALE (X,IX)
  DO 1 I=1,N
    D(I)=0.0
1  CONTINUE
  D(N+1)=1.0
  I=1
  NL=N-1
2  IF (ROOTS(2,I)) 3,7,3
3  T=ROOTS(1,I)*ROOTS(1,I)
  U=ROOTS(2,I)*ROOTS(2,I)
  T=T+U
  U=2.0*ROOTS(1,I)
  DO 5 J=1,NL
    IF (I+J-N) 5,4,4
4  D(J)=D(J+2)+T*D(J)
  D(J)=D(J)-U*D(J+1)
5  CONTINUE
  D(N)=T*D(N)
  D(N)=D(N)-U*D(N+1)
  D(N+1)=T*D(N+1)
  I=I+2
6  IF (N-I) 10,2,2
7  DO 9 J=1,N
    IF (J+I-N) 9,9,8
8  D(J)=D(J+1)-D(J)*ROOTS(1,I)
9  CONTINUE
  D(N+1)=-D(N+1)*ROOTS(1,I)
  I=I+1
  GO TO 6
10 NS=N+1
  DO 11 II=1,NS
    D(II)=D(II)*X
11 CONTINUE
  RETURN
  END

```

i. Subroutine QUADIV

(Divides out quadratic factors)

C

```

SUBROUTINE QUADIV (N,A,IA,R,D,B)
  DIMENSION A(51,3),IA(51,3),R(2),D(2),B(49)
  TYPE DOUBLE A,R,D,B,XN,YN,AA
  B(1)=A(1,1)$IB=IA(1,1)
  CALL UNSCALE (B(1),IB)
  IF (N-2) 4,4,1
1  AA=A(2,1)$IAA=IA(2,1)
  CALL UNSCALE (AA,IAA)
  B(2)=AA-B(1)*D(1)
  IF (N-3) 4,4,2
2  NT=N-1
  DO 3 I=3,NT
    XN=B(I-1)*D(1)
    YN=B(I-2)*D(2)
    AA=A(I,1)$IAA=IA(I,1)
    CALL UNSCALE (AA,IAA)
    B(I)=AA-(XN+YN)
3  CONTINUE
4  XN=B(N-1)*D(1)
  YN=B(N-2)*D(2)
  AA=A(N,1)$IAA=IA(N,1)
  CALL UNSCALE (AA,IAA)
  R(1)=AA-(XN+YN)
  AA=A(N+1,1)$IAA=IA(N+1,1)
  CALL UNSCALE (AA,IAA)
  R(2)=AA-B(N-1)*D(2)
  RETURN
END

```

j. Subroutine DOUBLOG

(Computes double-precision log for numbers of the form x times 64^{IX})

```
C      SUBROUTINE DOUBLOG (X,IX,Y,IY)
      TYPE DOUBLE X,Y,T
      T=64.0
      IF (X)1,5,3
1     WRITE (61,2)
2     FORMAT (46H0THE LOG OF A NON-POSITIVE NUMBER IS REQUESTED)
5     Y=0.0
      IY=0
      GO TO 4
3     Y=DLOG(X)+IX*DLOG(T)
      IY=0
      CALL SCALE (Y,IY)
4     RETURN
      END
```


k. Subroutine DOUBLEXP(Computes in double precision, e^t , where $t = x(64Ix)$)

C

```

SUBROUTINE DOUBLEXP (X,IX,Z,IZ)
  TYPE DOUBLE X,Y,Z
  Z=DEXP(X)$IZ=0
  IF (IX) 1,8,6
1  I=-IX$II=6*I
  DO 4 J=1,II
  K=XMODF(IZ,2)
  IF (K) 2,3,2
2  IZ=IZ-1
  Z=64.0*Z
3  IZ=IZ/2
  Z=DSQRT(Z)
  CALL SCALE (Z,IZ)
4  CONTINUE
5  RETURN
6  I=6*IX
  DO 7 J=1,I
  Z=Z*Z$IZ=IZ+IZ
  CALL SCALE (Z,IZ)
7  CONTINUE
  GO TO 5
8  CALL SCALE (Z,IZ)
  GO TO 5
END

```

1. Subroutine ADD

(Adds numbers of the form $x(64^{Ix})$ and $y(64^{Iy})$,
where x and y are in double precision)

C SUBROUTINE ADD (X,IX,Y,IY,Z,IZ)

```

      TYPE DOUBLE X,Y,Z,A,B
      IF (X) 3,1,3
1     Z=Y
      IZ=IY
2     RETURN
3     IF (Y) 5,4,5
4     Z=X
      IZ=IX
      GO TO 2
5     IDIFF=IX-IY
      IF (IDIFF) 6,7,7
6     IA=IY
      A=Y
      B=X
      IDIFF=-IDIFF
      GO TO 8
7     IA=IX
      A=X
      B=Y
8     IF (16-IDIFF) 9,9,10
9     Z=A
      IZ=IA
      GO TO 2
10    DO 11 I=1,IDIFF
      B=B/64.0
11    CONTINUE
      Z=A+B
      IZ=IA
      CALL SCALE (Z,IZ)
      GO TO 2
      END

```

m. Subroutine SUBTRACT

(Subtracts numbers of the form $x(64I_x)$ and $y(64I_y)$
where x and y are in double precision)

C SUBROUTINE SUBTRACT (X,IX,Y,IY,Z,IZ)
 TYPE DOUBLE X,Y,Z,W
 W=-Y
 CALL ADD (X,IX,W,IY,Z,IZ)
 RETURN
 END

n. Subroutine SCALE

(Scales double-precision numbers of the form x times 64^{IX} so that
 $1/64 \leq |x| \leq 64$ ($x \neq 0$))

C SUBROUTINE SCALE (X,IX)

```

      TYPE DOUBLE X,Y
      REC64=1.0/64.0
      IF (X) 1,11,2
1     Y=-X
      GO TO 3
2     Y=X
3     IF (64.0-Y) 4,5,5
4     Y=Y/64.0
      IX=IX+1
      GO TO 3
5     IF (Y-REC64) 6,7,7
6     Y=Y*64.0
      IX=IX-1
      GO TO 5
7     IF (X) 8,9,9
8     X=-Y
      GO TO 10
9     X=Y
10    RETURN
11    IX=0
      GO TO 10
      END

```

o. Subroutine UNSCALE

(Uncales numbers of the form $x(64^{\bar{I}x})$ to the form $\bar{X}(64^{\bar{I}x})$ such that $\bar{I}x = 0$)

SUBROUTINE UNSCALE (X,IX)

C

```

      TYPE DOUBLE X
      IF (IX+170) 3,4,4
3     X=0.0
      IX=0
      GO TO 2
4     IF (IX-170) 5,5,6
6     X=1.E+307
      IX=0
      WRITE (61,7)
7     FORMAT (*0EXP. OVERFLOW IN UNSCALE *)
      GO TO 2
5     IF (IX) 1,2,1
1     X=X*64.0**IX
      IX=0
2     RETURN
      END

```

APPENDIX C

Operating Instructions

1. Operating Instructions for IBM-704a. Input Information

The following cards are required.

- 1) Title card. Format (72H).
Column 1 should be a "1" to restore the printer. The remainder of the card through column 72 contains the title which will head the output.
- 2) The second card is transcribed either in standard form or in special form. The standard form should normally prove satisfactory, although occasionally the results may be insufficiently accurate for some polynomials or unnecessarily precise for others.
 - (a) Standard Form
NDEGRE. Format (16)
NDEGRE is the degree of the polynomial. If NDEGRE is ≤ 0 the program returns control to the monitor. Standard settings are given to other program parameters. (See b below.)
 - (b) Special Form
NDEGRE, M, MMAX, DELTA, EPSIL. Format (3I6, 2E12.6)
NDEGRE is the degree of the polynomial.
M is the minimum number of root squarings to be performed. The standard setting is $M = 10$.
MMAX is the maximum number of root squarings allowed. The standard setting is $MMAX = 15$.
DELTA is the pivotal bound. The standard setting is 10^{-4} .
EPSIL is the tolerable relative error in the modulus of a root. The standard setting is 10^{-6} .

For low-degree polynomials (up to 6), the user can decrease M and MMAX from the standard settings while using the standard settings for DELTA and EPSIL. For difficult polynomials, the user should increase M and MMAX and decrease EPSIL and DELTA from their standard settings.

- 3) IPI, IPO. Format (2I6).

$$IPI = \begin{cases} 2 & \rightarrow \text{double-precision coefficient input.} \\ 1 & \rightarrow \text{single-precision coefficient input.} \end{cases}$$

$$\text{IPO} = \begin{cases} 2 \rightarrow \text{double-precision output.} \\ 1 \rightarrow \text{single-precision output.} \end{cases}$$

- 4) NDEGRE + 1 cards containing the coefficients are required.
If the polynomial is

$$P(x) = \sum_{i=0}^{\text{NDEGRE}} a_i x^{\text{NDEGRE}-i},$$

then the j^{th} card must contain a_{j-1} .

If IPI = 1, the coefficient is read with format (E12.6).

If IPI = 2, the coefficient is read with format (6I6) and is transcribed as follows:

$$\text{Let } a_{j-1} = b_1 b_2 b_3 b_4 \dots b_{16} b_{17} b_{18} b_{19} b_{20} \times 10^k,$$

where $0 \leq b_i \leq 9$, i.e., a representation with up to 20 significant digits appearing to the left of the decimal point and none to the right. Then the six right-adjusted integers entered on the j^{th} card are, respectively:

$$b_1 b_2 b_3 b_4, b_5 b_6 b_7 b_8, b_9 b_{10} b_{11} b_{12}, b_{13} b_{14} b_{15} b_{16}, b_{17} b_{18} b_{19} b_{20}, k.$$

If the coefficient is negative, negative signs prefix each of the first five entries; k is negative if the coefficient is not an integer.

Further sets of data including title cards may follow and consecutive cases run.

b. Possible Output

For each polynomial equation solved, the following is printed:

1. The title.
2. NDEGRE, M, DELTA, EPSIL (EPSIL is designated EPSILON).
3. The input coefficients of defined (via IPI) precision.
4. The roots of the polynomial with specified (via IPO) precision.
5. The coefficients of the polynomial as reconstructed from the roots and leading coefficient.

c. Additional Information

If any roots cannot be found, an appropriate message is printed, and reconstruction of the coefficients is not attempted.

2. Operating Instructions for CDC-3600

The following set of cards must be presented in order for each polynomial to be solved.

- a. Title card. Format (10A8).

All 80 columns may be used.

- b. DEGREE, M, MMAX, DELTA, EPSILON. Format (3I6, 2E12.4).

If $M = 0$, the standard setting is made for M, MMAX, DELTA and EPSILON.

DEGREE = degree of the polynomial.

M = number of root squarings desired.

MMAX = maximum number of root squarings allowed.

DELTA = the pivotal coefficient bound.

EPSILON = the acceptance tolerance allowed on roots.

The general user is advised to use the standard setting; hence, only DEGREE will occur on this card.

- c. IPI, IPO. Format (2I6).

$$IPI = \begin{cases} 2 \rightarrow \text{double-precision coefficient input.} \\ 1 \rightarrow \text{single-precision coefficient input.} \end{cases}$$

$$IPO = \begin{cases} 2 \rightarrow \text{double-precision output.} \\ 1 \rightarrow \text{single-precision output.} \end{cases}$$

(In any case, internal computations are done in double precision.)

- d. A_i . Format $\begin{cases} E 18.19 & \text{if } IPI = 1 \\ D 36.25 & \text{if } IPI = 2 \end{cases}$

These are the coefficients of the polynomial

$$\sum_{i=0}^n A_i x^{n-i}$$

and are given one to a card beginning with A_0 , on through A_n .

ACKNOWLEDGMENTS

The authors wish to thank William L. Hafner, Jr., for his help in the final debugging of the CDC-3600 code and for his careful preparing and solving of numerous test polynomials, a few of which are discussed in Appendix A.

REFERENCES

1. E. H. Bareiss, Resultant Procedure and the Mechanization of the Graeffe Process, Journal ACM 7, 1960, S. 346-386.
2. E. H. Bareiss and M. A. Fisherkeller, Resultant Procedure, A Method for Finding the Zeros of Real Polynomials, ANL-5997 (1959).
3. W. Barth, Praktisches Vorgchen und Fehlerabschätzung beim Graeffe-Verfahren. Dissertation, Darnstadt (1963).
4. W. Börsch-Supan, A-posteriori Error Bounds for the Zeros of Polynomials, Num. Math. (im Druck).
5. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill Book Company, Inc., New York (1956).
6. A. Ralston, A First Course in Numerical Analysis, McGraw-Hill Book Company, Inc., New York (1965).
7. N. P. Salikov, A Modification of Lobatchevsky's Method for Calculating the Moduli of the Roots of an Algebraic Equation (Russian), Z. Vycisl. Mat. i Mat. Fiz. 3 (1963), 54-70.
8. J. H. Wilkinson, The Evaluation of the Zeros of Ill-conditioned Polynomials, Num. Math. 1, 1959, S. 150-180.
9. J. H. Wilkinson, Error Analysis of Floating-point Computation, Num. Math. 2, 1960, S. 319-340.
10. J. H. Wilkinson, Rounding Errors in Algebraic Processes, Her Majesty's Stationery Office, London, 1963.
11. NR NPRE Double Precision Arithmetic (FORTRAN) SHARE Dist. 1122.