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ANALYTICAL AND EXPERIMENTAL INVESTIGATION
OF A
NUCLEAR REACTOR SUPPORT STRUCTURE*

by

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TABLE OF CONTENTS

	<u>Page</u>
NOMENCLATURE	8
ABSTRACT	9
I. INTRODUCTION.	9
II. PROBLEM DEFINED	11
III. METHOD OF ANALYSIS	13
IV. PROCEDURE OF SOLUTION	15
A. Preliminary Investigation	15
B. Assumptions.	18
C. Basic Procedure	20
1. Strain Energy of Plates	23
2. Strain Energy of Tubes	25
3. Strain Energy of Supports.	29
4. Potential Energy of External Loads	30
5. Final Results	30
D. Simplified Procedure	37
E. Computer Solution	38
1. Description of Computer Program	38
2. Example Problem	44
V. EXPERIMENTAL WORK	47
A. Test Equipment.	47
1. Grid Model	47
2. Edge Support	48
3. Loading Arrangement	49
4. Support Table.	49
5. Instrumentation	51
B. Test Procedure.	52
C. Test Results.	52
VI. CONCLUDING REMARKS.	56

TABLE OF CONTENTS

	<u>Page</u>
APPENDICES	
A. Derived Equations	59
1. Deflection and Slope of Plates	59
2. Expressions for the Moment of Homogeneous Plates . . .	60
3. Deformation of Tubes	61
4. Strain Energy of Plates	64
5. Strain Energy of Tubes	69
6. Strain Energy of Supports	72
7. Potential Energy of External Loads	74
B. Computer Program	76
1. Fortran II Statements	76
2. Sample Results	92
C. Experimental Data	96
ACKNOWLEDGMENT	99
REFERENCES	100

LIST OF FIGURES

<u>No.</u>	<u>Title</u>	<u>Page</u>
1.	Experimental Breeder Reactor II	10
2.	Typical Grid Support Structure of a Nuclear Reactor	11
3.	Pictorial of Reactor Vessel Grid Assembly.	12
4.	Blanket Assembly in a Deformed Tube	13
5.	Convergence of Deflection Functions	17
6.	Effective Elastic Constants for Perforated Plates	24
7.	Deformed Grid Tube	26
8.	Stress Intensities in Perforated Plate Ligaments	35
9.	Maximum Local Stress in Perforated Plates	36
10.	Peak Stress at Perforations Adjacent to Rim.	36
11.	Segment of a Grid Plate	40
12.	Pictorial Convergence of Deflection.	45
13.	Pictorial Convergence of Radial Stress	45
14.	Variation of Slope.	46
15.	Structure of Test Model.	48
16.	Test Structure and Auxiliary Support Equipment	49
17.	Close-up of Test Loading Arrangement	50
18.	View of the Entire Test Set-up	50
19.	Key to the Location of Strain Gages	51
20.	Matching the Edge Deflections	53
21.	Analytical and Experimental Distributions of Radial Stress . . .	54
22.	Analytical and Experimental Distributions of Tangential Stress	54
23.	Variation in Top and Bottom Plate Thickness	55
24.	General Load Distribution in the Rods	55

LIST OF TABLES

<u>No.</u>	<u>Title</u>	<u>Page</u>
1.	Measured Grid-plate Deflection	96
2.	Average Experimental Strains	96
3.	Experimental Load Distribution in the Rods	98

NOMENCLATURE

a	Outside radius of the plates, in.
A	Undetermined coefficients of the deflection function
B	Total number of concentric regions in a plate
C	Constant or coefficient
d	Tube diameter, in.
D	$Eh^3/12(1-\nu^2)$, flexural rigidity of plate, lb-in.
E	Modulus of elasticity, lb/in. ²
f	Intensity of uniformly distributed load, lb/in. ²
h	Plate thickness, in.
i,j	Indices referring to location of holes, tubes, or loads
I	$\pi(d_o^4 - d_i^4)/64$, tube moment of inertia, in. ⁴
k	Constant indicating the degree of edge restraint, defined by Eq. (2)
K_ℓ, K_p, K_r	Stress multipliers, given by O'Donnell and Langer, Ref. 3
ℓ	Distance between the plates, in.
m,n	Indices of terms in the deflection function
M	Bending moment, lb-in.
N	Maximum number of terms in the series of the deflection function
p,q	Constants associated with the edge restraint, defined by Eq. (8)
P	Individual loads, lb
r, θ , z	Cylindrical coordinates of plates
s	Distance between center lines of perforations, in.
t	Minimum ligament width, in.
U	Strain energy, in.-lb
V	Potential energy due to external load, in.-lb
w	Plate deflection, in.
W	Total potential energy, in.-lb
x,y	Coordinates of the tubes
α	See Eq. (26)
β	σ_r/σ_θ or σ_θ/σ_r such that $-1 \leq \beta \leq 1$, biaxiality ratio
δ	See Eq. (26)
ζ	See Eq. (26)
η	See Eq. (14)
λ	Number of holes, tubes, or loads
ν	Poisson's ratio
ξ	See Eq. (9)
ρ	Weight density, lb/in. ³
σ	Unit stress, lb/in. ²
χ	Lagrangian multiplier

Subscripts

h	Hole
i	Inside
ℓ	Load
m,n	Indices designating the terms of a summation
o	Outside
p	Plate
r	Radial
s	Support
t	Tube
θ	Tangential

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ABSTRACT

A general procedure is presented for determining the deformation (slopes and deflections) of a grid support structure for a nuclear reactor. Use is made of the Rayleigh-Ritz perturbation method which involves an infinite series of polynomials.

The grid structure consists of two circular perforated plates separated and interconnected by a large number of tubes. The holes are distributed in a hexagonal pattern with respect to the center of the plates. Tubes pass through each hole and serve as supports for loads acting on the top end of every tube. The edges of the plates are peripherally supported and may have any degree of restraint ranging from fixed to simple supports.

The complexity introduced by the holes in the plates is circumvented by the use of an equivalent solid plate. This idea, developed and substantiated by a number of investigators, is also applied to approximate pertinent stresses in the perforated plates.

A model was constructed to yield some experimental data. Although the model was a simplification of what could be expected in practice, nonetheless it brought forth some interesting and unexpected results. The analytical predictions, provided by an IBM-704 Fortran II computer program, were compared with the experimental results. The results were discussed and appropriate conclusions were drawn.

I. INTRODUCTION

With the advent of the atomic and space age many challenging problems have arisen in the field of structural analysis and design. Advanced

designs have called for more complex structures. In turn, such structures have required analyses of increasingly greater accuracy. The solution of complex problems has, at the same time, been greatly simplified by the use of modern computers. Although the computers have provided only an approximate approach, the results obtained have, for the most part, been sufficiently accurate to serve their intended purpose.

The problem considered in this report, although fairly typical for nuclear reactors, originated with the newly operational Experimental Breeder Reactor II designed and operated by Argonne National Laboratory under the auspices of the U. S. Atomic Energy Commission. The facility is an unmoderated, heterogeneous, sodium-cooled reactor and power plant.^{(1)*} Heat is generated in an enriched uranium core, which is surrounded on all sides by a fertile blanket of depleted uranium. The fuel and blanket assemblies are of hexagonal shape, thus providing a close-packed reactor geometry. These assemblies are supported and properly oriented in position by a grid structure from below, as shown in Fig. 1. Great accuracy is attained in the location of the subassemblies at the expense of very close tolerances in the grid structure.

Fig. 1
Experimental Breeder
Reactor II

139-461

Deviations in assumed geometry may result in interferences and binding during loading operation. Differences in structural deflections resulting from temperature-dependent changes in flexural rigidity in proceeding from cold to hot operations may alter the nuclear reactivity characteristics. It is necessary to analyze the structure to determine whether its rigidity is sufficient to prevent binding and to determine the magnitude of the changes of nuclear reactivity.

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*Supercript numbers in the text designate references listed at the end of this report.

II. PROBLEM DEFINED

The grid support structure to be analysed consists of two parallel perforated plates positioned one above the other and interconnected by a large number of tubes. A typical structure is shown in Fig. 2. The perforations are symmetrical with respect to the center of the plates and arranged in an equilateral triangular pattern. Tubes extend through the holes and are rigidly connected to both plates. Fuel and blanket assemblies rest on the top plate of the structure, as shown in Fig. 1. The weight of these assemblies supplies the total external load to the system.

112-3159

Fig. 2. Typical Grid Support Structure of a Nuclear Reactor

The structure would ordinarily be supported by the wall of the reactor vessel, as shown in Fig. 3. The boundary conditions depend on the manner of anchoring. The analysis of the structure, therefore, has to be sufficiently general to accommodate any degree of bending edge restraint.

Under load, the top plate deflects. Because of the interconnecting tubes, a deflection is also induced in the bottom plate. In addition, the interconnecting tubes deform because of the deflections of the top and bottom plates. Excessive deformation of the tubes could cause contact of their inside walls with the protrusions of the blanket assemblies (see Fig. 4). Removal of the assemblies could become obstructed if tube

112-576

Fig. 3. Pictorial of Reactor Vessel Grid Assembly

deformations became excessive. Furthermore, a substantial deformation of the plates might sufficiently tilt the assemblies so as to affect the geometry of the reactor core. In view of the above-mentioned reasons and the fact that very close tolerances have to be preserved, it is obvious that a deformation analysis of the grid structure is of utmost importance to a safe and satisfactory operation of the reactor.

Fig. 4. Blanket Assembly in a Deformed Tube

Excessive deformation is only one factor to be considered. Maximum stress intensities in the structure also have to be kept under a pre-assigned limit. High stresses could damage the material or make it act "plastically" - a characteristic which would not be apparent from an elastic deformation analysis alone. It is desirable, therefore, to predict the stress intensities in the structure.

Once a method is developed for the analysis of a grid structure, it is desirable to substantiate it by an experimental test. Because actual structures do not easily lend themselves to experimental investigation, it becomes necessary to resort to model analysis. There is no intention of making the model simulate any particular grid structure, which would require unduly high cost. The model, nevertheless, should be within the domain of the assumptions made in the analytical method.

In summary, it is required to produce a general solution to the previously described structure so that any type of edge conditions can be considered. Of primary importance is the determination of accurate deformation of the grid structure. An approximate prediction of the pertinent stress intensities is also required, but is of secondary importance. Experimental evidence is desired to support the findings of the analytical procedure.

III. METHOD OF ANALYSIS

Very little theoretical work related to perforated plates is to be found in the technical literature. Probably the most extensive investigations on bending of circular plates thus far have been those by Olszak and Mroz.⁽²⁾ They analyzed a circular plate with one eccentrically located hole for both simple and fixed supports at the edges. All previous work on similar problems had been restricted to fixed edges.

The approximate analysis of perforated plates has recently received considerable attention because of their widespread use in heat-exchange

equipment. A proposed method involving an "equivalent plate" has gained rather wide acceptance. The basic principle of this method consists of using fictitious elastic constants for the plate as though it were without holes. The elastic constants consist of a modified modulus of elasticity E^* and a Poisson's ratio ν^* which depend on the characteristics of the perforations.

O'Donnell and Langer⁽³⁾ summarized such work done and suggested methods of analysis based on experimental evidence. They obtained average ligament stresses from purely theoretical considerations, but derived effective elastic constants and peak stresses from strain measurements and photoelastic tests.

In this report, the grid structure as a whole will be analyzed. Through the introduction of the simplifying assumption of the "equivalent plate," the problem reduces to the investigation of the same grid structure without holes in the plates.

The simplified structure is still complicated by the presence of the connecting tubes between the plates. The structure is redundant to a high degree and exact methods are again at a great disadvantage. But, even if an exact solution could be obtained for this modified structure, it is doubtful if it would be very meaningful. Uncertainties introduced by the initial assumptions of the "equivalent plate," not to mention the ordinary design approximations, would certainly tend to diminish the significance of the results. In view of this, the idea of an exact method of analysis was discarded in favor of an approximate approach.

The familiar energy methods appeared especially applicable to the solution of the grid-structure problem. Instead of solving the governing differential equations with respect to the boundary conditions, one seeks functions that satisfy the boundary conditions and minimize the potential energy of the system. As a consequence, one has to deal with algebraic rather than differential equations.

Of particular interest to the grid problem was the so called Rayleigh-Ritz perturbation method. This method, initiated by Rayleigh⁽⁴⁾ and improved by Ritz,⁽⁵⁾ has been found quite useful in the analysis of many engineering problems. Its procedure is based on an assumed shape or deflection function in the form of a series with undetermined coefficients. Each term of the series must satisfy the geometric boundary conditions. The total potential energy of the system may then be obtained in terms of the undetermined coefficients since it is a function of the deflection. When this has been done, one utilizes the principle of minimum potential energy. Minimizing the total potential energy of the system with respect to each of the undetermined coefficients yields a set of linear algebraic equations in which the number of unknown coefficients is equal to the number of equations. When the coefficients are evaluated, the deflection expression is known and the problem is considered solved.

IV. PROCEDURE OF SOLUTION

The amount of work in the Rayleigh-Ritz method depends, to a large degree, on the characteristics of the assumed shape or deflection function. The procedure for choosing the appropriate deflection function is described in the following section. Subsequent sections contain the general solution of the grid-structure problem using the chosen deflection function.

A. Preliminary Investigation

The Rayleigh-Ritz method is based on a deflection function in the form of a series with undetermined coefficients, each term of the series satisfying the given boundary conditions. Since a number of such deflection expressions could be applicable for any problem, it is desirable to select the best of these for the investigation. It is well known from the properties of the Rayleigh-Ritz method that the greater the number of terms in the series of any given deflection function, the closer the approximation to the theoretical value. This property - the "rate of convergence" to the exact solution - may be used as a measure of the function quality.

Ideally, such a preliminary investigation of the deflection function could best be made in cases where an exact solution is available; the "rate of convergence" of the approximate solutions could then be observed by comparison with the known answer. Hence, it should be most advantageous to consider a case that is similar to the given problem and yet sufficiently simple to justify such a preliminary investigation.

A simplification to the deflection of the grid structure was considered to be the deflection of a single circular plate. Thus, the preliminary investigation of the deflection functions was confined to a circular plate with fixed edges. Two types of loading were considered: a concentrated load acting through the center of the plate, and a uniformly distributed load.

Fixed edges required that the functions satisfy the following geometric boundary conditions:

$$w \Big|_{r=a} = 0; \quad \frac{dw}{dr} \Big|_{r=a} = 0.$$

In addition, the functions were selected so as to also satisfy the conditions

$$w \Big|_{r=0} \neq 0; \quad \frac{dw}{dr} \Big|_{r=0} = 0.$$

Symmetrical deflection of the plate required the use of only one parameter in the deflection functions - the radius r of the cylindrical coordinate system. The choice was further confined to two types of functions that are commonly used in the Rayleigh-Ritz method: the trigonometric functions and polynomials. At first sight, the trigonometric functions were favored because of their property of orthogonality. It was found in the initial stages of the investigation, however, that these functions introduced considerable difficulty in the determination of strain energy of the plate. Integration could be made only by numerical means. Since approximations at this stage of the analysis were undesirable, the use of trigonometric functions was discarded.

The polynomial deflection functions lack the desirable property of orthogonality, but exact solutions of the strain energy of the plate can be obtained. There are also a number of polynomial expressions which satisfy the necessary boundary conditions. The object, then, is to select the most suitable expression for the analysis of the plate.

In the process of investigating the "convergence" of the deflection functions, several polynomial series-expressions were considered. The initial investigation was conducted having assumed only the first term of the polynomial series. These functions were used in the Rayleigh-Ritz method* to approximate the maximum deflection of the two above-mentioned cases of loading. Comparison was made with the exact solution (neglecting shear deflection), which is⁽⁸⁾

$$w_{\max} = \frac{1}{16} \frac{Pa^2}{\pi D}$$

for the concentrated load P , and

$$w_{\max} = \frac{1}{64} \frac{fa^4}{D}$$

for the uniformly distributed load f . The following functions exhibited reasonable accuracy in this first-order approximation:

$$w = \left[1 - \left(\frac{r}{a} \right)^2 \right]^2 \sum_{\substack{n=0 \\ n \neq 1}}^N r^n A_n;$$

$$w = \left[1 - \left(\frac{r}{a} \right)^2 \right]^2 \sum_{n=0}^{N-1} r^{2n} A_n;$$

---

*The Rayleigh-Ritz method with just one term of the series in the deflection function is often referred to as the Rayleigh method.

$$w = \sum_{n=0}^{N-1} \left[1 - \left(\frac{r}{a} \right)^2 \right]^{n+2} A_n,$$

where N is the maximum number of terms in the series. It should be noted that these equations (when $n = 0$) correspond to exact deflections for the case of a uniformly loaded plate. This is expected because the exact solution has the identical form

$$w = \frac{fa^4}{64D} \left[1 - \left(\frac{r}{a} \right)^2 \right]^2.$$

It also should be pointed out that the first two of the three given deflection functions exclude the series term $n = 1$, because this term gives an infinite value for the strain energy of the plate. Since this is known to be false, the value of the coefficient A_1 must be excluded from the series.

In the process of reducing the number of deflection functions still further, the logical step was to consider two, three, four, etc., terms in each of the series, following through the Rayleigh-Ritz method, and observing the "rate of convergence" of each function. The final results of this procedure are plotted in Fig. 5.

Fig. 5
Convergence of Deflection
Functions

It should be noted that N can only have integral values and, therefore, the curves of Fig. 5 do not represent continuous functions. Nevertheless, this graphical representation was found to be helpful in comparing the "convergence" of the different deflection functions. It was quite apparent from Fig. 5 that the function

$$w = \left[1 - \left(\frac{r}{a} \right)^2 \right]^2 \sum_{\substack{n=0 \\ n \neq 1}}^N r^n A_n \quad (1)$$

was the best function for the case of the concentrated load in the center of the plate. As previously noted, this function also gave an exact solution when the plate was subjected to a uniformly distributed load. Thus, in all respects, Eq. (1) represented the best deflection function.

It is recalled that this preliminary investigation applies only to the condition of fixed supports. A similar investigation for the case of simple supports would also be of interest. However, it was discovered (as will be shown in the subsequent sections of this report) that Eq. (1) can be modified to accommodate any degree of edge restraint.* Furthermore, it was found that the first term in the series of the modified deflection function gives an exact solution for any degree of edge restraint provided the external load is uniformly distributed. This made the investigation of deflection functions for simple supports unnecessary.

B. Assumptions

Before any engineering analysis is undertaken, some simplifying assumptions usually have to be made. The worth of the analysis is largely dependent on the validity of the assumptions. It is of great importance, therefore, to state precisely the factors that do or might affect the analysis. Assumptions governing the analysis contained in this report may be enumerated as follows:

a) The material in the grid structure obeys Hooke's Law. Although this would be true under ordinary conditions, similar structures may very likely be exposed to high temperatures. When high temperatures are involved, there are two distinct considerations that have to be taken into account, namely, a temperature gradient across the plates and a uniform elevated temperature throughout the structure. The first of these - the temperature gradient - is not considered in this report. A uniform temperature usually causes complications in the analysis whenever the temperature is excessively high. It is important, therefore, to state the limiting temperature for which the analysis is still valid. Since such grid structures are usually made of stainless steel, Hooke's Law holds for ambient temperatures below 900°F.⁽⁶⁾

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*This modified deflection function is given by Eq. (2), p. 20.

b) Any radiation effect in the material is negligible. The effect of neutron irradiation on the structural qualities of steel is to increase the yield point and ultimate strength, to decrease the ductility of the material, and to raise the ductile-to-brittle transition temperature. As long as the stresses remain below the yield point and a large number of stress reversals is avoided, the assumption of negligible radiation damage is reasonable and valid.

c) Perforated plates of the grid structure can be treated as "equivalent solid plates." This assumption has been experimentally substantiated by bending tests of perforated plates⁽⁷⁾ and provides best results when the ratio of plate thickness h and the distance s between the centerlines of the holes is larger than two, i.e., $h/s > 2$.

d) Deformation* of both plates of the structure is the same. This statement implies that the plates are identical in shape and have the same edge restraint. It also assumes that the tubes have uniform walls. The veracity of this assumption lies in the relative stiffness of the plates and the tubes. The compressive stiffness is here of major importance because the difference in the deflection of the plates depends on the axial contraction of the tubes. Since the compressive stiffness of tubes is high and the number of the tubes is great, the assumption of identical plate deformation would be very nearly realized.

e) Plate deflection is independent of θ . This assumption is not strictly true because of the hexagonal tube distribution and the localized effects of the tubes. Nevertheless, it is believed that due to the great number of tubes the assumption would be valid.

f) Membrane stresses in the plates are negligible. This statement is equivalent to stating that the deflection of the plate is small as compared with its thickness.⁽⁸⁾

g) The individual load is taken as acting at the centerline of the tube, rather than along the periphery. It should be noted that, when the grid structure deforms, the load would not remain evenly distributed over the periphery (see Fig. 4). The line of action of the force would be displaced in the direction away from the center of the plate. This discrepancy, however, would be minor, and the assumption of the loads acting through the center of the tubes should be good.

h) In place of the concentrated moments exerted on the plates by the tubes, an equivalent average moment is assumed. This assumption should be valid as long as the number of tubes remains large.

---

*The term "deformation" as used here refers to the deformation pertaining to the deflection of the plates and the tubes.

C. Basic Procedure

The analysis is based on the general solution of a simplified problem. Since the number of loads on the structure is large, it is reasonable to base the solution on the deflection of a uniformly loaded circular plate. It is interesting to note that the general deflection (neglecting shear deflection) of the plate under a uniform load may be generalized to incorporate the complete range of edge restraints. This deflection may be represented by the following expression:

$$w = \frac{fa^4}{64D} \left[\frac{5 + \nu - 4k}{1 + \nu} - 2 \frac{3 + \nu - 2k}{1 + \nu} \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right], \quad (2)$$

where the constant k depends on the degree of bending edge restraint, and ranges from zero for simple supports to unity for fixed supports. The meaning of k may best be realized by observing the expressions of the radial edge moment and slope for the entire range of edge restraints; the moment may be shown to be in terms of k and the fixed edge moment ($k = 1$), and the slope in terms of k and the slope of a free edge ($k = 0$). The relations are as follows:

$$M_r|_{r=a} = kM_r|_{r=a, k=1};$$

$$\frac{dw}{dr}|_{r=a} = (1 - k) \frac{dw}{dr}|_{r=a, k=0}$$

When the load on the plate is other than uniformly distributed but still symmetrical with respect to the center, the deflection expression (2) must be modified. The modified deflection function may be taken in the form of the polynomial*

$$w = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n - 2q_n \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right] r^n A_n, \quad (3)$$

where the constants p_n and q_n depend on the given restraint at the boundaries. In the case of fixed edges, the values of p_n and q_n are determined from the following boundary conditions:**

*It should be noted that the term $n = 1$ is excluded from the series because it involves a discontinuity in subsequent calculations.

**For derivation, see Appendix A.

$$w|_{r=a} = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n - 2q_n + 1] a^n A_n = 0;$$

$$\left. \frac{dw}{dr} \right|_{r=a} = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n n - 2q_n(n+2) + (n+4)] a^{n-1} A_n = 0.$$

Since each term of the series has to satisfy the geometric boundary conditions, it follows that every term must satisfy the following algebraic equations:

$$p_n - 2q_n + 1 = 0;$$

$$p_n n - 2q_n(n+2) + n + 4 = 0.$$

The solution of these simultaneous equations shows that

$$p_n = q_n = 1. \quad (4)$$

Thus, in the case of fixed edges, Eq. (2) is identical with Eq. (1).

Similarly, for the case of zero bending edge restraint of simple supports, the corresponding boundary conditions are*

$$w|_{r=a} = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n - 2q_n + 1] a^n A_n = 0;$$

$$M_r|_{r=a} = -D^* \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) + (n+4)(n+3+\nu)] a^{n-2} A_n = 0,$$

where D^* is the flexural rigidity of the plate. The resulting simultaneous equations are

$$p_n - 2q_n + 1 = 0;$$

$$p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) + (n+4)(n+3+\nu) = 0.$$

Hence, the values of the constants for the case of simple supports are found to be

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*For derivation of the radial moment, see Appendix A.

$$\left. \begin{aligned} p_n &= 1 + \frac{4}{1 + \nu + 2n} \\ q_n &= 1 + \frac{2}{1 + \nu + 2n} \end{aligned} \right\}. \quad (5)$$

It would be reasonable to assume that any other degree of edge restraint would have values of p_n and q_n intermediate between those given by Eqs. (4) and (5). This may be seen by introducing constant k in these expressions as follows:

$$\left. \begin{aligned} p_n &= 1 + \frac{4(1-k)}{1 + \nu + 2n} \\ q_n &= 1 + \frac{2(1-k)}{1 + \nu + 2n} \end{aligned} \right\}, \quad (6)$$

where the boundary condition $w|_{r=a} = 0$ is still satisfied. Thus, Eq. (6) represents values of p_n and q_n for any edge restraint, where k extends from zero to one and is independent of n .

It is interesting to observe the relationship of k with the slope and moment at the edge of the plate. The previously shown expressions for the edge slope and moment, respectively, are

$$\left. \frac{dw}{dr} \right|_{r=a} = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n n - 2q_n(n+2) + n + 4] a^{n-1} A_n;$$

$$M_r|_{r=a} = -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} [p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) + (n+4)(n+3+\nu)] a^{n-2} A_n.$$

When the expressions of Eq. (6) are substituted, the equations simplify to*

$$\left. \frac{dw}{dr} \right|_{r=a} = -\frac{8(1-k)}{a} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \frac{a^n A_n}{1 + \nu + 2n};$$

$$M_r|_{r=a} = -\frac{8Dk}{a^2} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} a^n A_n.$$

---

*For the derivations, refer to Appendix A.

It is observed that because of the series in these expressions, neither the slope nor the moment is simply proportional to k . This is true for the intermediate cases of edge restraint. In order to have any control in the range $0 < k < 1$, however, it is necessary to make the edge moment directly proportional to k .

The maximum moment at the radial takes place when $k = 1$. Consequently, the edge moment for $0 \leq k \leq 1$ may be written as follows:

$$M_r \Big|_{r=a} = k M_r \Big|_{r=a} \Big|_{k=1} \quad (7)$$

By expressing the left side of this equation in terms of the series, the same condition is realized by

$$\sum_{\substack{n=0 \\ n \neq 1}}^{\infty} a^n A_n = - \frac{a^2}{8D} M_r \Big|_{r=a} \Big|_{k=1} \quad (8)$$

Hence, this expression represents an auxiliary condition to be satisfied in the final selection of the appropriate values of A_n .

Once the deflection expression to be used in the investigation is established, the procedure of the Rayleigh-Ritz method calls for determination of the potential energy of the system. The total potential energy of the grid structure (subject to the previously mentioned assumptions) consists of four parts: strain energy of plates, strain energy of tubes, strain energy of supports, and potential energy of external loads. The corresponding terms for the strain energy are analyzed in the subsequent sections.

1. Strain Energy of Plates

Since bending alone is considered in the deformation, the strain energy stored in a circular plate⁽⁸⁾ of radius a is given by

$$U_p = \frac{D}{2} \int_0^{2\pi} \int_0^a \xi dr d\theta, \quad (9)$$

where

$$\xi = r \left(\frac{d^2 w}{dr^2} \right)^2 + \frac{1}{r} \left(\frac{dw}{dr} \right)^2 + 2\nu \left(\frac{d^2 w}{dr^2} \right) \left(\frac{dw}{dr} \right).$$

In the design of the reactor support structures it is practically impossible to use plates that are uniform in thickness or solid throughout. It is often

necessary to make the plate thinner in some places and/or perforated in others.[†] To accommodate this possibility, the plate area is divided into several concentric zones as shown in Fig. 2. In this case the strain energy of the plate becomes

$$U_p = \frac{1}{2} \int_0^{2\pi} \left[\sum_{b=1}^B D_b \int_{r_{b-1}}^{r_b} \xi_b dr \right] d\theta; \quad r_B = a, \quad (10)$$

where each subscript refers to an appropriate effective elastic constant⁽³⁾ (D_b , ν_b) in the zone circumscribed by r_b ; B is the total number of such zones.

Fig. 6

Effective Elastic Constants
for Perforated Plates*

*This figure is given by
O'Donnell and Langer
in Ref. 3.

When the indicated mathematical operations are performed, the strain energy of both plates may be written in terms of the undetermined coefficients as follows:^{††}

$$U_p = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C'_{mn} A_m A_n, \quad (11)$$

[†]In perforated regions the values of E , ν , and D should be changed to E^* , ν^* , and D^* , respectively, by the use of the graphs given in Fig. 6.

^{††}Derivation of this expression for the strain energy is shown in Appendix A.

where $m \neq 1$, $n \neq 1$, and $m \leq n$ in Eq. (11) and in all m - and n -indexed equations that follow. Here,

$$\begin{aligned}
 C'_{mn} = \frac{4\pi CD}{a^2} a^{m+n} & \left\{ p_m p_n m n \left[\frac{1 + (m-1)(n-1)}{m+n-2} + \nu \right] - 2p_m q_n m(n+2) \left[\frac{1 + (m-1)(n+1)}{m+n} + \nu \right] \right. \\
 & - 2q_m p_n (m+2)n \left[\frac{1 + (m+1)(n-1)}{m+n} + \nu \right] + p_m m(n+4) \left[\frac{1 + (m-1)(n+3)}{m+n+2} + \nu \right] \\
 & + 4q_m q_n (m+2)(n+2) \left[\frac{1 + (m+1)(n+1)}{m+n+2} + \nu \right] + p_n (m+4)n \left[\frac{1 + (m+3)(n-1)}{m+n+2} + \nu \right] \\
 & - 2q_m (m+2)(n+4) \left[\frac{1 + (m+1)(n+3)}{m+n+4} + \nu \right] - 2q_n (m+4)(n+2) \left[\frac{1 + (m+3)(n+1)}{m+n+4} + \nu \right] \\
 & \left. + (m+4)(n+4) \left[\frac{1 + (m+3)(n+3)}{m+n+6} + \nu \right] \right\},
 \end{aligned}$$

with

$$C = 1 \quad \text{for } m \neq n$$

and

$$C = \frac{1}{2} \quad \text{for } m = n.$$

2. Strain Energy of Tubes

If the angular rotation of the tube at its ends (equal to the slope of the grid plate at that point) is known, it is quite easy to determine the strain energy of the tube in terms of the undetermined coefficients. The tubes may be considered as beams with shear forces and bending moments at its ends, as shown in Fig.7.

The strain energy for bending* of a uniform beam is given by⁽⁹⁾

$$U = \frac{1}{2E_t I} \int_{h/2}^{\ell + (h/2)} M^2 dx. \quad (12)$$

The moment M in terms of the curvature of the beam is

$$M = -E_t I \frac{d^2 y}{dx^2},$$

---

*Strain energy due to axial load is neglected in this analysis.

where x and y are the coordinate axes of the tube. The general expression for the moment, as it applies to the tubes, becomes

$$M = M_B - F_B \left(x - \frac{h}{2} \right).$$

Fig. 7
Deformed Grid Tube

With a double-integration method for the analysis of the tubes, there are four unknowns: F_B , M_B , and two constants of integration. On the other hand, there are four known boundary conditions (known in terms of undetermined coefficients) for the deflections and slopes at the ends of the tubes. The double-integration procedure yields the following expressions:*

$$F_B = \frac{12E_t I(l+h)}{l^3} \frac{dw}{dr}; \quad M_B = \frac{6E_t I(l+h)}{l^2} \frac{dw}{dr}.$$

*For the derivation of the expressions see Appendix A.

Hence,

$$M = \frac{6E_t I (\ell + h)}{\ell^3} (\ell + h - 2x) \frac{dw}{dr}.$$

Finally, by substituting this moment expression into Eq. (12) and integrating, the expression for the strain energy of one tube becomes

$$U_t = \frac{6E_t I (\ell + h)^2}{\ell^3} \left(\frac{dw}{dr} \right)^2.$$

The total strain energy of all the tubes is the sum of their individual strain energies:

$$U_t = \frac{6E_t I (\ell + h)^2}{\ell^3} \sum_{i,j}^{\lambda_t} \left(\frac{dw}{dr} \right)_{i,j}^2,$$

where the indices i and j refer to the coordinates of any convenient two-dimensional coordinate system.

At this point, however, the individual strain energies of the tubes are expressed in the terms of the undetermined coefficients. Consequently, if all of the separate strain energies are added together, they lose their identities in the final form. This addition is really an irreversible operation. In other words, the total sum of the tube strain energy does not indicate what fraction of the total is due to a particular tube in question.

In order to establish the correct configuration of the plate deflection, it is necessary to specify the correct tube distribution. Because of the large number of tubes, the "tube density" may be quite accurately approximated by assuming uniform tube distribution over the plates. In this way the strain energy absorbed by the tubes is essentially assumed to be absorbed by an imaginary medium between the plates.

Thus, for the representation of the total tube strain energy, a single expression is required which would represent the true magnitude of the energy and also give a correct distribution of it with respect to the plates. The variation of the tube strain energy with respect to r , as noted from the expression for the single tube strain energy, is proportional to the square of the slope. Consequently, the strain energy of the tubes may be expressed in the form of the following function:

$$U_t = \frac{6\lambda_t E_t I (\ell + h)^2}{\pi a^2 \ell^3} \int_0^{2\pi} \int_0^a \left(\frac{dw}{dr} \right)^2 r dr d\theta. \quad (13)$$

This revised expression implies a uniform variation of the tube strain energy over the region between the plates instead of the algebraic energy sum of the individual tubes.

It would not be unusual to conceive a structure with different sizes of tubes. For this reason, the area of the plates again may be imagined to be divided into several concentric regions with different sizes of tubes. In this case the modified expression for the strain energy of the tubes becomes

$$U_t = \int_0^{2\pi} \left[\sum_{b=1}^B \eta_b \int_{r_{b-1}}^{r_b} \left(\frac{dw}{dr} \right)^2 r dr \right] d\theta; \quad r_B = a, \quad (14)$$

where $\eta = 6\lambda_t E_t I(\ell+h)^2 / \pi \ell^3 (r_b^2 - r_{b-1}^2)$ and the subscripts refer to elastic constants in the appropriate regions of the plates. The strain energy of the tubes, as in the case of the plates, may be expressed in the form*

$$U_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn}'' A_m A_n, \quad (15)$$

where

$$\begin{aligned} C_{mn}'' = \frac{24C\lambda_t E_t I(\ell+h)^2}{\ell^3(r_b^2 - r_{b-1}^2)} & \left\{ p_m p_n \frac{mn}{m+n} \left[\frac{r_b^{m+n} - r_{b-1}^{m+n}}{a^0} \right] - 2p_m q_n \frac{m(n+2)}{m+n+2} \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^2} \right] \right. \\ & - 2q_m p_n \frac{(m+2)n}{m+n+2} \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^2} \right] + p_m \frac{m(n+4)}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] \\ & + 4q_m q_n \frac{(m+2)(n+2)}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] + p_n \frac{(m+4)n}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] \\ & - 2q_m \frac{(m+2)(n+4)}{m+n+6} \left[\frac{r_b^{m+n+6} - r_{b-1}^{m+n+6}}{a^6} \right] - 2q_n \frac{(m+4)(n+2)}{m+n+6} \left[\frac{r_b^{m+n+6} - r_{b-1}^{m+n+6}}{a^6} \right] \\ & \left. + \frac{(m+4)(n+4)}{m+n+8} \left[\frac{r_b^{m+n+8} - r_{b-1}^{m+n+8}}{a^8} \right] \right\}, \end{aligned}$$

with

$$C = 1 \quad \text{for } m \neq n$$

and

$$C = 1/2 \quad \text{for } m = n.$$

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*The derivation of the expression for the tube strain energy is given in Appendix A.

3. Strain Energy of Supports

The strain energy stored in the supports of the grid structure can also be included in the analysis. In the intermediate states of edge restraint, the supports deform due to the action of the moment at the edges and thus store strain energy. An indication of the amount of energy stored in the supports is given by the magnitudes of the slope and moment at the edges.

It is well known that for elastic systems the edge moment is always directly proportional to the angular rotation (slope at the edge). The factor of proportionality depends on the rigidity of the supporting structure. In addition, both the slope and moment are known for the initial and final configuration of the structure. Hence, the work done by the moment on the support structure (which is equal to the strain energy stored in the supports) may be expressed as

$$U_s = \frac{1}{2} \left(2\pi a M_r \Big|_{r=a} \right) \frac{dw}{dr} \Big|_{r=a}. \quad (16)$$

When the series expressions are used, Eq. (16) becomes*

$$U_s = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn}''' A_m A_n, \quad (17)$$

where

$$\begin{aligned} C_{mn}''' = \frac{4\pi CD_B}{a^2} a^{m+n} & \left\{ p_m p_n^{mn} \left[\frac{m+n-2}{2} + \nu_B \right] - 2p_m q_n m(n+2) \left[\frac{m+n}{2} + \nu_B \right] \right. \\ & - 2q_m p_n (m+2)n \left[\frac{m+n}{2} + \nu_B \right] + p_m^{m(n+4)} \left[\frac{m+n+2}{2} + \nu_B \right] \\ & + 4q_m q_n (m+2)(n+2) \left[\frac{m+n+2}{2} + \nu_B \right] + p_n (m+4)n \left[\frac{m+n+2}{2} + \nu_B \right] \\ & - 2q_m (m+2)(n+4) \left[\frac{m+n+4}{2} + \nu_B \right] - 2q_n (m+4)(n+2) \left[\frac{m+n+4}{2} + \nu_B \right] \\ & \left. + (m+4)(n+4) \left[\frac{m+n+6}{2} + \nu_B \right] \right\}, \end{aligned}$$

---

*This expression is derived in Appendix A.

with

$$C = 1 \quad \text{for } m \neq n$$

and

$$C = 1/2 \quad \text{for } m = n.$$

4. Potential Energy of External Loads

The part of potential energy that derives from a gravitational load is simply the product of the magnitude of the load and its corresponding deflection. Since there are many such loads acting on the grid structure, the total potential energy of external loads may be represented as a summation:

$$V = - \sum_{i,j}^{\lambda_\ell} P_{i,j} w_{i,j}. \quad (18)$$

This equation, expressed in terms of the undetermined coefficients, is*

$$V = - \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} C_m A_m, \quad (19)$$

where

$$C_m = \sum_{i,j}^{\lambda_\ell} P_{i,j} r_{i,j}^m \left[p_m - 2q_m \left(\frac{r}{a} \right)_{i,j}^2 + \left(\frac{r}{a} \right)_{i,j}^4 \right].$$

5. Final Results

The total potential energy W of the system is the algebraic sum of Eqs. (11), (15), (17), and (19), and may be combined into a single expression:

$$W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} A_m A_n - \sum_{m=0}^{\infty} C_m A_m \quad (20)$$

---

*The derivation of this equation is presented in Appendix A.

where

$$C_{mn} = C'_{mn} + C''_{mn} + C'''_{mn}.$$

The constant C in the expressions for C'_{mn} , C''_{mn} , and C'''_{mn} is no longer dependent on the combinations of m and n , but always equals to unity ($C = 1$). This potential energy in the case of the two extreme boundary conditions ($k = 0$) and ($k = 1$) may be directly subjected to the requirements of the principle of minimum potential energy, i.e., $\partial W / \partial A_m = 0$. For $0 < k < 1$, in addition to minimizing W , the moment restriction given by Eq. (8) must also be satisfied. The moment condition represented by Eq. (8) may be more conveniently written as

$$\sum_{n=0}^{\infty} a^n A_n = -\frac{a^2}{8D_B} M_r \Big|_{r=a} = C_N. \quad (21)$$

In order to include the additional condition into the analysis, the method of Lagrangian multipliers may be utilized. This method results in the modified energy expression

$$\bar{W} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} A_m A_n + \chi \sum_{m=0}^{\infty} a^m A_m - \sum_{m=0}^{\infty} C_m A_m, \quad (22)$$

where χ is the Lagrangian multiplier. The modified energy equation is subjected to the same requirements as in the previous case: $\partial \bar{W} / \partial A_m = 0$. The result of minimizing the total potential-energy expression with respect to a particular A_m yields

$$\sum_{n=0}^{\infty} C_{mn} A_n + a_m \chi = C_m \quad (m = 0, 2, 3, 4, \dots, \infty). \quad (23)$$

The evaluation of the coefficients A_n may now be accomplished by the solution of the simultaneous algebraic Eqs. (21) and (23), where, in Eq. (23),

$$C_m = \sum_{i,j}^{\lambda \ell} P_{i,j} r_{i,j}^m \left[p_m - 2q_m \left(\frac{r}{a} \right)_{i,j}^2 + \left(\frac{r}{a} \right)_{i,j}^4 \right], \quad (24)$$

$$\begin{aligned}
C_{mn} = & 4\pi \left(p_m p_n m n \left\{ \sum_{b=1}^B \left[D_b \alpha_1 (\delta_1 + \nu_b) + \zeta_b \alpha_2 / \phi_2 \right] - \gamma_b \left(\frac{\phi_1}{2} + \nu_B \right) \right\} \right. \\
& - 2 p_m q_n m (n+2) \left\{ \sum_{b=1}^B \left[D_b \alpha_2 (\delta_2 + \nu_b) + \zeta_b \alpha_3 / \phi_3 \right] - \gamma_b \left(\frac{\phi_2}{2} + \nu_B \right) \right\} \\
& - 2 q_m p_n (m+2) n \left\{ \sum_{b=1}^B \left[D_b \alpha_2 (\delta_3 + \nu_b) + \zeta_b \alpha_3 / \phi_3 \right] - \gamma_b \left(\frac{\phi_2}{2} + \nu_B \right) \right\} \\
& + p_m m (n+4) \left\{ \sum_{b=1}^B \left[D_b \alpha_3 (\delta_4 + \nu_b) + \zeta_b \alpha_4 / \phi_4 \right] - \gamma_b \left(\frac{\phi_3}{2} + \nu_B \right) \right\} \\
& + 4 q_m q_n (m+2) (n+2) \left\{ \sum_{b=1}^B \left[D_b \alpha_3 (\delta_5 + \nu_b) + \zeta_b \alpha_4 / \phi_4 \right] - \gamma_b \left(\frac{\phi_3}{2} + \nu_B \right) \right\} \\
& + p_n (m+4) n \left\{ \sum_{b=1}^B \left[D_b \alpha_3 (\delta_6 + \nu_b) + \zeta_b \alpha_4 / \phi_4 \right] - \gamma_b \left(\frac{\phi_3}{2} + \nu_B \right) \right\} \\
& - 2 q_m (m+2) (n+4) \left\{ \sum_{b=1}^B \left[D_b \alpha_4 (\delta_7 + \nu_b) + \zeta_b \alpha_5 / \phi_5 \right] - \gamma_b \left(\frac{\phi_4}{2} + \nu_B \right) \right\} \\
& - 2 q_n (m+4) (n+2) \left\{ \sum_{b=1}^B \left[D_b \alpha_4 (\delta_8 + \nu_b) + \zeta_b \alpha_5 / \phi_5 \right] - \gamma_b \left(\frac{\phi_4}{2} + \nu_B \right) \right\} \\
& \left. + (m+4) (n+4) \left\{ \sum_{b=1}^B \left[D_b \alpha_5 (\delta_9 + \nu_b) + \zeta_b \alpha_6 / \phi_6 \right] - \gamma_b \left(\frac{\phi_5}{2} + \nu_B \right) \right\} \right\} \quad (25)
\end{aligned}$$

and

$$\left. \begin{aligned}
r_B &= a; \\
\zeta_b &= a^2 \eta_b; \\
\gamma_b &= D_b a^{\phi_1}; \\
\phi_c &= m + n + 2(c-2), \quad c = 1, 2, \dots, 6; \\
\alpha_c &= (r_b^{\phi_c} - r_{b-1}^{\phi_c}) / a^{2(c-1)}, \quad c = 1, 2, \dots, 6;
\end{aligned} \right\} \begin{aligned}
\delta_1 &= [1 + (m-1)(n-1)] / \phi_1 \\
\delta_2 &= [1 + (m-1)(n+1)] / \phi_2 \\
\delta_3 &= [1 + (m+1)(n-1)] / \phi_2 \\
\delta_4 &= [1 + (m-1)(n+3)] / \phi_3 \\
\delta_5 &= [1 + (m+1)(n+1)] / \phi_3 \\
\delta_6 &= [1 + (m+3)(n-1)] / \phi_3 \\
\delta_7 &= [1 + (m+1)(n+3)] / \phi_4 \\
\delta_8 &= [1 + (m+3)(n+1)] / \phi_4 \\
\delta_9 &= [1 + (m+3)(n+3)] / \phi_5
\end{aligned} \quad (26)$$

For practical purposes it is frequently sufficiently accurate to deal with a finite number of equations. Unfortunately, the number of terms N needed in the series for a given problem cannot be predicted in advance. The more complex the load distribution and the tube arrangement, the greater the number of terms. For the selection of the appropriate N it becomes essential to follow through a number of solutions, increasing the equations by one in each succession. This type of a solution for a required quantity possesses the property of convergence. If the magnitude of at least two successive solutions are very close to each other, then the number of terms needed in the deflection series is assumed to be sufficient.

Once N is determined and the corresponding coefficients A_m are evaluated, Eq. (3) provides the deflection of the plates. It is important to note that because of the presence of the stiffening tubes in the structure, the maximum deflection does not necessarily take place in the center of the plates. The easiest way to ascertain this maximum would be by plotting w as a function of r .

The configuration of the tubes may be determined from elementary beam theory. Hence, the deflection of a tube in terms of the slope of the plate at the location of the tube is found to be

$$y = g(x) \frac{dw}{dr}, \quad (27)$$

where

$$g(x) = \frac{1}{4\ell^3} [8(\ell+h)x^3 - 12(\ell+h)^2x^2 + 2(2\ell^3 + 6\ell^2h + 9\ell h^2 + 3h^3)x - h^2(\ell+h)(3\ell+h)]$$

and dw/dr is found by differentiating Eq. (3). It should be recalled again that if reference is made to any particular region of the plates, the appropriate physical constants (ν , E , D , ℓ , h , d_o , and d_i) for that region should be used. The maximum deflection of the tubes takes place wherever $g(x)$ and dw/dr have their maximum values. Since the maximum value of $g(x)$ is independent of the tube location, the maximum tube deflection becomes

$$y_{\max} = \frac{(3)^{1/2}}{18} \left[\frac{(\ell + 3h)^3}{\ell + h} \right]^{1/2} \left(\frac{dw}{dx} \right)_{\max}, \quad (28)$$

where $(dw/dr)_{\max}$ is the maximum angular deflection at the end of the most deformed tube. The stress at the outside diameter of the tube is given by

$$\sigma_t = \frac{3E_t d_o (\ell + h)}{\ell^3} (\ell + h - 2x) \frac{dw}{dr}, \quad (29)$$

and the corresponding maximum stress is given by

$$(\sigma_t)_{\max} = \frac{6E_t d_o (\ell + h)}{\ell^2} \left(\frac{dw}{dr} \right)_{\max}. \quad (30)$$

There is no similar correspondence of the plate deformation with the stresses in the perforated plates. Reference has to be made to the previously mentioned concept of "equivalent plate." The results of the derived formulas for pertinent stresses in perforated plates are in good agreement with the experimental tests⁽⁷⁾ for the case of a uniformly distributed load. The formulas given by O'Donnell and Langer⁽³⁾ also pertain to that of a uniformly distributed load.

The assumption of the uniformly distributed load instead of the many individual concentrated loads is quite a reasonable approximation, especially since only approximate stresses are required. This equivalent distributed load, including the weight of plates and tubes, is

$$\begin{aligned} f = & \frac{1}{\pi(r_b^2 - r_{b-1}^2)} \sum_{i,j}^{\lambda_\ell} P_{i,j} + \frac{2h}{a^2} \rho_p \left(a^2 \frac{\lambda_h}{4} d_o^2 \right) \\ & + \frac{\lambda_t}{4a^2} \rho_t (\ell + 2h)(d_o^2 - d_1^2), \end{aligned} \quad (31)$$

where λ_h , λ_ℓ , and λ_t are the total number of holes, concentrated loads, and tubes, respectively, and ρ_p and ρ_t are the weight density of the plate and tube material, respectively. The first term of Eq. (31) represents the intensity of the equivalent uniformly distributed load due to the individual loads, the second that due to the weight of the plates, and the third the intensity of the distributed load due to the weight of the tubes.

Stresses in the perforated plates are dependent on the stresses of the "equivalent plate." Therefore, the latter have to be determined before proceeding further. From ordinary plate theory the radial and tangential stresses at the distance z from the neutral surface are

$$\begin{aligned} \sigma_r &= \frac{12zD}{h^3} \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right]; \\ \sigma_\theta &= \frac{12zD}{h^3} \left[\nu \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right]. \end{aligned}$$

In terms of the series, the radial and tangential stresses at the surface of the plates are found to be

$$\sigma_r = -\frac{6D}{h^2} \sum_{\substack{n=0 \\ n \neq 1}}^N \left[p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) \left(\frac{r}{a}\right)^2 \right. \\ \left. + (n+4)(n+3+\nu) \left(\frac{r}{a}\right)^4 \right] r^{n-2} A_n;$$

$$\sigma_\theta = \frac{6D}{h^2} \sum_{\substack{n=0 \\ n \neq 1}}^N \left[p_n n[\nu(n-1)+1] - 2q_n(n+2)[\nu(n+1)+1] \left(\frac{r}{a}\right)^2 \right. \\ \left. + (n+4)[\nu(n+3)+1] \left(\frac{r}{a}\right)^4 \right] r^{n-2} A_n.$$

In the evaluation of the critical stresses of the perforated plate, the maximum stresses of the homogeneous plate have to be known first. The latter again may be most easily determined graphically.

Fig. 8. Stress Intensities in Perforated Plate Ligaments*

*This figure is given by O'Donnell and Langer in Ref. 3.

Of primary interest in the design of perforated plates is the stress which is necessary to satisfy the laws of equilibrium. This is the stress intensity averaged across the minimum ligament section and through the thickness of the plate. It is analogous to the average stress intensity in the shell of the pressure vessel under internal pressure. With reference to Fig. 8, the stress intensity is given by[†]

$$\sigma_{ls} = K_t \frac{s}{t} |\sigma_1|, \quad (32)$$

where $\sigma_1 = \sigma_r$ or σ_θ , whichever has the larger magnitude.

Of secondary importance is the knowledge of the peak stress in localized regions. This stress is only important if it is of a repetitive nature likely to cause fatigue failure. The peak stress is given as[†]

[†]This equation is given by O'Donnell and Langer.(3)

Fig. 9

Maximum Local Stress in Perforated Plates*

*This figure is given by O'Donnell and Langer in Ref. 3.

given by Eq. (30). Pertinent stresses in the perforated plates may be approximated by Eqs. (32), (33), and (34).

$$\sigma_{ps} = K_p \sigma_1 + f, \quad (33)$$

where values of K_p are presented in Fig. 9, and f is given by Eq. (31). This formula gives peak stresses throughout the perforated portion of the plate.

Photoelastic tests have revealed the existence of high local stresses at the perforations adjacent to the rim of the perforated plates.⁽⁷⁾ These peak stresses appear to be due to the influence of the rim and may be approximated by[†]

$$\sigma_{rs} = K_r \sigma_{rim} + f, \quad (34)$$

where values of K_r are shown in Fig. 10, and σ_{rim} is the circumferential stress at the inside of the rim.

In summary, the general deformation of the grid structure may be obtained by the use of Eq. (3), whenever the coefficients A_m are known. Maximum deflection for the tubes is given by Eq. (28). The general stress in the tube is evaluated by Eq. (29), while the maximum is

Fig. 10

Peak Stress at Perforations Adjacent to Rim*

*This figure is given by O'Donnell and Langer in Ref. 3.

[†]This equation is given by O'Donnell and Langer.⁽³⁾

D. Simplified Procedure

The most laborious calculations of the grid-structure analysis are those having to do with the location of loads. The computation that comes under this heading is that of the change in potential energy due to external loads.

The amount of work involved in computing the potential energy depends to a large degree on the type of load distribution with respect to the plates. Because of symmetry, the analysis of a hexagonal load distribution may be confined to one-sixth of the total tubes. This reduces the amount of work to one-sixth, but still does not make efficient hand computations practical. Therefore, some simplifications have to be introduced to make such hand computations feasible.

An obvious simplification to the actual load distribution is the assumption that the loads be arranged in a concentric circular pattern. This assumption reduces the number of relevant radii to the number of such hexagons of loads. If the number of such loads is large, the simplification reduces the amount of work considerably. The radius of such an equivalent circular load distribution might be taken as the average distance from the center of the plate to all the loads in the corresponding hexagon.

The amount of work involved in the determination of potential energy of external loads may be further reduced by a different method of simplification, which involves the assumption of a uniformly distributed load in place of all the concentrated loads. The validity of this assumption depends to a great degree on the number of concentrated loads acting on the plates and certainly on the degree of variation of the loads with respect to the plates. Thus, if the loads are assumed to be uniformly distributed over the plates, then the potential energy of external loads is found to be independent of r and may be expressed as follows:*

$$V = - \sum_{\substack{m=0 \\ m \neq 1}}^N C_m A_m,$$

where

$$C_m = 2\pi f(r_b^{m+2} - r_{b-1}^{m+2}) \left[\frac{P_m}{m+2} - \frac{2q_m}{m+4} \left(\frac{r_b^2 - r_{b-1}^2}{a^2} \right) + \frac{1}{m+6} \left(\frac{r_b^4 - r_{b-1}^4}{a^4} \right) \right]$$

and r_b, r_{b-1} are the radii limiting the area of distributed loads.

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*For derivation see Appendix A.

E. Computer Solution

The solution of the grid-structure problem is presented in the form of series. The use of successive terms in the series can be interpreted as increasingly better approximations to the true solution. Convergence cannot be readily observed without expanding the series and carrying the calculations to the end. Only in this way can the accuracy of the solution be predicted. If the convergence of the series is slow, then the solution of the grid problem, even by the simplified procedure, may be too laborious for hand calculations. A computer may be applied here to a great advantage by virtually eliminating the most tedious algebraic calculations and thus making a solution feasible.

A computer program, written in Fortran II language, has been prepared for solving the grid-structure problem with an IBM-704 digital computer. A description of this computer program is given in the following section.

1. Description of Computer Program[†]

In order to take advantage of the computer, a multitude of constants must be stored in its memory. These constants are basically divided into two groups: those describing the actual grid-structure problem, and those dealing with details of the computer solution. The data that must be specified are as follows:

Symbol Used in the Analysis	Symbol Used in Fortran II Language	Explanation
a	A	} dimensions of the grid structure
h	H	
l	TL	
d_0	$2 \cdot R\emptyset$	
d_i	$2 \cdot RI$	
r_1	$R\emptyset NE$	} lower and upper bounds of the U_t integral
r_2	$RTW\emptyset$	
r_3	R3	} lower and upper bounds of load distribution
r_4	R4	

[†]A complete computer program for the solution of the grid-structure problem is given in Appendix B.

Symbol Used in the Analysis	Symbol Used in Fortran II Language	Explanation
E_p	EP	properties of the materials used in the structure
E_t	ET	
ρ_p	RHØP	
ρ_t	RHØT	
ν	VU	
s	S	constants describing the load dis- tribution (see Fig. 11)
i_a	IA	
i_b	IB	
i_d	IB2	
λ_c	IN(IB+1)	number of tubes in the incomplete rows (see Fig. 11)
λ_d	IN(IB2)	
$P_{i,j}$	P(I)	the individual concentrated loads* for each row i (see Fig. 11)
N	NM	total number of terms in the series of the deflection function ($NM \leq 9$)
k	Z	degree of edge restraint in bending ($0 \leq k \leq 1$)
None	NZ	number of solutions (of different k) to be obtained
None	NUMBR	number of divisions for maximum calculations of w, dw/dr, M_r , and M_θ along the radius a (NUMBR $\leq 1,000$)
None	RCALCN	number of intermediate points for calculations of w, dw/dr, M_r , and M_θ along the radius a (RCALCN ≤ 20)
None	MCK	this provides an option for a check on the solution of the simultaneous equations (a numeral 1 in place of MCK means that a check is desired; numeral 0 signifies the opposite)
None	K1	if K1 = 0, normal full set of solu- tions for the six combinations of loads. If K1 = 1, only the hexagonal load distribution (including weight of structure) is solved

*The computer program is arranged in such a way as to allow for load variations along i but not along j.

Symbol Used in the Analysis	Symbol Used in Fortran II Language	Explanation
None	HØ	= 1 if there is a hole in the center; otherwise HØ = 0.
None	TØ	= 1 if there is a tube in the center; otherwise TØ = 0.
None	PØ	= 1 if there is a load in the center; otherwise PØ = 0.

Fig. 11
Segment of a
Grid Plate

The distance from the center of the plate to the center of each tube or load ($r_{i,j}$) has to be known for later calculations. Rather than making separate hand computations at this point, it is convenient to let the computer calculate the individual distance and store it for reference. This may be done by considering just one-sixth of the total number of tubes, as shown in Fig. 11. The expression for the distance is obtained from geometry in terms of s and the coordinates i and j as follows:

$$r_{i,j} = \left\{ \left[\frac{(3)^{1/2}}{2} s_i \right]^2 + \left(\frac{1}{2} s_i - s_j \right)^2 \right\}^{1/2},$$

which reduces to

$$r_{i,j} = s(i^2 - ij + j^2)^{1/2}.$$

Thus, the values of $r_{i,j}$ are calculated and stored in the memory of the computer. When this is done, the average distance r_i is determined for each hexagonal distribution of tubes by the equation

$$r_i = \frac{1}{\lambda_i} \sum_{j=1}^i r_{i,j},$$

where λ_i is the number of tubes in the i^{th} row. These values are stored in the memory and printed out together with the corresponding values of load.*

To make computer calculations more efficient, additional calculations are performed, the results of which are conveniently used in the subsequent stages of the solution. Thus, the flexural rigidity D of the perforated plates, tube moment of inertia I , intensity of the equivalent distributed weight f_1 due to the concentrated loads, the intensity of the distributed load f_2 due to the weight of the structure, and the values of p_n and q_n are calculated by use of the respective formulas

$$D = \frac{1}{12} \frac{E_p h^3}{1 - \nu^2};$$

$$I = \frac{\pi}{64} (d_0^4 - d_1^4);$$

$$f_1 = \frac{6}{\pi(r_4^2 - r_3^2)} \sum_{i,j}^{\lambda \ell / 6} P_{i,j},$$

where i and j refer to Fig. 11;

$$f_2 = \frac{2h}{a^2} \rho_p \left(a^2 - \frac{\lambda h}{4} d_0^2 \right) + \frac{\lambda t}{4a^2} \rho_t (\ell + 2h)(d_0^2 - d_1^2);$$

$$\left. \begin{aligned} p_n &= 1 + \frac{4(1-k)}{1+\nu+2n} \\ q_n &= 1 + \frac{2(1-k)}{1+\nu+2n} \end{aligned} \right\} (n = 0, 2, 3, 4, \dots, N).$$

---

*One representative sample of the printed results is given in Appendix B.

When all these quantities are calculated, printed, and stored for reference, the constants for the different parts of the total potential energy of the grid structure are calculated.

The constants pertaining to the strain energy of the plate (C'_{mn}), tubes (C''_{mn}), and supports (C'''_{mn}), are the terms of a double series, the expressions of which are known. They are calculated, stored, and printed in the form of a triangular matrix.

The constants C_m pertaining to the change in potential energy of the external load depend on the distribution of the loads. Because of symmetry, the constants are computed for each of the loads contained in one-sixth of the plate area, enabling various groupings of individual values of C_m . The following exact and approximate load distributions are assumed:

- a) hexagonal load distribution,
- b) equivalent circular load distribution, and
- c) equivalent uniformly distributed load.

Each of the above cases may include or neglect the weight of the structure. Thus, constants C_m for all of these six load distributions are calculated, stored, and given in the output. Separate values are printed for every concentric set of load distributions.

For each of these combinations the computer program is set up in such a way as to differentiate partially the modified total potential energy* $\bar{W}$ of the system with respect to all A_m 's. The product of this operation is a set of N algebraic equations with $(N+1)$ unknowns. The simultaneous solution of these and the auxiliary condition, which represents a sum of $(N+1)$ equations with $(N+1)$ unknowns, enables the final evaluation of all A_m .

The solution for the values of A_m , however, is undertaken in a particular way. For the cases $k = 0$ and $k = 1$, there are N separate sets of solutions for A_m , which may be explained as follows:

The first solution takes into account the coefficients pertaining to $m = 0$ and calculates A_0 ; the second solution takes into account the coefficients pertaining to $m = 0$, $m = 2$, and calculates A_0 and A_2 . This procedure is repeated for $m = 0$, $m = 2$, $m = 3$, and so on, resulting in N such solutions. For the intermediate values of k , the same procedure is followed except that starting with the second solution, the auxiliary condition is introduced. Thus, for $N \geq 2$ there are $(N+1)$ simultaneous equations to be solved for a number N of unknown A_m values.

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*For the extreme boundary conditions ($k = 0$ and $k = 1$), where the auxiliary conditions do not have to be satisfied, the true total potential energy of the system is used.

Thus, having determined the values of A_m , additional essential quantities are calculated directly and printed in the output. These constitute the final results of the computer solution and may be enumerated as follows:

$$U_p, U_t,^* U_s, V, \text{ and } W.$$

For the plates:

$$\left. \begin{array}{l} w \\ \frac{dw}{dr} \\ \sigma_r \\ \sigma_\theta \\ \beta \end{array} \right\} \begin{array}{l} \text{Calculated and printed for even increments of } a, \text{ the number} \\ \text{of which equals RCALCN} \end{array}$$

$$\left. \begin{array}{l} w_{\max} \\ \left(\frac{dw}{dr} \right)_{\max} \\ \left(M_r \right)_{\max} \text{ and } \left(\sigma_r \right)_{\max} \\ \left(M_\theta \right)_{\max} \text{ and } \left(\sigma_\theta \right)_{\max} \end{array} \right\} \begin{array}{l} \text{Found, calculated, and printed for the} \\ \text{number of even increments equal to} \\ \text{NUMBR, with the corresponding loca-} \\ \text{tions of } r.^{**} \end{array}$$

For the tubes:

$$\left. \begin{array}{l} y_{\max} \\ M_{\max} \\ \sigma_{\max} \end{array} \right\} \begin{array}{l} \text{Calculated and printed, referring to the tube with the} \\ \text{largest deformation} \end{array}$$

---

*Provisions are incorporated in the program also to calculate the true U_t for comparison with the U_t used in this calculation.

**These maxima are obtained by evaluating the respective quantities at several even intervals over the radius a of the plates; the maxima of the calculated values are retained in the memory and printed in the output.

As a check on the accuracy of the solution of the simultaneous algebraic equations, the values of A_m are substituted back into the original equations and may be given in the output. The value of the determinant may also be obtained.

2. Example Problem

In anticipation of describing test results of the experimental support structure later in this report, it was thought well to illustrate the analytical method by considering data of that particular model. (This test model with its pertinent dimensions is shown in Fig. 15 below.)

For the sake of convenience, the solution to the problem was obtained by the use of a computer program constructed for this purpose. Before the program could be utilized, a number of constants, corresponding to the list in the previous section, had first to be supplied. The following data were used:

A = 17.0	IB = 2
H = 0.512	IB2 = 2
TL = 3.0	IN(IB+1) = 0
RØ = 0.25	IN(IB2) = 2
RI = 0.0	P(I) = 500; I = 1, 2
RØNE = 0.0	NM = 5
RTWO = 17.0*	Z = 4
R3 = 0.0	NZ = 3
R4 = 17.0	NUMBR = 34
EP = 3.0×10^7	RCALCN = 17
ET = 3.0×10^7	MCK = 0
RHØP = 0.285	K1 = 1
RHØT = 0.285	HØ = 0
VU = 0.3	TØ = 1
S = 7	PØ = 1
IA = 1	

When the computer program was executed subject to the instruction and data listed above, an output, which is given in Appendix B, was obtained. The same data for deflection and radial stress are plotted in Figs. 12** and 13, respectively.

It is noted in Fig. 12 that for all practical purposes the deflections for $N > 2$ are the same. Thus, no additional effort need be expended in using more than two terms in the series of the deflection function.

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*The values were obtained by trial and error through matching of the true U_t with that used in this analysis.

**The figures and sample calculations shown in this report are based on hexagonal load distribution including the weight of the structure.

Fig. 12. Pictorial Convergence of Deflection

Fig. 13. Pictorial Convergence of Radial Stress

As could be expected, no such good convergence occurred for the stress configurations (see Fig. 13). Several points brought out by the stress distributions should be mentioned, for they may play a part in the evaluation of the stresses. There is no reason in mechanics for the "wavy" stress pattern exhibited by solutions for $N > 3$; there is a point indicating that the magnitudes, at least for $r = 0$, are incorrect. Notice that a mirror image of the stress distribution, with respect to the abscissa would correspond to the other half of the plate. Because of this, a continuous stress distribution is expected along the diametral section, and the slope at $r = 0$ must be zero; a stress discontinuity here, which is displayed for solutions of $N > 3$, would be inconceivable. Thus, the variation of the configurations for $N > 3$ is attributed to the round-off error involved in the solution. Another point which may be significant is the fact that successive solutions seem to indicate oscillations about the distribution of $N = 2$ and $N = 3$. In fact, this reason strengthens the belief that $N = 2$ or $N = 3$ should be the configurations for the evaluation of stress.

Thus, having established the N to be used, advantage may be taken of the calculations for maxima given in the output. Because of the very small number of tubes in this example, one needs to be careful with

the maximum values which pertain to the tubes. The maximum slope of the plate may be in a location where there is no tube. For this reason it is best to plot the slope, as has been done in Fig. 14. When the location of each tube is known, the selection of the tube with the maximum slope may be made by use of Fig. 14. The maximum slope of the plate in this case occurs at $r = 14$ in. It just so happens that a tube is located at this point. Thus, by coincidence, the calculations of maxima given and pertaining to the tubes are also applicable.

Fig. 14
Variation of Slope .

V. EXPERIMENTAL WORK

It was also desired to produce experimental evidence to substantiate the theoretical analysis of the grid structure. Certain assumptions or their limitations called for some experimental verification. It was interesting to see, for example, how well the developed procedure applied to the composite grid structure as a whole.

For the purpose of making experimental and analytical comparisons it is imperative to know the edge rigidity of the plates. Great effort would have to be sacrificed to accomplish this experimentally, unless a simple support were chosen. The desire for results in the intermediate range of edge restraint led to the discard of this easy solution.

Two physical quantities can be readily measured: the transverse plate deflection, and the strain of the plate surface. Since intricate and expensive experimental instrumentation is undesirable for the determination of some intermediate edge restraint, a substitute procedure was followed.

It is recalled that the predicted analytical deflection of the plate, as anticipated by the Rayleigh-Ritz procedure, is very close indeed to the actual one. The predicted strain or stress values, however, would only be approximate. Also, the diversity of the analytical method makes possible the treatment of intermediate edge conditions. Thus, the possibility arises of matching several experimental deflections at the edge with the contours of analytical predictions. This would establish the edge rigidity of the experimental model. Since the experimental deflection data are used to establish the effective edge rigidity, they are lost as comparison data for experimental and analytical results. The remaining material for direct comparison is that from the strain (stress) measurements.

A. Test Equipment

A model was constructed so as to fall within the limits of the assumptions quoted in Section IV-B. Edge restraints as well as the external loads had to be simulated.

1. Grid Model

A steel test model (see Fig. 15) consisted of two circular plates interconnected by nineteen rods distributed in a hexagonal manner with respect to the center of the plates. The size of the model was mainly governed by the manufacturing equipment available. The thickness of the plates was restricted by construction limitations; plates could not be too thin for fear of warping during welding operations. Rods could not be made too stiff; otherwise they might introduce localized deformation in the

plates. Because of these construction limitations the final product was an inefficient design. As will be seen later, the limiting stress in the rods was reached much sooner than in the plates.

Fig. 15. Structure of Test Model

The grid model was stress-relieved after welding, and the top and bottom surfaces were ground flat.

2. Edge Support

A 6-in. steel armor plate was chosen to support the test model on the periphery of the plates. For ease of assembly, the entire support plate was first cut in two, then joined together again by bolts. A circular hole was torch-cut to rough dimensions in this assembly and then machined to fit the test specimen. A drawing of the finished support and mating equipment is shown in Fig. 16.

In order to align the two halves of the support, keys were put into the mating parts. Turnbuckles were provided to decrease possible support deflection under load.

Fig. 16. Test Structure and Auxiliary Support Equipment

3. Loading Arrangement

The final grid structure was so rigid that gravity loads were inadequate to produce measurable deflections. Thus, artificial means had to be developed to supply the necessary loading. Nineteen separate loads were applied, one to the top of each rod, by the use of rings cut from a section of a standard commercial pipe. A dial gage was fastened to each ring to measure the change in the diameter under load. A bolt was used as a screw jack to apply the load through each ring. With this arrangement, the rings were calibrated by means of an accurate tensile tester (load vs reading on the dial gage). For convenience, the dials were marked to show a specific load, such as 500 and 1000 lb. The rings were mounted on a flat plate which in turn was supported by four posts from the base support. The loading arrangement is shown in Fig. 17.

4. Support Table

The assembly of the test specimen, the support plate, and the loading equipment was self-contained. Since accessibility to the top and bottom of the test specimen was desired, some means of suspension of the assembly was necessary. A table was made of narrow steel channels to hold the structure in position. A picture of the entire assembly, as supported from the ground, is shown in Fig. 18.

112-3760

Fig. 17. Close-up of Test Loading Arrangement

112-3762

Fig. 18. View of the Entire Test Set-up

5. Instrumentation

The test structure was instrumented in two diametral directions with SR-4 strain gages. Rosettes were used on the plates to measure the radial and circumferential strains at the surface. One-directional gages were used for the rods. Three temperature-compensating gages were used: two between the plates, and one on the bottom of the structure. The location of the strain gages is shown in Fig. 19.

Fig. 19. Key to the Location of Strain Gages

Conventional procedure was used in applying the gages to the structure. The resistance to ground was checked and found well above the specified minimum. The strain gages were wired to three distributing boxes which made reading of the strains by the use of the galvanometers very convenient. One DC- and two AC-powered galvanometers were used.

Mechanical dial gages were used to measure deflection of the top and bottom plates. These gages were mounted on rods connected to mounts which in turn were rigidly fixed to the edges of the supports. Ball bearings were used in the mounts in order to minimize the effect of bending movement. The dial gages were arranged so that detailed deflection measurements could be made near the edge supports.

Four additional dial gages were mounted on the support table close to the narrowest sections of the support. These were used to indicate the relative deformation of the supports under load. The turnbuckles could then be utilized to correct the induced deflection.

B. Test Procedure

Numerous trial runs were made to determine the most efficient test procedure. The final procedure adopted is discussed in the following.

Before starting a test, it was necessary to ascertain that all the dial gages were functioning properly, and that sufficient travel was provided for the entire test. The step-by-step procedure that followed during the test was:

- a) Check to see that the load is zero and the turnbuckles are not applied.
- b) Zero the dial gages and the strain readings.
- c) Apply the load by means of the four large nuts. All four nuts should exert approximately the same load. The nuts are turned until the average reading on the load-ring indicators is the desired load.
- d) Obtain a finer adjustment of the load-ring indications by the use of the adjustment screws. This adjustment requires some time (approx 15-30 min), because adjusting one ring offsets the reading of the others.
- e) As the desired readings of the indicators are approached, activate the turnbuckles so that the support deflection is corrected.
- f) Make final load adjustment with the rings.
- g) Record the deflection readings and the strains.
- h) Release the load by reversing the rotation of the four large nuts. Turnbuckles are also brought back to their initial position.
- i) Record drift of the strain readings.

This procedure is repeated each time a new loading is applied.

C. Test Results

Much effort was expended in securing a definite rigidity at the edges. Very careful measurements were taken in order to ascertain that both plates

had the same edge rigidity during the test. The final measurements of edge deflection of the top plate superimposed on those of the bottom plate resulted in practically the same deflection configuration.

Edge rigidity was established by contour matching of the experimental deflections with those of the predicted deflections. The selected values of k for all loads are shown in Fig. 20. A certain characteristic

Fig. 20. Matching the Edge Deflections

of the effectiveness of the supports may be noted. The rigidity of the supports appeared to be not simply proportional to the load. It was found that the degree of edge restraint or the spring constant of the supports varied with the load.

Since the main body of the supports may also be considered an elastic structure, the deviation from simple proportionality must be attributed to side effects such as friction and minute permanent deformation.

Although the two plates had the same edge restraint, their overall deformation was not identical. In the region of the outermost rods the stress-strain relationships of the upper and lower plate were quite different as shown in Figs. 21 and 22. Contributing to this effect would be the difference in top and bottom plates exhibited in Fig. 23. This difference in deformation was also born out by the strain readings of the rods. The rods indicated axial tension instead of the anticipated compression. The general load distribution in the rods is shown in

Fig. 24. Thus, the results show that the transverse loads are not equally distributed between the top and bottom plates.

In order to explain this fact it is essential to take into account axial load in addition to bending in the rods. The rods deform into an S-shape with ends remaining perpendicular to the plates. This S-shape is most pronounced close to the edge of the plates where the slope is greatest. If it be assumed that the length of the rod remains constant and this S-shape

develops, then the distance between the end points of the rod has to diminish. Thus, if the ends were restrained from moving, a tensile load would result in the rod. The contraction of the rod due to the transmitted axial load offsets this effect to some extent. Predominance of one effect over the other should depend on the "density" of tubes, their stiffness, edge restraint, etc. Apparently, the effect of "geometric" deformation in the outermost rods of the tested structure was of such magnitude as to produce a resultant compressive stress.

Fig. 21. Analytical and Experimental Distributions of Radial Stress

Fig. 22. Analytical and Experimental Distributions of Tangential Stress

Fig. 23

Variation in Top and Bottom
Plate Thickness

Fig. 24

General Load Distribution in
the Rods

VI. CONCLUDING REMARKS

In this report a procedure has been developed for predicting the configuration and approximating the stresses of a symmetrically loaded grid structure. This analysis encompasses all the edge restraints ranging from simple to fixed supports. In other words, the initially stated problem has been solved. There remains, however, the task of reiterating the main parts of the analysis; outlining its advantages, and pointing out the discrepancies involved.

The procedure given in this analysis introduced one main modification in the Rayleigh-Ritz method. It was observed that for intermediate degrees of edge restraint neither the slope nor the radial moment at the edge was simply proportional to k . An example solution indicated also that, without any modification to the conventional Rayleigh-Ritz method, the radial moment at the edge could have any direction of rotation, there being nothing in the analysis to control the sign of the moment. To remedy this situation, a definite relationship was introduced between k and the moment which had to be satisfied in the analysis. This relationship performed a twofold function, namely, it provided a direct proportionality between the radial edge moment and k , and also controlled the sign of the moment. This relation was really the second boundary condition for the intermediate degrees of edge restraint.* The incorporation of this condition into the analysis involved the use of the Lagrangian multipliers in conjunction with the Rayleigh-Ritz procedure. It is interesting to note that the satisfaction of the geometrical boundary condition

$$\left. \frac{dw}{dr} \right|_{r=a} = (1-k) \left. \frac{dw}{dr} \right|_{\substack{r=a \\ k=0}}$$

in place of the moment restriction was insufficient. In other words, the geometrical boundary conditions of deflection and slope alone were unable to control the sign of the radial moment at the edge.

The procedure developed in this report is applicable both to hand and computer calculations, although a computer solution is much more favorable. The solution of the given problem is considered obtained whenever the unknown coefficients A_m of the deflection function are evaluated. This is accomplished by constructing the needed number of simultaneous equations and directly solving for A_m . A single expression is presented in the analysis for evaluating all the constants of the coefficient matrix and another expression for the terms of the column vector. The most laborious calculations appear in determining the terms of the constant vector which is related to the external potential energy of the system.

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*The first geometric boundary condition ($w|_{r=a} = 0$) was satisfied by the deflection function for $0 \leq k \leq 1$.

The calculations concerned with this potential energy of the external load, as discovered from the solution of the example problem, may be simplified considerably. A great deal depends on the accuracy of the solution desired and the load variation with respect to the plates. Naturally, greater accuracy requires the use of more terms in the series for the deflection function, resulting in a large set of equations to be evaluated. A marked variation of load induces a more complex deflection geometry, which also necessitates that more terms in the series be evaluated. This variation of load governs the kind of simplified procedure that can be used in any particular case.

Several additional example solutions tend to indicate that for hexagonal load distribution the equivalent circular approximation could be used without any noticeable discrepancy. This simplification may account for a considerable saving of labor. Still a greater simplification to the true load distribution is the assumption of the equivalent uniformly distributed load. This assumption is less accurate than that of the circular load distribution, but may be used where the variation of load is small.

The true indication of the quality of the analytical method is the resultant comparison of the analysis and experiment. The results of the directly measured experimental deflection were used to establish the edge fixity of the plate with reference to the analytical method. The indirectly measured stresses were used as the basis of comparison between theory and experiment. By the admission of the same analysis, the stress results are not expected to be very accurate. Thus, no true quantitative conclusion can be drawn. Nevertheless, a qualitative comparison, mainly illustrated by Figs. 21 and 22, shows quite good agreement. The deviation, small as it is under the circumstances, may partly be attributed to factors which divert the test model from the assumed "ideal structure" considered by the analysis. Factors, such as the very small number of rods per given plate area, certainly affect the agreement between theory and experiment.

It is also essential to criticize constructively the presented analysis and to suggest improvements. The basis for the evaluation of the analysis is the given numerical solution of the example.

The present analysis assumes that both plates have identical deflections. This implies that the plates are equally loaded; however, it is known for a fact that they are not. The shearing forces exerted by the tubes produce stresses on the neutral surfaces of the plates, opposite stresses for both of them. Although the contribution of these stresses to the overall configuration is believed to be small, the analysis fails to show their true effect. This change in the analysis would mean that the plates could not be assumed to deflect identically.

In order to improve the analysis still further, it might be possible to consider the nonaxisymmetric case. Here two parameters (r and θ) would be considered instead of the one in the present analysis. Because of the use of two parameters, the modified analysis may become rather complex.

These improvements are suggested for the analysis of the specific grid tested. If excessive accuracy is not necessary, problems of this type could be readily solved by using this method without the improvements suggested above.

APPENDICES

A. DERIVED EQUATIONS1. Deflection and Slope of Plates

The deflection of grid plates is expressed in the form of the following function:

$$w = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n - 2q_n \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right] r^n A_n,$$

where

$$p_n = 1 + \frac{4(1-k)}{1+\nu+2n}$$

and

$$q_n = 1 + \frac{2(1-k)}{1+\nu+2n}.$$

The slope or the first derivative of the deflection function becomes

$$\begin{aligned} \frac{dw}{dr} &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n - 2q_n \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right] nr^{n-1} A_n + \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[-4q_n \left(\frac{r}{a^2} \right) + 4 \left(\frac{r^3}{a^4} \right) \right] r^n A_n \\ &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n (n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right] r^{n-1} A_n. \end{aligned}$$

The slope at the edge ($r = a$) is

$$\left. \frac{dw}{dr} \right|_{r=a} = \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n (n+2) + n+4 \right] a^{n-1} A_n.$$

Substitution of the expressions for p_n and q_n yields

$$\begin{aligned}
 \left. \frac{dw}{dr} \right|_{r=a} &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ \left[1 + \frac{4(1-k)}{1+\nu+2n} \right] n - 2 \left[1 + \frac{2(1-k)}{1+\nu+2n} \right] (n+2) + (n+4) \right\} a^{n-1} A_n \\
 &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[n + \frac{4(1-k)n}{1+\nu+2n} - 2n - \frac{4(1-k)n}{1+\nu+2n} - 4 - \frac{8(1-k)}{1+\nu+2n} + n + 4 \right] a^{n-1} A_n \\
 &= -\frac{8(1-k)}{a} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \frac{a^n A_n}{1+\nu+2n}.
 \end{aligned}$$

2. Expressions for the Moment of Homogeneous Plates

From plate theory, the moments in a homogeneous circular plate are given by

$$\begin{aligned}
 M_r &= \frac{h^3}{12z} \sigma_r = -D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right]; \\
 M_\theta &= \frac{h^3}{12z} \sigma_\theta = -D \left[\nu \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right].
 \end{aligned}$$

In terms of the derivatives of the deflection function, these expressions yield

$$\begin{aligned}
 M_r &= -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) \left(\frac{r}{a} \right)^2 + (n+4)(n+3+\nu) \left(\frac{r}{a} \right)^4 \right] r^{n-2} A_n; \\
 M_\theta &= -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ p_n n[\nu(n-1)+1] - 2q_n(n+2)[\nu(n+1)+1] \left(\frac{r}{a} \right)^2 + (n+4)[\nu(n+3)+1] \left(\frac{r}{a} \right)^4 \right\} r^{n-2} A_n.
 \end{aligned}$$

The radial moment at the edge is

$$M_r \Big|_{r=a} = -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n(n-1+\nu) - 2q_n(n+2)(n+1+\nu) + (n+4)(n+3+\nu) \right] a^{n-2} A_n.$$

Substitution of the expressions for p_n and q_n yields

$$\begin{aligned}
 M_r|_{r=a} &= -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ \left[1 + \frac{4(1-k)}{1+\nu+2n} \right] (n^2 - n + n\nu) - 2 \left[1 + \frac{2(1-k)}{1+\nu+2n} \right] (n^2 + 3n + 2 + n\nu + 2\nu) \right. \\
 &\quad \left. + n^2 + 7n + n\nu + 4\nu + 12 \right\} a^{n-2} A_n \\
 &= -D \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ -\frac{4(1-k)}{1+\nu+2n} (4n + 2\nu + 2) + 8 \right\} a^{n-2} A_n \\
 &= -\frac{8Dk}{a^2} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} a^n A_n.
 \end{aligned}$$

3. Deformation of Tubes

With reference to Fig. 7, the boundary conditions of the tubes are

$$\begin{aligned}
 \text{a) } \frac{dy}{dx} \Big|_{x=h/2} &= \frac{dw}{dr} \Big|_B; & \text{c) } \frac{dy}{dr} \Big|_{x=(h/2)+\ell} &= \frac{dw}{dr} \Big|_A; \\
 \text{b) } y \Big|_{x=h/2} &= \frac{h}{2} \frac{dw}{dr} \Big|_B; & \text{d) } y \Big|_{x=(h/2)+\ell} &= -\frac{h}{2} \frac{dw}{dr} \Big|_A.
 \end{aligned}$$

The elastic beam equation is

$$\frac{d^2 y}{dx^2} = -\frac{M}{E_t I},$$

where

$$M = M_B - F_B \left(x - \frac{h}{2} \right)$$

or

$$\frac{d^2 y}{dx^2} = -\frac{1}{E_t I} \left[M_B - F_B \left(x - \frac{h}{2} \right) \right].$$

Integration along the length of the tube yields

$$\frac{dy}{dx} = -\frac{1}{E_t I} \left[M_B x - F_B \left(\frac{x^2}{2} - \frac{hx}{2} \right) \right] + C_1.$$

When the first boundary condition (a) is substituted, C_1 becomes

$$C_1 = \frac{1}{E_t I} \left[M_B \left(-\frac{h}{2} \right) + F_B \left(\frac{h^2}{8} \right) \right] + \frac{dw}{dr} \Big|_B.$$

The expression of the slope is as follows:

$$\frac{dy}{dx} = -\frac{1}{E_t I} \left[M_B \left(x - \frac{h}{2} \right) - F_B \left(\frac{x^2}{2} - \frac{xh}{2} + \frac{h^2}{8} \right) \right] + \frac{dw}{dr} \Big|_B.$$

Integration of this equation yields

$$y = -\frac{1}{E_t I} \left[M_B \left(\frac{x^2}{2} - \frac{xh}{2} \right) - F_B \left(\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{xh^2}{8} \right) \right] + x \frac{dw}{dr} \Big|_B + C_2,$$

and the second boundary condition (b) gives

$$C_2 = \frac{1}{E_t I} \left[M_B \left(-\frac{h^2}{8} \right) - F_B \left(\frac{h^3}{48} \right) \right].$$

Upon substitution for C_2 , the deflection becomes

$$y = -\frac{1}{E_t I} \left[M_B \left(\frac{x^2}{2} - \frac{xh}{2} + \frac{h^2}{8} \right) - F_B \left(\frac{x^3}{6} - \frac{x^2 h}{4} + \frac{xh^2}{8} - \frac{h^3}{48} \right) \right] + x \frac{dw}{dr} \Big|_B.$$

Similarly, when the two remaining boundary conditions are substituted into the corresponding slope and deflection expressions, the following two simultaneous equations result:

$$F_B \left(\frac{\ell^2}{2} \right) - M_B(\ell) = E_t I \left(\frac{dw}{dr} \Big|_A - \frac{dw}{dr} \Big|_B \right);$$

$$F_B \left(\frac{\ell^3}{6} \right) - M_B \left(\frac{\ell^2}{2} \right) = -E_t I \left[\frac{h}{2} \frac{dw}{dr} \Big|_A + \left(\ell + \frac{h}{2} \right) \frac{dw}{dr} \Big|_B \right].$$

Solution of these equations yields expressions for F_B and M_B :

$$F_B = \frac{6E_t I(\ell+h)}{\ell^3} \left[\frac{dw}{dr} \Big|_A + \frac{dw}{dr} \Big|_B \right];$$

$$M_B = \frac{E_t I}{\ell^2} \left[(2\ell + 3h) \frac{dw}{dr} \Big|_A + (4\ell + 3h) \frac{dw}{dr} \Big|_B \right].$$

If

$$\frac{dw}{dr} \Big|_A = \frac{dw}{dr} \Big|_B = \frac{dw}{dr},$$

the corresponding expressions become

$$F_B = \frac{12E_t I(\ell+h)}{\ell^3} \frac{dw}{dr};$$

$$M_B = \frac{6E_t I(\ell+h)}{\ell^2} \frac{dw}{dr}.$$

Finally, these equations are used to obtain the following general expressions pertaining to the tube:

$$y = \frac{1}{4\ell^3} [8(\ell+h) x^3 - 12(\ell+h)^2 x^2 + 2(2\ell^3 + 6\ell^2 h + 9\ell h^2 + 3h^3) x - h^2(\ell+h)(3\ell+h)] \frac{dw}{dr};$$

$$\frac{dy}{dx} = \frac{1}{2\ell^3} [12(\ell+h) x^2 - 12(\ell+h)^2 x + (2\ell^3 + 6\ell^2 h + 9\ell h^2 + 3h^3)] \frac{dw}{dr};$$

$$\frac{d^2 y}{dx^2} = -\frac{6(\ell+h)}{\ell^3} [\ell + h - 2x] \frac{dw}{dr};$$

$$M = \frac{6E_t I(\ell+h)}{\ell^3} [\ell + h - 2x] \frac{dw}{dr}.$$

To find the location of the extremum values of tube deflection along the length of the tube, it is required to set the slope to zero and solve for x . Thus,

$$\frac{dy}{dx} = \frac{1}{2\ell^3} [12(\ell+h) x^2 - 12(\ell+h)^2 x + (2\ell^3 + 6\ell^2 h + 9\ell h^2 + 3h^3)] \frac{dw}{dr} = 0.$$

Then,

$$x = \frac{1}{24(\ell+h)} \left\{ 12(\ell+h)^2 \pm [144(\ell+h)^4 - 48(\ell+h)(2\ell^3 + 6\ell^2h + 9\ell h^2 + 3h^3)]^{1/2} \right\},$$

which reduces to

$$x = \frac{1}{2} \left[(\ell+h) \pm \frac{\ell}{(3)^{1/2}} \left(\frac{\ell+3h}{\ell+h} \right)^{1/2} \right].$$

The magnitude of deformation of each tube is also dependent on the rotation of the ends of the tubes. Consequently, the absolute maximum deformation takes place in the tube whichever has the largest slope at its ends. Hence, when the above value of x is substituted into the general deflection equation, the maximum tube deflection becomes

$$y_{\max} = \frac{(3)^{1/2}}{18} \left[\frac{(\ell+3h)^3}{\ell+h} \right]^{1/2} \left(\frac{dw}{dr} \right)_{\max},$$

where $\left(\frac{dw}{dr} \right)_{\max}$ refers to the maximum angular deformation of the tube.

4. Strain Energy of Plates

The strain energy of one symmetrically loaded plate is

$$U_p = \pi \sum_{b=1}^B D_b \int_{r_{b-1}}^{r_b} \left[\frac{1}{r} \left(\frac{dw}{dr} \right)^2 + r \left(\frac{d^2w}{dr^2} \right)^2 + 2\nu_b \left(\frac{dw}{dr} \right) \left(\frac{d^2w}{dr^2} \right) \right] dr.$$

The individual terms of the integrand become

$$\begin{aligned} \frac{1}{r} \left(\frac{dw}{dr} \right)^2 &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right]^2 r^{2n-3} A_n^2 \\ &+ 2 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m < n}}^{\infty} \left[p_m m - 2q_m(m+2) \left(\frac{r}{a} \right)^2 + (m+4) \left(\frac{r}{a} \right)^4 \right] \\ &\times \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right] r^{m+n-3} A_m A_n; \end{aligned}$$

$$\begin{aligned}
r \left(\frac{d^2 w}{dr^2} \right)^2 &= \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n(n-1) - 2q_n(n+2)(n+1) \left(\frac{r}{a} \right)^2 + (n+4)(n+3) \left(\frac{r}{a} \right)^4 \right]^2 r^{2n-3} A_n^2 \\
&+ 2 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m < n}}^{\infty} \left[p_m m(m-1) - 2q_m(m+2)(m+1) \left(\frac{r}{a} \right)^2 + (m+4)(m+3) \left(\frac{r}{a} \right)^4 \right] \\
&\times \left[p_n n(n-1) - 2q_n(n+2)(n+1) \left(\frac{r}{a} \right)^2 + (n+4)(n+3) \left(\frac{r}{a} \right)^4 \right] r^{m+n-3} A_m A_n; \\
2\nu_b \left(\frac{dw}{dr} \right) \left(\frac{d^2 w}{dr^2} \right) &= 2\nu_b \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_m m - 2q_m(m+2) \left(\frac{r}{a} \right)^2 + (m+4) \left(\frac{r}{a} \right)^4 \right] \\
&\times \left[p_n n(n-1) - 2q_n(n+2)(n+1) \left(\frac{r}{a} \right)^2 + (n+4)(n+3) \left(\frac{r}{a} \right)^4 \right] r^{m+n-3} A_m A_n \\
&= 2\nu_b \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right] \\
&\times \left[p_n n(n-1) - 2q_n(n+2)(n+1) \left(\frac{r}{a} \right)^2 + (n+4)(n+3) \left(\frac{r}{a} \right)^4 \right] r^{2n-3} A_n^2 \\
&+ 2\nu_b \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m < n}}^{\infty} \left[p_m m - 2q_m(m+2) \left(\frac{r}{a} \right)^2 + (m+4) \left(\frac{r}{a} \right)^4 \right] \\
&\times \left[p_n n(n-1) - 2q_n(n+2)(n+1) \left(\frac{r}{a} \right)^2 + (n+4)(n+3) \left(\frac{r}{a} \right)^4 \right] r^{m+n-3} A_m A_n \\
&+ 2\nu_b \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m < n}}^{\infty} \left[p_m m(m-1) - 2q_m(m+2)(m+1) \left(\frac{r}{a} \right)^2 + (m+4)(m+3) \left(\frac{r}{a} \right)^4 \right] \\
&\times \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right] r^{m+n-3} A_m A_n.
\end{aligned}$$

The integrand may be combined into one expression as follows:

$$\begin{aligned}
& \frac{1}{r} \left(\frac{dw}{dr} \right)^2 + r \left(\frac{d^2w}{dr^2} \right)^2 + 2\nu_b \left(\frac{dw}{dr} \right) \left(\frac{d^2w}{dr^2} \right) \\
&= C_1 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m \leq n}}^{\infty} \left(p_m p_n m n \{1 + (m-1)(n-1) + \nu_b [(m-1) + (n-1)]\} \right. \\
&\quad - 2p_m q_n m(n+2) \{1 + (m-1)(n+1) + \nu_b [(m-1) + (n+1)]\} \left(\frac{r}{a} \right)^2 \\
&\quad - 2q_m p_n (m+2) n \{1 + (m+1)(n-1) + \nu_b [(m+1) + (n-1)]\} \left(\frac{r}{a} \right)^2 \\
&\quad + p_m m(n+4) \{1 + (m-1)(n+3) + \nu_b [(m-1) + (n+3)]\} \left(\frac{r}{a} \right)^4 \\
&\quad + 4q_m q_n (m+2)(n+2) \{1 + (m+1)(n+1) + \nu_b [(m+1) + (n+1)]\} \left(\frac{r}{a} \right)^4 \\
&\quad + p_n (m+4) n \{1 + (m+3)(n-1) + \nu_b [(m+3) + (n-1)]\} \left(\frac{r}{a} \right)^4 \\
&\quad - 2q_m (m+2)(n+4) \{1 + (m+1)(n+3) + \nu_b [(m+1) + (n+3)]\} \left(\frac{r}{a} \right)^6 \\
&\quad - 2q_n (m+4)(n+2) \{1 + (m+3)(n+1) + \nu_b [(m+3) + (n+1)]\} \left(\frac{r}{a} \right)^6 \\
&\quad \left. + (m+4)(n+4) \{1 + (m+3)(n+3) + \nu_b [(m+3) + (n+3)]\} \left(\frac{r}{a} \right)^8 \right) r^{m+n-3} A_m A_n,
\end{aligned}$$

where $C_1 = 2$ for $m \neq n$, and $C_1 = 1$ for $m = n$.

If integration is carried out over the annular region, extending from r_{b-1} to r_b , there is obtained for the strain energy of one plate

$$\begin{aligned}
 U_P &= \pi \sum_{b=1}^B D_b \int_{r_{b-1}}^{r_b} \left[\frac{1}{r} \left(\frac{dw}{dr} \right)^2 + r \left(\frac{d^2w}{dr^2} \right)^2 + 2\nu_b \left(\frac{dw}{dr} \right) \left(\frac{d^2w}{dr^2} \right) \right] dr \\
 &= 2\pi C \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m \leq n}}^{\infty} \left\{ \sum_{b=1}^B p_m p_n D_b m n \left[\frac{1 + (m-1)(n-1)}{m+n-2} + \nu_b \right] \left[r_b^{m+n-2} - r_{b-1}^{m+n-2} \right] \right. \\
 &\quad - 2 \sum_{b=1}^B p_m q_n D_b m(n+2) \left[\frac{1 + (m-1)(n+1)}{m+n} + \nu_b \right] \left[\frac{r_b^{m+n} - r_{b-1}^{m+n}}{a^2} \right] \\
 &\quad - 2 \sum_{b=1}^B q_m p_n D_b (m+2)n \left[\frac{1 + (m+1)(n-1)}{m+n} + \nu_b \right] \left[\frac{r_b^{m+n} - r_{b-1}^{m+n}}{a^2} \right] \\
 &\quad + \sum_{b=1}^B p_m D_b m(n+4) \left[\frac{1 + (m-1)(n+3)}{m+n+2} + \nu_b \right] \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^4} \right] \\
 &\quad + 4 \sum_{b=1}^B q_m q_n D_b (m+2)(n+2) \left[\frac{1 + (m+1)(n+1)}{m+n+2} + \nu_b \right] \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^4} \right] \\
 &\quad + \sum_{b=1}^B p_n D_b (m+4)n \left[\frac{1 + (m+3)(n-1)}{m+n+2} + \nu_b \right] \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^4} \right] \\
 &\quad - 2 \sum_{b=1}^B q_m D_b (m+2)(n+4) \left[\frac{1 + (m+1)(n+3)}{m+n+4} + \nu_b \right] \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^6} \right] \\
 &\quad - 2 \sum_{b=1}^B q_n D_b (m+4)(n+2) \left[\frac{1 + (m+3)(n+1)}{m+n+4} + \nu_b \right] \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^6} \right] \\
 &\quad \left. + \sum_{b=1}^B D_b (m+4)(n+4) \left[\frac{1 + (m+3)(n+3)}{m+n+6} + \nu_b \right] \left[\frac{r_b^{m+n+6} - r_{b-1}^{m+n+6}}{a^8} \right] \right\} A_m A_n,
 \end{aligned}$$

where $C = 1$ for $m \neq n$, and $C = \frac{1}{2}$ for $m = n$.

Thus, the strain energy of both plates may be written in the following form:*

$$U_p = \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} C'_{mn} A_m A_n, \\ m \leq n$$

where

$$\begin{aligned} C'_{mn} = 4\pi C \left[\sum_{b=1}^B p_m p_n m n D_b(\delta_1 + \nu_b) \alpha_1 - 2 \sum_{b=1}^B p_m q_n m(n+2) D_b(\delta_2 + \nu_b) \alpha_2 \right. \\ - 2 \sum_{b=1}^B q_m p_n (m+2) n D_b(\delta_3 + \nu_b) \alpha_2 + \sum_{b=1}^B p_m m(n+4) D_b(\delta_4 + \nu_b) \alpha_3 \\ + 4 \sum_{b=1}^B q_m q_n (m+2)(n+2) D_b(\delta_5 + \nu_b) \alpha_3 + \sum_{b=1}^B p_n (m+4) n D_b(\delta_6 + \nu_b) \alpha_3 \\ - 2 \sum_{b=1}^B q_m (m+2)(n+4) D_b(\delta_7 + \nu_b) \alpha_4 - 2 \sum_{b=1}^B q_n (m+4)(n+2) D_b(\delta_8 + \nu_b) \alpha_4 \\ \left. + \sum_{b=1}^B (m+4)(n+4) D_b(\delta_9 + \nu_b) \alpha_5 \right] \end{aligned}$$

and

$$C = 1 \quad \text{for } m \neq n;$$

$$C = \frac{1}{2} \quad \text{for } m = n;$$

$$r_B = a;$$

$$\phi_c = m + n + 2(c-2), \quad c = 1, 2, \dots, 6;$$

$$\alpha_c = \left(r_b^{\phi_c} - r_{b-1}^{\phi_c} \right) / a^{2(c-1)}, \quad c = 1, 2, \dots, 6;$$

$$\delta_1 = [1 + (m-1)(n-1)]/\phi_1; \quad \delta_6 = [1 + (m+3)(n-1)]/\phi_3;$$

$$\delta_2 = [1 + (m-1)(n+1)]/\phi_2; \quad \delta_7 = [1 + (m+1)(n+3)]/\phi_4;$$

$$\delta_3 = [1 + (m+1)(n-1)]/\phi_2; \quad \delta_8 = [1 + (m+3)(n+1)]/\phi_4;$$

$$\delta_4 = [1 + (m-1)(n+3)]/\phi_3; \quad \delta_9 = [1 + (m+3)(n+3)]/\phi_5.$$

$$\delta_5 = [1 + (m+1)(n+1)]/\phi_3;$$

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*It should be noted that p and q are functions of ν . Thus, in a strict sense, p and q should be also subscripted by b . However, in order to avoid the confusion of double subscripts, the subscript b has been discarded.

5. Strain Energy of Tubes

The bending strain energy of one tube (see Fig. 7) is

$$U_t = \frac{1}{2E_t I} \int_{h/2}^{(h/2)+\ell} M^2 dx,$$

where

$$M = \frac{6E_t I(\ell + h)}{\ell^3} [\ell + h - 2x] \frac{dw}{dr}.$$

When the integration is performed, the strain energy is obtained as

$$U_t = \frac{6E_t I(\ell + h)^2}{\ell^3} \left(\frac{dw}{dr} \right)^2.$$

In terms of a series expansion, the expression becomes

$$\begin{aligned} U_t = & \frac{6E_t I(\ell + h)^2}{\ell^3} \left\{ \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right]^2 r^{2n-2} A_n^2 \right. \\ & + 2 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1 \\ m < n}}^{\infty} \left[p_m m - 2q_m(m+2) \left(\frac{r}{a} \right)^2 + (m+4) \left(\frac{r}{a} \right)^4 \right] \\ & \times \left. \left[p_n n - 2q_n(n+2) \left(\frac{r}{a} \right)^2 + (n+4) \left(\frac{r}{a} \right)^4 \right] r^{m+n-2} A_m A_n \right\}. \end{aligned}$$

If the strain energy of the tubes is assumed to vary in a continuous manner from r_{b-1} to r_b , the expression becomes

$$U_t = \eta \int_0^{2\pi} \int_{r_{b-1}}^{r_b} \left(\frac{dw}{dr} \right)^2 r dr d\theta,$$

where

$$\eta = 6\lambda_t E_t I(\ell+h)^2/\pi \ell^3 (r_b^2 - r_{b-1}^2).$$

In terms of a series expansion, the equation becomes

$$\begin{aligned} U_t &= 2\pi\eta \int_{r_{b-1}}^{r_b} \left\{ \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \left(\frac{r}{a}\right)^2 + (n+4) \left(\frac{r}{a}\right)^4 \right]^2 r^{2n-1} A_n^2 \right. \\ &\quad + 2 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_m m - 2q_m(m+2) \left(\frac{r}{a}\right)^2 + (m+4) \left(\frac{r}{a}\right)^4 \right] \left[p_n n - 2q_n(n+2) \left(\frac{r}{a}\right)^2 \right. \\ &\quad \left. \left. + (n+4) \left(\frac{r}{a}\right)^4 \right] r^{m+n-1} A_m A_n \right\} dr \\ &= 2\pi\eta \left(\int_{r_{b-1}}^{r_b} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \left(\frac{r}{a}\right)^2 + (n+4) \left(\frac{r}{a}\right)^4 \right]^2 r^{2n-1} A_n^2 \right. \\ &\quad + 2 \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ \left[p_m p_n \frac{mn}{m+n} - 2p_m q_n \frac{m(n+2)}{m+(n+2)} \left(\frac{r_b}{a}\right)^2 - 2q_m p_n \frac{(m+2)n}{(m+2)+n} \left(\frac{r_b}{a}\right)^2 \right. \right. \\ &\quad + p_m \frac{m(n+4)}{m+(n+4)} \left(\frac{r_b}{a}\right)^4 + 4q_m q_n \frac{(m+2)(n+2)}{(m+2)+(n+2)} \left(\frac{r_b}{a}\right)^4 + p_n \frac{(m+4)n}{(m+4)+n} \left(\frac{r_b}{a}\right)^4 \\ &\quad - 2q_m \frac{(m+2)(n+4)}{(m+2)+(n+4)} \left(\frac{r_b}{a}\right)^6 - 2q_n \frac{(m+4)(n+2)}{(m+4)+(n+2)} \left(\frac{r_b}{a}\right)^6 \\ &\quad \left. \left. + \frac{(m+4)(n+4)}{(m+4)+(n+4)} \left(\frac{r_b}{a}\right)^8 \right] r_b^{m+n} - \left[p_m p_n \frac{m+n}{m+n} - 2p_m q_n \frac{m(n+2)}{m+(n+2)} \left(\frac{r_{b-1}}{a}\right)^2 \right. \right. \\ &\quad - 2q_m p_n \frac{(m+2)n}{(m+2)+n} \left(\frac{r_{b-1}}{a}\right)^2 + p_m \frac{m(n+4)}{m+(n+4)} \left(\frac{r_{b-1}}{a}\right)^4 + 4q_m q_n \frac{(m+2)(n+2)}{(m+2)+(n+2)} \left(\frac{r_{b-1}}{a}\right)^4 \\ &\quad + p_n \frac{(m+4)n}{(m+4)+n} \left(\frac{r_{b-1}}{a}\right)^4 - 2q_m \frac{(m+2)(n+4)}{(m+2)+(n+4)} \left(\frac{r_{b-1}}{a}\right)^6 - 2q_n \frac{(m+4)(n+2)}{(m+4)+(n+2)} \left(\frac{r_{b-1}}{a}\right)^6 \\ &\quad \left. \left. + \frac{(m+4)(n+4)}{(m+4)+(n+4)} \left(\frac{r_{b-1}}{a}\right)^8 \right] r_{b-1}^{m+n} \right\} A_m A_n \right). \end{aligned}$$

The expression giving the strain energy of the tubes for all regions of the plates may be presented in the following form:

$$\begin{aligned}
 U_t = & 4\pi C \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ \sum_{b=1}^B p_m p_n \eta_b \frac{mn}{m+n} \left[r_b^{m+n} - r_{b-1}^{m+n} \right] \right. \\
 & - 2 \sum_{b=1}^B p_m q_n \eta_b \frac{m(n+2)}{m+n+2} \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^2} \right] \\
 & - 2 \sum_{b=1}^B q_m p_n \eta_b \frac{(m+2)n}{m+n+2} \left[\frac{r_b^{m+n+2} - r_{b-1}^{m+n+2}}{a^2} \right] \\
 & + \sum_{b=1}^B p_m \eta_b \frac{m(n+4)}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] \\
 & + 4 \sum_{b=1}^B q_m q_n \eta_b \frac{(m+2)(n+2)}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] \\
 & + \sum_{b=1}^B p_n \eta_b \frac{(m+4)n}{m+n+4} \left[\frac{r_b^{m+n+4} - r_{b-1}^{m+n+4}}{a^4} \right] \\
 & - 2 \sum_{b=1}^B q_m \eta_b \frac{(m+2)(n+4)}{m+n+6} \left[\frac{r_b^{m+n+6} - r_{b-1}^{m+n+6}}{a^6} \right] \\
 & - 2 \sum_{b=1}^B q_n \eta_b \frac{(m+4)(n+2)}{m+n+6} \left[\frac{r_b^{m+n+6} - r_{b-1}^{m+n+6}}{a^6} \right] \\
 & \left. + \sum_{b=1}^B \eta_b \frac{(m+4)(n+4)}{m+n+8} \left[\frac{r_b^{m+n+8} - r_{b-1}^{m+n+8}}{a^8} \right] \right\} A_m A_n,
 \end{aligned}$$

where

$$C = 1 \quad \text{for } m \neq n; \quad C = \frac{1}{2} \quad \text{for } m = n;$$

$$\eta_b = \left[6\lambda_t E_t I (\ell + h)^2 / \pi \ell^3 (r_b^2 - r_{b-1}^2) \right]_b.$$

The same expression may be written in a slightly more compact way:

$$U_t = \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} C''_{mn} A_m A_n, \\ m \leq n$$

where

$$\begin{aligned} C''_{mn} = & 4\pi Ca^2 \left[\sum_{b=1}^B p_m p_n m n \eta_b \frac{\alpha_2}{\phi_2} - 2 \sum_{b=1}^B p_m p_n m(n+2) \eta_b \frac{\alpha_3}{\phi_3} \right. \\ & - 2 \sum_{b=1}^B q_m p_n (m+2) n \eta_b \frac{\alpha_3}{\phi_3} + \sum_{b=1}^B p_m m(n+4) \eta_b \frac{\alpha_4}{\phi_4} \\ & + 4 \sum_{b=1}^B q_m q_n (m+2)(n+2) \eta_b \frac{\alpha_4}{\phi_4} + \sum_{b=1}^B p_n (m+4) n \eta_b \frac{\alpha_4}{\phi_4} \\ & - 2 \sum_{b=1}^B q_m (m+2)(n+4) \eta_b \frac{\alpha_5}{\phi_5} - 2 \sum_{b=1}^B q_n (m+4)(n+2) \eta_b \frac{\alpha_5}{\phi_5} \\ & \left. + \sum_{b=1}^B (m+4)(n+4) \eta_b \frac{\alpha_6}{\phi_6} \right]. \end{aligned}$$

6. Strain Energy of Supports

The work done by a couple is proportional to the product of the moment and the angular displacement for the elastic system treated here, the work done on the plate by the supports is equal to the strain energy U_s stored in the supports, which may be written as

$$U_s = \frac{1}{2} \left[2\pi a M_r \Big|_{r=a} \right] \frac{dw}{dr} \Big|_{r=a}.$$

Substitution of the expression for $M_r \Big|_{r=a}$ and $\frac{dw}{dr} \Big|_{r=a}$ yields

$$\begin{aligned} U_s = & \pi a \left\{ \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \left[p_m m - 2q_m(m+2) + (m+4) \right] a^{m-1} A_m \right\} \left\{ -D_B \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n(n-1+\nu_B) \right. \right. \\ & \left. \left. - 2q_n(n+2)(n+1+\nu_B) + (n+4)(n+3+\nu_B) \right] a^{n-2} A_n \right\} \\ = & -\pi a D_B \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_m m - 2q_m(m+2) + (m+4) \right] \left[p_n n(n-1+\nu_B) \right. \\ & \left. - 2q_n(n+2)(n+1+\nu_B) + (n+4)(n+3+\nu_B) \right] a^{m+n-3} A_m A_n \end{aligned}$$

$$\begin{aligned}
U_s = & -\pi a D_B \left\{ \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) + (n+4) \right] \left[p_n n(n-1+\nu_B) - 2q_n(n+2)(n+1+\nu_B) \right. \right. \\
& + (n+4)(n+3+\nu_B) \left. \right] a^{2n-3} A_n^2 + \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_m m - 2q_m(m+2) + (m+4) \right] \left[p_n n(n-1+\nu_B) \right. \\
& - 2q_n(n+2)(n+1+\nu_B) + (n+4)(n+3+\nu_B) \left. \right] a^{m+n-3} A_m A_n + \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n n - 2q_n(n+2) \right. \\
& + (n+4) \left. \right] \left[p_m m(m-1+\nu_B) - 2q_m(m+2)(m+1+\nu_B) + (m+4)(m+3+\nu_B) \right] \left. \right\} a^{m+n-3} A_m A_n \\
= & -\pi C a D_B \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ p_m p_n m n \left[(m-1+\nu_B) + (n-1+\nu_B) \right] \right. \\
& - 2p_m q_n m(n+2) \left[(m-1+\nu_B) + (n+1+\nu_B) \right] - 2q_m p_n(m+2) n \left[(m+1+\nu_B) + (n-1+\nu_B) \right] \\
& + p_m m(n+4) \left[(m-1+\nu_B) + (n+3+\nu_B) \right] + 4q_m q_n(m+2)(n+2) \left[(m+1+\nu_B) + (n+1+\nu_B) \right] \\
& + p_n(m+4) n \left[(m+3+\nu_B) + (n-1+\nu_B) \right] - 2q_m(m+2)(n+4) \left[(m+1+\nu_B) + (n+3+\nu_B) \right] \\
& - 2q_n(m+4)(n+2) \left[(m+3+\nu_B) + (n+1+\nu_B) \right] + (m+4)(n+4) \left[(m+3+\nu_B) \right. \\
& \left. \left. + (n+3+\nu_B) \right] \right\} a^{m+n-3} A_m A_n.
\end{aligned}$$

The same expression for both supports may be written in terms of other symbols as follows:

$$U_s = \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} C_{mn}''' A_m A_n,$$

$m \leq n$

where

$$\begin{aligned}
C_{mn}''' = & -4CD_B a^{\phi_1} \left[p_m p_n m n \left(\frac{\phi_1}{2} + \nu_B \right) - 2p_m q_n m(n+2) \left(\frac{\phi_2}{2} + \nu_B \right) - 2q_m p_n(m+2) n \left(\frac{\phi_2}{2} + \nu_B \right) \right. \\
& + p_m m(m+4) \left(\frac{\phi_3}{2} + \nu_B \right) + 4q_m q_n(m+2)(n+2) \left(\frac{\phi_3}{2} + \nu_B \right) + p_n(m+4) n \left(\frac{\phi_3}{2} + \nu_B \right) \\
& \left. - 2q_m(m+2)(n+4) \left(\frac{\phi_4}{2} + \nu_B \right) - 2q_n(m+4)(n+2) \left(\frac{\phi_4}{2} + \nu_B \right) + (m+4)(n+4) \left(\frac{\phi_5}{2} + \nu_B \right) \right]
\end{aligned}$$

and

$$C = 1 \quad \text{for } m \neq n;$$

$$C = \frac{1}{2} \quad \text{for } m = n.$$

Note should be made of the fact that expressions for p and q should also be subscripted by b , for these pertain to the region at the edge.

A simplification of the expression for C_{mn}''' can be obtained when the equations of p and q are substituted:

$$C_{mn}''' = \frac{128 CDk(1-k)}{a^2} a^{m+n} \left[\frac{1}{1 + \nu_B + 2m} + \frac{1}{1 + \nu_B + 2n} \right].$$

7. Potential Energy of External Loads

The change in potential energy of a concentrated load is the product of the load and the corresponding deflection which may be represented as follows:

$$V = - Pw$$

or, in terms of a series expansion,

$$V = - P \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n - 2q_n \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right] r^n A_n.$$

If there is more than one load, the total change in potential energy may be represented as follows:

$$V = - \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} C_n A_n,$$

where

$$C_n = \sum_{i,j}^{\lambda_\ell} P_{i,j} r_{i,j}^n \left[p_n - 2q_n \left(\frac{r}{a} \right)_{i,j}^2 + \left(\frac{r}{a} \right)_{i,j}^4 \right].$$

In the case of uniformly distributed load, the change in potential energy due to the load is obtained by integration. Thus,

$$V = - \int_0^{2\pi} \int_{r_{C-1}}^{r_C} f r w d r d\theta = - 2\pi \int_{r_{C-1}}^{r_C} f r w d r,$$

where f is the load intensity, and r_C and r_{C-1} are the limiting radii of the distribution. When the series is introduced, the equation becomes

$$\begin{aligned} V &= - 2\pi f \int_{r_{C-1}}^{r_C} \left\{ \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left[p_n - 2q_n \left(\frac{r}{a} \right)^2 + \left(\frac{r}{a} \right)^4 \right] r^{n+1} A_n \right\} d r \\ &= - 2\pi f \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} \left\{ p_n \frac{1}{n+2} \left[r_C^{n+2} - r_{C-1}^{n+2} \right] - 2q_n \frac{1}{n+4} \left[\frac{r_C^{n+4} - r_{C-1}^{n+4}}{a^2} \right] + \frac{1}{n+6} \left[\frac{r_C^{n+6} - r_{C-1}^{n+6}}{a^4} \right] \right\} A_n \end{aligned}$$

or

$$V = - \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} C_n A_n,$$

where

$$C_n = 2\pi f \left\{ p_n \frac{1}{n+2} \left[r_C^{n+2} - r_{C-1}^{n+2} \right] - 2q_n \frac{1}{n+4} \left[\frac{r_C^{n+4} - r_{C-1}^{n+4}}{a^2} \right] + \frac{1}{n+6} \left[\frac{r_C^{n+6} - r_{C-1}^{n+6}}{a^4} \right] \right\}.$$

B. COMPUTER PROGRAM1. Fortran II Statements

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C      VERSION TO USE K AS PROPORTIONALITY FACTOR WITH MAX MOMENTS
C
C      MAIN CODE ORGANIZES INPUT, INITIALIZES VARIOUS
C      QUANTITIES AND SERVES AS A MASTER CONTROL ROUTINE OVER
C      SUBROUTINES WHICH PERFORM THE ARITHMETIC REQUIRED.
C      SUBROUTINES REQUIRED ARE GETOUT, KCALC, UPLATE, SUMVUT,
C      DIFFER, MATINV, CHECK AND MAXIM
C
      DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
      1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
      2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
      3UT(10,10),V(10)
      DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
      1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
      2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
      DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
      1RA(1000),RB(1000),AUT(10,10,2)
      3,AM1(6,10),TEMP 1D(10)
      COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
      1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,ID,IA,IB,IB1,IB2,
      2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
      3DETERM,DWA,AMR,WA,AMO,WO,IAVE
      COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
      1T3D,T1D2,T2D2,T3D2,Y
      2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
      3,NR,RINT,WDEFL,SLOPE
      COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
      1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
      2,LINES
      DIMENSIONVCK(10),DCK(10,10),ZU(10)
C
      PI=3.14159265
      9 READ INPUT TAPE 7,10,ID,(BXCX(I),I=1,11)
      10 FORMAT(I6,11A6)
      READ INPUT TAPE 7,11,IA,IB,IB2,NZ,NM,(IN(I),I=IB,IB2)
      1,NUMBR,MCK,K1
      PT=0.
      HT=0.
      TT=0.
      CR=NUMBR
      11 FCRMAT(12I6)
      IB1=IB+1
      IF(IB-IN(IB))12,13,12
      12 IP=321
      CALL GETOUT(1)
      CALL SYSTEM
      13 READ INPUT TAPE 7,14,EP,ET,H,VU,A,RO,RI,TL,S,RHOP,RHOT,RCALCN,
      1RCNE,RTWO,R3,R4,HO,TO,PO,(ZU(I),I=1,NZ)
      14 FORMAT(6E12.6)
      READ INPUT TAPE 7,14,(P(I,1),I=1,IB2)
      KZ=1
      Z=ZU(1)

```

```

NR=RCALCN+1.0
FRACT=A/(RCALCN+1.0)
DO 15 I=1,NR
FI=I
15 RINT(I)=FI*FRACT
IF (NUMBR) 16,16,142
142 ASQ=A**2
CRD=A/CR
RAB(1)=CRD
RA(1)=RAB(1)**2/ASQ
RB(1)=RA(1)**2
C RAB=ACTUAL DISTANCE, RA=(RELATIVE DISTANCE)**2
DO 143 I=2,NUMBR
RAB(I)=RAB(I-1)+CRD
RA(I)=RAB(I)**2/ASQ
143 RB(I)=RA(I)**2
C KEEP TRACK OF PICTURE
16 DO 20 I=1,IB
IN(I)=I
ICOL(I)=1
20 ICOLR(I)=I
IF (IB-IB2) 21,23,23
21 DO 22 I=IB1,IB2
ICOL(I)=I/2-(IN(I)-1)/2
22 ICOLR(I)=ICOL(I)+IN(I)-1
23 DO 30 I=1,IB2
RAVE(I)=0.0
FI=I
ICL=ICOL(I)
ICLR=ICOLR(I)
DO 29 J=ICL,ICLR
FJ=J
R(I,J)=S*SQRTF(FI**2-FI*FJ+FJ**2)
RAVE(I)=RAVE(I)+R(I,J)
PT=PT+P(I,1)
29 P(I,J)=P(I,1)
FI=IN(I)
HT=HT+FI
IF (I-IA) 30,295,295
295 TT=TT+FI
30 RAVE(I)=RAVE(I)/FI
HT=6.*HT+HO
TT=6.*TT+TO
PTD=6.*PT+PO
PT=PTD/(PI*(R4**2-R3**2))
F1=((2.0*H)*RHOP*(A**2-HT*RO**2)+TT*RHOT*
1 (TL+(2.0*H))*(RO**2-RI**2))/A**2
D = (EP*H**3)/(12.0*(1.0-VU**2))
AI = 0.78539816*(RO**4-RI**4)
CALL GETOUT(1)
PT=2.0*PT
F1=2.0*F1
Z=1.0
C RETURN TO 220 FOR NEW VALUES OF K,Z
220 CALL KCALC
CALL UPLATE
CALL GETOUT(2)
C AUT(1) IS INTEGRAL FROM R1 TO R2
K=1
XU=(RONE/A)**2

```

```

221 XU2=XU**2
    IK=0
    JK=0
    DO 226 I=1,NM
        FM=IK
        C1(I)=PK(I)*FM
        C2(I)=2.0*Q(I)*(FM+2.)
        C3(I)=FM+4.
    DO 225 J=1,NM
        IF (I-J) 223,222,222
222 FT1=1.0
    JK=IK
    GO TO 224
223 FT1=2.0
224 FN=JK
    AUT(I,J,K)=PK(I)*PK(J)*FM*FN/(FM+FN)-2.*PK(I)*Q(J)*FM*(FN+2.)*
1XU/((FM+FN)+2.)-2.*Q(I)*PK(J)*(FM+2.)*FN*XU/((FM+FN)+2.)
2+PK(I)*FM*(FN+4.)*XU2/((FM+FN)+4.)+4.*Q(I)*Q(J)*(FM+2.)*
3(FN+2.)*XU2/((FM+FN)+4.)
    AUT(I,J,K)=FT1*(AUT(I,J,K)+PK(J)*(FM+4.)*FN*XU2/((FM+FN)+4.)
1-2.*Q(I)*(FM+2.)*(FN+4.)*(XU*XU2)/((FM+FN)+6.)
2-2.*Q(J)*(FM+4.)*(FN+2.)*(XU*XU2)/((FM+FN)+6.)
3+(FM+4.)*(FN+4.)*XU2**2/((FM+FN)+8.))*SQRTF(XU*A**2)**
4(IK+JK)
    JK=J+1
225 CONTINUE
    IK=I+1
226 CCNTINUE
    IF(K-1)227,227,228
227 K=2
    XU=(RTWO/A)**2
    GO TO 221
228 FT=12.*ET*AI*(TL+H)**2*TT/(TL**3*(RTWO**2-RONE**2))
    DO 2290 I=1,NM
    DO 2290 J=1,NM
        AUT(I,J,1)=FT*(AUT(I,J,2)-AUT(I,J,1))
2290 CONTINUE
    CALL GETOUT(3)
    IAVE=0
    KOF6=1
C    RETURN TO 230 FOR AVERAGE R CALCULATIONS AT Z=0, Z=1
230 DO 300 I=1,IB2
    DO 240 I1=1,NM
        SUMV(I1)=0.0
    DO 240 I2=I1,NM
240 SUMUT(I1,I2)=0.0
    IP=I
    IF(IAVE)260,260,250
250 ICL=1
    ICLR=1
286 FLJ=IN(I)*6
    GO TO 280
260 ICL=ICOL(I)
    ICLR=ICOLR(I)
285 FLJ = 6.0
280 DO 299 J=ICL,ICLR
    JP=J
    CALL SUMVUT
289 DO 290 I1=1,NM
    SUMV(I1)=SUMV(I1)+FLJ*V(I1)

```

```

      AA(I1)=AA(I1)+FLJ*V(I1)
      IF(I-IA) 290,2891,2891
2891 DO 2895 I2=I1,NM
      SUMUT(I1,I2)=SUMUT(I1,I2)+FLJ*UT(I1,I2)
      C(I1,I2)=C(I1,I2)+UT(I1,I2)*FLJ
2895 CONTINUE
      290 CONTINUE
      299 CONTINUE
      300 CONTINUE
      DO 500 I=1,NM
      SUMV(I)=AA(I)
      DO 500 J=I,NM
      UT(I,J)=C(I,J)
      500 C(I,J)=AUT(I,J,1)+UP(I,J)+US(I,J)
C      DIFFERENTIATE OMEGA, USING DIFF, C(OMEGA) = DI(I,J)
      CALL DIFFER
      CALL GETOUT(5)
      KV=0
      IF (IAVE) 540,540,560
      540 FI=0.
      I=0
      DO 550 I1=1,NM
      546 TEMP1D(I1)=(PK(I1)/(FI+2.))-2.0*Q(I1)*(RAVE(IB2)/A)**2/(FI+4.)
      1+(RAVE(IB2)/A)**4/(FI+6.))-2.0*RAVE(IB2)**I
      TEMP1(I1)=(PK(I1)/(FI+2.))-2.*Q(I1)/(FI+4.))+1./(FI+6.))
      1*A**(I+2)
      I=I1+1
      550 FI=I
      560 KV=KV+1
      IP=KV
      I=0
      DO 6045 I1=1,NM
      GO TO (601,602,603,604), KV
      601 V(I1)=-(SUMV(I1)+F1*TEMP1(I1)*PI)
      GO TO 6045
      602 V(I1)=-SUMV(I1)
      GO TO 6045
      603 V(I1)=-PTD*TEMP1D(I1)-PI*F1*TEMP1(I1)
      GO TO 6045
      604 V(I1)=-PTD*TEMP1D(I1)
      6045 CCNTINUE
      CALL GETOUT(6)
C      SOLVE FOR A(I) USING EACH ORDER OF ARRAYS UP TO NM
      DO 700 IO=1,NM
      N=0
      DO 651 I=1,IO
C      STORE CORRECT PORTIONS OF DI, V IN DD,AA. AA(I) WILL=A(I)
      DO 650 I1=1,IO
      650 DD(I,I1)=DI(I,I1)
      DD(IO+1,I)=A**N
      DD(I,IO+1)=DD(IO+1,I)
      N=I+1
      651 AA(I)=-V(I)
      DD(IO+1,IO+1)=0.
      AA(IO+1)=AM1(KOF6,IO)
      IF (Z) 6512,6512,6501
      6501 IF (Z-1.0) 6502,6512,6502
      6502 IF (IO-2) 6512,6511,6511
      6511 IP=IO+1
      GO TO 6513

```



```

6512 IP=IO
6513 CALLMATINV(DD,IP,AA,1,DETERM)
      IF (MCK) 653,653,6514
6514 N=0
      DO6510I=1,IO
      DO6500I1=1,IO
6500 DD(I,I1)=DI(I,I1)
      DD(IO+1,I)=A**N
      DC(I,IO+1)=DD(IO+1,I)
      N=I+1
      VCK(I)=-V(I)
6510 CONTINUE
      DC(IO+1,IO+1)=0.
      VCK(IO+1)=AM1(KOF6,IO)
      IF(Z)15120,15120,15010
15010 IF(Z-1.0)15020,15120,15020
15020 IF(IO-2) 15120,15110,15110
15110 IP=IO+1
      GOTO15130
15120 IP=IO
15130 DO6515I=1,IP
6515 DCK(I)=0.0
      DO6516I=1,IP
      DO6516J=1,IP
6516 DCK(I)=DCK(I)+DD(I,J)*AA(J)
      WRITEOUTPUTTAPE6,6517,DETERM,(VCK(I),I=1,IP)
6517 FORMAT(22H0CHECK MATRIX SOLUTION4X7HDETERM=E17.8/(5E17.8))
      WRITEOUTPUTTAPE6,6518,(DCK(I),I=1,IP)
6518 FORMAT(1H0/(5E17.8))
653 WO=AA(1)*PK(1)
      IF(IO-1) 670,670,680
670 AA(2)=0.0
680 AMO=2.*D*(1.+VU)*((2.*AA(1)*Q(1))/A**2-AA(2)*PK(2))
      IP=IO
      CALL GETOUT(7)
690 CALL CHECK
6908 IF (Z-1.0) 695,691,691
691 AM1(KOF6,IO)=-AMRP(NR)*A**2/(8.0*D*Z)
695 IF (NUMBR) 700,700,698
698 CALL MAXIM
700 CONTINUE
      IF (K1) 7050,7050,807
7050 KOF6=KOF6+1
      IF (KV-2) 560,7001,7001
C      DETERMINE IF AVERAGE VALUE CALCULATION IS TO BE
C      OR HAS BEEN DONE AND SHIFT VALUES ACCORDINGLY
7001 IF (IAVE) 701,701,7002
7002 IF (KV-4) 560,701,701
701 DO 704 I1=1,NM
      AA(I)=0.0
      DO 704 I2=I1,NM
704 C(I1,I2)=0.0
      IF (IAVE) 705,705,805
705 IAVE=1
      DO 750 I=1,I82
      TEMP(I)=R(I,1)
750 R(I,1)=RAVE(I)
      GO TO 230
805 DO 806 I=1,I82
806 R(I,1)=TEMP(I)

```

```

807 KZ=KZ+1
    Z=ZU(KZ)
    IF (KZ-NZ) 220,220,9

```

```

SUBROUTINE GETOUT(KIND)

```

```

C
C
C
C

```

```

    GETOUT WRITES EACH TYPE OF OUTPUT REQUESTED BY THE MAIN
    CODE, AS REQUESTED BY THE VALUE OF KIND

```

```

    DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
    1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
    2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
    3UT(10,10),V(10)
    DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
    1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
    2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
    DIMENSION AMPR(20),AMTR(20),C1(10),C2(10),C3(10),RAB(1000),
    1RA(1000),RB(1000),AUT(10,10,2)
    COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
    1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,1D,1A,1B,1B1,1B2,
    2D,1A,1E,1E,1H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,1P,1P,
    3DETERM,DWA,AMR,WA,AMO,WO,IAVE
    COMMON US,TT,PT,F1,KV,RHOP,RHOT,P1,TEMP1,T1F,T2,T3,T10,T2D,
    1T3D,T1D2,T2D2,T3D2,Y
    2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
    3,NR,RINT,WDEFL,SLOPE
    COMMON AMPR,AMTR,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
    1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
    2,LINES
    DIMENSION BC(10),CBC(2,5)

```

```

C

```

```

    GO TO (100,200,300,400,500,600,700),KIND

```

```

C

```

```

    GETOUT(1) INPUT

```

```

100 WRITE OUTPUT TAPE 6,150,(BXCX(I),I=1,11)

```

```

150 FORMAT(11H11540/RE278//1H05X11A6)

```

```

    WRITE OUTPUT TAPE 6,151,1A,1B,1B2,S,A,RO,RI,EP,ET,VU,H,TL,
    1RHOP,RHOT

```

```

151 FORMAT(7H0 INPUT/1H 5X2H1A10X2H1B9X3H1B210X1H511X1HA10X4HR(0)8X4HR
    1(I)/1B,2I12,E15.4,3E12.4/1H05X2HEP10X2HET10X2HNU10X1HH10X4HL(T)7X6
    2HRHO(P)6X6HRHO(T)/1PE11.4,6E12.4)

```

```

C

```

```

    IF INPUT ERROR OCCURS PRINT INPUT, RETURN TO SYSTEM

```

```

    IF (IP-321) 102,101,102

```

```

101 WRITE OUTPUT TAPE 6,153,IN(1B)

```

```

153 FORMAT(29H INPUT ERROR IN FIRST 2 CARDS/20X9H IN(1B) =16)

```

```

    RETURN

```

```

102 WRITE OUTPUT TAPE 6,1515,NZ,NM,NUMBR,NR,RONE,RTWO

```

```

1515 FORMAT(1H05X2HNZ10X2HNM8X5HNUMBR7X6HRCALCN5X6HR(ONE)

```

```

    16X6HR(TWO)/1B,3I12,1PE15.4,E12.4)

```

```

    WRITE OUTPUT TAPE 6,152,D,1A, PT,F1,(I,RAVE(I),P(I,1),I=1,1B2)

```

```

152 FORMAT(14H0 CALCULATIONS//6H D =1PE12.5,1H,2X3HI =E12.5,1H,2X

```

```

    16HF(1) =1PE12.5,9H, F(2) =1PE12.5///

```

```

    240H AVERAGE RADIUS AND LOAD TO EACH ROW I/1H04X1HI6X4HR(I)10X4HP
    3(I)10X1HI6X4HR(I)10X4HP(I)//(16,2E14.5,17,2E14.5))

```

```

    BC(1)=6HP(N) =

```

```

    BC(2)=6HQ(N) =

```

```

    BC(3)=6HDISTRI

```

```

    BC(4)=6HBUTION

```

```

    BC(5)=6H CALCU

```

```

BC(6)=6HLATION
BC(7)=6HS K =
BC(8)=6H POTE
BC(9)=6HNTIAL
BC(10)=6HENERGY
CBC(1,1)=6H HEXA
CBC(2,1)=6HGONAL
CBC(1,2)=6H CIR
CBC(2,2)=6HCULAR
CBC(1,3)=6H UN
CBC(2,3)=6HIFORM
CBC(1,4)=6H INCL
CBC(2,4)=6HUDING
CBC(1,5)=6H NEGL
CBC(2,5)=6HECTING
RETURN

```

```

C   GETOUT(2)    K,U(P) COEFFICIENTS
200 WRITE OUTPUT TAPE 6,150,(BXCX(I),I=1,11)
    WRITE OUTPUT TAPE 6,250,Z
250 FORMAT(6H0 K =E12.5,4X17HN =0,2,3,4,...,NM)
    WRITE OUTPUT TAPE 6,253,BC(1),(PK(I),I=1,NM)
253 FORMAT(A10,1PE12.6,4E13.6/4X5E15.6)
    WRITE OUTPUT TAPE 6,253,BC(2),(Q(I),I=1,NM)
    WRITE OUTPUT TAPE 6,254
254 FORMAT(24H0STRAIN ENERGY OF PLATES)
    WRITE OUTPUT TAPE 6,255,((UP(I,J),J=1,5),I=1,5)
255 FORMAT(1H03X1P5E14.5/18X4E14.5/32X3E14.5/46X2E14.5/60XE14.5)
    IF (NM-5) 210,210,205
205 WRITE OUTPUT TAPE 6,260,((I,J,UP(I,J),I=1,J),J=6,NM)
260 FORMAT(46H0 NM HIGHER THAN 5, REMAINDER OF MATRIX BELOW/
1(5X1H(I2,1H,I2,1H)1PE18.5))
210 WRITE OUTPUT TAPE 6,264
264 FORMAT(26H0STRAIN ENERGY OF SUPPORTS)
    WRITE OUTPUT TAPE 6,255,((US(I,J),J=1,5),I=1,5)
    IF (NM-5) 220,220,215
215 WRITE OUTPUT TAPE 6,260,((I,J,US(I,J),I=1,J),J=6,NM)
220 RETURN
300 WRITE OUTPUT TAPE 6,350
350 FORMAT(44H0STRAIN ENERGY OF TUBES USED IN CALCULATIONS)
    WRITE OUTPUT TAPE 6,255,((AUT(I,J,1),J=1,5),I=1,5)
C   IF (NM-5) 400,400,301
301 WRITE OUTPUT TAPE 6,260,((I,J,AUT(I,J,1),I=1,J),J=6,NM)
400 RETURN
500 IHAVE=IAVE+1
    WRITE OUTPUT TAPE 6,540,CBC(1,IHAVE),CBC(2,IHAVE),(BC(I),I=3,7),Z
540 FORMAT(1H17A6,F6.3)
    WRITE OUTPUT TAPE 6,541
541 FORMAT(30H0STRAIN ENERGY OF TUBES, EXACT)
    WRITE OUTPUT TAPE 6,255,((UT(I,J),J=1,5),I=1,5)
    IF (NM-5) 502,502,501
501 WRITE OUTPUT TAPE 6,260,((I,J,UT(I,J),I=1,J),J=6,NM)
502 WRITE OUTPUT TAPE 6,550
550 FORMAT(31HOMATRIX OF COEFFICIENTS, C(I,J))
    WRITE OUTPUT TAPE 6,255,((C(I,J),J=1,5),I=1,5)
    IF (NM-5)510,510,505
505 WRITE OUTPUT TAPE 6,260,((I,J,C(I,J),J=1,NM),I=6,NM)
510 WRITE OUTPUT TAPE 6,555
555 FORMAT(34H0DERIVATIVE MATRIX OF COEFFICIENTS)
    WRITE OUTPUT TAPE 6,556,((DI(I,J),J=1,5),I=1,5)

```

```

556 FORMAT(1H03X1P5E14.5/(E18.5,4E14.5))
    IF(NM-5)610,610,6051
6051 WRITE OUTPUT TAPE 6,260,((I,J,DI(I,J),J=6,NM),I=1,5)
    WRITE OUTPUT TAPE 6,260,((I,J,DI(I,J),J=1,NM),I=6,NM)
610 RETURN
600 GO TO (601,602,604,605),KV
601 IHAVE1=4
    GO TO 603
602 IHAVE1=5
603 IHAVE=IAVE+1
    GO TO 607
604 IHAVE1=4
    GO TO 606
605 IHAVE1=5
606 IHAVE=3
607 WRITE OUTPUT TAPE 6,651,CBC(1,IHAVE),CBC(2,IHAVE),BC(3),BC(4),
    IBC(8),BC(9),BC(10),CBC(1,IHAVE1),CBC(2,IHAVE1),Z
651 FORMAT(1H19A6,24H WEIGHT OF STRUCTURE, K=F4.2)
626 WRITE OUTPUT TAPE 6,657,(V(I),I=1,NM)
657 FORMAT(13H0V(I), I=1,NM1P5E13.5/16X5E13.5)
    RETURN
700 IF(IP-1) 701,701,702
701 WRITE OUTPUT TAPE 6,750,IP,AA(1)
750 FORMAT(43H0SOLUTIONS OF SIMULTANECUS LINEAR EQUATIONS//
    18H ORDERI2,4X6HA(0) =1PE15.6)
    LINES = 14
    GOTO 706
702 IF (LINES-53) 704,704,703
703 WRITE OUTPUT TAPE 6,752
752 FORMAT(1H1)
    LINES=1
704 WRITE OUTPUT TAPE 6,751,IP,(AA(I),I=1,IP)
751 FORMAT(8H0 ORDERI2/8H A(I)(2X1P5E14.6))
    LINES=LINES+9
706 RETURN

```

SUBROUTINE KCALC

C
C
C
C

KCALC DETERMINES P(I) AND Q(I) (FOR I=0,2,3,4, ,NM) AS
FUNCTIONS OF K (OR Z)

```

    DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
    1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
    2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
    3UT(10,10),V(10)
    DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
    1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
    2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
    DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
    1RA(1000),RB(1000),AUT(10,10,2)
    COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
    1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,ID,IA,IB,IB1,IB2,
    2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
    3DETERM,CWA,AMR,WA,AMO,WO,IAVE
    COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
    1T3D,T1D2,T2D2,T3D2,Y
    2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
    3,NR,RINT,WDEFL,SLOPE

```

```
COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
2,LINES
```

```
C
  YP=0.0
  DO 30 N=1,NM
  DEN=0.5/(2.*YP+VU+1.)
  PK(N)=((YP+4.)*(YP+3.+VU)-8.*Z)/DEN
  Q(N)=PK(N)/2.
  YP=N+1
30 CONTINUE
  RETURN
```

```
SUBROUTINE UPLATE
```

```
C
C CALCULATE U(PLATES) AND U(SUPPORTS)
```

```
C
  DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
3UT(10,10),V(10)
  DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
  DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
1RA(1000),RB(1000),AUT(10,10,2)
3,AM1(6,10),TEMP 1D(10)
  COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,ID,IA,IB,IB1,IB2,
2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
3DETERM,DWA,AMR,WA,AMO,WO,IAVE
  COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
1T3D,T1D2,T2D2,T3D2,Y
2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
3,NR,RINT,WDEFL,SLOPE
  COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
2,LINES
  DIMENSIONVCK(10),DCK(10,10)
```

```
C
  IK=-1
  JK=-1
  XA=0.
  YA=0.
  DO 40 I=1,NM
  X1=XA+1.
  X3=XA+3.
  X4=XA+4.
  X2=XA+2.
  X5=XA-1.
  DO 38 J=I,NM
  FP=2.*PI*D
  FS=-8.*PI*D*Z/A**2
  Y1=YA+1.
  Y2=YA+2.
  Y3=YA+3.
  Y4=YA+4.
  Y5=YA-1.
```

```

      IF (I-J) 2,3,2
2  FP=2.*FP
   FS=2.*FS
3  UP(I,J)=FP*A**((IK+JK)*(X4*Y4*(1.+X3*Y3+VU*(X3+Y3))
   1/(X3+Y3)-(2.*X2*Y4*(1.+X1*Y3+VU*(X1+Y3))/(X1+Y3))*Q(I)
   2-2.*X4*Y2*((1.+X3*Y1+VU*(X3+Y1))/(X3+Y1))*Q(J)+4.*X2*Y2*
   3*((1.+X1*Y1+VU*(X1+Y1))/(X1+Y1))*Q(I)*Q(J)+X4*YA*((1.+X3*Y5+
   4VU*(X3+Y5))/(X3+Y5))*PK(J)+XA*Y4*((1.+X5*Y3+VU*(X5+Y3))/
   5(X5+Y3))*PK(I))
   UP(I,J)=UP(I,J)+FP*A**((IK+JK)*((-2.*X2*YA*(1.+X1*Y5
   1+VU*(X1+Y5))/(X1+Y5))*Q(I)*PK(J)-2.*XA*Y2*((1.+X5*Y1
   2+VU*(X5+Y1))/(X5+Y1))*PK(I)*Q(J)+XA*YA*((1.+X5*Y5
   3+VU*(X5+Y5))/(X5+Y5))*PK(I)*PK(J))
   US(I,J)=FS*A**((IK+JK+2)*((XA*(XA+2.)-(8.-8.*Z))/((1.+VU)+2.*XA)
   1+(YA*(YA+2.)-(8.-8.*Z))/((1.+VU)+2.*YA))
   JK=J
38 YA=J+1
   IK=I
   JK=I
   YA=I+1
40 XA=I+1
   RETURN

```

SUBROUTINE SUMVUT

```

C
C   SUMVUT CALCULATES POINT BY POINT OR ROW BY ROW VALUES OF
C   COEFFICIENTS OF V (NEGLECTING WEIGHT OF SYSTEM) AND
C   EXACT COEFFICIENTS FOR U(TUBES)
C
   DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
   1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
   2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
   3UT(10,10),V(10)
   DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
   1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
   2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
   DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
   1RA(1000),RB(1000),AUT(10,10,2)
   3,AM1(6,10),TEMP 1D(10)
   COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
   1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,ID,IA,IB,IB1,IB2,
   2D,A1,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
   3DETERM,DWA,AMR,WA,AMD,WO,IAVE
   COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
   1T3D,T1D2,T2D2,T3D2,Y
   2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
   3,NR,RINT,WDEFL,SLOPE
   COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
   1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
   2,LINES
   DIMENSIONVCK(10),DCK(10,10)
C
   M=IP
   N=JP
   X=(R(M,N)/A)**2
   IK=-1
   XA=0.
   DO 40 I=1,NM

```

```

      V(I)=P(M,N)*R(M,N)**(IK+1)*(PK(I)-2.*Q(I)*X+X**2)
      IF (IP-IA) 40,15,15
15   DO 35 J=1,NM
      FT=6.*ET*AI*(TL+H)**2 /TL**3
      IF (I-J) 20,25,20
20   FT=2.*FT
      GO TO 30
25   JK=IK
      YA=XA
30   UT(I,J)=FT*R(M,N)**(IK+JK)*(PK(J)*YA-2.*Q(J)*(YA+2.)*X
      1+(YA+4.)*(X**2))*(PK(I)*XA-2.*Q(I)*(XA+2.)*X
      2+(XA+4.)*(X**2))
      JK=J
35   YA=J+1
      XA=I+1
40   IK=I
      RETURN

```

SUBROUTINE DIFFER

C
C
C
C

DIFFER DIFFERENTIATES THE TRIANGULAR COEFFICIENT MATRIX
WITH RESPECT TO A(I)

```

      DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
      1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
      2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
      3UT(10,10),V(10)
      DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
      1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
      2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
      DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
      1RA(1000),RB(1000),AUT(10,10,2)
      3,AM1(6,10),TEMP 1D(10)
      COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
      1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,1D,IA,IB,IB1,IB2,
      2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
      3DETERM,DWA,AMR,WA,AMO,WO,IAVE
      COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
      1T3D,T1D2,T2D2,T3D2,Y
      2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
      3,NR,RINT,WDEFL,SLOPE
      COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
      1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
      2,LINES
      DIMENSIONVCK(10),DCK(10,10)

```

C

```

      DO 80 I1=1,NM
      DO 80 I2=1,NM
      IF(I2-I1)20,40,60
20   DI(I1,I2)=C(I2,I1)
      GO TO 80
40   DI(I1,I2)=2.0*C(I1,I2)
      GO TO 80
60   DI(I1,I2)=C(I1,I2)
80   CONTINUE
      RETURN

```

```

      SUBROUTINE MATINV(A,N,B,M,DETERM)
C
C
C      MATINV INVERTS THE DERIVATIVE MATRIX AND IN THE PROCESS
C      DETERMINES THE SOLUTION VECTOR A(I)
C
      DIMENSION IPIVOT(10),A(10,10),B(10,1),INDEX(10,2),PIVOT(10)
      COMMON PIVOT, INDEX, IPIVOT
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)
C
C      INITIALIZATION
C
      10 DETERM=1.0
      15 DO 20 J=1,N
      20 IPIVOT(J)=0
      30 DO 550 I=1,N
C
C      SEARCH FOR PIVOT ELEMENT
C
      40 AMAX=0.0
      45 DO 105 J=1,N
      50 IF (IPIVOT(J)-1) 60, 105, 60
      60 DO 100 K=1,N
      70 IF (IPIVOT(K)-1) 80, 100, 740
      80 IF (ABSF(AMAX)-ABSF(A(J,K))) 85, 100, 100
      85 IROW=J
      90 ICOLUM=K
      95 AMAX=A(J,K)
      100 CONTINUE
      105 CONTINUE
      110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C
C      INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
      130 IF (IROW-ICOLUM) 140, 260, 140
      140 DETERM=-DETERM
      150 DO 200 L=1,N
      160 SWAP=A(IROW,L)
      170 A(IROW,L)=A(ICOLUM,L)
      200 A(ICOLUM,L)=SWAP
      205 IF(M) 260, 260, 210
      210 DO 250 L=1, M
      220 SWAP=B(IROW,L)
      230 B(IROW,L)=B(ICOLUM,L)
      250 B(ICOLUM,L)=SWAP
      260 INDEX(I,1)=IROW
      270 INDEX(I,2)=ICOLUM
      310 PIVOT(I)=A(ICOLUM,ICOLUM)
      320 DETERM=DETERM*PIVOT(I)
C
C      DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
      330 A(ICOLUM,ICOLUM)=1.0
      340 DO 350 L=1,N
      350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
      355 IF(M) 380, 380, 360
      360 DO 370 L=1,M
      370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
C
C      REDUCE NON-PIVOT ROWS

```



```

380 DO 550 L1=1,N
390 IF(L1-ICOLUM) 400, 550, 400
400 T=A(L1,ICOLUM)
420 A(L1,ICOLUM)=0.0
430 DO 450 L=1,N
450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
455 IF(M) 550, 550, 460
460 DO 500 L=1,M
500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
550 CONTINUE

```

```

C
C   INTERCHANGE COLUMNS
C

```

```

600 DO 710 I=1,N
610 L=N+1-I
620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
630 JROW=INDEX(L,1)
640 JCOLUM=INDEX(L,2)
650 DO 705 K=1,N
660 SWAP=A(K,JROW)
670 A(K,JROW)=A(K,JCOLUM)
700 A(K,JCOLUM)=SWAP
705 CONTINUE
710 CONTINUE
740 RETURN

```

SUBROUTINE CHECK

```

C
C   CALCULATE TOTAL STRAINS AND POTENTIAL ENERGY AS WELL AS
C   THE DEFLECTION, SLOPE AND MOMENTS AT EACH OF NR POINTS
C

```

```

    DIMENSION  RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
    1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
    2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
    3UT(10,10),V(10)
    DIMENSION  US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
    1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
    2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
    DIMENSION  AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
    1RA(1000),RB(1000),AUT(10,10,2)
    3,AM1(6,10),TEMP 1D(10)
    COMMON  RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
    1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,1D,IA,IB,IB1,IB2,
    2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
    3DETERM,DWA,AMR,WA,AMO,WO,IAVE
    COMMON  US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T2D,
    1T3D,T1D2,T2D2,T3D2,Y
    2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
    3,NR,RINT,WDEFL,SLOPE
    COMMON  AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
    1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
    2,LINES
    DIMENSION BETA(20)

```

```

C
    FSTS=6./H**2
    ACTUT=0.
    STRP=0.
    STRS=0.

```

```

      STRTT=0.
      VT=0.
      OMEGAL=0.
      DO 10 I=1,NR
      WDEFL(I)=0.
      SLOPE(I)=0.
      AMRP(I)=0.
10  AMTP(I)=0.
      N=0
      FN=0.0
      DO 100 I=1,IP
      DO 50 J=I,IP
      STRP=STRP+UP(I,J)*AA(I)*AA(J)
      STRS=STRS+US(I,J)*AA(I)*AA(J)
      STRTT=STRTT+UT(I,J)*AA(I)*AA(J)
      OMEGAL=OMEGAL+C(I,J)*AA(I)*AA(J)
      ACTUT=ACTUT+AUT(I,J,1)*AA(I)*AA(J)
50  CONTINUE
      VT=VT+V(I)*AA(I)
      DO 80 K=1,NR
      RA2=(RINT(K)/A)**2
      RA4=RA2**2
      WDEFL(K)=WDEFL(K)+(PK(I)-2.*Q(I)*RA2+RA4)*AA(I)
      1*RINT(K)**N
      SLOPE(K)=SLOPE(K)+(C1(I)-C2(I)*RA2+C3(I)*RA4)
      1*AA(I)*RINT(K)**(N-1)
      AMRP(K)=AMRP(K)-D*(C1(I)*((FN+VU)-1.)-C2(I)*((FN+VU)
      1+1.)*RA2+C3(I)*((FN+VU)+3.)*RA4)*AA(I)*RINT(K)**(N-2) *FSTS
      AMTP(K)=AMTP(K)-D*(C1(I)*((VU*FN)-VU+1.)-C2(I)*((VU*FN)
      1+VU+1.)*RA2+C3(I)*((VU*FN)+3.*VU+1.)*RA4)
      2*AA(I)*RINT(K)**(N-2)*FSTS
      BETA(K)=AMRP(K)/AMTP(K)
      IF (ABSF(BETA(K))-1.) 80,80,75
75  BETA(K)=AMTP(K)/AMRP(K)
80  CONTINUE
C    OMEGAL=SUM(C(I,J)*A(I)*A(J)), WHICH IS =SUM(UP+US+UT)
      N=I+1
      FN=N
100  CONTINUE
      OMEGAT=OMEGAL+VT
      TEST=STRP+STRTT+STRS+VT
      WRITE OUTPUT TAPE 6,101,STRP,STRS,ACTUT,STRTT,OMEGAL,VT,OMEGAT,
      XTEST
101  FORMAT(1H05X10HU (PLATES)7X12HU (SUPPORTS)7X9HU (TUBES)8X12HU (T),
      1 EXACT/1P4E18.7//6X9HU (TOTAL)9X9HV (TOTAL)8X13HOMEGA (TOTAL)6X11H
      2OMEGA (SUM)/4E18.7)
      RAD=0.
      ONE = 1.
      AMO=AMO*FSTS
      WRITE OUTPUT TAPE 6,102,RAD,WO,RAC,AMO,AMC,ONE
102  FORMAT(1H05X6HRADIUS5X10HDEFLECTICN9X5HSLOPE 11X7HSTR (R)
      110X7HSTR (T)10X8HBETA (R)/F12.3,1P4E17.7,OPF14.6)
      WRITE OUTPUT TAPE 6,103,(RINT(K),WDEFL(K),SLOPE(K),AMRP(K),
      1AMTP(K),BETA(K),K=1,NR)
103  FORMAT(OPF12.3,1P4E17.7,OPF14.6)
      AMTP(NR)=AMTP(NR)/FSTS
      AMRP(NR)=AMRP(NR)/FSTS
      RETURN

```

SUBROUTINE MAXIM

C
C
C
C
C
C
MAXIM DETERMINES THE VALUES AND POSITIONS OF MAXIMUM
DEFLECTION, SLOPE, TUBE SLOPE, AND RADIAL AND TANGENTIAL
MOMENTS BY COMPARING VALUES OF EACH AT NUMBER POINTS ALONG RADIUS

```

    DIMENSION RESERV(40),AA(10),BXCX(11),C(10,10),DD(10,10),
    1DI(10,10),PK(10),Q(10),ICOL(30),ICOLR(30),IN(30),P(30,30),
    2R(30,30),RAVE(30),SUMUT(10,10),SUMV(10),TEMP(30),UP(10,10),
    3UT(10,10),V(10)
    DIMENSION US(10,10),TEMP1(10),T1F(3,10),T2(3,10),T3(3,10),
    1T1D(3,10),T2D(3,10),T3D(3,10),T1D2(3,10),T2D2(3,10),T3D2(3,10),
    2Y(5),W(10),RINT(20),WDEFL(20),SLOPE(20)
    DIMENSION AMRP(20),AMTP(20),C1(10),C2(10),C3(10),RAB(1000),
    1RA(1000),RB(1000),AUT(10,10,2)
    COMMON RESERV,AA,BXCX,C,DD,DI,PK,Q,ICOL,ICOLR,IN,
    1P,R,RAVE,SUMUT,SUMV,TEMP,UP,UT,V,ID,IA,IB,IB1,IB2,
    2D,AI,EP,ET,H,VU,A,RO,RI,TL,S,NZ,FNZ,NM,Z,IP,JP,
    3DETERM,DWA,AMR,WA,AMO,WO,IAVE
    COMMON US,TT,PT,F1,KV,RHOP,RHOT,PI,TEMP1,T1F,T2,T3,T10,T20,
    1T30,T1D2,T2D2,T3D2,Y
    2,X1,Y1,X2,Y2,X3,Y3,X4,Y4,X5,Y5,XA,YA,W,HT
    3,NR,RINT,WDEFL,SLOPE
    COMMON AMRP,AMTP,C1,C2,C3,RAB,RA,RB,CR,NUMBR,KIB2,
    1STRP,STRS,STRTT,OMEGAL,VT,OMEGAT,TEST,UTCH,RONE,RTWO,AUT
    2,LINES

```

C

```

    MD=0
    MSL=0
    MSR=0
    MRM=0
    MTM=0
    SL=0.
    WM=WO
    RM=AMO
    TM=AMO
10 DO 80 M=1,NUMBR
    WM1=0.
    SL1=0.
    RM1=0.
    TM1=0.
    I=-2
    BI=0.
    DO 20 N=1,IP
    WM1=WM1+(PK(N)-2.*Q(N)*RA(M)+RB(M))*AA(N)*RAB(M)**(I+2)
    RA1=C2(N)*RA(M)
    RB1=C3(N)*RB(M)
    SL1=SL1+(C1(N)-RA1+RB1)*AA(N)*RAB(M)**(I+1)
    RM1=RM1-(C1(N)*(BI-1.+VU)-RA1*(BI+1.+VU)
    1+RB1*(BI+3.+VU))*D*RAB(M)**I*AA(N)
    TM1=TM1-(C1(N)*(VU*(BI-1.)+1.))-RA1*(VU*(BI+1.)+1.)
    1+RB1*(VU*(BI+3.)+1.))*D*AA(N)*RAB(M)**I
    I=N-1
20 BI=N+1
    IF (ABSF(SL)-ABSF(SL1)) 30,40,40
30 SL=SL1
    MSL=M
    IF(RAB(MSL)-RTWO) 35,35,40
35 MSR=MSL
C    SR IS MAXIMUM SLOPE IN TUBE REGION

```

```

      SR=ABSF(SL)
40 IF (ABSF(RM)-ABSF(RM1)) 50,60,60
50 RM=RM1
   MRM=M
60 IF (ABSF(TM)-ABSF(TM1)) 70,75,75
70 TM=TM1
   MTM=M
75 IF (ABSF(WM)-ABSF(WM1)) 77,80,80
77 WM=WM1
   MC=M
80 CONTINUE
   RSMAX=6.*RM/H**2
   TSMAX=6.*TM/H**2
   DEFMAX=SQRTF(3.*(TL+3.*H)**3/(TL+H))*SR/18.
   TMAXM=6.*ET*AI*(TL+H)**2*SR/TL**3
   TMAXS=RO*TMAXM/AI
   IF (MD) 90,95,90
90 RADIUS=RAB(MD)
   GO TO 96
95 RADIUS=0.0
96 IF (LINES-55) 98,98,97
97 WRITE OUTPUT TAPE 6,752
752 FORMAT(1H1)
   LINES=1
98 LINES=LINES+7
100 WRITE OUTPUT TAPE 6,110,WM,RADIUS,SL,RAB(MSL),RM,RAB(MRM),RSMAX,
   1RAB(MRM),TM,RAB(MTM),TSMAX,RAB(MTM),SR,RAB(MSR),DEFMAX,
   2RAB(MSR),TMAXM,RAB(MSR),TMAXS,RAB(MSR)
110 FORMAT(1H018X7HMAXIMUM6X8HPOSITION19X7HMAXIMUM6X8HPOSITION/
   112X1HWE17.7,F10.5,8X5HDW/DRE17.7,F10.5/9X4HM(R)E17.7,F10.5,7X6HSTR
   2(R)E17.7,F10.5/9X4HM(T)E17.7,F10.5,7X6HSTR(T)E17.7,F10.5/2X11HDW/D
   3R(TUBE)E17.7,F10.5,6X7HY(TUBE)E17.7,F10.5/6X7HM(TUBE)E17.7,F10.5,4
   4X9HSTR(TUBE)E17.7,F10.5)
   RETURN

```

2. Sample Results

1540/RE278

```

INPUT
  IA      IB      IB2      S      A      R(C)      R(I)
  1        2        2    0.7000E 01  0.1700E 02  0.2500E-00  0.

  EP      ET      NU      H      L(T)      RHO(P)      RHO(T)
3.0000E 07 3.0000E 07 3.0000E-01 5.1200E-01 3.0000E 00 2.8500E-01 2.8500E-01

  NZ      NM      NUMBR      RCALCN      R(ONE)      R(TWC)
  3        5      34      17      0.      1.7000E 01

```

CALCULATIONS

D = 3.68730E 05, I = 3.06796E-03, F(1) = 1.04635E 01, F(2) = 2.95353E-01

K = 0.50000E 00 N = 0,2,3,4,...,NM

P(N) = 2.538462E 00 1.377358E 00 1.273973E 00 1.215054E 00 1.176991E 00

Q(N) = 1.769231E 00 1.188679E 00 1.136986E 00 1.107527E 00 1.088496E 00

STRAIN ENERGY OF PLATES

```

1.84176E 05  7.69352E 06  3.43107E 07 -1.07515E 08 -7.22819E 09
              3.02203E 09  7.33300E 10  8.81260E 11  1.06100E 13
              5.37624E 11  1.47953E 13  1.98971E 14
                              1.12411E 14  3.27457E 15
                              2.54092E 16

```

STRAIN ENERGY OF SUPPORTS

```

9.86659E 04  3.55086E 07  5.71070E 08  9.39260E 09  1.56208E 11
              2.02130E 09  5.93099E 10  9.17061E 11  1.45884E 13
                              4.24113E 11  1.28693E 13  2.01750E 14
                              9.62097E 13  2.98165E 15
                              2.28834E 16

```

STRAIN ENERGY OF TUBES USED IN CALCULATIONS

```

1.68666E 05  9.90445E 06  8.38253E 07  7.94391E 08  8.12470E 09
              5.25289E 08  1.13730E 10  1.27837E 11  1.48396E 12
                              6.73049E 10  1.62218E 12  1.99221E 13
                              1.03390E 13  2.66024E 14
                              1.77927E 15

```

HEXAGONAL DISTRIBUTION CALCULATIONS K = 0.500

STRAIN ENERGY OF TUBES, EXACT

```

1.12416E 06  6.51305E 07  5.60266E 08  5.34548E 09  5.38518E 10
              7.79176E 09  1.84396E 11  2.23791E 12  2.79263E 13
                              1.25702E 12  3.37543E 13  4.55123E 14
                              2.44887E 14  7.02171E 15
                              5.28865E 16

```

MATRIX OF COEFFICIENTS, C(I,J)

```

4.51509E 05  5.31065E 07  6.89206E 08  1.00795E 10  1.57105E 11
              5.56862E 09  1.44013E 11  1.92616E 12  2.66823E 13
                              1.02904E 12  2.92868E 13  4.20643E 14
                              2.18960E 14  6.52224E 15
                              5.00719E 16

```

DERIVATIVE MATRIX OF COEFFICIENTS

```

9.03017E 05  5.31065E 07  6.89206E 08  1.00795E 10  1.57105E 11
5.31065E 07  1.11372E 10  1.44013E 11  1.92616E 12  2.66823E 13
6.89206E 08  1.44013E 11  2.05808E 12  2.92868E 13  4.20643E 14
1.00795E 10  1.92616E 12  2.92868E 13  4.37920E 14  6.52224E 15
1.57105E 11  2.66823E 13  4.20643E 14  6.52224E 15  1.00144E 17

```

HEXAGONAL DISTRIBUTION POTENTIAL ENERGY INCLUDING WEIGHT CF STRUCTURE, K=0.50

V(I), I=1,NM -1.09854E 04 -4.79306E 05 -4.64079E 06 -4.98128E 07 -5.72341E 08

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

ORDER 1 A(0) = 1.216520E-02

U (PLATES) U (SUPPORTS) U (TUBES) U (T), EXACT
2.7256661E 01 1.4601783E 01 2.4961296E 01 1.6636630E 02

U (TOTAL) V (TOTAL) OMEGA (TOTAL) OMEGA (SUM)
6.6819740E 01 -1.3363948E 02 -6.6819742E 01 7.4585262E 01

RADIUS	DEFLECTION	SLOPE	STR (R)	STR (T)	BETA (R)
0.	3.0880902E-02	0.	3.2683547E 03	3.2683547E 03	1.000000
1.000	3.0732099E-02	-2.9731430E-04	3.2521284E 03	3.2590123E 03	0.997888
2.000	3.0287438E-02	-5.9113290E-04	3.2034498E 03	3.2309852E 03	0.991478
3.000	2.9552164E-02	-8.779607E-04	3.1223186E 03	3.1842733E 03	0.980544
4.000	2.8535014E-02	-1.1543002E-03	3.0087351E 03	3.1188767E 03	0.964685
5.000	2.7248224E-02	-1.4166574E-03	2.8626990E 03	3.0347953E 03	0.943292
6.000	2.5707526E-02	-1.6615361E-03	2.6842106E 03	2.9320292E 03	0.915479
7.000	2.3932143E-02	-1.8854406E-03	2.4732697E 03	2.8105784E 03	0.879986
8.000	2.1944801E-02	-2.0848751E-03	2.2298764E 03	2.6704429E 03	0.835021
9.000	1.9771715E-02	-2.2563440E-03	1.9540306E 03	2.5116226E 03	0.777995
10.000	1.7442600E-02	-2.3963515E-03	1.6457323E 03	2.3341175E 03	0.705077
11.000	1.4990665E-02	-2.5014020E-03	1.3049817E 03	2.1379278E 03	0.610396
12.000	1.2452614E-02	-2.5679996E-03	9.3177856E 02	1.9230533E 03	0.484531
13.000	9.8686482E-03	-2.5926488E-03	5.2612301E 02	1.6894940E 03	0.311409
14.000	7.2824643E-03	-2.5718538E-03	8.8014982E 01	1.4372499E 03	0.061238
15.000	4.7412536E-03	-2.5021190E-03	-3.8254538E 02	1.1663213E 03	-0.327993
16.000	2.2957045E-03	-2.3799486E-03	-8.8555832E 02	8.7670777E 02	-0.990006
17.000	-0.	-2.2018468E-03	-1.4210238E 03	5.6840950E 02	-0.400000

	MAXIMUM	POSITION		MAXIMUM	POSITION
W	0.3088090E-01	0.	DW/DR	-0.2592649E-02	13.00000
M(R)	0.3268355E 04	0.	STR(R)	0.7480670E 05	0.
M(T)	0.3268355E 04	0.	STR(T)	0.7480670E 05	0.
DW/DR(TUBE)	0.2592649E-02	13.00000	Y(TUBE)	0.1286068E-02	13.00000
M(TUBE)	0.6540506E 03	13.00000	STR(TUBE)	0.5329684E 05	13.00000

ORDER 2

A(I) 1.258895E-02 1.868235E-05

U (PLATES) U (SUPPORTS) U (TUBES) U (T), EXACT
3.2052818E 01 2.4693542E 01 2.9243313E 01 1.9619588E 02

U (TOTAL) V (TOTAL) OMEGA (TOTAL) OMEGA (SUM)
8.5989671E 01 -1.4724910E 02 -6.1259425E 01 1.0569314E 02

RADIUS	DEFLECTION	SLOPE	STR (R)	STR (T)	BETA (R)
0.	3.1956573E-02	0.	2.8175608E 03	2.8175608E 03	1.000000
1.000	3.1828165E-02	-2.5681942E-04	2.8178300E 03	2.8177220E 03	0.999962
2.000	3.1442921E-02	-5.1366951E-04	2.8179174E 03	2.8178661E 03	0.999982
3.000	3.0800856E-02	-7.7041991E-04	2.8156618E 03	2.8169734E 03	0.999534
4.000	2.9902260E-02	-1.0266181E-03	2.8074613E 03	2.8133451E 03	0.997909
5.000	2.8748108E-02	-1.2813285E-03	2.7882731E 03	2.8046024E 03	0.994178
6.000	2.7340617E-02	-1.5329713E-03	2.7516139E 03	2.7876873E 03	0.987060
7.000	2.5683973E-02	-1.7791613E-03	2.6895593E 03	2.7588617E 03	0.974880
8.000	2.3785217E-02	-2.0165471E-03	2.5927446E 03	2.7137085E 03	0.955425
9.000	2.1655292E-02	-2.2406498E-03	2.4503637E 03	2.6471301E 03	0.925668
10.000	1.9310251E-02	-2.4457023E-03	2.2501703E 03	2.5533503E 03	0.881262
11.000	1.6772625E-02	-2.6244879E-03	1.9784770E 03	2.4259127E 03	0.815560
12.000	1.4072954E-02	-2.7681793E-03	1.6201558E 03	2.2576814E 03	0.717619
13.000	1.1251476E-02	-2.8661778E-03	1.1586378E 03	2.0408409E 03	0.567726
14.000	8.3599832E-03	-2.9059519E-03	5.7591341E 02	1.7668961E 03	0.325946
15.000	5.4638302E-03	-2.8728764E-03	-1.4746757E 02	1.4266724E 03	-0.103365
16.000	2.6441132E-03	-2.7500715E-03	-1.0323966E 03	1.0103153E 03	-0.978612
17.000	8.0454351E-11	-2.5182416E-03	-2.1012055E 03	5.0729095E 02	-0.241429

	MAXIMUM	POSITION		MAXIMUM	POSITION
W	0.3195657E-01	0.	DW/DR	-0.2905952E-02	14.00000
M(R)	0.2817561E 04	0.	STR(R)	0.6448885E 05	0.
M(T)	0.2817561E 04	0.	STR(T)	0.6448885E 05	0.
DW/DR(TUBE)	0.2905952E-02	14.00000	Y(TUBE)	0.1441481E-02	14.00000
M(TUBE)	0.7330879E 03	14.00000	STR(TUBE)	0.5973738E 05	14.00000

ORDER 3

A(I) 1.262456E-02 1.973724E-05 -2.146839E-07

U (PLATES) U (SUPPORTS) U (TUBES) U (T), EXACT
 3.2069300E 01 2.3581023E 01 2.9282249E 01 1.9618979E 02

U (TOTAL) V (TOTAL) OMEGA (TOTAL) OMEGA (SUM)
 8.4932570E 01 -1.4714959E 02 -6.2217014E 01 1.0469053E 02

RADIUS	DEFLECTION	SLOPE	STR (R)	STR (T)	BETA (R)
0.	3.2046965E-02	0.	2.7952456E 03	2.7952456E 03	1.000000
1.000	3.1919295E-02	-2.5563049E-04	2.8120519E 03	2.8068576E 03	0.998153
2.000	3.1535117E-02	-5.1301130E-04	2.8286571E 03	2.8185604E 03	0.996431
3.000	3.0892763E-02	-7.7190436E-04	2.8410821E 03	2.8283997E 03	0.995536
4.000	2.9991048E-02	-1.0315567E-03	2.8440560E 03	2.8338056E 03	0.996396
5.000	2.8829856E-02	-1.2905626E-03	2.8311295E 03	2.8316429E 03	0.999819
6.000	2.7410856E-02	-1.5467390E-03	2.7947911E 03	2.8182632E 03	0.991671
7.000	2.5738334E-02	-1.7970133E-03	2.7265805E 03	2.7895543E 03	0.977425
8.000	2.3820138E-02	-2.0373248E-03	2.6172044E 03	2.7409931E 03	0.954838
9.000	2.1668704E-02	-2.2625385E-03	2.4566509E 03	2.6676947E 03	0.920889
10.000	1.9302174E-02	-2.4663723E-03	2.2343045E 03	2.5644652E 03	0.871256
11.000	1.6745569E-02	-2.6413370E-03	1.9390610E 03	2.4258517E 03	0.799332
12.000	1.4032023E-02	-2.7786891E-03	1.5594412E 03	2.2461933E 03	0.694260
13.000	1.1204058E-02	-2.8683965E-03	1.0837076E 03	2.0196726E 03	0.536576
14.000	8.3148779E-03	-2.8991176E-03	4.9997771E 02	1.7403663E 03	0.287283
15.000	5.4296892E-03	-2.8581935E-03	-2.0366052E 02	1.4022969E 03	-0.145234
16.000	2.6270155E-03	-2.7316517E-03	-1.0390345E 03	9.9948243E 02	-0.961934
17.000	8.4997157E-11	-2.5042246E-03	-2.0177712E 03	5.2598886E 02	-0.260678

	MAXIMUM	POSITION		MAXIMUM	POSITION
W	0.3204696E-01	0.	DW/DR	-0.2899118E-02	14.00000
M(R)	0.2795246E 04	0.	STR(R)	0.6397810E 05	0.
M(T)	0.2795246E 04	0.	STR(T)	0.6397810E 05	0.
DW/DR(TUBE)	0.2899118E-02	14.00000	Y(TUBE)	0.1438091E-02	14.00000
M(TUBE)	0.7313638E 03	14.00000	STR(TUBE)	0.5959689E 05	14.00000

ORDER 4

A(I) 1.249411E-02 3.291928E-05 -2.603022E-06 1.047528E-07

U (PLATES) U (SUPPORTS) U (TUBES) U (T), EXACT
 3.1530432E 01 2.4417818E 01 2.8879164E 01 1.9340753E 02

U (TOTAL) V (TOTAL) OMEGA (TOTAL) OMEGA (SUM)
 8.4827411E 01 -1.4616905E 02 -6.1341639E 01 1.0318673E 02

RADIUS	DEFLECTION	SLOPE	STR (R)	STR (T)	BETA (R)
0.	3.1715830E-02	0.	2.3617961E 03	2.3617961E 03	1.000000
1.000	3.1604905E-02	-2.2509238E-04	2.5506347E 03	2.4938998E 03	0.977757
2.000	3.1259487E-02	-4.6858819E-04	2.7176670E 03	2.6146786E 03	0.962104
3.000	3.0662699E-02	-7.2712249E-04	2.8459496E 03	2.7152184E 03	0.954064
4.000	2.9801877E-02	-9.9571723E-04	2.9229919E 03	2.7886745E 03	0.954048
5.000	2.8669929E-02	-1.2682995E-03	2.9410340E 03	2.8304181E 03	0.962389
6.000	2.7266162E-02	-1.5382257E-03	2.8968824E 03	2.8379935E 03	0.979672
7.000	2.5596612E-02	-1.7987594E-03	2.7912980E 03	2.8108853E 03	0.993032
8.000	2.3673929E-02	-2.0434552E-03	2.6279409E 03	2.7500991E 03	0.955580
9.000	2.1516960E-02	-2.2663939E-03	2.4118683E 03	2.6575499E 03	0.907553
10.000	1.9150182E-02	-2.4622234E-03	2.1475898E 03	2.5352645E 03	0.847087
11.000	1.6603213E-02	-2.6259500E-03	1.8366751E 03	2.3843930E 03	0.770290
12.000	1.3910686E-02	-2.7524324E-03	1.4749176E 03	2.2040319E 03	0.669191
13.000	1.1112787E-02	-2.8355273E-03	1.0490542E 03	1.9898585E 03	0.527200
14.000	8.2568505E-03	-2.8668358E-03	5.3303763E 02	1.7325755E 03	0.307656
15.000	5.4004284E-03	-2.8339997E-03	-1.1613405E 02	1.4161676E 03	-0.082006
16.000	2.6163081E-03	-2.7184973E-03	-9.6303575E 02	1.0159680E 03	-0.947900
17.000	2.7213589E-10	-2.4928892E-03	-2.0988747E 03	4.9653684E 02	-0.236573

	MAXIMUM	POSITION		MAXIMUM	POSITION
W	0.3171583E-01	0.	DW/DR	-0.2866836E-02	14.00000
M(R)	0.2361796E 04	0.	STR(R)	0.5405722E 05	0.
M(T)	0.2361796E 04	0.	STR(T)	0.5405722E 05	0.
DW/DR(TUBE)	0.2866836E-02	14.00000	Y(TUBE)	0.1422077E-02	14.00000
M(TUBE)	0.7232201E 03	14.00000	STR(TUBE)	0.5893327E 05	14.00000

ORDER 5

A(I) 1.252136E-02 4.048048E-05 -4.926916E-06 3.363174E-07 -7.361113E-09

U (PLATES)	U (SUPPORTS)	U (TUBES)	U (T), EXACT
3.1771137E 01	2.4042493E 01	2.9056008E 01	1.9465383E 02

U (TOTAL)	V (TOTAL)	OMEGA (TCTAL)	OMEGA (SUM)
8.4869639E 01	-1.4662962E 02	-6.1759980E 01	1.0383783E 02

RADIUS	DEFLECTION	SLOPE	STR (R)	STR (T)	BETA (R)
0.	3.1784998E-02	0.	2.1405932E 03	2.1405932E 03	1.000000
1.000	3.1681422E-02	-2.1289620E-04	2.4760778E 03	2.3778661E 03	0.960336
2.000	3.1349010E-02	-4.5630662E-04	2.7363203E 03	2.5731134E 03	0.940355
3.000	3.0761625E-02	-7.2124270E-04	2.9079955E 03	2.7187799E 03	0.934933
4.000	2.9902616E-02	-9.9791799E-04	2.9889473E 03	2.8126867E 03	0.941029
5.000	2.8765047E-02	-1.2768584E-03	2.9857064E 03	2.8569663E 03	0.956881
6.000	2.7350951E-02	-1.5496953E-03	2.9104217E 03	2.8567364E 03	0.981554
7.000	2.5669987E-02	-1.8095908E-03	2.7774815E 03	2.8186240E 03	0.985403
8.000	2.3737854E-02	-2.0512769E-03	2.6001063E 03	2.7492576E 03	0.945749
9.000	2.1574855E-02	-2.2707219E-03	2.3871925E 03	2.6538402E 03	0.899524
10.000	1.9204973E-02	-2.4644697E-03	2.1406887E 03	2.5349192E 03	0.844480
11.000	1.6655729E-02	-2.6287302E-03	1.8537815E 03	2.3914660E 03	0.775165
12.000	1.3959039E-02	-2.7583282E-03	1.5101739E 03	2.2183821E 03	0.680755
13.000	1.1153125E-02	-2.8456548E-03	1.0847352E 03	2.0065458E 03	0.540598
14.000	8.2854158E-03	-2.8797934E-03	5.4580149E 02	1.7435128E 03	0.313047
15.000	5.4161319E-03	-2.8460260E-03	-1.4059042E 02	1.4149881E 03	-0.099358
16.000	2.6221002E-03	-2.7259564E-03	-1.0042623E 03	1.0071805E 03	-0.997103
17.000	4.3715081E-10	-2.4985222E-03	-2.0619548E 03	5.1015766E 02	-0.247415

	MAXIMUM	POSITION		MAXIMUM	POSITION
W	0.3178500E-01	0.	DW/DR	-0.2879793E-02	14.00000
M(R)	0.2140593E 04	0.	STR(R)	0.4899429E 05	0.
M(T)	0.2140593E 04	0.	STR(T)	0.4899429E 05	0.
DW/CR(TUBE)	0.2879793E-02	14.00000	Y(TUBE)	0.1428505E-02	14.00000
M(TUBE)	0.7264889E 03	14.00000	STR(TUBE)	0.5919964E 05	14.00000

C. EXPERIMENTAL DATA

Table 1
MEASURED GRID-PLATE DEFLECTION

Load/Ring (lb)	Deflection Reading (in.)					
	Gage A	Gage B	Gage C	Gage D	Gage E	Gage F
250	0.0018	0.0053	0.0090	0.0016	0.0038	0.0065
500	0.0031	0.0085	0.0156	0.0024	0.0065	0.0115
750	0.0038	0.0114	0.0211	0.0034	0.0091	0.0160
1000	0.0040	0.0135	0.0300	0.0036	0.0094	0.0180

Gage Locations

Gage	A	B	C	D	E	F
Plate	Top	Top	Top	Bottom	Bottom	Bottom
Distance from Edge, in.	5/16	2 $\frac{5}{16}$	4 $\frac{1}{4}$	3/16	1 $\frac{5}{8}$	3 $\frac{3}{8}$

Table 2
AVERAGE EXPERIMENTAL STRAINS

Gage	Experimental Strain Reading (μ in./in.)			
	250 lb/Ring	500 lb/Ring	750 lb/Ring	1000 lb/Ring
1 ALR	-38	-91	-162	-208
1 ALT	41	71	101	110
1 AUR	39	100	160	210
1 AUT	-43	-75	-101	-127
1 BLR	-8	-21	-43	-87
1 BLT	35	50	57	74
1 BUR	5	21	44	89
1 BUT	-34	-58	-82	-87
2 ALR	23	32	33	30
2 ALT	35	65	92	114
2 AUR	-18	-24	-36	-30
2 AUT	-47	-85	-117	-151
2 BLR	12	12	14	16
2 BLT	34	59	74	100
2 BUR	-15	-17	-15	-11
2 BUT	-42	-74	-110	-137
3 ALR	46	78	113	132

Table 2 (Contd.)

Gage	Experimental Strain Reading (μ in./in.)			
	250 lb/Ring	500 lb/Ring	750 lb/Ring	1000 lb/Ring
3 ALT	43	85	115	156
3 AUR	-43	-75	-115	-136
3 AUT	-41	-75	-98	-137
3 BLR	51	89	132	156
3 BLT	55	103	136	180
3 BUR	-53	-90	-130	-150
3 BUT	-42	-74	-110	-137
4 ALR	-26	-65	-122	-148
4 ALT	39	65	91	107
4 AUR	30	72	120	150
4 AUT	-38	-67	-90	-116
4 BLR	-9	-25	-59	-105
4 BLT	23	35	36	55
4 BUR	5	21	71	104
4 BUT	-26	-47	-56	-83
5 ALR	39	63	94	97
5 ALT	33	59	78	100
5 AUR	-39	-64	-90	-100
5 AUT	-36	-65	-94	-120
5 BLR	35	56	77	91
5 BLT	43	70	100	129
5 BUR	-37	-57	-75	-85
5 BUT	-29	-55	-80	-102
6 ALR	46	78	110	138
6 ALT	48	81	123	156
6 AUR	-42	-76	-114	-136
6 AUT	-46	-82	-120	-153
6 BLR	55	98	140	168
6 BLT	51	96	139	172
6 BUR	-51	-94	-137	-156
6 BUT	-50	-89	-131	-163
7 AI	-439	-722	-1021	-1234
7 AO	457	776	1104	1296
7 BI	157	267	388	436
7 BO	-147	-234	-330	-417
8 AI	-335	-606	-910	-1155
8 AO	281	492	751	949
8 BI	109	189	304	411
8 BO	-163	-303	-462	-618
9 AI	-480	-829	-1218	-1446
9 AO	475	826	1190	1387
9 BI	161	279	402	472
9 BO	-176	-296	-434	-554

Table 3
EXPERIMENTAL LOAD DISTRIBUTION IN THE RODS

Load/Ring (lb)	Axial Load in Rods (lb)					
	Rod 7		Rod 8		Rod 9	
	Top	Bottom	Top	Bottom	Top	Bottom
250	53	29	159	159	24	45
500	100	98	336	336	63	57
750	170	170	469	465	83	94
1000	182	57	663	610	174	242

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