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A METHOD OF SOLVING THE TIME DEPENDENT
NEUTRON THERMALIZATION PROBLEM

by

J. U. Koppel

This is a preprint of a paper to be submitted to
Nuclear Science and Engineering

Project 48.01

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ABSTRACT

The method of singular eigenfunctions introduced first by Van Kampen and developed later by Case and Mika in connection with a one velocity transport problem, has been adapted in order to solve the time and energy dependent infinite medium problem. The expansion of neutron density and scattering kernel in series of Hermite functions reduces the Boltzmann equation to a system of homogeneous linear equations. The resulting set of regular and singular eigenfunctions is shown to be complete and explicit formulae are found for the normalization integrals and Green's function.

I. INTRODUCTION

The present paper will be concerned with the time and energy dependent Boltzmann equation describing the approach to thermal equilibrium with a moderator, of a neutron spectrum of arbitrary initial shape. This problem which is of practical interest because of its connection to the well known pulsed neutron diffusion experiments has been considerably clarified in several recent papers.^{1,2,3} A standard method of solving it has been based in the past on the belief that a general solution can be obtained by separation of the energy and time variable. Thus it was assumed that the general solution could be expressed as a sum of terms of the form

$$\phi(E)e^{-\lambda t}$$

representing discrete energy modes or eigenfunctions, each decaying in time with a particular decay constant or eigenvalue λ . A necessary condition for this to be true is that the set of eigenfunctions be complete in order to be able to satisfy the arbitrary initial condition of the problem. However, as it has been shown in the references just mentioned this assumption is generally not correct. It turns out that in fact there is a discrete set of regular eigenfunctions corresponding to a sequence of eigenvalues λ not exceeding a limiting value $\lambda_{\infty} = (\Sigma v)_{\min}$. But, in addition, for $\lambda > \lambda_{\infty}$, there also is a continuum of singular solutions of a special kind, all orthogonal to the discrete set. Thus it is seen that this set is not complete and that besides a linear combination of the regular eigenfunctions, the general solution must contain an essentially non-separable term given by an integral over the singular continuum. These conclusions seem to

hold for all reasonable scattering models, unless $1/\lambda_\infty$, which is the maximum average time between collisions, is assumed to be zero, as for the physically unrealistic model of the heavy gas.

The present paper is a generalization of reference (2). However, for the sake of consistency it seems to be worthwhile to restate here the basic formalism concerning the homogeneous Boltzmann equation. This is done in Section II. In Section III the whole set of regular and singular eigenfunctions is shown to be complete in the interval $0 < E < \infty$. Their normalization integrals are given in Section IV. Finally, the non-homogeneous Boltzmann equation is dealt with in Section V.

II. SOLUTION OF THE HOMOGENEOUS BOLTZMANN EQUATION

The homogeneous Boltzmann equation for the infinite medium reads:

$$\frac{\partial n(w, \tau)}{\partial \tau} = -n(w, \tau) w \left[\Sigma_s(w) + \Sigma_a(w) \right] + \int_0^{\infty} n(w', \tau) w' \Sigma(w' \rightarrow w) dw' \quad (1)$$

where n is the neutron density, τ the time, w the neutron velocity, and $\Sigma(w \rightarrow w')$, Σ_a and Σ_s are the scattering kernel, the absorption and the scattering cross sections of the medium. It is convenient to introduce the non-dimensional quantities

$$\begin{aligned} x &= \frac{w}{w_0} & t &= \tau w_0 \Sigma_{s0} \\ v(x) &= x \frac{\Sigma_s(x) + \Sigma_a(x)}{\Sigma_{s0}} & G(x \rightarrow x') &= x' \frac{\Sigma(w' \rightarrow w)}{\Sigma_{s0}} \end{aligned}$$

w_0 being the most probable velocity of a maxwellian spectrum at room temperature and Σ_{s0} the free atom scattering cross section (at some fixed energy). Eq. (1) then becomes

$$\frac{\partial n(x, t)}{\partial t} = -n(x, t) v(x) + \int_0^{\infty} n(x') G(x' \rightarrow x) dx' \quad (2)$$

We are interested in solutions of this equation of the form

$$n(x, t) = n(x) e^{-\lambda t} \quad (3)$$

which substituted into (2) lead to the homogeneous integral equation

$$n(x) \left[V(x) - \lambda \right] = \int_0^{\infty} n(x') G(x' \rightarrow x) dx' \quad (4)$$

Owing to the principle of detailed balance the kernel $G(x' \rightarrow x)$ can be symmetrized making

$$n(x) = N(x) \sqrt{M(x)} \quad (5)$$

$$G(x' \rightarrow x) = S(x|x') \sqrt{\frac{M(x)}{M(x')}} \quad (6)$$

with

$$M(x) = x^2 e^{-x^2}$$

Then we have

$$N(x) \left[V(x) - \lambda \right] = \int_0^{\infty} N(x') S(x|x') dx' \quad (7)$$

In order to solve this eigenvalue problem let us assume the kernel $S(x|x')$ can be approximated by a truncated double series of functions which are orthonormal in the interval $0 \leq x < \infty$:

$$S(x|x') = \sum_0^K \sum_0^K S_{ij} \mu_i(x) \mu_j(x') \quad (8)$$

or

$$S(x|x') = \sum_0^K f_j(x) \mu_j(x') \text{ with } f_j(x) = \sum_{i=0}^K S_{ij} \mu_i(x) \quad (9)$$

Expanding $N(x)$ in an infinite series of the same functions

$$N(x) = \sum_0^{\infty} \psi_k \mu_k(x) \quad (10)$$

it follows that

$$N(x) = \frac{1}{V(x) - \lambda} \sum_0^K f_j(x) \psi_j \quad (11)$$

A convenient set of μ_i are the (normalized) odd Hermite Polynomials times $e^{-x^2/2}$. For this particular set we have the relation:

$$\int_0^\infty G(x \rightarrow x') dx' = \int_0^\infty \frac{\mu_0(x')}{\mu_0(x)} S(x|x') dx' \quad (12)$$

or else

$$V(x) = \frac{f_0(x)}{\mu_0(x)} + x \frac{\Sigma_a(x)}{\Sigma_{s0}} \quad (13)$$

Furthermore, in the important case of zero or $1/v$ absorption μ_0 becomes a solution of (7) and (11) (see Appendix B). Now we shall assume that in the interval $0 \leq x < \infty$ the function $V(x)$ has a finite minimum value $V_{\min} > 0$. This implies that the total cross section $\Sigma = \Sigma_s + \Sigma_a$ increases at least like $1/v$ as $v \rightarrow 0$ and appears to hold for any actual moderator. The kernel $S(x|x')$ being symmetric the eigenvalues λ must be real and if we are looking only for regular solutions of (7) and (11) it follows that*

$$\lambda \leq V_{\min} \quad (14)$$

Let us first find these regular solutions. Because of the orthogonality of the μ_i :

$$\psi_k = \sum_{j=0}^K \psi_j \int_0^\infty \frac{f_j(x) \mu_k(x)}{V(x) - \lambda} dx = \sum_{j=0}^K K_{kj}(\lambda) \psi_j \quad (15)$$

with:

$$K_{kj}(\lambda) = \sum_{i=0}^K L_{ki}(\lambda) S_{ij} \quad k = 0, 1, \dots, K \quad (16)$$

and

$$L_{ki}(\lambda) = \int_0^\infty \frac{\mu_i \mu_k}{V - \lambda} dx \quad (17)$$

* Generally the minimum of the approximate $V(x)$ as given by (13) approaches the exact $V_{\min}$ for values of K which are either all odd or all even. Only these approximations should be considered here.

In matrix notation equation (16) can be written

$$K(\lambda) = L(\lambda) S \quad (18)$$

For the non trivial solution of the system (15) its determinant must vanish:

$$\det \left[K_{kj}(\lambda) - \delta_{kj} \right] = 0 \quad k, j = 0, 1, \dots, K \quad (19)$$

In the appendix it will be shown that (19) can be satisfied at most by $K+1$ values of λ . If an arbitrary normalization condition (for instance $\psi_1 = 1$) is added to the equations (15), this system can be solved for each one of the eigenvalues λ_n . The eigenfunctions $N_n(x)$ are then obtained by simply substituting the resulting $\psi_0, \psi_1, \dots, \psi_K$ into equation (11).

In the absence of degeneration the eigenfunctions will be orthogonal to each other. To prove it write eq.(7) for two different eigenvalues λ_m and λ_n .

$$N_n(x) \left[V(x) - \lambda_n \right] = \int_0^\infty N_n(x') S(x|x') dx' \quad (20)$$

$$N_m(x) \left[V(x) - \lambda_m \right] = \int_0^\infty N_m(x') S(x|x') dx \quad (21)$$

Multiply (20) by $N_m(x)$ and (21) by $N_n(x)$, then integrate over x and subtract:

$$(\lambda_n - \lambda_m) \int_0^\infty N_n(x) N_m(x) dx = 0 \quad (22)$$

Having assumed $\lambda_n \neq \lambda_m$ it follows that $N_n(x)$ and $N_m(x)$ must be orthogonal to each other.

Now let us investigate what happens for $\lambda > V_{\min}$. In this case there are no regular solutions neither of (7) nor of (11) because of the singularity at $V(x) = \lambda$. However, it still is possible to find a continuum of solutions which are distributions in the sense of L. Schwartz. These singular solutions first introduced by Van Kampen⁴ in connection with a problem of plasma oscillations were used later on by Case⁵ and Mika⁶

for solving the one velocity space dependent Boltzmann equation. In our case and making the not too restrictive assumption that $V(x)$ is a monotonically increasing function* of x , the singular solutions of equation (11) will be:

$$N(x, \lambda) = P \frac{1}{V-\lambda} \sum_{j=0}^K f_j(x) \psi_j(\lambda) + \omega(\lambda) \delta(x-x_\lambda) \quad (23)$$

where $f_j(x)$ and $\psi_j(\lambda)$ have the same meanings as above and x_λ is defined by

$$V(x_\lambda) - \lambda = 0 \quad (24)$$

The notation $P \frac{1}{V-\lambda}$ indicates that when $\frac{1}{V-\lambda}$ appears as a factor of an integrand the Cauchy principal value of the integral is meant. Multiplying equation (23) by $V(x)-\lambda$ and proceeding as before it is easy to show that the singular solutions (23) are orthogonal to each other as well as to the discrete eigenfunctions. It will be shown in section III below that the set of discrete modes complemented with the continuum is complete for the interval $0 \leq x < \infty$.

In order to find the functions $\psi_k(\lambda)$ and $\omega(\lambda)$ multiply equation (23) by $\mu_k(x)$ and integrate. It follows:

$$\psi_k(\lambda) = \sum_{j=0}^K \psi_j(\lambda) K_{kj}(\lambda) + \omega(\lambda) \mu_k(x_\lambda) \quad \text{with } k = 0, 1, \dots, K \quad (25)$$

$K_{kj}(\lambda)$ is now defined as the principal value

$$K_{kj}(\lambda) = \oint_0^\infty \frac{\mu_k(x) f_j(x)}{V(x)-\lambda} dx \quad (26)$$

An arbitrary normalization condition (for instance $\psi_j=1$) added to the $K+1$ equation (25) determines uniquely all $K+2$ variables $\psi_0, \psi_1, \dots, \psi_K$ and ω . Substituting them into (23) we get the singular eigenfunctions $N(x, \lambda)$.

* For the infinite medium only the inelastic part of $\Sigma(w' \rightarrow w)$ and Σ_s has to be considered. Thus even for crystalline moderators the assumption of a monotone $V(x)$ is not unreasonable.

III COMPLETENESS OF THE SET OF REGULAR AND SINGULAR EIGENVECTORS

In order to prove completeness of our set of eigenvectors it is sufficient to show that an arbitrary function $F(x)$, orthogonal to the discrete spectrum, can be expressed as a linear combination of the singular solutions (23).

In the following it is convenient to change to the variable

$$v = V(x) - V(o) \quad (27)$$

Because of the assumed monotonic behavior of $V(x)$, $V(o) = V_{\min}$ and therefore

$$0 \leq v < \infty$$

for

$$0 \leq x < \infty$$

Furthermore let

$$\begin{aligned} h_i(v) &= \mu_i(x) & \phi(\alpha) &= \psi(\lambda) \\ g_i(v) &= f_i(x) & \Omega(\alpha) &= \omega(\lambda) \\ D(v) &= \frac{dv}{dx} & Q_{ij}(\alpha) &= K_{ij}(\lambda) - \delta_{ij} \\ \alpha &= \lambda - V(o) \end{aligned} \quad (28)$$

With these notations the singular solutions of equation (11) become:

$$N(v, \alpha) = P \frac{1}{v - \alpha} \sum_{j=0}^K g_j(v) \phi_j(\alpha) + D(v) \Omega(\alpha) \delta(v - \alpha) \quad (29)$$

while calling the discrete eigenvalues

$$\alpha_n = V(o) - \lambda_n \quad (30)$$

the regular eigenfunctions can be written as

$$N_n(v) = \frac{1}{v + \alpha_n} \sum_{j=0}^K g_j(v) \phi_j(-\alpha_n) \quad (31)$$

The completeness theorem to be proved here requires that if $F(v)$ is an arbitrary function satisfying the orthogonality relations

$$\int_0^{\infty} F(v) N_n(v) \frac{dv}{D(v)} = 0 \quad (32)$$

it should be possible to express $F(v)$ by the integral

$$F(v) = \int_0^{\infty} A(\alpha) N(v, \alpha) d\alpha \quad (33)$$

or,

$$F(v) = \Omega(v) D(v) A(v) - \sum_{j=0}^K g_j(v) \int_0^{\infty} \frac{\phi_j(\alpha) A(\alpha) d\alpha}{\alpha - v} \quad (34)$$

In other words, we must prove that a solution $A(\alpha)$ of this singular integral equation exists for any $F(v)$ satisfying the condition (32).

First let us define the functions of the complex variable z (z not on the positive real axis)

$$A_j(z) = \frac{1}{2\pi i} \int_0^{\infty} \frac{A(t) \phi_j(t) dt}{t - z} \quad (35)$$

which behave like $\frac{\text{const}}{z}$ for $|z| \rightarrow \infty$ and are analytical in the complex plane cut along the positive real axis. For the limits when z approaches the cut from above (+) or below (-) the Plemelj formulae⁷ yield:

$$A_j^+(\alpha) = \frac{1}{2\pi i} \int_0^{\infty} \frac{A(t) \phi_j(t) dt}{t - \alpha} + \frac{1}{2} A(\alpha) \phi_j(\alpha) \quad (36)$$

Hence:

$$A_j^+(\alpha) + A_j^-(\alpha) = \frac{1}{\pi i} \int_0^{\infty} \frac{A(t) \phi_j(t) dt}{t - \alpha} \quad (37)$$

and

$$A_j^+(\alpha) - A_j^-(\alpha) = A(\alpha) \phi_j(\alpha) \quad (38)$$

Similarly, for the sectionally holomorphic function $\theta_{ij}(z)$ defined by:

$$\theta_{ij}(z) = \int_0^\infty \frac{h_i(v) g_j(v)}{v - z} \frac{dv}{D(v)} - \delta_{ij} \quad (39)$$

we find the relations:

$$\theta_{ij}^+(\alpha) + \theta_{ij}^-(\alpha) = 2 \left[\int_0^\infty \frac{h_i(v) g_j(v)}{v - \alpha} \frac{dv}{D(v)} - \delta_{ij} \right] = 2 Q_{ij}(\alpha) \quad (40)$$

$$\theta_{ij}^+(\alpha) - \theta_{ij}^-(\alpha) = 2\pi i \frac{h_i(\alpha) g_j(\alpha)}{D(\alpha)} \quad (41)$$

In our new notation equation (25) reads:

$$\sum_{j=0}^K Q_{nj}(\alpha) \phi_j(\alpha) + \Omega(\alpha) h_n(\alpha) = 0 \quad \text{with } n = 0, 1, \dots, K \quad (42)$$

Substituting this relation into equation (34) we have:

$$- \frac{F(v) h_n(v)}{D(v)} = \sum_{j=0}^K \left\{ A(v) Q_{nj}(v) \phi_j(v) + \frac{h_n(v) g_j(v)}{D(v)} \int_0^\infty \frac{\phi_j(t) A(t) dt}{t - v} \right\} \quad (43)$$

Which can now be written (dropping the argument v):

$$-2 \frac{F h_n}{D} = \sum_{j=0}^K \left[(A_j^+ + A_j^-) (\theta_{nj}^+ + \theta_{nj}^-) + (A_j^+ - A_j^-) (\theta_{nj}^+ - \theta_{nj}^-) \right] \quad (44)$$

In order to solve for A_j^+ we need another set of relations which can be obtained dividing equation (43) by $v - \alpha$, integrating over v and taking the principal value

$$\oint_0^\infty \frac{F(v) h_n(v) dv}{(v-\alpha) D(v)} = \oint_0^\infty \frac{A(v) dv}{v-\alpha} \sum_{j=0}^K \phi_j(v) Q_{nj}(v) + \sum_{j=0}^K \oint_0^\infty \frac{h_n(v) g_j(v) dv}{(v-\alpha) D(v)} \oint_0^\infty \frac{\phi_j(t) A(t) dt}{t-v} \quad (45)$$

If the order of integration is inverted in the last term of the right side of this equation, the Bertrand-Poincare formula⁷ yields:

$$\begin{aligned} \sum_{j=0}^K \oint_0^\infty \frac{h_n(v) g_j(v) dv}{(v-\alpha) D(v)} \oint_0^\infty \frac{\phi_j(t) A(t) dt}{t-v} &= \oint_0^\infty A(t) dt \sum_{j=0}^K \phi_j(t) \oint_0^\infty \frac{h_n(v) g_j(v) dv}{(v-\alpha)(t-v) D(v)} \\ &- \pi^2 A(\alpha) \sum_{j=0}^K h_n(\alpha) g_j(\alpha) \phi_j(\alpha) \frac{1}{D(\alpha)} \end{aligned} \quad (46)$$

Introducing this result into (45) and noting that:

$$\frac{1}{(v-\alpha)(t-v)} = \frac{1}{t-\alpha} \left(\frac{1}{v-\alpha} + \frac{1}{t-v} \right) \quad (47)$$

it follows:

$$- \oint_0^\infty \frac{F(v) h_n(v) dv}{(v-\alpha) D(v)} = \sum_{j=0}^K Q_{nj}(\alpha) \oint_0^\infty \frac{A(v) \phi_j(v) dv}{v-\alpha} - \pi^2 \frac{h_n(\alpha)}{D(\alpha)} \sum_{j=0}^K g_j(\alpha) \phi_j A(\alpha) \quad (48)$$

and thus:

$$- \frac{2}{\pi i} \oint_0^\infty \frac{F(v) h_n(v) dv}{(v-\alpha) D(v)} = \sum_{j=0}^K \left[(A_j^+ + A_j^-)(\theta_{nj}^+ + \theta_{nj}^-) + (A_j^+ - A_j^-)(\theta_{nj}^+ - \theta_{nj}^-) \right] \quad (49)$$

Adding and subtracting (44) and (49) and recalling again the Flemelj formulae we find:

$$\sum_{j=0}^K A_j^\pm \theta_{nj}^\pm = - H_n^\pm \quad \text{with } n = 0, 1, \dots, K \quad (50)$$

where now $H_n(z)$ stands for the sectionally holomorphic function:

$$H_n(z) = \frac{1}{2\pi i} \int_0^\infty \frac{F(t) h_n(t) dt}{(t-z) D(t)} \quad (51)$$

Solving for $A_j^\pm$ we get

$$A_j^\pm = - \frac{\sum_{n=0}^K H_n^\pm \Delta_{nj}^\pm}{\Delta^\pm} \quad (52)$$

where $\Delta^\pm$ are the limiting values for z approaching the cut from above or below of the determinant:

$$\Delta(z) = \det [\theta_{ij}(z)] \quad i, j = 0, 1, \dots, K \quad (53)$$

the $\Delta_{nj}(z)$ being the cofactors corresponding to the n^{th} row and the j^{th} column.

Now let us consider the functions

$$a_j(z) = - \frac{\sum_{n=0}^K H_n(z) \Delta_{nj}(z)}{\Delta(z)} \quad (54)$$

According to their definition the $\theta_{ij}(z)$ and the $H_n(z)$ are analytic in the cut plane and approach the limits

$$\lim_{z \rightarrow \infty} \theta_{ij} = -\delta_{ij} + \frac{\text{const}}{z} \quad (55)$$

$$\lim_{z \rightarrow \infty} H_n(z) = \frac{\text{const}}{z} \quad (56)$$

Therefore it can be seen that the $a_j(z)$ vanish at infinity with the same degree as the $H_n(z)$. Their only singularities, besides the cut along the

positive real axis are poles at the zeros of $\Delta(z)$. But according to the discussion in Section II, $\Delta = 0$ is just the equation (19) which defines the set of discrete eigenvalues α_n^* . In consequence, provided the function $F(t)$ obeys the conditions:

$$\sum_{m=0}^K \Delta_{mj}(-\alpha_n) \int_0^{\infty} \frac{F(t) h_m(t) dt}{(t+\alpha_n) D(t)} = 0 \quad n = 0, 1, \dots, L \quad (57)$$

where L is the number of discrete eigenvalues (j is arbitrary) we can establish the identity:

$$a_j(z) = A_j(z) \quad (58)$$

as in this case both sets of functions will be analytical in the cut plane, satisfy equation (52) and vanish at infinity. Thus we have proved that if the conditions (57) are satisfied it is always possible to find a set of $A_j(z)$ having the analytical properties required by their definition according to equation (35). Therefore the function $A(\alpha)$ exists and can be determined using equation (38). It will be shown in the appendix that the condition (57) and the orthogonality relations (32) are equivalent. This completes our proof of completeness.

* Also see Appendix D.

IV NORMALIZATION INTEGRALS

Owing to the completeness theorem just proved it is possible to represent an arbitrary function $G(v)$ by the expression:

$$G(v) = \sum_{n=0}^L A_n N_n(v) + \int_0^{\infty} A(\alpha) N(v, \alpha) d\alpha \quad (59)$$

where L is the number of discrete eigenvalues and A_n are constants which can be determined using the orthogonality properties of the eigenfunctions $N_n(v)$ and $N(v, \alpha)$

$$A_n R_n = \int_0^{\infty} G(v) N_n(v) \frac{dv}{D(v)} \quad (60)$$

The R_n appearing in this expression are normalization integrals and according to (31)

$$R_n = \int_0^{\infty} N_n^2(v) \frac{dv}{D(v)} = \int_0^{\infty} \left[\frac{\sum_{j=0}^K g_j(v) \phi_j(-\alpha_n)}{v + \alpha_n} \right]^2 \frac{dv}{D(v)} \quad (61)$$

The function $A(\alpha)$ in the second term of (59) can be determined by the expressions found in the previous section. However it is usually much simpler to resort again to the orthogonality of the eigenfunctions writing:

$$\int_0^{\infty} G(v) N(v, \alpha) \frac{dv}{D(v)} = R(\alpha) A(\alpha) \quad (62)$$

Here the normalization integral $R(\alpha)$ can be calculated using the Bertrand-Poincaré formula. After some straightforward algebra it is found:

$$R(\alpha) = \Omega^2(\alpha) D(\alpha) + \frac{\pi^2}{D(\alpha)} \left[\sum_{j=0}^K g_j(\alpha) \phi_j(\alpha) \right]^2 \quad (63)$$

V NONHOMOGENEOUS BOLTZMANN EQUATION

The solution of the nonhomogeneous equation:

$$\frac{\partial N(v, t)}{\partial t} = -N(v, t) V(v) + \int_0^{\infty} N(v, t) S(v|v') dv' + Q(v, t) \quad (64)$$

can always be expressed as:

$$N(v, t) = \int_0^t \int_0^{\infty} N_G(v, v_0, t_0) Q(v_0, t-t_0) dv_0 dt_0 \quad (65)$$

where $N_G(v, v_0, t)$ is called Green's function and is the solution of:

$$\frac{\partial N_G(v, v_0, t)}{\partial t} = -N_G(v, v_0, t) V(v) + \int_0^{\infty} N_G(v, v_0, t) S(v|v') dv' + \delta(v-v_0) \delta(t) \quad (66)$$

The only assumption made is that $Q(v, t) = N(v, t) = 0$ for $t < 0$. If we integrate equation (66) over the time interval $-\epsilon \leq t \leq \epsilon$ it is:

$$\lim_{\epsilon \rightarrow 0} N_G(v, v_0, \epsilon) = \delta(v - v_0) \quad (67)$$

Now because of the completeness theorem proved in Section III we can make the expansion:

$$N_G(v, v_0, t) = e^{-V(v)t} \left[\sum_{n=0}^L A_n(v_0) N_n(v) e^{+\alpha_n t} + \int_0^{\infty} A(\alpha, v_0) N(v, \alpha) e^{-\alpha t} d\alpha \right] \quad (68)$$

and determine $A_n(v_0)$ and $A(\alpha, v_0)$ from the initial condition (67) using the formulae of Section IV with

$$G(v) = \delta(v - v_0) \quad (69)$$

The final result is:

$$N_G(v, v_o, t) = \left[\sum_{n=0}^L \frac{N_n(v_o) N_n(v)}{R_n} e^{\alpha_n t} + \int_0^{\infty} \frac{N(v_o, \alpha) N(v, \alpha)}{R(\alpha)} e^{-\alpha t} d\alpha \right] \frac{e^{-v(o)t}}{D(v_o)} \quad (70)$$

VI CONCLUSIONS

It has been shown that in general the time and energy dependent Boltzmann equation can be solved by separation of the variables only if a continuum of singular eigenvectors is introduced. Besides this continuum there also is a discrete spectrum of regular eigenfunctions. All regular and singular eigenvectors are orthogonal to each other. When the scattering kernel is approximated by $K+1$ terms of a double expansion in orthogonal functions the number of regular eigenfunctions is at most $K+1$. The value $\lambda = (\Sigma w)_{\min}$ separates the discrete spectrum from the continuum. The completeness of the whole set of eigenvectors has been proved for an arbitrary value of K .

Concerning the accuracy of the present method, it should be emphasized that only the expansion of the scattering kernel is truncated after a finite number of terms. The solution corresponding to any truncated kernel is exact.

APPENDIX A

NUMBER OF DISCRETE EIGENVALUES ($\lambda < v_{\min}$)

Let us assume the symmetrized kernel $S(x|x')$ is positive definite (hence the truncated kernel (9) has the same property) and consider the eigenvalue problem

$$X(x) = \frac{\gamma}{V(x) - \lambda} \int_0^{\infty} S(x|x') X(x') dx \quad (A-1)$$

where λ is just a parameter, γ now being the eigenvalue. If $S(x|x')$ is again approximated by the truncated kernel, equation (A-1) becomes equivalent to the system

$$\eta_i = \sum_{j=0}^K K_{ij}(\lambda) \eta_j \quad \text{with } i = 0, 1, \dots, K \quad (A-2)$$

$$\text{where } \eta_j = \int_0^{\infty} X(x) \mu_j(x) dx$$

It is known that there are at most $K+1$ values γ_i which satisfy equation (A-2).

For $\lambda = 0$ the lowest one must be $\left[\gamma_0 \right]_{\lambda=0} \geq 1 \quad (A-3)$

for

$$\Sigma_a \geq 0$$

For all eigenvalues it is

$$\gamma_i = \frac{\int_0^{\infty} X_i^2(x) [V(x) - \lambda] dx}{\int_0^{\infty} \int_0^{\infty} X_i(x) X_i(x') S(x|x') dx dx'} \quad (A-4)$$

and from the variational principle

$$\left[\delta \gamma_i \right]_{\lambda=\text{const}} = 0 \quad (\text{A-5})$$

it follows that

$$\delta \gamma_i = - \frac{\int_0^\infty x_i^2(x) dx}{\int_0^\infty \int_0^\infty x_i(x) x_i(x') S(x|x') dx dx'} \delta \lambda \quad (\text{A-6})$$

Hence the γ_i decreases monotonically for increasing values of λ , and therefore for each γ_i there can be only one λ_i such that $\gamma_i = 1$. It follows that $K+1$ is an upper bound for the number N_Δ of discrete eigenvalues λ_i corresponding to the truncated kernel (9).

Another interesting relation concerning the number N_Δ can be derived from the fact that the function $\Delta(z)$ (see equation 53) is analytic in the cut plane and approaches the constant limit

$$\lim_{z \rightarrow \infty} \Delta(z) = (-1)^{K+1} \quad (\text{A-7})$$

Thus the number of its roots within the contour of Figure 1 is equal to the integrated change of its argument after one complete turn along this contour divided by $2\pi i$ (it is assumed that $\Delta^+(v) \neq 0$ for $0 \leq v < \infty$. See Appendix D for the case when this condition is violated). Owing to equation (A-7) and due to the fact that

$$\lim_{v \rightarrow 0} \Delta^+(v) = \lim_{v \rightarrow 0} \Delta^-(v) \quad (\text{A-8})$$

the change of argument along the arcs DA and BC vanishes as $R \rightarrow \infty$ and $r \rightarrow 0$.

Thus we are left only with the contributions along the cut:

$$N_\Delta = \frac{1}{2\pi i} \left\{ \int_0^\infty d \left[\arg \Delta^+(x) \right] - \int_0^\infty d \left[\arg \Delta^-(v) \right] \right\} \quad (\text{A-9})$$

From the definition of $\Delta(z)$ and the $\theta_{ij}(z)$ it follows that Δ^+ and Δ^- are complex conjugates. This leads to the final result:

$$N_{\Delta} = \frac{1}{\pi i} \int_0^{\infty} d \left[\arg \Delta^+(v) \right] \quad (\text{A-10})$$

APPENDIX B

1/v Absorption ($\Sigma_a = \frac{\beta}{x}$)

As 1/v absorption occurs quite frequently it seems worthwhile to point out some simplifications which can be made in this case.

With the choice of the set of μ_i made in Section II, the lowest regular eigenfunction now becomes

$$N_o(x) = \mu_o(x) \quad (B-1)$$

Hence eq. (22) yields

$$(\lambda_n - \lambda_o) \psi_o = 0 \quad (B-2)$$

where $1/\lambda_o$ is known to be the neutron lifetime in the medium. Eq. (B-2) means that only $N_o(x)$ has a component ψ_o . For all the other regular and singular eigenvectors $\psi_o = 0$. This reduces by one the number of variables of the systems (15) and (25). However, it is easily verified that all K+1 equations of both systems are still compatible. In fact from the definition of the K_{ij} now it follows that

$$\sum_{i=0}^K (K_{ij} - \delta_{ij}) S_{oi} = (\lambda - \lambda_o) K_{oj} \quad \text{for } j=0, 1, \dots, K \quad (B-3)$$

On the other hand eq. (24) becomes equivalent to

$$\sum_{j=0}^K S_{oj} \mu_j(x_\lambda) = (\lambda - \lambda_o) \mu_o(x_\lambda) \quad (B-4)$$

From these relations it is seen that when $\psi_o = 0$ the first equation of (25) becomes a linear combination of the remaining K equations. The same applies to the system (15).

APPENDIX C

Equivalence of conditions (32) and (57). Let us write eq. (57) for $j=0$ in the form:

$$\begin{vmatrix} H_0 & \theta_{01} & \dots & \theta_{0K} \\ H_1 & \theta_{11} & \dots & \theta_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ H_K & \theta_{K1} & \dots & \theta_{KK} \end{vmatrix} = 0 \quad (C-1)$$

where, for simplicity we have dropped the argument $z = -\alpha_n$ of H_i and θ_{ij} .

As

$$\Delta(-\alpha_n) = 0 \quad (C-2)$$

it is seen that if (57) holds for $j=0$, it must hold for $j=1, 2, \dots, K$ too. Now let us multiply eq. (C-1) by $\det [S_{ij}]$

$$\begin{vmatrix} S_{00} & S_{01} & \dots & S_{0K} \\ S_{10} & S_{11} & \dots & S_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ S_{K0} & S_{K1} & \dots & S_{KK} \end{vmatrix} \cdot \begin{vmatrix} H_0 & \theta_{01} & \dots & \theta_{0K} \\ H_1 & \theta_{11} & \dots & \theta_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ H_K & \theta_{K1} & \dots & \theta_{KK} \end{vmatrix} = \begin{vmatrix} G_0 & T_{01} & \dots & T_{0K} \\ G_1 & T_{11} & \dots & T_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ G_K & T_{K1} & \dots & T_{KK} \end{vmatrix} = 0 \quad (C-3)$$

where

$$G_i = \sum_{j=0}^K S_{ij} H_j = \int_0^{\infty} \frac{F(v) g_i(v) dv}{(v + \alpha_n) D(v)} \quad (C-4)$$

and

$$T_{ij} = \sum_{k=0}^K S_{ik} \theta_{kj} \quad (C-5)$$

The T_{ij} are the elements of the matrix

$$T = S\theta \quad (C-6)$$

and from eq. (18) and the definition of the θ_{ij} it follows that

$$T = SIS - SE \quad (C-7)$$

where E is the unit matrix. S and L being symmetric it is found

$$T_{ij} = T_{ji} \quad (C-8)$$

Hence (C-3) is equivalent to

$$\begin{vmatrix} G_o & G_1 & \dots & G_K \\ T_{1o} & T_{11} & \dots & T_{1K} \\ \dots & \dots & \dots & \dots \\ T_{Ko} & T_{K1} & \dots & T_{KK} \end{vmatrix} = 0 \quad (C-9)$$

Now let us calculate the product $P_o = \Delta_{oo} \det [S_{ij}]$:

$$P_o = \begin{vmatrix} S_{oo} & S_{o1} & \dots & S_{oK} \\ S_{1o} & S_{11} & \dots & S_{1K} \\ \dots & \dots & \dots & \dots \\ S_{Ko} & S_{K1} & \dots & S_{KK} \end{vmatrix} \cdot \begin{vmatrix} 1 & \theta_{o1} & \dots & \theta_{oK} \\ 0 & \theta_{11} & \dots & \theta_{1K} \\ \dots & \dots & \dots & \dots \\ 0 & \theta_{K1} & \dots & \theta_{KK} \end{vmatrix} = \begin{vmatrix} S_{oo} & T_{o1} & \dots & T_{oK} \\ S_{1o} & T_{11} & \dots & T_{1K} \\ \dots & \dots & \dots & \dots \\ S_{Ko} & T_{K1} & \dots & T_{KK} \end{vmatrix} \quad (C-10)$$

In general calling $\tau = \det [T_{ij}]$ and denoting with τ_{ij} the cofactor corresponding to its element T_{ij} it is

$$P_\mu = \Delta_{o\mu} \det [S_{ij}] = \sum_{i=0}^K S_{io} \tau_{i\mu} \quad (C-11)$$

and therefore

$$\frac{P_\mu}{P_\nu} = \frac{\Delta_{o\mu}}{\Delta_{o\nu}} = \frac{\tau_{o\mu}}{\tau_{o\nu}} \frac{\sum_{i=0}^K S_{io} \frac{\tau_{i\mu}}{\tau_{o\mu}}}{\sum_{j=0}^K S_{jo} \frac{\tau_{j\nu}}{\tau_{o\nu}}} \quad (C-12)$$

Owing to the fact that

$$\tau = \Delta \det [S_{ij}] = 0 \quad (C-13)$$

it is

$$\frac{\tau_{i\mu}}{\tau_{j\mu}} = \frac{\tau_{i\nu}}{\tau_{j\nu}} \quad (C-14)$$

and introducing this relation into (C-12)

$$\frac{\Delta_{o\mu}}{\Delta_{ov}} = \frac{\tau_{o\mu}}{\tau_{ov}} \quad (C-15)$$

It follows that eq. (C-9) is equivalent to

$$\begin{vmatrix} G_o & G_1 & \dots & G_K \\ \theta_{1o} & \theta_{11} & \dots & \theta_{1K} \\ \theta_{Ko} & \theta_{K1} & \dots & \theta_{KK} \end{vmatrix} = 0$$

Now let us compare this result with eq. (32), which according to (31) and (C-4) can be written

$$\sum_{j=0}^K G_j (-\alpha_n) \phi_j (-\alpha_n) = 0 \quad (C-17)$$

As $\phi_j (-\alpha_n) \equiv \psi_j (\lambda_n)$ is a solution of the system (15) and noting that $\theta_{ij} (-\alpha_n) \equiv K_{ij} (\lambda_n)$ we have (dropping again the argument $-\alpha_n$)

$$\frac{\phi_j}{\phi_i} = \frac{\Delta_{oj}}{\Delta_{oi}}$$

Taking into account this relation it is seen immediately that (C-17) and (C-16), and therefore (32) and (57) are equivalent.

APPENDIX D

Throughout this paper it was assumed that

$$\Delta^+_{-}(\alpha) \neq 0 \quad (D-1)$$

for $0 \leq \alpha \leq \infty$

However, for special values of the S_{ij} it could happen that simultaneously

$$\sum g_i(\alpha_c) \phi_i(\alpha_c) = 0 \quad (D-2)$$

and

$$\det [Q_{ij}(\alpha_c)] = 0 \quad (D-3)$$

for some $0 < \alpha_c < \infty$. It can be shown that this pair of equations is equivalent to $\Delta^+_{-}(\alpha_c) = 0$. It follows from them that

$$\Omega(\alpha_c) = 0 \quad (D-4)$$

and furthermore that an integral over v of the expression

$$\frac{1}{v - \alpha_c} \sum_{j=0}^K g_j(v) \phi_j(\alpha_c)$$

will now be defined in the ordinary sense. Thus in this case it is seen from eq. (29) that there would be a regular eigenfunction $N(x, \alpha_c)$ with $\alpha = \alpha_c$ within the continuum of singular solutions. Being a regular solution of the original Boltzmann equation, the expression

$$e^{-[V(0) + \alpha_c]t} N(x, \alpha_c)$$

would describe a persistent infinite medium spectrum decaying at a rate

$$-\frac{dn}{ndt} = V(0) + \alpha_c.$$

This rate, however, does not seem physically possible as it exceeds the assumed minimum collision rate $V_{\min} = V(0)$. Consequently it is believed that the condition (D-1) actually holds for all physically realistic scattering kernels. Nevertheless, if it were violated the present method would still apply. Only one additional orthogonality condition (eq. 32) for each additional discrete modes within the continuum would now be required. Obviously there would be as many more terms in the sums of eq. (59) and (70).

$$Z = v + iu$$

Figure 1

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